ASC Report No. 03/2011

Sums, Couplings, and Completions of Almost Pontryagin Spaces

Henk de Snoo, Harald Woracek

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-04-9

Most recent ASC Reports

02/2011 *Ansgar Jüngel, Josipa-Pina Milišić*
Full Compressible Navier-Stokes Equations for Quantum Fluids: Derivation and Numerical Solution

01/2011 *Sofi Esterhazy, Jens Markus Melenk*
On Stability of Discretizations of the Helmholtz Equation (extended version)

36/2010 *Tobias Wurzer*
Stability of the Trace of the Polynomial L^2-projection an Triangles

35/2010 *Markus Aurada, Michael Feischl, Michael Karkulik, Dirk Praetorius*
Adaptive Coupling of FEM and BEM: Simple Error Estimators and Convergence

34/2010 *Michael Feischl, Marcus Page, Dirk Praetorius*
Convergence and Quasi-Optimality of Adaptive FEM with Inhomogeneous Dirichlet Data

33/2010 *Michael Feischl, Marcus Page, Dirk Praetorius*
Convergence of Adaptive FEM for some Elliptic Obstacle Problem with Inhomogeneous Dirichlet Data

32/2010 *Othmar Koch, Friedrich Kupka, Bernhard Löw-Baselli, A. Mayrhofer, F. Zaussinger*
SDIRK Methods for the ANTARES Code

31/2010 *Roland Donninger, Birgit Schörkhuber, Peter C. Aichelburg*
On Stable Self-similar Blow Up for Equivalent Wave Maps: The Linearized Problem

30/2010 *Matthias Langer, Harald Woracek*
The Exponential Type of the Fundamental Solution of an Indefinite Hamiltonian System

29/2010 *Harald Woracek*
An Addendum to M.G. Krein's Inverse Spectral Theorem for Strings

Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: `admin@asc.tuwien.ac.at`
WWW: `http://www.asc.tuwien.ac.at`
FAX: `+43-1-58801-10196`

ISBN 978-3-902627-04-9

ASC TU WIEN

Sums, couplings, and completions of almost Pontryagin spaces

Henk de Snoo, Harald Woracek

Abstract

An almost Pontryagin space can be written as the direct and orthogonal sum of a Hilbert space, a finite-dimensional anti-Hilbert space, and a finite-dimensional neutral space. In this paper various constructions with such spaces are considered. In particular, orthogonal sums of almost Pontryagin spaces and completions to almost Pontryagin spaces are treated in detail.

AMS MSC 2010: Primary 46 C 20, Secondary 15 A 63
Keywords: almost Pontryagin space, degenerate space, orthogonal coupling, completion.

1 Introduction

The notion of an almost Pontryagin space was introduced in [KWW] as a generalization of the more familiar notion of a Pontryagin space. A Pontryagin space is an inner product space which can be written as the direct and orthogonal sum of a Hilbert space and a finite-dimensional anti-Hilbert space, whereas an almost Pontryagin space can be written as the direct and orthogonal sum of a Hilbert space, a finite-dimensional anti-Hilbert space, and a finite-dimensional neutral space.

The introduction of these more general objects was motivated by several classical interpolation and extrapolation problems. The following example may be illuminating. Let the continuous function $f : [-2a, 2a] \to \mathbb{C}$ be hermitian (i.e., $f(-t) = \overline{f(t)}$) with κ negative squares (i.e., the kernel $f(t-s)$, $s, t \in (-a, a)$ has κ negative squares). Then f has exactly one continuous hermitian extension to $\mathbb{R}$ with κ negative squares or it has infinitely many continuous hermitian extensions to $\mathbb{R}$ with κ negative squares. In the latter case f has also infinitely many continuous hermitian extensions to $\mathbb{R}$ with κ_1 negative squares for every $\kappa_1 \geq \kappa$. This result originates from the usual operator theoretic considerations involving the Pontryagin space induced by the problem. However, in the first case of the alternative it turns out that there exists a number $0 < \Delta \leq \infty$ such that f has no continuous hermitian extensions to $\mathbb{R}$ with κ_1 negative squares for $\kappa < \kappa_1 < \kappa + \Delta$, and infinitely many continuous hermitian extensions to $\mathbb{R}$ with κ_1 negative squares for $\kappa_1 \geq \kappa + \Delta$, cf. [KW1]. This addition to the case where f has a unique extension originates from operator theoretic considerations involving an almost Pontryagin space induced by the problem. For other appearances of almost Pontryagin spaces (sometimes only implicitly), see [W], [KW2], [KW3], [PT].

