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M. Huber, A. Pechstein and J. Schöberl

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
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Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

**E-Mail:** admin@asc.tuwien.ac.at
**WWW:** http://www.asc.tuwien.ac.at
**FAX:** +43-1-58801-10196

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# Hybrid Domain Decomposition Solvers for Scalar and Vectorial Wave Equation

M. Huber[1], A. Pechstein[2] and J. Schöberl[1]

[1]Institute for Analysis and Scientific Computing,
Wiedner Hauptstrasse 8-10, A-1040 Wien,
e-mail: martin.huber@tuwien.ac.at, joachim.schoeberl@tuwien.ac.at
[2]Institute for Technical Mechanics,
Altenbergerstrasse 69, A-4040 Linz,
e-mail: astrid.pechstein@jku.at

**Abstract**

We present hybrid finite element methods, which are equivalent to a discontinuous Galerkin method based on the ultra weak variational formulation (UWVF) by Cessenat and Despres. When solving a scalar or vectorial wave equation with hybrid finite elements, normal and tangential continuity of the flux field, respectively, is broken across element interfaces and reinforced again by introducing hybrid variables, which are supported only on the element facets. Using this technique, coupling between degrees of freedom belonging to the interior of different elements is avoided, and the large number of volume unknowns on the element can be eliminated easily. Consequently, the linear system of equations needs only to be solved for the introduced facet degrees of freedom. It is shown by numerical experiments, that iterative solvers using Schwarz preconditioners show good convergence properties for these systems of equations.

## 1 Introduction

The solution of the Helmholtz equation is of high interest for time harmonic wave propagation problems in optics and acoustics. When the wave equation for high wave numbers $\kappa$ is solved with a standard finite element method (FEM), the oscillatory behavior of the solution results in a large number of degrees of freedom, which is necessary to resolve the wave. It is well known, that due to the pollution error ([11, 10]) the number of unknowns needed to solve the Helmholtz equation

in $d$ dimensions grows faster than $O(\kappa^d)$.
To overcome this difficulty, many methods have been developed during the last years. Apart from *hp* FEM ([11]), Galerkin Least Square Methods ([9]) or Discontinuous Galerkin Methods ([8]), some methods make use of problem adapted functions like plane waves. The most popular among them are the Partition of Unity Method ([13, 12]), the Discontinuous Enrichment Approach ([7, 16]) or the UWVF ([3, 14]). All these techniques end up with large, complex valued, indefinite possible symmetric system matrices. Although some advances in solving such systems have been made ([6, 5]), efficient preconditioners for wave type problems are still a challenge.
In our work the hybrid FEM from [15], which is equivalent to a discontinuous Galerkin Method, based on the UWVF of [3], is used in order to solve the scalar wave equation, and this method is extended to the vectorial case. Motivated by hybrid FEMs for the Laplace equation ([4]), the tangential continuity of the flux field is broken across element interfaces. In order to reinforce continuity again, Lagrange multipliers supported only on the facets are introduced. These multipliers can be interpreted as the tangential component of the unknown field. By adding a second set of Lagrange multipliers, representing the tangential component of the flux field, it is possible to eliminate the volume degrees of freedom. Because, after hybridization, there is no coupling between volume basis functions of different elements, elimination of inner degrees of freedom can be done cheaply element by element, and the linear system of equations is reduced onto the smaller system for the Lagrange multipliers. Numerical tests show, that a preconditioned cg iteration has good convergence properties. Apart from a Multiplicative (MS) or Additive Schwarz (AS) block preconditioner with blocks containing the degrees of freedom related to one facet and element, respectively, a domain decomposition (DD) preconditioner, which directly inverts the degrees of freedom belonging to one subdomain, can be used.
The paper is organized as follows. In Sec. 2 the mixed hybrid formulation for the vectorial wave equation is derived. The section concludes with a short discussion of the discrete finite element spaces. Sec. 3 is devoted to the solution of the system of linear equations. There, we present different precondition strategies for the system of equation for the facet unknowns obtained by static condensation. The effectiveness of these preconditioners is tested by numerical examples in Sec. 4.

