ASC Report No. 10/2011

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www.asc.tuwien.ac.at ISBN 978-3-902627-04-9

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ISBN 978-3-902627-04-9

Global estimates of fundamental solutions for higher-order Schrödinger equations

JinMyong Kim, Anton Arnold and Xiaohua Yao

Abstract. In this paper we first establish global pointwise time-space estimates of the fundamental solution for Schrödinger equations, where the symbol of the spatial operator is a real non-degenerate elliptic polynomial. Then we use such estimates to establish related $L^p - L^q$ estimates on the Schrödinger solution. These estimates extend known results from the literature and are sharp. This result was lately already generalized to a degenerate case (cf. [4]).

Mathematics Subject Classification (2000). 42B20; 42B37; 35Q41; 35B65.

Keywords. Oscillatory integral, higher-order Schrödinger equation, fundamental solution estimate.

1. Introduction

In this paper we are interested in L^p-L^q estimates of solutions for the following Schrödinger equation:

$$\frac{\partial u}{\partial t} = iP(D)u, u(0, \cdot) = u_0 \in L^p(\mathbf{R}^n), \tag{1.1}$$

where $D = -i(\partial/\partial x_1, \cdots, \partial/\partial x_n)$, $P : \mathbf{R}^n \longrightarrow \mathbf{R}$ is a non-degenerate real elliptic polynomial of the even order m. In the sequel, we may assume without loss of generality that $P_m(\xi) > 0$ for $\xi \neq 0$ where $P_m(\xi)$ is the principal part of $P(\xi)$. The non-degeneracy condition on the polynomial P reads as follows.

This work was supported by the Postdoctoral Science Foundation of Huazhong University of Science and Technology in China and the Eurasia-Pacific Uninet scholarship for post-docs in Austria. The second author was supported by the FWF (project I 395-N16). The third author was supported by NSFC (No. 10801057), the Key Project of Chinese Ministry of Education (No. 109117), NCET-10-0431, and CCNU Project (No. CCNU09A02015).

(a) For any $\xi \in \mathbf{R}^n \backslash \{0\}$ the Hessian

$$\left(\frac{\partial^2}{\partial \xi_i \partial \xi_j} P_m(\xi) \right)$$

is non-degenerate.

For an elliptic polynomial P, condition (a) is equivalent to the following condition (see [1]):

(b) For any $z \in \mathbf{S}^{n-1}$(the unit sphere of $\mathbf{R}^n$), the function on $\mathbf{S}^{n-1}$ $\psi(\omega) := < z, \omega > (P_m(\omega))^{-1/m}$, where $\omega \in \mathbf{S}^{n-1}$, is non-degenerate at its critical points. This means, if $d_\omega \psi$, the differential of ψ at a point $\omega \in \mathbf{S}^{n-1}$ vanishes, then $d^2_\omega \psi$, the second order differential of ψ at this point is non-degenerate.

For every initial data $u_0 \in S(\mathbf{R}^n)$ (the Schwarz space), the solution of the Cauchy problem (1.1) is given by

$$u(t, \cdot) = e^{itP(D)} u_0 = \mathcal{F}^{-1}(e^{itP}) * u_0,$$

where $\mathcal{F}$ denotes the Fourier transform, $\mathcal{F}^{-1}$ its inverse, and $\mathcal{F}^{-1}(e^{itP})$ is understood in the distributional sense. From the ellipticity assumption on P, it is easy to find that $I(t, x) := \mathcal{F}^{-1}(e^{itP})(x)$ is an infinitely differentiable function in the x variable for every fixed $t \neq 0$ (see [3]).

When the symbol P is homogeneous, Miyachi [7] and Zheng et al. [11] considered the pointwise estimates of the oscillatory integral I and the $L^p - L^q$ estimates of the operator $e^{itP(D)}(t \neq 0)$. Dropping the homogeneity of P, Balabane et al. [1] and Cui [2, 3] studied the same estimates under the above non-degeneracy condition. We remark that the results of Balabane et al. are not sharp, while those of Cui are sharp estimates, but under the assumption of local t, i.e. $0 < |t| < T$. Here, sharpness means that the decay rate in the spatial variable is identical with that in the homogeneous case, namely, the decay rate is $-\frac{n(m-2)}{2(m-1)}$ (see [11]).

The purpose of this paper is to prove global pointwise time-space estimates and $L^p - L^q$ estimates of the fundamental solution of (1.1) for all $|t| > 0$. Our proof depends heavily on a decay estimate for the oscillatory integral $\mathcal{F}^{-1}(e^{itP})$. Compared with previous papers (see [1, 3, 6, 10, 11]), we estimate the oscillatory integral with two parameters, i.e. both the time variable and the spatial variable simultaneously. So we obtain the sharp decay in the spatial variable, even for $|t|$

large. Recently, our result was already generalized in [4]. But since the method applied there is different, this paper provides an alternative approach.

This paper is organized as follows. In Section 2, we make some pretreatment of the oscillatory integral $\mathcal{F}^{-1}(e^{itP})$, review the method of Balabane et al. [1] and Cui [3], and present some necessary lemmata. In Section 3, we prove global pointwise time-space estimates of the fundamental solution of (1.1) which is our main result. Finally, in §4 we use them to obtain the related $L^p - L^q$ estimates for the Schrödinger solution.

2. Preliminaries

Throughout this paper, we assume that $P : \mathbf{R}^n \to \mathbf{R}$ is always a non-degenerate elliptic inhomogeneous polynomial of order m where $n \geq 2$ and m is even. It is clear that P is non-degenerate if and only if $\det(\partial_i \partial_j P(\xi))_{n\times n}$ is an elliptic polynomial of order $n(m-2)$, which is also equivalent to $(H2)$ in [1], i.e. our condition (b).