In order to treat a broad range of classical problems involving degenerate cases it is necessary to develop an extension theory for symmetric operators or relations in almost Pontryagin spaces. The theory of such extensions depends on various geometric operations within the class of almost Pontryagin spaces. It is the purpose of the present paper to make available some such constructions. Although we are mainly having in mind our needs in the forthcoming treatment of exit space extensions of symmetric relations in [SW1], [SW2], [SW3], we believe that the general geometric theory discussed in the present paper is also of independent interest.

The paper is organized in six sections. After this introduction, in Section 2, we recall some facts about almost Pontryagin spaces. In Section 3 we deal with direct (but not necessarily orthogonal) sums of general inner product spaces, the topic considered in Section 4 is orthogonal coupling of inner product spaces. The problem to associate a Pontryagin space with a given almost Pontryagin space can be solved via factorization or by extension. This topic is treated in Section 5. In Section 6 we investigate almost Pontryagin space completions, a topic which has already been adressed in [KWW]. In the present paper we use a different approach, which gives more complete and structured results.

Our standard reference for the geometry of inner product spaces is [B]. For Pontryagin space theory, we also refer the reader to [IKL].

2 Preliminaries on almost Pontryagin spaces

We start with recalling the definition of almost Pontryagin spaces and their morphisms.

An inner product space is a pair consisting of a linear space $\mathcal{L}$ and an inner product $[.,.]$ on $\mathcal{L}$. We will usually not mention the inner product $[.,.]$ explicitly, and speak of an inner product space $\mathcal{L}$. The negative index of an inner product space $\mathcal{L}$ is defined as

$$\operatorname{ind}_- \mathcal{L} := \sup \Big\{ \dim \mathcal{N} : \mathcal{N} \text{ negative subspace of } \mathcal{L} \Big\} \in \mathbb{N}_0 \cup \{\infty\} ,$$

where a subspace $\mathcal{N}$ of $\mathcal{L}$ is called negative, if $[x,x] < 0$, $x \in \mathcal{N} \setminus \{0\}$. Moreover, $\mathcal{L}^\circ$ denotes the isotropic part of $\mathcal{L}$, i.e. $\mathcal{L}^\circ := \mathcal{L} \cap \mathcal{L}^\perp$, and $\operatorname{ind}_0 \mathcal{L} := \dim \mathcal{L}^\circ$ is called the degree of degeneracy of $\mathcal{L}$. If $\operatorname{ind}_- \mathcal{L} = 0$, we speak of a nondegenerated inner product space, otherwise we call $\mathcal{L}$ degenerated.

2.1 Definition. An almost Pontryagin space (aPs, for short) is a triple $\langle \mathcal{A}, [.,.], \mathcal{T} \rangle$ consisting of a linear space $\mathcal{A}$, an inner product $[.,.]$ on $\mathcal{A}$, and a topology $\mathcal{T}$ on $\mathcal{A}$, such that

- **(aPs1)** $\mathcal{T}$ is a Banach space topology on $\mathcal{A}$;
- **(aPs2)** $[.,.]$ is $\mathcal{T}$-continuous;
- **(aPs3)** There exists a $\mathcal{T}$-closed linear subspace $\mathcal{M}$ of $\mathcal{A}$ with finite codimension such that $\langle \mathcal{M}, [.,.] \rangle$ is a Hilbert space.

//

Again we will often suppress explicit notation of the inner product $[.,.]$ and the topology $\mathcal{T}$, and speak of an almost Pontryagin space $\mathcal{A}$.

Note that the subspace $\mathcal{M}$ in (aPs3) is complemented in the Banach space $\mathcal{A}$. With help of the open mapping theorem, one can easily deduce that the topology $\mathcal{T}$ is actually induced by some Hilbert space inner product on $\mathcal{A}$.

2.2 Remark. In order to provide a more concrete picture of almost Pontryagin spaces, let us recall [KWW, Proposition 2.5].

(*i*) Let $\mathcal{A}$ be an almost Pontryagin space. Then there exist closed subspaces $\mathcal{A}_+$ and $\mathcal{A}_-$ of $\mathcal{A}$, such that $\langle \mathcal{A}_+, [.,.]\rangle$ is a Hilbert space, $\langle \mathcal{A}_-, -[.,.]\rangle$ is a negative subspace with $\dim \mathcal{A}_- = \operatorname{ind}_- \mathcal{A} < \infty$, and

$$\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{A}_-[\dot{+}]\mathcal{A}^\circ .$$

where '$[\dot{+}]$' denotes a direct and orthogonal sum.