# 2 Hybridization of the Helmholtz Equation

In this section we discuss the mixed hybrid formulation of the following settings. As computational domain, we consider a Lipschitz polyhedron $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ and the boundary $\Gamma = \partial\Omega$. In the scalar case, we search for a function $u : \Omega \to \mathbb{C}$,

which fulfills the Helmholtz equation with absorbing boundary conditions,

$$\begin{aligned} \Delta u + \omega^2 u &= 0 && \text{in } \Omega \\ \frac{1}{i\omega}\frac{\partial u}{\partial n} + u &= g && \text{on } \Gamma, \end{aligned}$$

where $\omega$ is the angular frequency. From [12] we know, that the solution $u$ exists and is unique. Introducing incoming and outgoing impedance traces,

$$\frac{1}{i\omega}\frac{\partial u}{\partial n} + u \qquad \text{and} \qquad -\frac{1}{i\omega}\frac{\partial u}{\partial n} + u,$$

respectively, the absorbing boundary condition can be interpreted as a prescription of the amplitude of incoming waves.
In the vectorial case, we search for a vector valued function $\mathbf{E} : \Omega \to \mathbb{C}^3$, which solves

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\mathbf{E} - \omega^2\mathbf{E} &= 0 && \text{in } \Omega \\ -\frac{1}{i\omega}\mathbf{n} \times \operatorname{curl}\mathbf{E} + \mathbf{E}_{\|} &= \mathbf{g} && \text{on } \Gamma, \end{aligned}$$

where $\mathbf{n}$ is the outer normal vector on $\Gamma$, and $\mathbf{E}_{\|}$ represents the tangential component of $\mathbf{E}$, i.e. $\mathbf{n}\times\mathbf{E}\times\mathbf{n}$. Like in the scalar case, the absorbing boundary condition is equivalent to prescribing the incoming impedance trace.
When deriving the hybrid formulation, we use a regular finite element mesh $\mathcal{T}$ with elements $T$ and the set of facets is called $\mathcal{F}$. The vector $\mathbf{n}_T$ is the outer normal vector of the element $T$, and $\mathbf{n}_F$ represents the normal vector onto a facet $F$. Furthermore, we denote a volume integral as $\big(u, v\big)_T := \int_T uv \, d\mathbf{x}$, and a surface integral as $\big\langle u, v\big\rangle_{\partial T} := \int_{\partial T} uv \, ds$.

## 2.1 The Mixed Hybrid Formulation for the Scalar Problem

The mixed hybrid formulation for the scalar Helmholtz problem was already introduced in [15]. For completness, we repeat the problem formulation:
Find $(u, \mathbf{v}, u^F, v^F) \in L^2(\Omega) \times H(\operatorname{div}, T) \times L^2(\mathcal{F}) \times L^2(\mathcal{F}) =: X \times \tilde{Y} \times X^F \times Y^F$, such that for all $(\sigma, \mathbf{w}, \sigma^F, w^F) \in X \times \tilde{Y} \times X^F \times Y^F$

$$\begin{aligned} \sum_{T\in\mathcal{T}} \Big( \big(i\omega u, \sigma\big)_T - \big(i\omega\mathbf{v}, \mathbf{w}\big)_T - \big(\operatorname{div}\mathbf{v}, \sigma\big)_T - \big(u, \operatorname{div}\mathbf{w}\big)_T + \big\langle u^F, \mathbf{n}_T\cdot\mathbf{w}\big\rangle_{\partial T} \\ + \big\langle \mathbf{n}_T\cdot\mathbf{v}, \sigma^F\big\rangle_{\partial T} + \big\langle \mathbf{n}_F\cdot\mathbf{v} - v^F, \mathbf{n}_F\cdot\mathbf{w} - w^F\big\rangle_{\partial T}\Big) + \big\langle u^F, \sigma^F\big\rangle_{\Gamma} = \big\langle g, \sigma^F\big\rangle_{\Gamma}. \end{aligned}$$

