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Convergence of adaptive FEM for elliptic obstacle problems

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We treat the convergence of adaptive lowest-order FEM for some elliptic obstacle problem with affine obstacle. For error estimation, we use a residual error estimator which is an extended version of the estimator from [2] and additionally controls the data oscillations. The main result states that an appropriately weighted sum of energy error, edge residuals, and data oscillations satisfies a contraction property that leads to convergence. In addition, we discuss the generalization to the case of inhomogeneous Dirichlet data and non-affine obstacles $\chi \in H^2(\Omega)$ for which similar results are obtained.

1 Introduction and Model Problem

In the past decades, adaptive finite element methods for elliptic boundary value problems have been intensively studied and are now a popular tool in science and engineering, see [1] and the references therein. In recent years, the analysis has been extended to cover more general applications, such as mixed methods, non-conforming elements, and obstacle problems [2]. The latter is a classic introductory example to study nonlinear problems characterized by variational inequalities. The aim of our work is twofold: First, we provide a numerical scheme for variational inequalities that arise from many physical phenomena [5]. Second, by extending the mathematical analysis to new problems, we contribute to the understanding of the method itself.

Throughout, we consider the following model problem: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary $\Gamma := \partial \Omega$. We prescribe an obstacle on $\Omega$ by an affine function $\chi$ with $\chi \leq 0$ on $\Gamma$. The set $\mathcal{A}$ of admissible functions reads

\begin{equation}
\mathcal{A} := \{ v \in H^1_0(\Omega) : v \geq \chi \text{ a.e. in } \Omega \}.
\end{equation}

It is closed, convex, and non-empty. For given $f \in L^2(\Omega)$, we consider the energy functional $\mathcal{J}(v) = \langle v, v \rangle / 2 - (f, v)$, where the energy scalar product reads $\langle u, v \rangle = \int_\Omega \nabla u \cdot \nabla v \, dx$ for all $u, v \in H^1_0(\Omega)$ and where $(f, v) = \int_\Omega fv \, dx$ denotes the $L^2$-scalar product. By $\| \cdot \|$, we denote the energy norm on $H^1_0(\Omega)$ induced by $\langle \cdot, \cdot \rangle$. The obstacle problem then reads: Find $u \in \mathcal{A}$ such that

\begin{equation}
\mathcal{J}(u) = \min_{v \in \mathcal{A}} \mathcal{J}(v).
\end{equation}

It is well known, that this problem admits a unique solution that is equivalently characterized by the variational inequality

\begin{equation}
\langle u, v - u \rangle \leq (f, u - v) \quad \text{for all } v \in \mathcal{A}.
\end{equation}

For discretization of (3), we consider conforming and shape regular triangulations $\mathcal{T}_h$ of $\Omega$ and denote the standard P1-FEM space of globally continuous and piecewise affine functions by $S^1(\mathcal{T}_h)$. The finite dimensional problem then reads: Find $U_h \in \mathcal{A}_h := \mathcal{A} \cap S^1(\mathcal{T}_h)$ such that $\mathcal{J}(U_h) = \min_{V_h \in \mathcal{A}_h} \mathcal{J}(V_h)$. Again, this problem can equivalently be stated in terms of a variational inequality (3).

2 Reliable Error Estimator and Convergence of adaptive FEM

Now, let $\mathcal{E}^I_h$ (resp. $\mathcal{E}^o_h$) denote the set of all interior (resp. all) edges of $\mathcal{T}_h$. For $E \in \mathcal{E}^I_h$, the patch is defined by $\Omega_{h,E} := T^+ \cup T^-$ with $T^+ \in \mathcal{T}_h$ and $T^+ \cap T^- = E$. To steer the adaptive mesh-refinement, we use some residual-based error estimator that has basically been introduced in [2]

\begin{equation}
\eta^2_h = \rho^2_h + \text{osc}^2_h \quad \text{with} \quad \rho^2_h = \sum_{E \in \mathcal{E}^I_h} \rho(E)^2 \quad \text{and} \quad \text{osc}^2_h = \sum_{E \in \mathcal{E}^o_h} \text{osc}(E)^2.
\end{equation}