We denote by $\mathbf{S}^{n-1}$ the unit sphere in $\mathbf{R}^n$, and by $(\rho,\omega) \in [0,\infty) \times \mathbf{S}^{n-1}$ the polar coordinates in $\mathbf{R}^n$. By the conditions on P, we know that $P_m(\xi) > 0$ for $\xi \neq 0$, which implies that there exists a large constant $a > 0$ with: For each fixed $s \geq a$ and $\omega \in \mathbf{S}^{n-1}$, the equation $P(\rho\omega) = s$ has an unique positive solution $\rho = \rho(s,\omega) \in C^\infty([a,\infty) \times \mathbf{S}^{n-1})$. By Lemma 2 in [1] we have

$$\rho(s,\omega) = s^{\frac{1}{m}} (P_m(\omega))^{-\frac{1}{m}} + \sigma(s,\omega), \tag{2.1}$$

where σ lies in the symbol class $S^0_{1,0}([a,\infty) \times \mathbf{S}^{n-1})$ (cf. [4]), i.e. $\sigma \in C^\infty([a,\infty) \times \mathbf{S}^{n-1})$. Moreover for every $k \in \mathbf{N}_0 := \{0,1,2,\cdots\}$ and every differential operator L_ω on $\mathbf{S}^{n-1}$ there exists a constant C_{kL} such that

$$|\partial_s^k L_\omega \sigma(s,\omega)| \leq C_{kL}(1+s)^{-k} \quad \text{for } s \geq a \text{ and } \omega \in \mathbf{S}^{n-1}. \tag{2.2}$$

We now recall two lemmata (see [1, 3]) on the estimates of the following phase function

$$\phi(s,\omega) := s^{-\frac{1}{m}} \rho(s,\omega) \langle u,\omega\rangle \quad \text{for } s \geq a \text{ and } \omega \in \mathbf{S}^{n-1},$$

with any fixed $u \in \mathbf{S}^{n-1}$. Since for every fixed $u_0 \in \mathbf{S}^{n-1}$ there exists a sufficiently small neighborhood $U_{u_0} \subset \mathbf{S}^{n-1}$ of u_0 such that the following lemmata always hold uniformly in $u \in U_{u_0}$(i.e. the constants in Lemma 2.1 and Lemma 2.2 are independent of u) we do not put the variable u in the function ϕ. Clearly, $\phi \in S^0_{1,0}([a,\infty)\times \mathbf{S}^{n-1})$.

Lemma 2.1. *There exists a constant $a_0 \geq a$ and an open cover $\{\Omega_0, \Omega_+, \Omega_-\}$ of $\mathbf{S}^{n-1}$ with $\Omega_+ \cap \Omega_- = \emptyset$ such that for $s \geq a_0$,*

(a) The function $\Omega_0 \ni \omega \mapsto \phi(s,\omega)$ has no critical points, and

$$\|d_\omega \phi(s,\omega)\| \geq c > 0 \quad \text{for } \omega \in \Omega_0, \tag{2.3}$$

where the constant c is independent of s.

(b) The function $\Omega_\pm \ni \omega \mapsto \phi(s,\omega)$ has a unique critical point $\omega_\pm \in C^\infty([a_0,\infty);\Omega'_\pm)$ for some open subset $\Omega'_\pm$ with $\overline{\Omega'}_\pm \subset \Omega_\pm$, respectively. Furthermore

$$\|(d^2_\omega \phi(s,\omega))^{-1}\| \leq c_0 \quad \text{for } \omega \in \Omega_\pm, \tag{2.4}$$

where the constant c_0 is independent of s. Moreover, $\lim_{s\to\infty}\omega_\pm(s)$ exist and

$$|\omega_\pm^{(k)}(s)| \leq c_k(1+s)^{-k-\frac{1}{m}} \quad \text{for } k \in \mathbf{N}.$$

Lemma 2.2. *Let $\phi_\pm(t,r,s) = st + rs^{\frac{1}{m}}\phi(s,\omega_\pm(s))$ for t, $r > 0$ and $s \geq a$. Then there exist constants $a_1 \geq a_0$ and $c_2 > c_1 > 0$ such that for $s \geq a_1$, $t > 0$, and $r > 0$,*

$$c_1 \leq \pm\phi(s,\omega_\pm(s)) \leq c_2, \tag{2.5}$$

$$\partial_s \phi_+(t,r,s) \geq t + c_1 r s^{\frac{1}{m}-1}, \tag{2.6}$$

$$t - c_2 r s^{\frac{1}{m}-1} \leq \partial_s \phi_-(t,r,s) \leq t - c_1 r s^{\frac{1}{m}-1}, \tag{2.7}$$

$$c_1 r s^{\frac{1}{m}-2} \leq |\partial_s^2 \phi_-(t,r,s)| \leq c_2 r s^{\frac{1}{m}-2}, \tag{2.8}$$

and

$$|\partial_s^k \phi_\pm(t,r,s)| \leq c_2 r s^{\frac{1}{m}-k} \quad \text{for } k = 2,3,\cdots. \tag{2.9}$$