## 2.2 The Mixed Hybrid Formulation for the Vectorial Problem

We will now concentrate on the derivation of the mixed hybrid formulation for the vectorial wave equation. We start from the mixed Helmholtz equation, i.e. we introduce a flux field $\mathbf{H} : \Omega \to \mathbb{C}^3$, which leads to

$$\begin{aligned} \operatorname{curl} \mathbf{H} + i\omega \mathbf{E} &= 0 && \text{in } \Omega \\ \operatorname{curl} \mathbf{E} - i\omega \mathbf{H} &= 0 && \text{in } \Omega \\ -\mathbf{n} \times \mathbf{H} + \mathbf{E}_{\|} &= \mathbf{g} && \text{on } \Gamma. \end{aligned}$$

By multiplying the first equation with a testfunction $\mathbf{e} \in U := \left(L^2(\Omega)\right)^3$ and the second one with a function $\mathbf{h} \in V := H(\operatorname{curl}, \Omega)$, integrating over the domain $\Omega$ and performing integration by parts elementwise, we obtain

$$\begin{aligned} \sum_{T \in \mathcal{T}} \left( \left( \operatorname{curl} \mathbf{H}, \mathbf{e} \right)_T + \left( i\omega \mathbf{E}, \mathbf{e} \right)_T \right) &= 0 && \forall \mathbf{e} \in U \\ \sum_{T \in \mathcal{T}} \left( \left( \mathbf{E}, \operatorname{curl} \mathbf{h} \right)_T - \left( i\omega \mathbf{H}, \mathbf{h} \right)_T - \left\langle \mathbf{E}, \mathbf{n}_T \times \mathbf{h} \right\rangle_{\partial T} \right) &= 0 && \forall \mathbf{h} \in V. \end{aligned}$$

Note that for a tangential continuous field $\mathbf{E}$ the boundary integrals for inner facets cancel due to the tangential continuity of $\mathbf{h}$, and inserting the absorbing boundary condition into the boundary facet integrals would lead to the standard mixed finite element formulation for our problem.
Next, the tangential continuity of the flux field $\mathbf{H}$ is broken across element interfaces, thus we search for $\mathbf{H} \in \tilde{V}$ with

$$\tilde{V} := \left\{ \mathbf{v} \in \left( L^2(\Omega) \right)^3 \ : \ \mathbf{v}|_T = H(\operatorname{curl}, T) \ \forall T \in \mathcal{T} \right\}.$$

In order to reinforce continuity, Lagrange multipliers $\mathbf{E}^F$, which are only supported on the elementfacets, i.e. they are from the space $U^F := \left( L^2(\mathcal{F}) \right)^3$, are introduced. The continuity of the tangential fluxes is reached via an additional equation, which forces the jump of $[\mathbf{n} \times \mathbf{H}] := \mathbf{n}_{T_1} \times \mathbf{H}|_{T_1} + \mathbf{n}_{T_2} \times \mathbf{H}|_{T_2}$ for inner facets $F \in \mathcal{F}_I$ with adjacent elements $T_1$ and $T_2$ to zero, thus

$$\sum_{F \in \mathcal{F}_I} \left\langle [\mathbf{n} \times \mathbf{H}], \mathbf{e} \right\rangle_F = \sum_{T \in \mathcal{T}} \left( \left\langle \mathbf{n}_T \times \mathbf{H}, \mathbf{e} \right\rangle_{\partial T} - \left\langle \mathbf{n}_T \times \mathbf{H}, \mathbf{e} \right\rangle_{\partial T \cap \Gamma} \right) = 0, \quad \forall \mathbf{e} \in U^F.$$

The resulting system of equations for $(\mathbf{E}, \mathbf{H}, \mathbf{E}^F) \in U \times \tilde{V} \times U^F$ reads as

$$\begin{aligned}
\sum_{T\in\mathcal{T}} \left( \left( \operatorname{curl} \mathbf{H}, \mathbf{e} \right)_T + \left( i\omega \mathbf{E}, \mathbf{e} \right)_T \right) &= 0 && \forall \mathbf{e} \in U \\
\sum_{T\in\mathcal{T}} \left( \left( \mathbf{E}, \operatorname{curl} \mathbf{h} \right)_T - \left( i\omega \mathbf{H}, \mathbf{h} \right)_T - \left\langle \mathbf{E}^F, \mathbf{n}_T \times \mathbf{h} \right\rangle_{\partial T} \right) &= 0 && \forall \mathbf{h} \in \tilde{V} \\
\sum_{T\in\mathcal{T}} \left( -\left\langle \mathbf{n}_T \times \mathbf{H}, \mathbf{e}^F \right\rangle_{\partial T} \right) + \left\langle \mathbf{E}^F, \mathbf{e}^F \right\rangle_\Gamma &= \left\langle \mathbf{g}, \mathbf{e}^F \right\rangle_\Gamma && \forall \mathbf{e}^F \in U^F.
\end{aligned}$$

