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Convergence and quasi-optimality of adaptive FEM with inhomogeneous Dirichlet data

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We consider an adaptive mesh-refining algorithm for a lowest-order finite element method (AFEM). Contrary to prior works, where the Poisson equation with homogeneous Dirichlet data is analyzed, our focus is on the case of inhomogeneous Dirichlet data $g \neq 0$. As is usually done in practice, we use nodal interpolation to discretize g. Besides convergence of AFEM, which is proven by means of an appropriate contraction quantity, we also discuss quasi-optimality of the proposed algorithm.

1 Short history of AFEM

A posteriori error estimation and related adaptive mesh-refinement have themselves proven to be effective tools in scientific computing. Although adaptive mesh-refining algorithms for FEM have successfully been used for the last decades, the mathematical understanding of convergence and of the empirically observed quasi-optimal convergence behaviour has only recently been achieved. In the literature, mainly the homogeneous Dirichlet problem

$$-\Delta u = f \text{ in } \Omega \quad \text{subject to homogeneous Dirichlet conditions} \quad u = 0 \text{ on } \Gamma = \partial\Omega \tag{1}$$

has been considered, and adaptivity is driven by a residual error estimator

$$\rho_\ell^2 = \sum_{T \in \mathcal{T}_\ell} \rho_\ell(T)^2 \quad \text{with} \quad \rho_\ell(T)^2 = |T|\,\|f\|_{L^2(T)}^2 + |T|^{1/2}\|[\partial_n U_\ell]\|_{L^2(\partial T\cap\Omega)}^2. \tag{2}$$

Here, $\mathcal{T}_\ell$ denotes the regular triangulation of Ω in the ℓ-th step of the adaptive loop, U_ℓ is the corresponding lowest-order FE solution, and $[\cdot]$ denotes the jump over interior edges of $\mathcal{T}_\ell$. Moreover, we stress the scaling $|T|^{1/2} \simeq \mathrm{diam}(T)$ so that, up to shape regularity, this definition coincides with those found in the literature, cf. e.g. [2].

The first convergence result for an algorithm of the type

$$\boxed{\text{solve}} \longrightarrow \boxed{\text{estimate}} \longrightarrow \boxed{\text{mark}} \longrightarrow \boxed{\text{refine}} \tag{3}$$

has been derived by DÖRFLER [5]. He proved that, provided the initial mesh $\mathcal{T}_0$ satisfies $\|h_0 f\|_{L^2(\Omega)} \le \varepsilon$, the adaptive algorithm leads to convergence $\lim_\ell |||u - U_\ell||| \lesssim \varepsilon$ within the energy norm $|||\cdot||| = \|\nabla(\cdot)\|_{L^2(\Omega)}$.

His result has been generalized by MORIN, NOCHETTO, and SIEBERT [6]. Using newest vertex bisection, where marked elements are refined by five bisections, they proved

$$|||u - U_{\ell+1}|||^2 \le \kappa\,|||u - U_\ell|||^2 + \mathrm{osc}_{\Omega,\ell}^2 \text{ with } \mathrm{osc}_{\Omega,\ell}^2 = \sum_{T\in\mathcal{T}_\ell} \mathrm{osc}_{\Omega,\ell}(T)^2, \quad \mathrm{osc}_{\Omega,\ell}(T)^2 = |T|\|f - f_T\|_{L^2(T)}^2 \tag{4}$$

with some constant $0 < \kappa < 1$ and $f_T = |T|^{-1}\int_T f\,dx$ the local integral mean of f. Vanishing data oscillations $\lim_\ell \mathrm{osc}_{\Omega,\ell} = 0$ thus lead to convergence $\lim_\ell |||u - U_\ell||| = 0$.

The next milestone was that BINEV, DAHMEN, and DEVORE [3] provided a link between AFEM and approximation theory. They proved that if the data oscillations $\mathrm{osc}_{\Omega,\ell}$ are treated properly and if the adaptive loop (3) is extended by a coarsening step, the algorithm from [6] asymptotically leads to the best possible algebraic convergence rate.

STEVENSON [7] showed that the coarsening step is, in fact, not necessary to prove quasi-optimality, but still his analysis relied on the mesh-refinement of [6].