Next, we consider the following oscillatory integral

$$\Phi(\lambda,s) = \int_{\mathbf{S}^{n-1}} e^{i\lambda\phi(s,\omega)} b(s,\omega)\, d\omega,$$

where $b(s,\omega) := s^{1-\frac{n}{m}}\rho^{n-1}\partial_s\rho \in S^0_{1,0}([a,\infty)\times\mathbf{S}^{n-1})$. Let φ_+, φ_-, φ_0 be a partition of unity of $\mathbf{S}^{n-1}$, subordinate to the open cover given in Lemma 2.1. Then

$$\Phi(\lambda,s) = \Phi_+(\lambda,s) + \Phi_-(\lambda,s) + \Psi_0(\lambda,s),$$

where

$$\Phi_\pm(\lambda, s) = \int_{\mathbf{S}^{n-1}} e^{i\lambda\phi(s,\omega)} b(s,\omega)\varphi_\pm(\omega)d\omega$$

and

$$\Psi_0(\lambda, s) = \int_{\mathbf{S}^{n-1}} e^{i\lambda\phi(s,\omega)} b(s,\omega)\varphi_0(\omega)d\omega.$$

By using the stationary phase method for Ψ_0, and Lemma 2.1 and Corollary 1.1.8 in [8] for $\Phi_\pm$, one has the following result.

Lemma 2.3. *For $\lambda > 0$ and $s > a_1$ we have*

$$\Phi(\lambda, s) = \lambda^{-\frac{n-1}{2}} e^{i\lambda\phi(s,\omega_+(s))}\Psi_+(\lambda, s) + \lambda^{-\frac{n-1}{2}} e^{i\lambda\phi(s,\omega_-(s))}\Psi_-(\lambda, s) + \Psi_0(\lambda, s), \tag{2.10}$$

where $\Psi_\pm$, $\Psi_0 \in C^\infty((0,\infty)\times[a_0,\infty))$ and

$$|\partial_\lambda^k \partial_s^j \Psi_\pm(\lambda, s)| \le c_{k,j}(1+\lambda)^{-k} s^{-j} \quad \text{for } k, j \in \mathbf{N}_0, \tag{2.11}$$

$$|\partial_\lambda^k \partial_s^j \Psi_0(\lambda, s)| \le c_{k,j,l}(1+\lambda)^{-l} s^{-j} \quad \text{for } k, j, l \in \mathbf{N}_0. \tag{2.12}$$

3. Estimates on the oscillatory integral

In this section we establish the global pointwise time-space estimates of the fundamental solution for the Schrödinger equation (1.1).

Theorem 3.1. *If the inhomogeneous polynomial P is elliptic and non-degenerate, then the fundamental solution of* (1.1) *satisfies that there exists a constant $C > 0$ such that*

$$|I(t,x)| = |\mathcal{F}^{-1}(e^{itP})(x)| \le \begin{cases} C|t|^{-\frac{n}{m}}(1+|t|^{-\frac{1}{m}}|x|)^{-\mu} & \text{for } \ 0 < |t| \le 1, \\ C|t|^{-\frac{1}{m}}(1+|t|^{-1}|x|)^{-\mu} & \text{for } \ |t| \ge 1, \end{cases} \tag{3.1}$$

where $\mu = \frac{n(m-2)}{2(m-1)}$.

Proof. We first consider

Case (i): $t \ge 1$ and $r := |x| \ge 1$.

Let $\psi \in C^\infty(\mathbf{R})$ such that $\psi(s) = \begin{cases} 0 & \text{for } s \le a_1 \\ 1 & \text{for } s > 2a_1 \end{cases}$, where a_1 is given in Lemma 2.2. We write

$$\begin{aligned} I(t,x) &= \mathcal{F}^{-1}(e^{itP})(x) = \int_{\mathbf{R}^n} e^{i(\langle x,\xi\rangle + tP(\xi))}\psi(P(\xi))d\xi \\ &+ \int_{\mathbf{R}^n} e^{i(\langle x,\xi\rangle + tP(\xi))}(1-\psi(P(\xi)))d\xi =: I_1(t,x) + I_2(t,x). \end{aligned}$$

First we rewrite I_2 as the Fourier transform of a measure, supported on the graph $S := \{z = P(\xi);\, \xi \in \mathbf{R}^n\} \subset \mathbf{R}^{n+1}$:

$$I_2(t,x) = \int_{\mathbf{R}^{n+1}} e^{i(\langle x,\xi\rangle + tz)}(1-\psi(P(\xi)))\delta(z - P(\xi))\, d\xi\, dz\,. \tag{3.2}$$

Since the polynomial P is of order m, the supporting manifold of the above integrand is of type m (in the sense of § VIII.3.2, [9]). Then, Theorem 2 of § VIII.3 in [9] implies

$$|I_2(t,x)| \le C(1+|t|+|x|)^{-\frac{1}{m}} \qquad \forall t,\, x. \tag{3.3}$$

This can be generalized: Since $f(t,\xi) := e^{itP}(1-\psi(P)) \in C_c^\infty(\mathbf{R}^n)$ for every $t > 0$, an integration by parts in I_2 yields

$$I_2(t,x) = i\int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle}\frac{x}{|x|^2}\cdot\nabla_\xi f(t,\xi)d\xi.$$

Proceeding recursively, a simple estimate yields

$$|I_2(t,x)| \le C_k t^k r^{-k} \quad \text{for } k \in \mathbf{N}_0,$$

and hence also $\forall\, k \ge 0$. But proceeding as in (3.2) yields the improvement

$$|I_2(t,x)| \le C_k|t|^{-\frac{1}{m}}(1+|t|^{-1}|x|)^{-(k+\frac{1}{m})} \text{ for } \ |t|\ge 1,\ x\in\mathbf{R}^n,\ \forall\ k \ge 0. \tag{3.4}$$