(*ii*) Let $\langle \mathcal{A}_+, [.,.]_+\rangle$ be a Hilbert space, let $\langle \mathcal{A}_-, [.,.]_-\rangle$ be a finite dimensional negative inner product space, and let $\mathcal{A}_0$ be a finite dimensional linear space. Let $\mathcal{A}_0$ be endowed with the euclidean topology, and let $\mathcal{A}_+$ and $\mathcal{A}_-$ carry their natural topologies induced by the inner product.

We define a linear space $\mathcal{A}$ as

$$\mathcal{A} := \mathcal{A}_+ \times \mathcal{A}_- \times \mathcal{A}^\circ ,$$

an inner product on $\mathcal{A}$ as

$$\big[(x_+, x_-, x_0), (y_+, y_-, y_0)\big] := [x_+, y_+]_+ + [x_-, y_-], \qquad (x_+, x_-, x_0), (y_+, y_-, y_0) \in \mathcal{A} ,$$

and a topology on $\mathcal{A}$ as the product topology of $\mathcal{A}_+$, $\mathcal{A}_-$, and $\mathcal{A}_0$. Then $\mathcal{A}$ is an almost Pontryagin space.

//

2.3 Remark. Pontryagin spaces form a subclass of almost Pontryagin spaces. In fact, if $\langle \mathcal{A}, [.,.], \mathcal{T}\rangle$ is an aPs, then $\langle \mathcal{A}, [.,.]\rangle$ is a Pontryagin space if and only if $\operatorname{ind}_0 \mathcal{A} = 0$. Conversely, let $\langle \mathcal{A}, [.,.]\rangle$ be a Pontryagin space. If $\mathcal{T}$ denotes the natural topology of $\mathcal{A}$, then $\langle \mathcal{A}, [.,.], \mathcal{T}\rangle$ is an aPs. These facts have been shown in [KWW, Corollar 2.7]. //

2.4 Definition. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces. A map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is called a morphism from $\mathcal{A}_1$ to $\mathcal{A}_2$, if it is linear, isometric, continuous, and maps closed subspaces of $\mathcal{A}_1$ onto closed subspaces of $\mathcal{A}_2$.

A morphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is called an isomorphism, if there exists a morphism $\psi : \mathcal{A}_2 \to \mathcal{A}_1$, such that $\psi \circ \phi = \operatorname{id}_{\mathcal{A}_1}$ and $\phi \circ \psi = \operatorname{id}_{\mathcal{A}_2}$. //

Next we recall some basic results concerning almost Pontryagin spaces. Proofs of these facts can be found in [KWW, §3].

2.5 Remark.

(*i*) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces, and let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ be linear and isometric. Then $\phi^{-1}\big([\operatorname{ran} \phi]^\circ\big) = \mathcal{L}_1^\circ$. In particular, $\ker \phi \subseteq \mathcal{L}_1^\circ$. Hence, if $\mathcal{L}_1$ is nondegenerated, then ϕ is injective.

(*ii*) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Pontryagin spaces with $\operatorname{ind}_- \mathcal{A}_1 = \operatorname{ind}_- \mathcal{A}_2$, and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a map. Then ϕ is a morphism if and only if ϕ is linear and isometric.

(*iii*) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be aPs and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a map. If ϕ is linear, isometric, continuous and surjective, then ϕ is a morphism.

(*iv*) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be aPs and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a map. Then ϕ is an isomorphism if and only if ϕ is linear, isometric, continuous, and bijective.

(*v*) Let $\mathcal{A}$ be an aPs and let $\mathcal{A}_0$ be a closed subspace of $\mathcal{A}$. Then $\mathcal{A}_0$ is, with the inner product and topology naturally inherited from $\mathcal{A}$, an aPs. The set-theoretic inclusion map $\subseteq: \mathcal{A}_0 \to \mathcal{A}$ is a morphism.

(*vi*) Let $\mathcal{A}$ be an aPs and let $\mathcal{B}$ be a linear subspace of $\mathcal{A}^\circ$. Then $\mathcal{A}/_{\mathcal{B}}$ is, with the inner product and topology naturally inherited from $\mathcal{A}$, an aPs. The canonical projection $\pi : \mathcal{A} \to \mathcal{A}/_{\mathcal{B}}$ is a morphism.

(*vii*) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be aPs and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism. Then there exits a unique isomorphism $\tilde{\phi} : \mathcal{A}_1/_{\ker\phi} \to \operatorname{ran}\phi$, such that

$$
\begin{array}{ccc}
\mathcal{A}_1 & \xrightarrow{\ \phi\ } & \mathcal{A}_2 \\
{\scriptstyle \pi}\downarrow & & \uparrow{\scriptstyle \subseteq} \\
\mathcal{A}_1/_{\ker\phi} & \xrightarrow[\ \tilde{\phi}\ ]{} & \operatorname{ran}\phi
\end{array}
$$

//

3 Direct sums of inner product spaces

In this section we formalize decompositions of an inner product space into a direct, but not necessarily orthogonal, sum. When considering just the inner product structure, this construction is completely elementary, one might say trivial, and is carried out only to provide the appropriate machinery. Things change, however, when turning to almost Pontryagin spaces; including topological aspects into the discussion makes matters significantly more involved.