Finally, CASCÓN, KREUZER, NOCHETTO, and SIEBERT [4] proved that convergence of AFEM holds for each mesh-refinement strategy which preserves uniform shape regularity. Moreover, AFEM is quasi-optimal for each fixed variant of newest vertex bisection.

The incorporation of Neumann boundary conditions, i.e. $\Gamma = \Gamma_D \cup \Gamma_N$ with $|\Gamma_D| > 0$ and $\Gamma_N \cap \Gamma_D = \emptyset$,

$$-\Delta u = f \text{ in } \Omega \quad \text{subject to the boundary conditions} \quad u = 0 \text{ on } \Gamma_D \quad \text{and} \quad \partial_n u = \phi \text{ on } \Gamma_N \tag{5}$$

into the analysis of [4] is straight-forward. However, the incorporation of inhomogeneous Dirichlet conditions, i.e. $u = g$ on Γ_D, is technically more demanding since an adaptive resolution of g is involved and ansatz and test space now differ.

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2 Model problem with inhomogeneous Dirichlet data

In this work, we comprise our manuscript [8] and consider the model problem

$$-\Delta u = f \text{ in } \Omega \subset \mathbb{R}^2 \quad \text{subject to inhomogeneous Dirichlet conditions} \quad u = g \text{ on } \Gamma, \tag{6}$$

where $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$ are given. We recall that $H^{1/2}(\Gamma)$ is characterized as the trace space of $H^1(\Omega)$. The weak form of (6) reads: Find $u \in H^1(\Omega)$ which satisfies $u|_\Gamma = g$ (in the sense of traces) such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx =: \langle\!\langle \nabla u \, , \, \nabla v \rangle\!\rangle = (f \, , \, v)_\Omega := \int_\Omega f v \, dx \quad \text{for all } v \in H^1_0(\Omega) := \big\{ w \in H^1(\Omega) \, : \, w|_\Gamma = 0 \big\}. \tag{7}$$

On the analytical side, unique solvability of the weak form is derived by means of a linear and continuous lifting operator $\mathcal{L}$ which maps Dirichlet data $g \in H^{1/2}(\Gamma)$ to a function $\mathcal{L}g \in H^1(\Omega)$. Then, (7) is reformulated for $u_0 := u - \mathcal{L}g \in H^1_0(\Omega)$, and existence and uniqueness of u_0 which follow from the Lax-Milgram lemma, prove unique solvability of (7).

Given a regular triangulation $\mathcal{T}_\ell$ of Ω into triangles, we stress that an FE function $U_\ell \in \mathcal{S}^1(\mathcal{T}_\ell) := \big\{ U_\ell \in C(\overline{\Omega}) \, : \, U_\ell|_T \text{ affine } \forall T \in \mathcal{T}_\ell \big\}$ can (in general) not satisfy the prescribed Dirichlet conditions, i.e. $U_\ell|_\Gamma \neq g$. Moreover, the lifting $\mathcal{L}g$ is hardly accessible numerically so that (6) cannot be transferred into the form (1) in practice. As is usually done in practice, we assume additional regularity $g \in H^1(\Gamma)$. According to the Sobolev embedding on the one-dimensional manifold Γ, g then is continuous. Let g_ℓ denote the nodal interpolant of g which belongs to the discrete trace space $\mathcal{S}^1(\mathcal{T}_\ell|_\Gamma)$ of $\mathcal{S}^1(\mathcal{T}_\ell)$, i.e. $g_\ell \in \mathcal{S}^1(\mathcal{T}_\ell|_\Gamma)$. Moreover, as is shown in [1], it holds that

$$\|g - g_\ell\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(g - g_\ell)'\|_{L^2(\Gamma)} = \min_{W_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)} \|h_\ell^{1/2}(g - W_\ell|_\Gamma)'\|_{L^2(\Gamma)} =: \mathrm{osc}_{\Gamma,\ell}, \tag{8}$$

where $(\cdot)'$ denotes the arclength derivative and where $h_\ell \in L^\infty(\Gamma)$ is defined by $h_\ell|_{T\cap\Gamma} = |T|^{1/2}$ for all $T \in \mathcal{T}_\ell$. The multiplicative constant hidden in the symbol $\lesssim$ depends only on the shape regularity of the mesh $\mathcal{T}_\ell$ and on Ω.