To estimate I_1, we shall derive an ε–uniform estimate of its regularization

$$J_\varepsilon(t,x) := \int_{\mathbf{R}^n} e^{-\varepsilon P(\xi)+i(\langle x,\xi\rangle+tP(\xi))}\psi(P(\xi))d\xi \quad \text{for } \varepsilon > 0.$$

By the polar coordinate transform and by the change of variables $\rho = \rho(s,\omega)$ we have

$$\begin{aligned} J_\varepsilon(t,x) &= \int_0^\infty\int_{\mathbf{S}^{n-1}} e^{-\varepsilon P(\rho\omega)+i(\rho\langle x,\omega\rangle+tP(\rho\omega))}\psi(P(\rho\omega))\rho^{n-1}d\omega d\rho \\ &= \int_0^\infty\int_{\mathbf{S}^{n-1}} e^{-\varepsilon s+its+ir\rho\langle u,\omega\rangle}\psi(s)\rho^{n-1}\partial_s\rho d\omega ds \\ &= \int_0^\infty e^{-\varepsilon s+its}s^{\frac{n}{m}-1}\psi(s)\Phi(rs^{\frac{1}{m}},s)ds, \end{aligned}$$

where $u = x/|x|$.

Due to the compactness of $\mathbf{S}^{n-1}$ we may assume without loss of generality that $u \in U_{u_0}$ (see section 2 for the definition of U_{u_0}). Thus by Lemma 2.3

$$\begin{aligned} J_\varepsilon(t,x) &= r^{-\frac{n-1}{2}} \int_0^\infty e^{-\varepsilon s + i\phi_+(t,r,s)} s^{\frac{n+1}{2m}-1} \psi(s) \Psi_+(rs^{\frac{1}{m}}, s) ds \\ &\quad + r^{-\frac{n-1}{2}} \int_0^\infty e^{-\varepsilon s + i\phi_-(t,r,s)} s^{\frac{n+1}{2m}-1} \psi(s) \Psi_-(rs^{\frac{1}{m}}, s) ds \\ &\quad + \int_0^\infty e^{-\varepsilon s + its} s^{\frac{n}{m}-1} \psi(s) \Psi_0(rs^{\frac{1}{m}}, s) ds \\ &= R_\varepsilon^+(t,x) + R_\varepsilon^-(t,x) + R_\varepsilon^0(t,x), \end{aligned}$$

where $\phi_\pm$ is the same as in Lemma 2.2. In the sequel, we denote by C a generic positive constant independent of t, r, s and ε, and put $\mu := \frac{n(m-2)}{2(m-1)}$ and $\nu := \frac{n}{2(m-1)}$.

We first estimate the integral $R_\varepsilon^0(t,x)$. Let $v_0(s) := s^{\frac{n}{m}-1}\psi(s)\Psi_0(rs^{\frac{1}{m}}, s)$. By the Leibniz rule and (2.12) one has

$$|v_0^{(k)}(s)| \leq C(rs^{\frac{1}{m}})^{-j} s^{\frac{n}{m}-1-k} \quad \text{for } j,k \in \mathbf{N}_0,$$

where $r \geq 1$ and $s \geq a_1$. Choosing $j \geq \mu$ and $k \geq \nu$, it follows by integration by parts that

$$|R_\varepsilon^0(t,x)| \leq Ct^{-k} \int_{a_1}^\infty (rs^{\frac{1}{m}})^{-j} s^{\frac{n}{m}-1-k} ds \leq Ct^{-k} r^{-j} \leq Ct^{-\nu} r^{-\mu}. \tag{3.5}$$

To estimate the integral $R_\varepsilon^+(t,x)$, for given $t, r \geq 1$ we set

$$\begin{cases} u_+(s) := -\varepsilon s + i\phi_+(t,r,s) \\ v_+(s) := s^{\frac{n+1}{2m}-1}\psi(s)\Psi_+(rs^{\frac{1}{m}}, s) \end{cases}$$

for $s \geq a_1$. Since $u_+'(s) \neq 0$ for $s \geq a_1$, we can define $D_* f = (gf)'$ for $f \in C^1(0,\infty)$ where $g = -1/u_+'$. It is not hard to show

$$D_*^j v_+ = \sum_\alpha c_\alpha g^{(\alpha_1)} \cdots g^{(\alpha_j)} v_+^{(\alpha_{j+1})} \quad \text{for } j \in \mathbf{N} \tag{3.6}$$

where the sum runs over all $\alpha = (\alpha_1, \cdots \alpha_{j+1}) \in \mathbf{N}_0^{j+1}$ such that $|\alpha| = j$ and $0 \leq \alpha_1 \leq \cdots \leq \alpha_j$. Since (2.6) and (2.9) imply, respectively, that $|g(s)| \leq Cr^{-1}s^{1-\frac{1}{m}}$ and

$$|u_+^{(k)}(s)| \leq Crs^{\frac{1}{m}-k} \quad \text{for } k = 2, 3, \cdots,$$

by induction on k we find that

$$|g^{(k)}(s)| \leq Cr^{-1}s^{1-\frac{1}{m}-k} \quad \text{for } k \in \mathbf{N}_0,$$

which shall yield the spatial decay of I_1. To derive the time decay of I_1 we note that (2.6) also implies $|g(s)| \le t^{-1}$. Hence, it follows that

$$|g^{(k)}(s)| \le Ct^{-1}s^{-k} \quad \text{for } k \in \mathbf{N}_0.$$

The novel key step is now to interpolate these two inequalities, which will allow to derive estimates also for large time. We have for any $\theta \in [0,1]$,

$$|g^{(k)}(s)| \le Ct^{\theta-1}r^{-\theta}s^{\theta(1-\frac{1}{m})-k} \quad \text{for } k \in \mathbf{N}_0. \tag{3.7}$$

