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Convergence of adaptive FEM for elliptic obstacle problems

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We treat the convergence of adaptive lowest-order FEM for some elliptic obstacle problem with affine obstacle. For error estimation, we use a residual error estimator which is an extended version of the estimator from [2] and additionally controls the data oscillations. The main result states that an appropriately weighted sum of energy error, edge residuals, and data oscillations satisfies a contraction property that leads to convergence. In addition, we discuss the generalization to the case of inhomogeneous Dirichlet data and non-affine obstacles $\chi \in H^2(\Omega)$ for which similar results are obtained.

1 Introduction and Model Problem

In the past decades, adaptive finite element methods for elliptic boundary value problems have been intensively studied and are now a popular tool in science and engeneering, see [1] and the references therein. In recent years, the analysis has been extended to cover more general applications, such as mixed methods, non-conforming elements, and obstacle problems [2]. The latter is a classic introductory example to study nonlinear problems characterized by variational inequalities. The aim of our work is twofold: First, we provide a numerical scheme for variational inequalities that arise from many physical phenomena [5]. Second, by extending the mathematical analysis to new problems, we contribute to the understanding of the method itself.

Throughout, we consider the following model problem: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary $\Gamma := \partial\Omega$. We prescribe an obstacle on $\overline{\Omega}$ by an affine function χ with $\chi \leq 0$ on Γ. The set $\mathcal{A}$ of admissible functions reads

$$\mathcal{A} := \{v \in H^1_0(\Omega) \ : \ v \geq \chi \text{ a.e. in } \Omega\}. \tag{1}$$

It is closed, convex, and non-empty. For given $f \in L^2(\Omega)$, we consider the energy functional $\mathcal{J}(v) = \langle\!\langle v \,,\, v\rangle\!\rangle/2 - (f, v)$, where the energy scalar product reads $\langle\!\langle u \,,\, v\rangle\!\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx$ for all $u, v \in H^1_0(\Omega)$ and where $(f, v) = \int_\Omega fv \, dx$ denotes the L^2-scalar product. By $|\!|\!| \cdot |\!|\!|$, we denote the energy norm on $H^1_0(\Omega)$ induced by $\langle\!\langle \cdot \,,\, \cdot\rangle\!\rangle$. The obstacle problem then reads: *Find $u \in \mathcal{A}$ such that*

$$\mathcal{J}(u) = \min_{v \in \mathcal{A}} \mathcal{J}(v). \tag{2}$$

It is well known, that this problem admits a unique solution that is equivalently characeterized by the variational inequality

$$\langle\!\langle u \,,\, u - v\rangle\!\rangle \leq (f, u - v) \quad \text{for all } v \in \mathcal{A}. \tag{3}$$

For discretization of (3), we consider conforming and shape regular triangulations $\mathcal{T}_\ell$ of Ω and denote the standard P1-FEM space of globally continuous and piecewise affine functions by $\mathcal{S}^1(\mathcal{T}_\ell)$. The finite dimensional problem then reads: *Find $U_\ell \in \mathcal{A}_\ell := \mathcal{A} \cap \mathcal{S}^1(\mathcal{T}_\ell)$ such that* $\mathcal{J}(U_\ell) = \min_{V_\ell \in \mathcal{A}_\ell} \mathcal{J}(V_\ell)$. Again, this problem can equivalently be stated in terms of a variational inequality (3).

2 Reliable Error Estimator and Convergence of adaptive FEM

Now, let $\mathcal{E}_\ell^\Omega$ (resp. $\mathcal{E}_\ell$) denote the set of all interior (resp. all) edges of $\mathcal{T}_\ell$. For $E \in \mathcal{E}_\ell^\Omega$, the patch is defined by $\Omega_{\ell,E} := T^+ \cup T^-$ with $T^\pm \in \mathcal{T}_\ell$ and $T^+ \cap T^- = E$. To steer the adaptive mesh-refinement, we use some residual-based error estimator that has basically been introduced in [2]

$$\eta_\ell^2 := \rho_\ell^2 + \mathrm{osc}_\ell^2 \quad \text{with} \quad \rho_\ell^2 = \sum_{E \in \mathcal{E}_\ell^\Omega} \rho_\ell(E)^2 \quad \text{and} \quad \mathrm{osc}_\ell^2 = \sum_{E \in \mathcal{E}_\ell} \mathrm{osc}_\ell(E)^2. \tag{4}$$

First, $\rho_\ell(E)^2 := h_E \|[\partial_n U_\ell]\|^2_{L^2(E)}$ for $E \in \mathcal{E}_\ell$ denotes the weighted L^2-norm of the normal jump, where $h_E = \mathrm{diam}(E)$ and $[\cdot]$ the jump over an interior edge. Second, $\mathrm{osc}_\ell(E)^2 := |\Omega_{\ell,E}| \|f - f_{\Omega_{\ell,E}}\|^2_{L^2(\Omega_{\ell,E})}$ are the oscillations of f over E, for