The lowest-order FE formulation now reads as follows: Find $U_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ which satisfies $U_\ell|_\Gamma = g_\ell$ such that

$$\langle\!\langle \nabla U_\ell \, , \, \nabla V_\ell \rangle\!\rangle = (f \, , \, V_\ell)_\Omega \quad \text{for all } V_\ell \in \mathcal{S}^1_0(\mathcal{T}_\ell) := \big\{ W_\ell \in \mathcal{S}^1(\mathcal{T}_\ell) \, : \, W_\ell|_\Gamma = 0 \big\}. \tag{9}$$

To prove unique solvability of (9), let $\mathcal{P}_\ell : H^1(\Omega) \to \mathcal{S}^1(\mathcal{T}_\ell)$ denote the usual Scott-Zhang projection which satisfies H^1-stability and first-order L^2-approximation property as well as $\mathcal{P}_\ell W_\ell = W_\ell$ for all $W_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$. Moreover, it preserves discrete Dirichlet data, i.e. for all $w \in H^1(\Omega)$ with $w|_\Gamma \in \mathcal{S}^1(\mathcal{T}_\ell|_\Gamma)$ holds $(\mathcal{P}_\ell w)|_\Gamma = w|_\Gamma$.

With the help of the Scott-Zhang projection, the operator $\mathcal{P}_\ell\mathcal{L}$ provides a discrete lifting, i.e. $\mathcal{P}_\ell\mathcal{L}g \in \mathcal{S}^1(\mathcal{T}_\ell)$ and $(\mathcal{P}_\ell\mathcal{L}g_\ell)|_\Gamma = g_\ell$. First, this allows to follow the lines of the continuous proof to see that (9) admits a unique FE solution $U_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$. Moreover, we can prove the following Céa-type estimate for the computed FE solution

$$\|u - U_\ell\|_{H^1(\Omega)} \lesssim |||u - U_\ell||| + \mathrm{osc}_{\Gamma,\ell} \lesssim \min_{W_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)} |||u - W_\ell||| + \mathrm{osc}_{\Gamma,\ell}. \tag{10}$$

3 Residual error estimator & adaptive algorithm

For a posteriori error estimation, we extend the residual error estimator from (2). The local refinement indicators now read

$$\varrho_\ell(T)^2 = \rho_\ell(T)^2 + \mathrm{osc}_{\Gamma,\ell}(T)^2 \quad \text{with} \quad \mathrm{osc}_{\Gamma,\ell}(T)^2 = |T|^{1/2}\|(g - g_\ell)'\|^2_{L^2(\partial T\cap\Gamma)}, \tag{11}$$

i.e. $\varrho_\ell^2 = \rho_\ell^2 + \mathrm{osc}^2_{\Gamma,\ell}$. Since the seminal work of DÖRFLER [5], who introduced the appropriate marking criterion (12), the adaptive loop (3) reads as follows: Let $\ell = 0$ and $\mathcal{T}_0$ be a given initial triangulation and fix some adaptivity parameter $0 < \theta < 1$, where large $\theta \approx 1$ corresponds to uniform mesh-refinement, whereas small $\theta \approx 0$ leads to highly adapted meshes.

(i) Compute discrete solution $U_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$.

(ii) Compute refinement indicators $\varrho_\ell(T)$ for all $T \in \mathcal{T}_\ell$.

(iii) Determine subset $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ of marked elements such that

$$\theta \sum_{T \in \mathcal{T}_\ell} \varrho_\ell(T)^2 \leq \sum_{T \in \mathcal{M}_\ell} \varrho_\ell(T)^2. \tag{12}$$

(iv) Refine at least marked elements $T \in \mathcal{M}_\ell$ to obtain a regular triangulation $\mathcal{T}_{\ell+1}$, increase counter $\ell \mapsto \ell + 1$, and iterate.

For mesh-refinement, either red-green-blue refinement or newest vertex bisection is chosen. We stress that both refinement strategies ensure uniform shape regularity of the created sequence of meshes $\mathcal{T}_\ell$.