From this system of equation, one can see, that the Lagrange parameter $\mathbf{E}^F$ plays the role of the tangential component of $\mathbf{E}$, evaluated on the facets. Due to the fact, that there is no coupling between volume unknowns of different elements, it possible to eliminate them cheaply element by element by static condensation (compare [2]). The resulting system of equations needs now to be solved only for the Lagrange multipliers on the facets.
In order to eliminate the inner degrees of freedom, one has to solve the first two equations of the system from above for some function $\mathbf{E}^F$ element by element. But this is equivalent to solving a Dirichlet problem, and uniqueness of the solution can not be guaranteed. This drawback can be compensated by adding a new facet unknown $\mathbf{H}^F \in V^F := \left( L^2(\mathcal{F}) \right)^3$, which represents $\mathbf{n}_F \times \mathbf{H}$ on the facets via a consistent stabilization term $\sum_T \left\langle \mathbf{n}_F \times \mathbf{H} - \mathbf{H}^F, \mathbf{n}_F \times \mathbf{h} - \mathbf{h}^F \right\rangle_{\partial T}$. We obtain

$$\begin{aligned}
\sum_{T\in\mathcal{T}} \left( \left( \operatorname{curl} \mathbf{H}, \mathbf{e} \right)_T + \left( i\omega \mathbf{E}, \mathbf{e} \right)_T \right) &= 0 && \forall \mathbf{e} \in U \\
\sum_{T\in\mathcal{T}} \Big( \left( \mathbf{E}, \operatorname{curl} \mathbf{h} \right)_T - \left( i\omega \mathbf{H}, \mathbf{h} \right)_T - \left\langle \mathbf{E}^F, \mathbf{n}_T \times \mathbf{h} \right\rangle_{\partial T} & \\
- \left\langle \mathbf{n}_T \times \mathbf{H}, \mathbf{n}_T \times \mathbf{h} \right\rangle_{\partial T} + \left\langle \mathbf{H}^F, \mathbf{n}_F \times \mathbf{h} \right\rangle_{\partial T} \Big) &= 0 && \forall \mathbf{h} \in \tilde{V} \\
\sum_{T\in\mathcal{T}} \left( \left\langle \mathbf{n}_F \times \mathbf{H}, \mathbf{h}^F \right\rangle_{\partial T} - \left\langle \mathbf{H}^F, \mathbf{h}^F \right\rangle_{\partial T} \right) &= 0 && \forall \mathbf{h}^F \in U^F \\
\sum_{T\in\mathcal{T}} \left( -\left\langle \mathbf{n}_T \times \mathbf{H}, \mathbf{e}^F \right\rangle_{\partial T} \right) + \left\langle \mathbf{E}^F, \mathbf{e}^F \right\rangle_\Gamma &= \left\langle \mathbf{g}, \mathbf{e}^F \right\rangle_\Gamma && \forall \mathbf{e}^F \in U^F.
\end{aligned}$$

Now, by static condensation the Helmholtz equation with absorbing boundary conditions has to be solved on the element level, which guarantees uniqueness, and the resulting system only contains the facet unknowns $\mathbf{E}^F$ and $\mathbf{H}^\mathbf{F}$. Elimination of the inner degrees of freedom can be also seen as calculating for a given incoming impedance trace the resulting outgoing impedance trace on the element level. By exchanging the Dirichlet and Neumann traces $\mathbf{E}^\mathbf{F}, \mathbf{H}^F$ by incoming and outgoing impedance traces $\mathbf{E}^F + \mathbf{H}^F$ and $\mathbf{E}^F - \mathbf{H}^F$, one obtains an equivalent formulation, which fits well into the context of the UWVF of [3].