In order to motivate the below definition, consider an inner product space $\mathcal{L}$ and two linear subspaces $\mathcal{L}_1, \mathcal{L}_2$ of $\mathcal{L}$. Then $\mathcal{L}_1$ and $\mathcal{L}_2$ are themselves inner product spaces, namely with the inner product inherited from $\mathcal{L}$. Each element $x_1 \in \mathcal{L}_1$ gives rise to a linear functional on $\mathcal{L}_2$, namely by $[.,x_1]_{\mathcal{L}} : x_2 \mapsto [x_2,x_1]_{\mathcal{L}}$. Moreover, the map

$$
c : \begin{cases} \mathcal{L}_1 & \to & \mathcal{L}_2^* \\ x_1 & \mapsto & [.,x_1]_{\mathcal{L}} \end{cases} \tag{3.1}
$$

where $\mathcal{L}_2^*$ denotes the algebraic dual of $\mathcal{L}_2$, is conjugate linear. Clearly, the inner product of arbitrary elements of $\mathcal{L}_1 + \mathcal{L}_2$ can be recovered as

$$
\begin{aligned}
\big[x_1+x_2, y_1+y_2\big]_{\mathcal{L}} = [x_1,y_1]_{\mathcal{L}_1} + \overline{c(x_1)y_2} + c(y_1)x_2 + [x_2,y_2]_{\mathcal{L}_2},& \\
x_1,y_1 \in \mathcal{L}_1,\ x_2,y_2 \in \mathcal{L}_2\,.&
\end{aligned} \tag{3.2}
$$

3.1 Definition. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two inner product spaces, whose inner products are denoted by $[.,.]_1$ and $[.,.]_2$, respectively. Moreover, let

$$
c : \mathcal{L}_1 \to \mathcal{L}_2^*
$$

be a conjugate linear map of $\mathcal{L}_1$ into the algebraic dual space of $\mathcal{L}_2$. Denote by $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$ the inner product space whose carrier vector space is equal to $\mathcal{L}_1 \times \mathcal{L}_2$ and whose inner product is defined as

$$\big[(x_1,x_2),(y_1,y_2)\big]_c := [x_1,y_1]_1 + \overline{c(x_1)y_2} + c(y_1)x_2 + [x_2,y_2]_2, \qquad x_1,y_1 \in \mathcal{L}_1,\ x_2,y_2 \in \mathcal{L}_2$$

//

The fact that $[.,.]_c$ actually is an inner product follows with a straightforward computation using that c is conjugate linear.

3.2 Example. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces. The zero map $0 : \mathcal{L}_1 \to \mathcal{L}_2^*$, $0(x_1)x_2 := 0$, is conjugate linear. We have

$$\mathcal{L}_1 \ltimes_0 \mathcal{L}_2 = \mathcal{L}_1[\dot+]\mathcal{L}_2 .$$

//

We have natural embeddings of $\mathcal{L}_j$ into $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$, namely the maps $\iota_{c,j}$ defined as

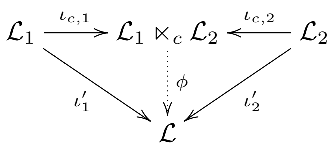
$$\iota_{c,1}(x) := (x,0),\ x \in \mathcal{L}_1, \quad \iota_{c,2}(x) := (0,x),\ x \in \mathcal{L}_2 . \tag{3.3}$$

These are injective and isometric, and

$$\mathcal{L}_1 \ltimes_c \mathcal{L}_2 = \operatorname{ran}\iota_{c,1} \dot+ \operatorname{ran}\iota_{c,2} ,$$

where '$\dot+$' denotes a direct sum. Hence, we may consider $\mathcal{L}_1$ and $\mathcal{L}_2$ as summands in a direct sum decomposition of $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$. Remembering our preliminary computation (3.2), conversely, each decomposition of an inner product space $\mathcal{L}$ into a direct sum gives rise to a representation $\mathcal{L} = \mathcal{L}_1 \ltimes_c \mathcal{L}_2$ where c is as in (3.1). This fact can be formulated in a slightly more general way.