4 Convergence of AFEM

In general, adaptive mesh-refinement does not guarantee that the local mesh-size $h_\ell \in L^\infty(\Omega)$ defined by $h_\ell|_T = |T|^{1/2}$ tends to zero everywhere, i.e. $h_\ell \not\to 0$ a.e. in Ω. Therefore, the question of convergence $\lim_\ell \|u - U_\ell\|_{H^1(\Omega)} = 0$ is by no means obvious, but of utter importance. Our proof only needs the following properties of the error estimator ϱ_ℓ and the model problem which are proven in [8]:

(a) reliability $\|u - U_\ell\|_{H^1(\Omega)} \le C_{\rm rel}\,\varrho_\ell$,

(b) estimator reduction $\varrho_{\ell+1}^2 \le q\,\varrho_\ell^2 + C_{\rm red}|\!|\!|U_{\ell+1} - U_\ell|\!|\!|^2$,

(c) quasi-Galerkin orthogonality $(1-\alpha)|\!|\!|u - U_{\ell+1}|\!|\!|^2 \le |\!|\!|u - U_\ell|\!|\!|^2 - |\!|\!|U_{\ell+1} - U_\ell|\!|\!|^2 + \alpha^{-1}C_{\rm gal}\|h_\ell^{1/2}(g_{\ell+1} - g_\ell)'\|_{L^2(\Gamma)}^2$,

(d) orthogonality for Dirichlet data approximation $\|h_\ell^{1/2}(g - g_{\ell+1})'\|_{L^2(\Gamma)}^2 = \|h_\ell^{1/2}(g - g_\ell)'\|_{L^2(\Gamma)}^2 - \|h_\ell^{1/2}(g_{\ell+1} - g_\ell)'\|_{L^2(\Gamma)}^2$.

Here, $0 < \alpha < 1$ is arbitrary and $0 < q < 1$ depends only on the choice of θ. The constants $C_{\rm rel}, C_{\rm red}, C_{\rm gal} > 0$ additionally depend on the shape regularity of $\mathcal{T}_\ell$. In the spirit of [4], our convergence theorem reads as follows.

Theorem 4.1 *There are sufficiently small constants $0 < \gamma, \lambda < 1$ and some $0 < \kappa < 1$ such that*

$$\Delta_{\ell+1} \le \kappa\,\Delta_\ell \quad \textit{with} \quad \Delta_\ell := |\!|\!|u - U_\ell|\!|\!|^2 + \gamma\varrho_\ell^2 + \lambda\|h_\ell^{1/2}(g - g_\ell)'\|_{L^2(\Gamma)}^2 \simeq \varrho_\ell^2. \tag{13}$$

In particular, this proves $\lim_{\ell\to\infty} \varrho_\ell = 0 = \lim_{\ell\to\infty} \|u - U_\ell\|_{H^1(\Omega)}$. The constants γ, λ, κ depend only on $C_{\rm rel}, C_{\rm red}, C_{\rm gal}, q > 0$. □

In the homogeneous case (1) considered in [4], the contraction (13) holds with $\lambda = 0$. Compared to this, the additional λ-weighted term allows to dominate the consistency error from the quasi-Galerkin orthogonality.

5 Optimality of Dörfler marking

One essential step in the proof of quasi-optimality of AFEM is the observation that the Dörfler marking (12) is not only sufficient, but in some sense also necessary for contraction (13). To prove this, the essential ingredients read as follows:

(e) contraction quantity $\Delta_\ell \simeq \varrho_\ell^2$ with $\Delta_{\ell+1} \le \kappa\,\Delta_\ell$ and $0 < \kappa < 1$, cf. Theorem 4.1,

(f) Céa-type estimate $|\!|\!|u - U_\ell|\!|\!| + {\rm osc}_{\Gamma,\ell} \le C_{\text{Céa}}\big(\min_{W_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)} |\!|\!|u - W_\ell|\!|\!| + {\rm osc}_{\Gamma,\ell}\big)$,

(g) efficiency $C_{\rm eff}^{-1}\,\varrho_\ell \le |\!|\!|u - U_\ell|\!|\!| + {\rm osc}_{\Omega,\ell} + {\rm osc}_{\Gamma,\ell}$,

(h) estimator dominates data oscillations ${\rm osc}_{\Omega,\ell} + {\rm osc}_{\Gamma,\ell} \le \varrho_\ell$,

(i) discrete local reliability $\|U_\star - U_\ell\|_{H^1(\Omega)}^2 \le C_{\rm rel}^2 \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{T}_\star} \varrho_\ell(T)^2$,

(j) converse quasi-Galerkin orthogonality $|\!|\!|u - U_\ell|\!|\!|^2 \le (1+\alpha)|\!|\!|u - U_\star|\!|\!|^2 + |\!|\!|U_\star - U_\ell|\!|\!|^2 + \alpha^{-1}C_{\rm gal}\|h_\ell^{1/2}(g_\star - g_\ell)'\|_{L^2(\Gamma)}^2$.