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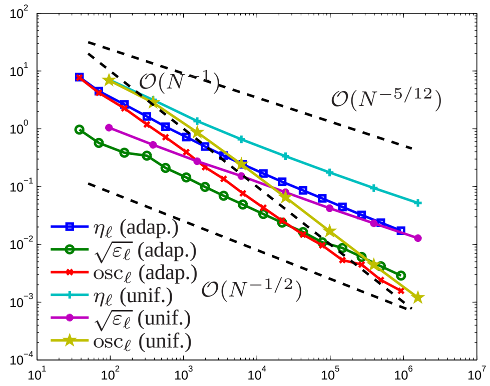

Fig. 1 Numerical results for uniform and adaptive mesh refinement with adaptivity parameter $\theta = 0.6$

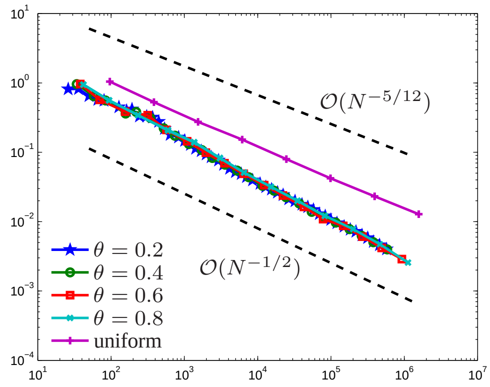

Fig. 2 Numerical results for $\sqrt{\varepsilon_\ell}$ for uniform and adaptive mesh refinement with $\theta \in \{0.2, 0.4, 0.6, 0.8\}$

$E \in \mathcal{E}_\ell$, where $f_{\Omega_{\ell,E}}$ denotes the corresponding integral mean. Finally, for edges E on the boundary, η_ℓ involves the weighted element residuals $\mathrm{osc}_\ell(E)^2 := |T| \|f\|_{L^2(T)^2}$ for $E \in \mathcal{E}_\ell \backslash \mathcal{E}_\ell^\Omega$, where $T \in \mathcal{T}_\ell$ is the unique element with $E \subseteq \partial T \cap \Gamma$. It is already observed in [2] that η_ℓ is reliable.

We can now state our main result from [6] for a standard P1-AFEM algorithm of the form

$$\boxed{\texttt{Solve}} \longmapsto \boxed{\texttt{Estimate}} \longmapsto \boxed{\texttt{Mark}} \longmapsto \boxed{\texttt{Refine}}$$

Theorem 2.1 *Using the strategy proposed by Dörfler [4] for marking, i.e. determine (minimal) set $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell$ s.t.*

$$\theta \eta_\ell^2 \le \sum_{E \in \mathcal{E}_\ell^\Omega \cap \mathcal{M}_\ell} \rho_\ell(E)^2 + \sum_{E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell} \mathrm{osc}_\ell(E)^2 \tag{5}$$

for some fixed adaptivity parameter $0 < \theta < 1$ and halving at least the marked edges $E \in \mathcal{M}_\ell$, the adaptive algorithm guarantees the contraction property

$$\Delta_{\ell+1} \le \kappa\, \Delta_\ell \quad \textit{for all } \ell \in \mathbb{N}, \quad \textit{where } \Delta_\ell := \mathcal{J}(U_\ell) - \mathcal{J}(u) + \gamma \eta_\ell^2. \tag{6}$$

The constants $0 < \gamma, \kappa < 1$ depend only on θ and the shape of elements in $\mathcal{T}_0$. In particular, this implies $\lim_{\ell\to\infty} \mathcal{J}(U_\ell) = \mathcal{J}(u)$ as well as $\lim_{\ell\to\infty} \|u - U_\ell\| = 0 = \lim_{\ell\to\infty} \eta_\ell$.

Remark 2.2 In the case of non-homogeneous Dirichlet boundary data or non-affine obstacles $\chi \in H^2(\Omega)$, we get the slightly weaker result $\widetilde{\Delta}_{\ell+1} \le \kappa\, \widetilde{\Delta}_\ell + \alpha_\ell$ for a certain zero sequence $\alpha_\ell \ge 0$ with $\lim_\ell \alpha_\ell = 0$. Elementary calculus then also proves $\lim \widetilde{\Delta}_\ell = 0$. Here, $\widetilde{\Delta}_\ell$ denotes a similar combined error quantity that additionally involves estimator terms that control the approximation of the given Dirichlet data, see [7].

3 Numerical Experiment

We consider an example from [2,6] with constant obstacle $\chi \equiv 0$ on the L-shaped domain $\Omega := (-2,2)^2 \backslash [0,2) \times (-2,0]$ with a corner singularity at the origin. In Figure 1, we compare error $\varepsilon_\ell := \big(\mathcal{J}(U_\ell) - \mathcal{J}(u)\big)$, estimator η_ℓ, and oscillations osc_ℓ of uniform and adaptive refinement for $\theta = 0.6$. Figure 2 additionally shows a comparison of the errors of adaptive refinement, where θ varies between 0.2 and 0.8, and uniform refinement. We can see that the convergence rate for adaptive refinement almost coincides for all choices of θ.

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