3.3 Proposition. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces. Moreover, let $\mathcal{L}$ be an inner product space together with isometric maps $\iota_j' : \mathcal{L}_j \to \mathcal{L}$, $j = 1,2$. Then there exists a unique conjugate linear map $c : \mathcal{L}_1 \to \mathcal{L}_2^*$ such that*



(3.4)

with some isometric linear map $\phi : \mathcal{L}_1 \ltimes_c \mathcal{L}_2 \to \mathcal{L}$. Explicitly, c is given as

$$c : \begin{cases} \mathcal{L}_1 \to \mathcal{L}_2^* \\ x_1 \mapsto \big(x_2 \mapsto [\iota_2'(x_2), \iota_1'(x_1)]_{\mathcal{L}}\big) \end{cases} \tag{3.5}$$

The map ϕ in the diagram (3.4) *is uniquely determined. Explicitly, ϕ is given as*

$$\phi : \begin{cases} \mathcal{L}_1 \ltimes_c \mathcal{L}_2 \to \mathcal{L} \\ (x_1,x_2) \mapsto \iota_1'(x_1) + \iota_2'(x_2) \end{cases} \tag{3.6}$$

Moreover, we have

$$\ker\phi = \big\{(x_1,x_2) :\ \iota_1'(x_1) = -\iota_2'(x_2)\big\}, \quad \operatorname{ran}\phi = \operatorname{ran}\iota_1' + \operatorname{ran}\iota_2' . \tag{3.7}$$

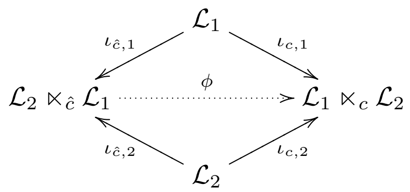
Proof. Let c and ϕ be defined by (3.5) and (3.6). A short calculation will show that ϕ is isometric. By the definition of $\iota_{c,j}$, the diagram (3.4) commutes. We have $\phi(x_1, x_2) = 0$ if and only if $\iota'_1(x_1) = -\iota'_2(x_2)$. Hence, the kernel of ϕ has the asserted form. Moreover, clearly, $\operatorname{ran}\phi = \operatorname{ran}\iota'_1 + \operatorname{ran}\iota'_2$.

It remains to show uniqueness of c and ϕ. Assume that $c' : \mathcal{L}_1 \to \mathcal{L}_2^*$ is conjugate linear and that there exists an isometric map ϕ' of $\mathcal{L}_1 \ltimes_{c'} \mathcal{L}_2$ into $\mathcal{L}$ which makes the diagram (3.4) commute. Then

$$c'(x_1)x_2 = \big[\iota_{c',2}(x_2), \iota_{c',1}(x_1)\big]_{c'} = \big[\phi'(\iota_{c',2}(x_2)), \phi'(\iota_{c',1}(x_1))\big]_{\mathcal{L}} =$$
$$= \big[\iota'_2(x_2), \iota'_1(x_1)\big]_{\mathcal{L}} = c(x_1)x_2, \quad x_1 \in \mathcal{L}_1, x_2 \in \mathcal{L}_2\,,$$

i.e. $c' = c$. The map ϕ is uniquely determined by (3.4) since the ranges of $\iota_{c,1}$ and $\iota_{c,2}$ jointly span $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$. ❑

3.4 Corollary. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces. Moreover, let $c : \mathcal{L}_1 \to \mathcal{L}_2^*$ be conjugate linear. Then there exists a unique conjugate linear map $\hat{c}$: $\mathcal{L}_2 \to \mathcal{L}_1^*$ such that*



(3.8)

with some isometric linear map ϕ. Explicitly, c is given as

$$\hat{c}(x_2)x_1 = \overline{c(x_1)x_2}\,. \tag{3.9}$$

The map ϕ in the diagrams (3.8) *is uniquely determined. Explicitly, ϕ is given as*

$$\phi\big((x_2, x_1)\big) = (x_1, x_2)\,. \tag{3.10}$$

The map ϕ is bijective.