On the other hand, by the Leibniz rule and (2.11),

$$|v_+^{(k)}(s)| \le Cs^{\frac{n+1}{2m}-1-k} \quad \text{for } k \in \mathbf{N}_0. \tag{3.8}$$

It thus follows from (3.6) - (3.8) that

$$|D_*^j v_+(s)| \le Ct^{j(\theta-1)}r^{-j\theta}s^{j\theta(1-\frac{1}{m})+\frac{n+1}{2m}-1-j} \quad \text{for } j \in \mathbf{N}_0, \tag{3.9}$$

where $D_*^0 v_+ = v_+$. Particularly ($\theta = \frac{\mu}{n} = \frac{m-2}{2(m-1)}$, $j = n$)

$$|D_*^n v_+(s)| \le Ct^{\mu-n}r^{-\mu}s^{-\frac{nm+n-1}{2m}-1}.$$

Noting that $\mu - n < -\nu$, by integration by parts one gets that

$$|R_\varepsilon^+(t,x)| = r^{-\frac{n-1}{2}}\Big|\int_0^\infty e^{u_+}(D_*^n v_+)ds\Big| \le Ct^{\mu-n}r^{-\frac{n-1}{2}-\mu} \le Ct^{-\nu}r^{-\mu}.$$

We now turn to the integral $R_\varepsilon^-(t,x)$. Here we put

$$\begin{cases} u_-(s) := -\varepsilon s + i\phi_-(t,r,s) \\ v_-(s) := s^{\frac{n+1}{2m}-1}\psi(s)\Psi_-(rs^{\frac{1}{m}}, s) \end{cases}$$

for $s \ge a_1$, and write

$$\begin{aligned} R_\varepsilon^-(t,x) &= r^{-\frac{n-1}{2}}\Big\{\int_0^{c_1' s_0} + \int_{c_1' s_0}^{c_2' s_0} + \int_{c_2' s_0}^\infty\Big\}e^{u_-(s)}v_-(s)ds \\ &= R_{\varepsilon 1}^-(t,x) + R_{\varepsilon 2}^-(t,x) + R_{\varepsilon 3}^-(t,x), \end{aligned}$$

where $s_0 = (r/t)^{\frac{m}{m-1}}$, $c_1' = (c_1/2)^{\frac{m}{m-1}}$, and $c_2' = (2c_2)^{\frac{m}{m-1}}$ (c_1 and c_2 are given in Lemma 2.2).

By integration by parts one gets

$$R_{\varepsilon 3}^-(t,x) = r^{-\frac{n-1}{2}}\Big(\frac{e^{u_-(c_2' s_0)}}{u_-'(c_2' s_0)}\sum_{j=0}^{n-1}(D_*^j v_-)(c_2' s_0) + \int_{c_2' s_0}^\infty e^{u_-}(D_*^n v_-)ds\Big).$$

Since (2.7) implies that $|u_-'(s)| \ge c_2 r s^{\frac{1}{m}-1}$ for $s \ge c_2' s_0$, we find that $v_-(s)$ still satisfies (3.9) (with $\theta = 1$) for $s \ge c_2' s_0$.

If $c_2's_0 \le a_1$, then $(D_*^j v_-)(c_2's_0) = 0$ for $j = 0, ..., n-1$ (note that $\psi \equiv 0$ on $[0, a_1]$). Integration by parts then yields

$$|R_{\varepsilon 3}^-(t,x)| = \left| r^{-\frac{n-1}{2}} \int_{c_2's_0}^{\infty} e^{u_-}(D_*^n v_-) ds \right| \le Ct^{-\nu} r^{-\mu},$$

exactly as done for $R_\varepsilon^+(t,x)$. If $c_2's_0 > a_1$, then

$$\begin{aligned} |R_{\varepsilon 3}^-(t,x)| &\le Cr^{-\frac{n-1}{2}} \Big((rs_0^{\frac{1}{m}-1})^{-1} \sum_{j=0}^{n-1} r^{-j} s_0^{-\frac{2j-n-1}{2m}-1} + \int_{c_2's_0}^{\infty} r^{-n} s^{-\frac{n-1}{2m}-1} ds \Big) \\ &\le Cr^{-\frac{n-1}{2}} (r^{-1} s_0^{\frac{n-1}{2m}} \sum_{j=0}^{n-1} (rs_0^{\frac{1}{m}})^{-j} + r^{-n} s_0^{-\frac{n-1}{2m}}). \end{aligned}$$

Noting that $r \ge 1$, $s_0 \ge a_1/c_2'$, and $t \ge 1$ it follows that

$$|R_{\varepsilon 3}^-(t,x)| \le Cr^{-\frac{n+1}{2}} s_0^{\frac{n+1}{2m}} = Ct^{-\frac{n+1}{2(m-1)}} r^{-\frac{(n+1)(m-2)}{2(m-1)}} \le Ct^{-\nu} r^{-\mu}.$$

Since $|u_-'(s)| \ge \frac{1}{2} c_1 r s^{\frac{1}{m}-1}$ for $a_1 \le s \le c_1' s_0$, a slight modification of the above method yields the same estimate for $R_{\varepsilon 1}^-(t,x)$.