As in the scalar case, it can be shown, that the presented method is energy conserving. Because, at least, a fixed number of degrees of freedom is needed to resolve the solution of the Helmholtz equation (compare [1]), high frequency parts of the solutions, which are not resolved by the mesh together with the polynomial order cause spurious modes. These spurious modes can be damped out by a consistent stabilization term (compare Fig. 1)

$$\sum_{T\in\mathcal{T}} \alpha\big(\mathbf{E}_\parallel - \mathbf{E}^F, \mathbf{e}_\parallel - \mathbf{e}^F\big)_{\partial T}.$$

### 2.3 Finite Element Discretization

Discretizing the scalar problem, we end up with the discrete versions of our volume spaces

$$\begin{aligned} X_h &:= \left\{u \in L^2(\Omega) \ : \ u|_T = P^p(T)\ \forall T \in \mathcal{T}\right\}, \\ \tilde{Y}_h &= \left\{\mathbf{v} \in \left(L^2(\Omega)\right)^d \ : \ \mathbf{v}|T = RT^p(T)\ \forall T \in \mathcal{T}\right\}, \end{aligned}$$

where $P^p$ is a polynomial of order $p$ and $RT^p$ is a Raviar Thomas basis function of order $p$. The fact that functions of $X_h$ and the normaltrace of functions of $\tilde{Y}_h$ are polynomials of order $p$ on the facets leads to the definition of the trace spaces

$$X_h^F = Y_h^F := \left\{v \in L^2(\mathcal{F}) \ : \ v|_F = P^p(F)\ \forall F \in \mathcal{F}\right\}.$$

Equivalently we define the discrete spaces for the vectorial problem

$$\begin{aligned} U_h &:= \left\{\mathbf{u} \in \left(L^2(\Omega)\right)^3 \ : \ \mathbf{u}|_T = \left(P^p(T)\right)^3\ \forall T \in \mathcal{T}\right\}, \\ \tilde{V}_h &= \left\{\mathbf{v} \in \left(L^2(\Omega)\right)^3 \ : \ \mathbf{v}|T = ND^p_{II}(T)\ \forall T \in \mathcal{T}\right\}, \end{aligned}$$

with the second order Nedelec polynomials $ND^p_{II}$ of order $p$. According to this the tracespaces are defined as

$$U_h^F = V_h^F := \left\{\mathbf{v} \in \left(L^2(\mathcal{F})\right)^3 \ : \ \mathbf{v}|_F = \left(P^p(F)\right)^2\ \forall F \in \mathcal{F}\right\}.$$

## 3 Iterative Solvers

In this section, we focus on solution strategies for Helmholtz problems. As already mentioned, the volume degrees of freedom can be eliminated cheaply element by element, and the resulting system of equation just has to be solved for the much smaller number of facet degrees of freedom. Because in the hybrid formulation, there is only coupling between facet degrees of freedom of the same facet, and

the volume degrees of freedom of one element couple apart from themselves only with facet degrees of freedom of surrounding facets, the Schur complement matrix obtained by static condensation is sparse, and it just has nonzero entries between facet degrees of freedom belonging to facets of the same element.
Due to the fact, that the Schur complement matrix is complex symmetric, a preconditioned cg-iteration together with an AS or MS block preconditioner is used, although convergence for complex symmetric matrices is not guaranteed. One option of choosing blocks is to collect the degrees of freedom belonging to one element, i.e. the degrees of freedom of the facets surrounding the element, in one block. Note that this strategy leads to overlapping blocks. Another possibility, which leads to nonoverlapping blocks, is to use blocks containing the degrees of freedom of one single facet.
Apart from an AS or MS Preconditioner a DD preconditioner compareable to [17] was used. The computational domain is divided into several subdomains, and all degrees of freedom belonging to facets of the interior of such a subdomain are collected in one subdomain block. In each preconditioner step, first a forward block Gauss Seidel iteration is carried out, followed by a direct inversion of each subdomain block. In order to keep symmetry, each precondition step is finished by a backward block Gauss Seidel step. Schematically, the DD preconditioner con be written as