Proof. Applying Proposition 3.3 with the spaces $\mathcal{L}_2$ and $\mathcal{L}_1$, and

$$\mathcal{L} := \mathcal{L}_1 \ltimes_c \mathcal{L}_2, \quad \iota'_1 := \iota_{c,2}, \ \iota'_2 := \iota_{c,1}\,,$$

gives the mappings $\hat{c}$ and ϕ as asserted in (3.9) and (3.10). ❑

The next result gives some of information about the isotropic part of $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$. For a linear space $\mathcal{L}$ and a subset M of $\mathcal{L}^*$, we denote by ${}^\perp M$ its left annihilator with respect to the natural duality between $\mathcal{L}$ and $\mathcal{L}^*$, i.e.

$${}^\perp M := \big\{x \in \mathcal{L} :\ f(x) = 0, f \in M\big\}\,.$$

3.5 Proposition. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces and let $c : \mathcal{L}_1 \to \mathcal{L}_2^*$ be conjugate linear. Then*

$$\iota_{c,1}(\mathcal{L}_1) \cap \big(\mathcal{L}_1 \ltimes_c \mathcal{L}_2\big)^\circ = \iota_{c,1}\big(\mathcal{L}_1^\circ \cap \ker c\big)\,,$$

$$\iota_{c,2}(\mathcal{L}_2) \cap \big(\mathcal{L}_1 \ltimes_c \mathcal{L}_2\big)^\circ = \iota_{c,2}\big(\mathcal{L}_2^\circ \cap {}^\perp\operatorname{ran} c\big)\,.$$

Proof. Let $y_1 \in \mathcal{L}_1$, then

$$\big[(x_1,x_2),(y_1,0)\big]_c = [x_1,y_1]_1 + c(y_1)x_2, \quad x_1 \in \mathcal{L}_1, x_2 \in \mathcal{L}_2 \,.$$

Hence $(y_1,0) \in (\mathcal{L}_1 \ltimes_c \mathcal{L}_2)^\circ$ if and only if

$$[x_1,y_1]_1 = 0,\ x_1 \in \mathcal{L}_1 \ \text{ and }\ c(y_1) = 0\,.$$

Let $y_2 \in \mathcal{L}_2$, then

$$\big[(x_1,x_2),(0,y_2)\big]_c = \overline{c(x_1)y_2} + [x_2,y_2]_2, \quad x_1 \in \mathcal{L}_1, x_2 \in \mathcal{L}_2 \,.$$

Hence $(0,y_2) \in (\mathcal{L}_1 \ltimes_c \mathcal{L}_2)^\circ$ if and only if $c(x_1)y_2 = 0$, $x_1 \in \mathcal{L}_1$, and $[x_2,y_2]_2 = 0$, $x_2 \in \mathcal{L}_2$. ❑

In general not much information on $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$ can be obtained. Concerning negative indices and degrees of degeneracy, we only have the following weak estimates:

$$\operatorname{ind}_- \mathcal{L}_1 \ltimes_c \mathcal{L}_2 \geq \max\big\{\operatorname{ind}_- \mathcal{L}_1, \operatorname{ind}_- \mathcal{L}_2\big\}\,,$$

$$\operatorname{ind}_0 \mathcal{L}_1 \ltimes_c \mathcal{L}_2 \geq \max\big\{\dim(\mathcal{L}_1^\circ \cap \ker c), \dim(\mathcal{L}_2^\circ \cap {}^\perp\operatorname{ran} c)\big\}\,.$$

It is easy to give examples which show that negative indices or degrees of degeneracy may increase arbitrarily.

3.6 Example. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two linear spaces with the same dimension. Choose bases $\{b_j^1 : j \in J\}$ and $\{b_j^2 : j \in J\}$ of $\mathcal{L}_1$ and $\mathcal{L}_2$, and let $\mathcal{L}_1 \times \mathcal{L}_2$ be endowed with inner products $[.,.]$ and $[.,.]'$ given by the Gram-matrices

$$G := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad G' := \begin{pmatrix} I & I \\ I & I \end{pmatrix}$$

Explicitly, this means that

$$\Big[\sum \lambda_i b_i^1 + \sum \mu_i b_i^2, \sum \lambda_j' b_j^1 + \sum \mu_j' b_j^2\Big] = \sum \big(\lambda_i \overline{\mu_i'} + \mu_i \overline{\lambda_i'}\big)$$

$$\Big[\sum \lambda_i b_i^1 + \sum \mu_i b_i^2, \sum \lambda_j' b_j^1 + \sum \mu_j' b_j^2\Big]' = \sum \big(\lambda_i \overline{\lambda_i'} + \lambda_i \overline{\mu_i'} + \mu_i \overline{\lambda_i'} + \mu_i \overline{\mu_i'}\big)$$

Define inner products $[.,.]_j$ and $[.,.]_j'$ on $\mathcal{L}_j$ by $[x_1,x_2]_j := 0$, $j = 1,2$, and

$$\Big[\sum \lambda_i b_i^j, \sum \mu_i b_i^j\Big]_j' := \sum \lambda_i \overline{\mu_i}, \quad j = 1,2\,.$$