To estimate $R_{\varepsilon 2}^-(t,x)$, it suffices to estimate the integral

$$\begin{aligned} R_{02}^-(t,x) &= r^{-\frac{n-1}{2}} \int_{c_1's_0}^{c_2's_0} e^{i\phi_-(t,r,s)} v_-(s) ds \\ &= r^{-\frac{n-1}{2}} s_0 \int_{c_1'}^{c_2'} e^{i\phi_-(t,r,s_0\tau)} v_-(s_0\tau) d\tau. \end{aligned}$$

We note by (2.8) that

$$|\partial_\tau^2 \phi_-(t,r,s_0\tau)| \ge c_1 r s_0^2 (s_0\tau)^{\frac{1}{m}-2} \ge Crs_0^{\frac{1}{m}}$$

for $\tau \in [c_1', c_2']$. Since $v_-(s)$ also satisfies (3.8), Van der Corput's lemma (cf. [9]) implies

$$\begin{aligned} |R_{02}^-(t,x)| &\le Cr^{-\frac{n-1}{2}} s_0 (rs_0^{\frac{1}{m}})^{-\frac{1}{2}} \Big(|v_-(c_2's_0)| + \int_{c_1'}^{c_2'} |s_0 v_-'(s_0\tau)| d\tau \Big) \\ &\le Cr^{-\frac{n-1}{2}} s_0 (rs_0^{\frac{1}{m}})^{-\frac{1}{2}} s_0^{\frac{n+1}{2m}-1} \\ &= Ct^{-\nu} r^{-\mu}. \end{aligned}$$

Since the dominated convergence theorem implies that $J_\varepsilon(t,\cdot)$ converges (as $\varepsilon \to 0$) uniformly for x in compact subsets of $\{x \in \mathbf{R}^n;\ |x| \ge 1\}$, summarizing the above estimates yields

$$|I_1(t,x)| \le Ct^{-\nu}|x|^{-\mu} \quad \text{for } t \ge 1 \text{ and } |x| \ge 1.$$

If $t \geq 1$, $|x| \geq 1$ and $t^{-1}|x| \geq 1$, then

$$|I_1(t,x)| \leq Ct^{-\nu}|x|^{-\mu} \leq Ct^{-\frac{n}{2}}(1+t^{-1}|x|)^{-\mu} \leq Ct^{-\frac{1}{m}}(1+t^{-1}|x|)^{-\mu}. \quad (3.10)$$

Combining this with the estimate (3.4) on I_2 (put $k = \mu - \frac{1}{m}$), we have

$$|I(t,x)| \leq Ct^{-\frac{1}{m}}(1+t^{-1}|x|)^{-\mu} \quad \text{for } t \geq 1,\ |x| \geq 1 \quad \text{and } t^{-1}|x| \geq 1.$$

If $t \geq 1$, $|x| \geq 1$ and $t^{-1}|x| < 1$, then

$$|I_1(t,x)| \leq Ct^{-\frac{1}{m}} \leq Ct^{-\frac{1}{m}}(1+t^{-1}|x|)^{-\mu}.$$

Combining this with (3.4) yields again

$$|I(t,x)| \leq Ct^{-\frac{1}{m}}(1+t^{-1}|x|)^{-\mu}.$$

Case (ii): $t \geq 1$, $|x| \leq 1$.
For I_1 we shall prove now that

$$|I_1(t,x)| \leq C|t|^{-n/2} \quad \text{for } |t| \geq 1 \text{ and } |x| \leq |t|. \quad (3.11)$$

To this end we write the integral $I_1(t,x)$ as follows:

$$I_1(t,x) = \int_{\mathbf{R}^n} e^{it(P(\xi)+\langle x/t,\xi\rangle)}\psi(P(\xi))d\xi =: \int_{\mathbf{R}^n} e^{it\Phi(\xi,x,t)}\psi(P(\xi))d\xi.$$

Note that this integral and the subsequent integrations by parts can be made meaningful by inserting a series of smooth cut-off functions $\phi(\epsilon\xi)$ for any $0 < \epsilon < 1$. However, this is just a technical procedure, and we refer to [4] for the details in a similar situation.

Since $|x/t| \leq 1$ and $|\nabla P(\xi)| \geq c|\xi|^{m-1}$ for large $|\xi|$, the possible critical points satisfying

$$\nabla_\xi \Phi(\xi,x,t) = \nabla P(\xi) + x/t = 0$$

must be located in some bounded ball. In order to apply later the stationary phase principle, let $\Omega \subset \mathbf{R}^n$ be some open set such that $\mathrm{supp}\psi(P) \subset \Omega$ and $|\nabla P(\xi)| \geq c|\xi|^{m-1}$ on Ω. Note that the constant a_1 (from the definition of ψ and Lemma 2.2) could be increased, if necessary, such that both of those conditions can hold. Then we decompose Ω into $\Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \{\xi \in \Omega;\ \ |\nabla P(\xi) + \frac{x}{t}| < \frac{1}{2}|\nabla P(\xi)| + 1\}$$

and

$$\Omega_2 = \{\xi \in \Omega;\ \ |\nabla P(\xi) + \frac{x}{t}| > \frac{1}{4}|\nabla P(\xi)|\}.$$

Since $|\frac{x}{t}| \le 1$ and $|\nabla P(\xi)| \to \infty$ as $|\xi| \to \infty$, Ω_1 must be a bounded domain and includes all critical points of Φ inside Ω. Now we choose smooth functions $\eta_1(\xi)$ and $\eta_2(\xi)$ such that $\mathrm{supp}\eta_j \subset \Omega_j$ and $\eta_1(\xi) + \eta_2(\xi) = 1$ in Ω (e.g. see [4] for a similar construction). And we decompose I_1 as

$$I_1(t,x) = I_{11}(t,x) + I_{12}(t,x), \quad I_{1j}(t,x) := \int_{\mathbf{R}^n} e^{it\Phi(\xi,x,t)}\eta_j(\xi)\psi(P(\xi))d\xi, \ j = 1,2.$$