$$\left(\prod_{i=N}^{1}(I-\mathcal{P}_{GS})\right)\ (I-\mathcal{P}_{DD})\ \left(\prod_{i=1}^{N}(I-\mathcal{P}_{GS})\right),$$

where $\mathcal{P}_{DD}$ is a projector on the subdomain blocks, and $\mathcal{P}_{GS}$ projects to the Gauss Seidel blocks. In the blocks of the Gauss Seidel iteration, we only take care of degrees of freedom belonging to facets close to the surface of the subdomains. Possibilities are to collect the degrees of freedom of each subdomain surface facet in one single block, or to use blocks, where each one contains the degrees of freedom of an element sharing a boundary with a subdomain boundary.

# 4 Numerical Results

We start the result section with an example showing the effect of the damping term. On a cylindric domain an incoming wave with gaussian amplitude is prescribed by absorbing boundary conditions. Mesh size and polynomial order are chosen such that the wave is not resolved. Figure 1 shows the absolute value of the solution $\mathbf{E}$ without (left) and with (right) damping term. It can be seen, that spurious modes, which are present on the lefthand plot are damped out by the damping term.

In order to demonstrate the dependence of the number of iterations on polynomial order, wavelength and meshsize $h$ for the presented preconditioners, we choose

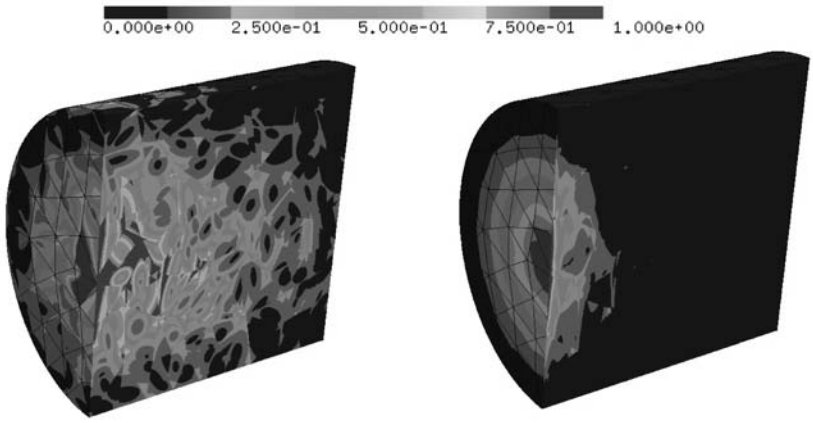


Figure 1: Absolute value for an unresolved solution **E** of the vectorial wave equation without (left) and with (right) damping term.

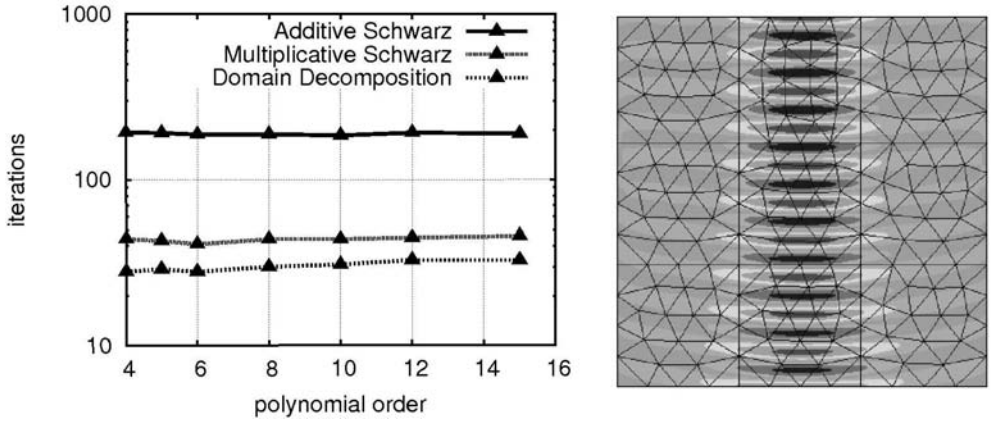


Figure 2: Dependence of the number of iterations on the polynomial order (left) for the two dimensional model problem (right).