Then $\langle \mathcal{L}_1 \times \mathcal{L}_2, [.,.]\rangle$ and $\langle \mathcal{L}_1 \times \mathcal{L}_2, [.,.]'\rangle$ can be realized as $\langle \mathcal{L}_1, [.,.]_1\rangle \ltimes_c \langle \mathcal{L}_2, [.,.]_2\rangle$ and $\langle \mathcal{L}_1, [.,.]_1'\rangle \ltimes_{c'} \langle \mathcal{L}_2, [.,.]_2'\rangle$, respectively, with some appropriate mappings c, c'. We see that

$$\operatorname{ind}_-\langle \mathcal{L}_1, [.,.]_1\rangle = \operatorname{ind}_-\langle \mathcal{L}_2, [.,.]_2\rangle = 0, \quad \operatorname{ind}_-\langle \mathcal{L}_1, [.,.]_1\rangle \ltimes_c \langle \mathcal{L}_2, [.,.]_2\rangle = |J|$$

$$\operatorname{ind}_0\langle \mathcal{L}_1, [.,.]_1'\rangle = \operatorname{ind}_0\langle \mathcal{L}_2, [.,.]_2'\rangle = 0, \quad \operatorname{ind}_0\langle \mathcal{L}_1, [.,.]_1'\rangle \ltimes_c \langle \mathcal{L}_2, [.,.]_2'\rangle = |J|$$

//

If one of the spaces $\mathcal{L}_1$ or $\mathcal{L}_2$ is finite dimensional, at least a rough upper estimate on $\operatorname{ind}_- \mathcal{L}_1 \ltimes_c \mathcal{L}_2$ and $\operatorname{ind}_0 \mathcal{L}_1 \ltimes_c \mathcal{L}_2$ can be given.

3.7 Remark. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces, and let $c : \mathcal{L}_1 \to \mathcal{L}_2^*$ be conjugate linear.

(i) Assume that $\dim \mathcal{L}_1 < \infty$. Since, for each subspace $\mathcal{K}$ of $\mathcal{L}_1 \ltimes_c \mathcal{L}_2$ we have $\dim(\mathcal{K} \cap \mathcal{L}_2) \geq \dim \mathcal{K} - \dim \mathcal{L}_1$, it follows that

$$\operatorname{ind}_- \mathcal{L}_1 \ltimes_c \mathcal{L}_2 \leq \operatorname{ind}_- \mathcal{L}_2 + \dim \mathcal{L}_1, \quad \operatorname{ind}_0 \mathcal{L}_1 \ltimes_c \mathcal{L}_2 \leq \operatorname{ind}_0 \mathcal{L}_2 + \dim \mathcal{L}_1$$

(ii) Assume that $\dim \mathcal{L}_2 < \infty$. Then it is seen from Corollary 3.4 that analogous inequalities hold.

//

Let us now turn our attention to the almost Pontryagin space situation. Assume that $\langle \mathcal{A}_1, [.,.]_1, \mathcal{T}_1 \rangle$ and $\langle \mathcal{A}_2, [.,.]_2, \mathcal{T}_2 \rangle$ are aPs, and let $c : \mathcal{A}_1 \to \mathcal{A}_2^*$ be conjugate linear. Since in general neither $\operatorname{ind}_-(\mathcal{A}_1 \ltimes_c \mathcal{A}_2) < \infty$ nor $\operatorname{ind}_0(\mathcal{A}_1 \ltimes_c \mathcal{A}_2) < \infty$ needs to hold, already the geometry of $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ will in general be far from an aPs. Also topologically $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ does not behave that simple. Of course, $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ carries a natural Banach space topology, namely the product topology $\mathcal{T} := \mathcal{T}_1 \times \mathcal{T}_2$. However, the inner product $[.,.]_c$ will in general not be continuous.

If $\mathcal{A}$ is an aPs, we denote by $\mathcal{A}'$ its topological dual space. Moreover, τ_{w^*} denotes the weak-* topology on $\mathcal{A}'$.

3.8 Proposition. *Let $\langle \mathcal{A}_1, [.,.]_1, \mathcal{T}_1 \rangle$ and $\langle \mathcal{A}_2, [.,.]_2, \mathcal{T}_2 \rangle$ be aPs, and let $c : \mathcal{A}_1 \to \mathcal{A}_2^*$ be conjugate linear. Then the inner product $[.,.]_c : (\mathcal{A}_1 \ltimes_c \mathcal{A}_2)^2 \to \mathbb{C}$ is $\mathcal{T}$-continuous if and only if $c(\mathcal{A}_1) \subseteq \mathcal{A}_2'$ and c is $\mathcal{T}_1$-to-τ_{w^*}–continuous.*