First we estimate I_{11}: Note that the determinant of the Hessian matrix

$$\det(\partial_{\xi_i}\partial_{\xi_j}\Phi)_{n\times n}(\xi,x,t) = \det(\partial_{\xi_i}\partial_{\xi_j}P)_{n\times n}(\xi)$$

is an elliptic polynomial according to our assumption (a) and the remarks in the first paragraph of Section 2. Hence, it is nonzero on Ω (if necessary, we can increase the value of a_1 to satisfy the requirement), that is, the Hessian matrix is non-degenerate on Ω. Moreover, $|\partial_\xi^\alpha \Phi| \le C_\alpha$ on Ω_1 for any multi-index $\alpha \in \mathbf{N}_0^n$. Hence we obtain by the stationary phase principle that

$$|I_{11}(t,x)| \le C|t|^{-n/2}.$$

Next we estimate I_{22}: Note that $|\nabla_\xi \Phi| = |\nabla P(\xi) + \frac{x}{t}| \ge \frac{1}{4}|\nabla P(\xi)| \ge c|\xi|^{m-1}$ for $\xi \in \Omega_2$ and $|\partial_\xi^\alpha \Phi| \le C_\alpha |\xi|^{m-\alpha}$ for $|\alpha| \ge 2$. Now we define the operator L by

$$Lf := \frac{\langle \nabla_\xi \Phi, \nabla_\xi \rangle}{it|\nabla_\xi \Phi|^2} f.$$

Since $Le^{it\Phi} = e^{it\Phi}$, we obtain by N iterated integrations by parts:

$$\begin{aligned} |I_{12}(t,x)| &= \left| \int_{\mathbf{R}^n} e^{it\Phi(\xi,x,t)} (L^*)^N (\eta_2(\xi)\psi(P(\xi))) d\xi \right| \\ &\le C_N |t|^{-N} \int_{\mathrm{supp}\psi(P)} |\xi|^{-mN} d\xi \le C_N' |t|^{-N}, \end{aligned}$$

where $N > n$ and L^* is the adjoint operator of L. Combining the two cases yields the claimed estimate $|I_1| \le C|t|^{-n/2}$ for $|t| > 1$ and $|x| \le |t|$.

Together with the estimate (3.4) (with $k = \mu - \frac{1}{m}$) on I_2 this yields

$$|I(t,x)| \le Ct^{-\frac{1}{m}}(1 + t^{-1}|x|)^{-\mu} \quad \text{for } t \ge 1 \text{ and } x \in \mathbf{R}^n. \tag{3.12}$$

Case (iii): $t \in (0,1)$ and $x \in \mathbf{R}^n$.
Here, we observe that

$$\int_{\mathbf{R}^n} e^{i(\langle x,\xi\rangle + tP(\xi))} d\xi = t^{-\frac{n}{m}} \int_{\mathbf{R}^n} e^{i(\langle t^{-\frac{1}{m}}x,\xi\rangle + tP(t^{-\frac{1}{m}}\xi))} d\xi.$$

Let $P_t(\xi) = tP(t^{-\frac{1}{m}}\xi)$, $\rho_t(s,\omega) = t^{\frac{1}{m}}\rho(\frac{s}{t},\omega)$, and $\sigma_t(s,\omega) = t^{\frac{1}{m}}\sigma(\frac{s}{t},\omega)$, then (2.1) still holds with P, ρ, σ replaced respectively by P_t, ρ_t, σ_t. Since it is easy to check that σ_t also satisfies (2.2) with the same constants C_{kL}, we can deduce from (3.12) (with $t = 1$) that

$$|I(t,x)| \leq Ct^{-\frac{n}{m}}(1 + t^{-\frac{1}{m}}|x|)^{-\mu} \quad \text{for } t \in (0,1) \text{ and } x \in \mathbf{R}^n. \tag{3.13}$$

And the proof for negative t is analogous. This completes the proof of the theorem. □

Remark 3.2. If P is homogeneous and non-degenerate, then by scaling the estimates (3.1), one recovers the following sharp form in the (t,x)-variables (see [11]):

$$|\mathcal{F}^{-1}(e^{itP(\xi)})(x)| \leq Ct^{-\frac{n}{m}}(1 + |t|^{-1/m}|x|)^{-\mu} \text{ for } t \neq 0.$$

In particular, we remark that the index $\mu = \frac{n(m-2)}{2(m-1)}$ is optimal by testing the special case $e^{i|\xi|^m}$. In fact, from Proposition 5.1(ii) in [7], p. 289, there exists a positive constant c such that

$$|\mathcal{F}^{-1}(e^{i|\cdot|^m})(x)| \geq c(1 + |x|)^{-\mu} \text{ for } x \in \mathbf{R}^n.$$

Remark 3.3. The decay estimate (3.3) on I_2 can be improved under the additional assumption that $P(\xi)$ has only non-degenerate critical points (or, equivalently, for a nonzero Gaussian curvature of the hypersurface S) inside the support of $(1 - \psi(P(\xi))$. Then, Theorem 1 of § VIII.3 in [9] implies:

$$|I_2(t,x)| \leq C(1 + |t| + |x|)^{-\frac{n}{2}} \qquad \forall t,\, x.$$

E.g., this assumption holds if $m = 2$ or in the example $P(\xi) = |\xi|^4 + |\xi|^2$.