Here, $\mathcal{T}_\star$ is an arbitrary refinement of $\mathcal{T}_\ell$ with corresponding Galerkin solution $U_\star \in \mathcal{S}^1(\mathcal{T}_\star)$. As above, $0 < \alpha < 1$ is arbitrary, and $C_{\rm rel}, C_{\rm eff}, C_{\rm gal}, C_{\text{Céa}} > 0$ depend on the shape regularity of $\mathcal{T}_\ell$. We stress that by choice of the Scott-Zhang projection $\mathcal{P}_\ell$ in the analysis, the constant $C_{\rm rel} > 0$ in (a) and (i) coincides.

First, combining (e), (f), (g), and (h), we see equivalence

$$\Delta_\ell \simeq \varrho_\ell^2 \simeq \min_{W_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)} |\!|\!|u - W_\ell|\!|\!|^2 + {\rm osc}_{\Omega,\ell}^2 + {\rm osc}_{\Gamma,\ell}^2 \tag{14}$$

Second, the combination of this with (i) and (j) allows to verify the so-called optimality of the Dörfler marking. So far, we have proven that Dörfler marking implies contraction (13) of Δ_ℓ. The following proposition states the converse implication.

Proposition 5.1 *There are constants $0 < \kappa_\star, \theta_\star < 1$ such that for all $0 < \theta \le \theta_\star$ and all refinements $\mathcal{T}_\star$ of $\mathcal{T}_\ell$ holds*

$$\Delta_\star \le \kappa_\star\,\Delta_\ell \quad \Longrightarrow \quad \theta \sum_{T \in \mathcal{T}_\ell} \varrho_\ell(T)^2 \le \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{T}_\star} \varrho_\ell(T)^2, \tag{15}$$

i.e. the refined elements $\mathcal{T}_\ell \setminus \mathcal{T}_\star$ satisfy the Dörfler marking (12)*. The constants $\kappa_\star, \theta_\star$ depend only on $C_{\rm rel} > 0$ and the equivalence constants in* (14)*.* □

6 Quasi-optimality of AFEM

From now on, we restrict to newest vertex bisection for mesh-refinement. Let $\mathbb{T}$ denote the set of all triangulations which can be obtained from the initial mesh $\mathcal{T}_0$. Under certain conditions on $\mathcal{T}_0$, it has been observed in [3] that

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \lesssim \sum_{j=0}^{\ell-1} \#\mathcal{M}_j, \tag{16}$$

i.e. the mesh-closure to avoid hanging nodes in step (iv) of the adaptive loop generates, up to some multiplicative constant, no extra elements. Since newest vertex bisection is a binary refinement rule, the coarsest common refinement $\mathcal{T}' \oplus \mathcal{T}''$ of two meshes $\mathcal{T}', \mathcal{T}'' \in \mathbb{T}$ is just the overlay and satisfies

$$\#(\mathcal{T}' \oplus \mathcal{T}'') \le \#\mathcal{T}' + \#\mathcal{T}'' - \#\mathcal{T}_0, \tag{17}$$

see [7].