| $h$ | $\lambda$ =0.64 | $\lambda$ =0.32 | $\lambda$ =0.16 | $\lambda$ =0.08 | $\lambda$ =0.04 | $\lambda$ =0.02 | $\lambda$ =0.01 |
|---|---|---|---|---|---|---|---|
| 0.16 | 35/40 | 35/38 | 32/33 | 31/31 | | | |
| 0.08 | 52/42 | 48/38 | 50/36 | 47/33 | 50/38 | | |
| 0.04 | 88/55 | 76/47 | 74/43 | 76/39 | 65/35 | 97/59 | |
| 0.02 | 147/75 | 129/55 | 113/48 | 117/44 | 118/42 | 115/38 | 199/82 |
| 0.01 | 246/107 | 236/80 | 226/60 | 203/53 | 228/49 | 271/50 | 291/45 |

Table 1: Iterations depending on wavelength and mesh size for the MS/DD preconditioner ($p$=6)

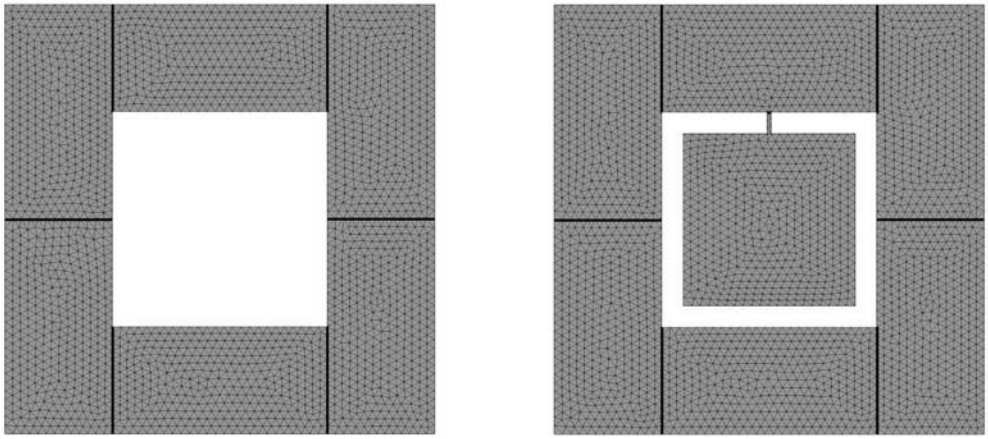

Figure 3: A Resonator (right) is compared with the domain without cavity (left).

a simple two dimensional model problem with a wave of Gaussian amplitude and wavelength $\lambda$ propagating through a unit square domain (compare Fig. 2). For a meshsize $h = \lambda = 0.1$ the lefthand plot shows the number of iterations for different polynomial orders. For the three preconditioners, the degrees of freedom of an element were collected in one block. In addition, for the DD preconditioner, the computational domain was divided into nine subdomains. If the polynomial order is large enough to resolve the wave, i.e. larger than four, the number of iterations stays constant or is only slightly growing with growing polynomial order, while the number of facet unknowns grows linearely in 2D.

Next we investigate the dependence on $h$ and $\lambda$ for a fixed polynomial order of six. The results are presented in Table 1. If the wavelength is smaller than $\frac{h}{2}$, which corresponds to less than three unknowns per wavelength, the solution can not be resolved, and the solvers show large iteration numbers, or do not converge at all. If the wavelength grows, the iteration number decreases, and it reaches a minimum at $h \approx \lambda$, which is equivalent to six unknowns per wavelength. A

| $h$ | dofs | MS | DD |
|---|---|---|---|
| 0.16 | 69980 | 0.35 | 0.37 |
| 0.08 | 217900 | 1.73 | 1.33 |
| 0.04 | 701228 | 9.30 | 5.15 |
| 0.02 | 2518524 | 53.5 | 22.4 |
| 0.01 | 9857920 | 367 | 111 |

Table 2: Iteration times for $\lambda = 0.08$ and a polynomial order of 6.