An intermediate decay result for I_2 holds, if the Hessian of P has at least rank k $(1 \leq k \leq n)$ inside the support of $(1 - \psi(P(\xi))$ (or, eqivalently, if S has at least k nonzero principal curvatures there). Then we have $I_2 = \mathcal{O}\left((1 + |t| + |x|)^{-k/2}\right)$ by Littman's Theorem (cf. § VIII.5.8 in [9]).

Remark 3.4. An analogous method as above leads to

$$|\partial^\alpha I(t,x)| = |\mathcal{F}^{-1}(\xi^\alpha e^{itP(\xi)})(x)| \leq \begin{cases} C|t|^{-\frac{n}{m}}(1 + |t|^{-\frac{1}{m}}|x|)^{-\mu} \text{ for } 0 < |t| \leq 1, \\ C|t|^{-\frac{1}{m}}(1 + |t|^{-1}|x|)^{-\mu} \text{ for } |t| \geq 1, \end{cases}$$

where $\alpha \in \mathbf{Z}_+^n$, $|\alpha| = b$, $0 \leq b \leq \frac{mn-2n}{2}$ and $\mu = \frac{mn-2n-2b}{2(m-1)}$.

4. Decay/growth estimates for Schrödinger equations

Here we shall apply Theorem 3.1 to establish $L^p - L^q$ estimates for (1.1). Since $P(D)$ is self-adjoint in $L^2(\mathbf{R}^n)$, we have $\|e^{itP(D)}u_0\|_{L^2} = \|u_0\|_{L^2}$ for all $0 \le |t| < \infty$ by Stone's theorem. Next we define the following set of admissible index pairs:

$\triangle := \{(p,q);\ (\frac{1}{p}, \frac{1}{q})$ lies in the closed quadrilateral $ABCD$ subtracting the apex $A\}$,

where $A = (\frac{1}{2}, \frac{1}{2})$, $B = (1, \frac{1}{\tau})$, $C = (1, 0)$, and $D = (\frac{1}{\tau'}, 0)$ for $\tau = \frac{2(m-1)}{m-2}$ and $\frac{1}{\tau} + \frac{1}{\tau'} = 1$. Moreover, we denote by H^1 the Hardy space on $\mathbf{R}^n$ and by BMO the space of functions with bounded mean oscillation on $\mathbf{R}^n$.

Theorem 4.1. *Let the assumption of Theorem* 3.1 *be satisfied. Then*

$$\|e^{itP(D)}u_0\|_{L^q} \le \begin{cases} C|t|^{\frac{n}{m}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^p} & \text{for } \ 0 < |t| \le 1, \\ C|t|^{n|\frac{1}{q}-\frac{1}{p'}|-\frac{1}{m}}\|u_0\|_{L^p} & \text{for } \ |t| \ge 1, \end{cases} \tag{4.1}$$

where $(p,q) \in \triangle$, *but* $(p,q) \ne (1,\tau),\ (\tau', \infty)$. *When* $(p,q) = (1,\tau)$ *(resp.* (τ',∞)*),* (4.1) *still holds if* L^1 *(resp.* L^∞*) is replaced by* H^1 *(resp. BMO).*

Proof. When $(\frac{1}{p}, \frac{1}{q})$ lies in the edge BC, but $(\frac{1}{p}, \frac{1}{q}) \ne$ B (i.e., $p = 1$ and $\tau < q \le \infty$), it follows from Young's inequality and Theorem 3.1 that

$$\|e^{itP(D)}u_0\|_{L^q} \le \|\mathcal{F}^{-1}(e^{itP})\|_{L^q}\|u_0\|_{L^1} \le \begin{cases} C|t|^{\frac{n}{m}(\frac{1}{q}-1)}\|u_0\|_{L^1} & \text{for } \ 0 < |t| \le 1, \\ C|t|^{\frac{n}{q}-\frac{1}{m}}\|u_0\|_{L^1} & \text{for } \ |t| \ge 1. \end{cases} \tag{4.2}$$

When $(\frac{1}{p}, \frac{1}{q}) =$ B (i.e., $p = 1$ and $q = \tau$), this estimate (with L^1 replaced by H^1) follows from the boundedness of the Riesz potential $I_{n/\tau'}$ (cf. [9], p.136). This proves the points $(1, \frac{1}{q})$ in the side $\overline{CB}$. Now in view of (4.2), by the Marcinkiewicz interpolation theorem (see [5], p.56), we can conclude the proof of (4.1) for the points in the closed triangle ABC. Next, by duality the desired arguments for the triangle ADC follow immediately from the results in the triangle ABC. This completes the proof of the theorem. □

Remark 4.2. Let $\Omega = \{\xi \in \mathbf{R}^n : |\xi| > a\}$ for some sufficiently large a with $\text{supp}\mathcal{F}u_0 \subset \Omega$. Also let $(p,q) \in \triangle$, but $(p,q) \ne (1,\tau),\ (\tau',\infty)$. First we note that (3.10) and (3.11) combine into

$$|I_1(t,x)| \le Ct^{-\frac{n}{2}}(1 + |t|^{-1}|x|)^{-\mu} \le C|t|^{-\frac{n}{m}}(1 + |t|^{-\frac{1}{m}}|x|)^{-\mu} \text{ for } \ |t| \ge 1.$$

Similarly to the above proof, this estimate implies

$$\|e^{itP(D)}u_0\|_{L^q} = \|I_1(t,\cdot) * u_0\|_{L^q} \le C|t|^{\frac{n}{m}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^p} \text{ for } |t| > 0. \tag{4.3}$$

When $(p,q) = (1,\tau)$ (resp. (τ',∞)), (4.3) still holds if L^1 (resp. L^∞) is replaced by H^1 (resp. BMO).

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