First, $\rho(E)^2 := h_E \| \partial_h U_h \|_{L^2(E)}^2$ for $E \in \mathcal{E}_I$ denotes the weighted $L^2$-norm of the normal jump, where $h_E = \text{diam}(E)$ and $[\cdot]$ the jump over an interior edge. Second, $\text{osc}(E)^2 := \| \Omega_{h,E} \| f - f_{\Omega_{h,E}} \|_{L^2(\Omega_{h,E})}^2$ are the oscillations of $f$ over $E$, for

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E ∈ ℰₜ, where f₀,ₜ,E denotes the corresponding integral mean. Finally, for edges E on the boundary, ηₑ involves the weighted element residuals oscₑ(E)² := |T|∥f∥₁(E,T)² for E ∈ ℰₜ∩ℰ₁, where T ∈ ℰₑ is the unique element with E ⊆ ∂T ∩ Γ. It is already observed in [2] that ηₑ is reliable.

We can now state our main result from [6] for a standard P₁-AFEM algorithm of the form

\[ \text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine} \]

**Theorem 2.1** Using the strategy proposed by Dörfler [4] for marking, i.e. determine (minimal) set ℳₜ ⊆ ℰₜ s.t.

\[ \theta\etaₜ² ≤ \sum_{E \in \mathcal{E} ∈ \mathcal{M}_t} \rho(E)² + \sum_{E \in \mathcal{E} \cap \mathcal{M}_t} \text{osc}_E(E)² \]  \hspace{1cm} (5)

for some fixed adaptivity parameter 0 < θ < 1 and halving at least the marked edges E ∈ ℳₜ, the adaptive algorithm guarantees the contraction property

\[ \Deltaₜ₊₁ ≤ \gamma \Deltaₜ \quad \text{for all} \quad \ell \in \mathbb{N}, \]  \hspace{1cm} (6)

The constants 0 < γ, κ < 1 depend only on θ and the shape of elements in ℰ₀. In particular, this implies \( \lim_{\ell \to \infty} \mathcal{J}(Uₜ) = \mathcal{J}(u) \) as well as \( \lim_{\ell \to \infty} \|u - Uₜ\|₀ = \lim_{\ell \to \infty} \etaₜ \).

**Remark 2.2** In the case of non-homogeneous Dirichlet boundary data or non-affine obstacles χ ∈ H²(Ω), we get the slightly weaker result \( \Deltaₜ₊₁ ≤ \gamma \Deltaₜ + \alphaₜ \) for a certain zero sequence \( \alphaₜ \geq 0 \) with \( \lim_{\ell \to \infty} \alphaₜ = 0 \). Elementary calculus then also proves \( \lim \Deltaₜ = 0 \). Here, \( \Deltaₜ \) denotes a similar combined error quantity that additionally involves estimator terms that control the approximation of the given Dirichlet data, see [7].

### 3 Numerical Experiment

We consider an example from [2,6] with constant obstacle \( χ \equiv 0 \) on the L-shaped domain \( \Omega := (-2, 2)² \setminus [0, 2) × (-2, 0] \) with a corner singularity at the origin. In Figure 1, we compare error \( \varepsilonₜ := (\mathcal{J}(Uₜ) - \mathcal{J}(u)) \), estimator \( \etaₜ \), and oscillations \( \text{osc}_E \) of uniform and adaptive refinement for \( \theta = 0.6 \). Figure 2 additionally shows a comparison of the errors of adaptive refinement, where \( \theta \) varies between 0.2 and 0.8, and uniform refinement. We can see that the convergence rate for adaptive refinement almost coincides for all choices of \( \theta \).

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