| | cavity | | square | |
|---|---|---|---|---|
| | its. | time(s) | its. | time(s) |
| DD (element) | 35 | 40.4 | 34 | 31.2 |
| DD (facet) | 64 | 69.7 | 61 | 59.7 |
| MS (element) | 1612 | 1720 | 102 | 88.9 |
| AS (element) | $> 10^5$ | $>$ 1h | 575 | 186 |

Table 3: Iteration numbers and computational times for the cavity and the square.

further increase of the wavelength leads to an increase in the iteration numbers for both solvers. For $h = 0.16$ every subdomain consists only of a small number of elements, and there is no big advantage if the degrees of freedom are inverted subdomain by subdomain, compared to an inversion element by element. Therfore the MS and the DD preconditioner show about the same performance. If the mesh size decreases, and the number of elements per subdomain increases, it is more and more advantageous to collect the unknowns in subdomain blocks. While the iteration number almost doubles for the MS preconditioner if the mesh size is divided by two, the increase is much less for the DD preconditioner. Table 2 shows, that the DD preconditioner also performs better than the MS preconditioner with respect to time, although one iteration is more expensive.

Now we compare the preconditioners for a resonator, and the square without cavity (compare Fig. 3). From the top of the square an incident wave with wavelength 0.01 is prescribed. The DD-preconditioner uses, depending on the presence of the cavity a division of the domain into six and seven subdomains, respectively. All the cavity degrees of freedom are collected in one single block, containing additionally the degrees of freedom of the cavity boundary. Table 3 shows the iteration numbers and computational times for different preconditioners and for the two examples. For the domain without cavity the performance of the preconditioners is compareable. When the cavity is added reflections inside the cavity lead to an enormous increase in iteration numbers and computational times

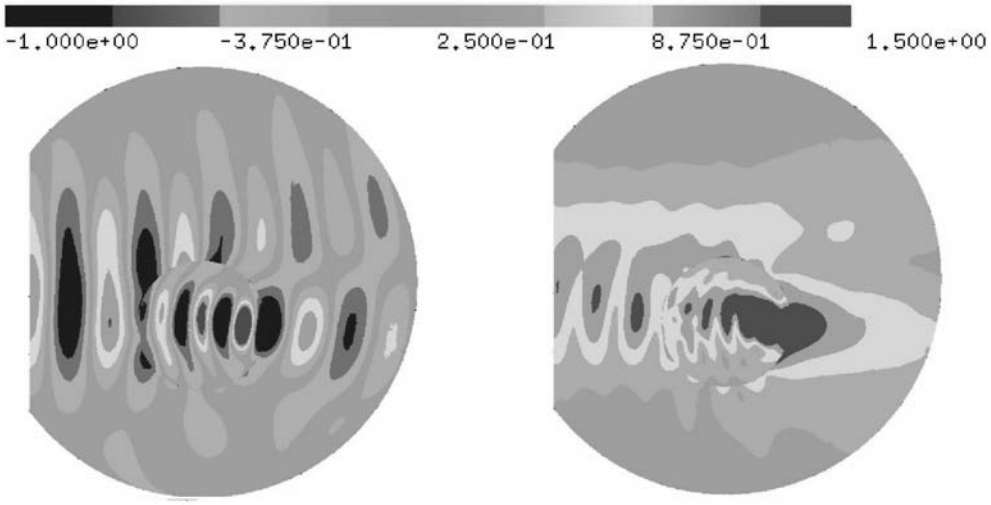


Figure 4: Real part of the $y$ component of $E$ (left) and $|E|$ (right) evaluated at a cross section parallel to the $xy$ plane.

for the AS and the MS preconditioner. Because of direct inversion of the cavity degrees of freedom, the DD preconditioner does not suffer from internal reflections and the iteration number stays almost constant, which leads together with a larger number of unknowns to a moderat increase in computational time.

We finish the numerical results section with an example from optics. A small sphere with radius 0.3 and refractive index two is placed (not exactly in the center) in a spherical computational domain with radius one and background refractive index one. We prescribe an incident wave from the left with a Gaussian amplitude and wavelength 0.35, such that the diameter of the computational domain is approximately six wavelength in free space. In order to resolve the wave we used 3256 elements with a polynomial order of six, which results in 1.66 millions of unknowns. The solution was obtained by 258 cg-iterations with a Block AS preconditioner.

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