Suppose that $\mathcal{T}_\star \in \mathbb{T}_N = \big\{\mathcal{T} \in \mathbb{T} \,:\, \#\mathcal{T} - \#\mathcal{T}_0 \le N\big\}$ is an arbitrary mesh with Galerkin solution $U_\star$ and corresponding estimator $\varrho_\star$. Conceptually, a convergence behaviour $\varrho_\star = \mathcal{O}(N^{-s})$ is possible if and only if

$$\sup_{N\in\mathbb{N}} \big(N^s \inf_{\mathcal{T}_\star \in \mathbb{T}_N} \varrho_\star\big) \simeq \sup_{N\in\mathbb{N}} \big(N^s \inf_{\mathcal{T}_\star \in \mathbb{T}_N} \big[\inf_{W_\star \in \mathcal{S}^1(\mathcal{T}_\star)} \|\!|u - W_\star|\!\| + \mathrm{osc}_{\Omega,\star} + \mathrm{osc}_{\Gamma,\star}\big]\big) < \infty, \tag{18}$$

where $\simeq$ denotes the equivalence from (14) now stated on $\mathcal{T}_\star$. In case of (18), we write $(u, f, g) \in \mathbb{A}_s$. We say that *AFEM is quasi-optimal* if and only if, for $(u, f, g) \in \mathbb{A}_s$, the adaptive algorithm leads to $\varrho_\ell = \mathcal{O}(\#\mathcal{T}_\ell^{-s})$, i.e. each possible algebraic convergence rate $s > 0$ is asymptotically obtained by AFEM. In view of (18), the convergence behaviour is characterized by the smoothness of the solution and the given data f and g.

Along the lines of the original proof of [7], we now obtain the following result. Besides the properties (16)–(17), the proof relies only on Theorem 4.1, Proposition 5.1, and the definition of the approximation class $\mathbb{A}_s$.

Theorem 6.1 *Suppose that the adaptivity parameter satisfies* $0 < \theta \le \theta_\star$*, so that Dörfler marking* (12) *is optimal in the sense of Proposition 5.1. Moreover, assume that in each step of the adaptive loop, the set* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ *has minimal cardinality. Then, for all* $s > 0$*, the following implication holds*

$$(u, f, g) \in \mathbb{A}_s \quad \Longrightarrow \quad \|u - U_\ell\|_{H^1(\Omega)} \lesssim \varrho_\ell \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}, \tag{19}$$

i.e. each possible algebraic convergence order is, in fact, regained by AFEM. □

7 Extensions of the analysis

We have seen that, for a Dirichlet problem in 2D, nodal interpolation of the given Dirichlet data g and an appropriate residual error estimator lead to a convergent adaptive scheme with quasi-optimal convergence behaviour. The result can be extended to mixed boundary value problem (5) with inhomogeneous Dirichlet data $u = g$ on Γ_D.

For lowest-order FEM it is well-known that the volume residuals can be replaced by the edge oscillations of f, i.e. replace

$$\|h_\ell f\|^2_{L^2(\Omega)} \quad \rightsquigarrow \quad \sum_{E \in \mathcal{E}_\ell^\Omega} \|h_\ell (f - f_{\omega_E})\|^2_{L^2(\omega_E)}$$

in the definition of ϱ_ℓ. Here, $\mathcal{E}_\ell^\Omega$ denotes the set of all interior edges of $\mathcal{T}_\ell$ and $\omega_E = T_+ \cup T_-$ is the edge patch of some edge $E = T_+ \cap T_- \in \mathcal{E}_\ell^\Omega$ with corresponding integral mean f_{ω_E}. Then, of course, the error estimator and the adaptive algorithm may be reformulated in an edge-based formulation $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell$ and $\varrho_\ell^2 = \sum_{E \in \mathcal{E}_\ell} \varrho_\ell(E)^2$. Our results then hold accordingly.

In 3D, $g \in H^1(\Gamma)$ is *not* sufficient to guarantee continuity of g, and therefore nodal interpolation is *not* allowed. Instead, one may use the Scott-Zhang projection $\mathcal{P}_\ell$ or the L^2-orthogonal projection onto $\mathcal{S}^1(\mathcal{T}_\ell|_\Gamma)$ to obtain g_ℓ. In either case, the Dirichlet oscillations read $\mathrm{osc}_{\Gamma,\ell} = \|h_\ell^{1/2} \nabla_\Gamma (g - g_\ell)\|_{L^2(\Gamma)}$ with the surface gradient ∇_Γ. Convergence of AFEM with these approaches is also found in [8] for lowest-order FEM. In case of the Scott-Zhang projection, it seems to be possible to prove also quasi-optimality simultaneously in 2D and 3D and for FEM of arbitrary but fixed polynomial order. This is currently investigated.

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