Proof. Assume first that c maps $\mathcal{A}_1$ $\mathcal{T}_1$-to-τ_{w^*}–continuously into $\mathcal{A}_2'$. Choose norms $\|.\|_1$, $\|.\|_2$, which induce $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, and put $\|.\| := \max\{\|.\|_1, \|.\|_2\}$. Let $M_1, M_2 > 0$ be such that

$$\big|[x_j, y_j]_j\big| \leq M_j \|x_j\|_j \|y_j\|_j, \quad x_j, y_j \in \mathcal{A}_j, j = 1, 2\,.$$

Since c is $\mathcal{T}_1$-to-τ_{w^*}–continuous, for each fixed $x_2 \in \mathcal{A}_2$ there exists $M_{x_2} > 0$ such that

$$|c(y_1)x_2| \leq M_{x_2}, \quad y_1 \in \mathcal{A}_1, \|y_1\|_1 \leq 1\,.$$

The Principle of Uniform Boundedness implies

$$M := \sup\big\{\|c(y_1)\| : y_1 \in \mathcal{A}_1, \|y_1\|_1 \leq 1\big\} < \infty\,.$$

For $x_1 + x_2, y_1 + y_2 \in \mathcal{A}_1 \ltimes_c \mathcal{A}_2$ with $\|x_1 + x_2\|, \|y_1 + y_2\| \leq 1$, we thus obtain the estimate

$$\big|[x_1{+}x_2, y_1{+}y_2]_c\big| \leq |[x_1, y_1]_1| + |c(x_1)y_2| + |c(y_1)x_2| + |[x_2, y_2]_2| \leq M_1 + 2M + M_2\,.$$

This shows that $[.,.]_c$ is $\mathcal{T}$-continuous.

Conversely, assume that $[.,.]_c$ is $\mathcal{T}$-continuous. We have

$$c(y_1)x_2 = [0 + x_2, y_1 + 0]_c, \quad y_1 \in \mathcal{A}_1, x_2 \in \mathcal{A}_2\,.$$

Keeping y_1 fixed and letting x_2 vary through $\mathcal{A}_2$ shows that the functional $c(y_1)$ belongs to $\mathcal{A}_2'$. Keeping x_2 fixed and letting y_1 vary through $\mathcal{A}_1$ shows that c is $\mathcal{T}_1$-to-τ_{w^*}–continuous. ❑

Let us explicitly point out the following fact: Proposition 3.8 says that, under the stated conditions, $[.,.]_c$ is continuous. It does not claim that $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an aPs. However, if one of the summands $\mathcal{A}_1$ or $\mathcal{A}_2$ is finite dimensional, matters simplify. Then we can conclude that $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an aPs.

3.9 Corollary. *Let $\langle \mathcal{A}_1, [.,.]_1, \mathcal{T}_1\rangle$ and $\langle \mathcal{A}_2, [.,.]_2, \mathcal{T}_2\rangle$ be aPs, and let $c : \mathcal{A}_1 \to \mathcal{A}_2^*$ be conjugate linear.*

- *(i) Assume that $\dim \mathcal{A}_1 < \infty$. Then $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space if and only if $c(\mathcal{A}_1) \subseteq \mathcal{A}_2'$.*
- *(ii) Assume that $\dim \mathcal{A}_2 < \infty$. Then $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space if and only if c is continuous. Note here that, under the present hypothesis, $\mathcal{A}_2' = \mathcal{A}_2^*$ and the topology on $\mathcal{A}_2'$ is just the euclidean topology.*

Proof. Consider the case that $\mathcal{A}_1$ is finite dimensional. Assume first that $c(\mathcal{A}_1) \subseteq \mathcal{A}_2'$. Since c is conjugate linear, $\dim \mathcal{A}_1 < \infty$ implies that c $\mathcal{T}_1$-to-τ_{w^*}–continuous. By Proposition 3.8, $[.,.]_c$ is $\mathcal{T}$-continuous. Let $\mathcal{M}$ be a $\mathcal{T}_2$-closed subspace of $\mathcal{A}_2$ which is a Hilbert space and has finite codimension in $\mathcal{A}_2$. Then $\mathcal{M}$ is also $\mathcal{T}$-closed and has finite codimension in $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$. Moreover, $[.,.]_c|_{\mathcal{M}\times\mathcal{M}} = [.,.]_2|_{\mathcal{M}\times\mathcal{M}}$, and hence $\mathcal{M}$ is a Hilbert space with respect to $[.,.]_c$. We see that $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an aPs. Conversely, if $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an aPs, then Proposition 3.8 yields $c(\mathcal{A}_1) \subseteq \mathcal{A}_2'$.

The case that $\mathcal{A}_2$ is finite dimensional is settled in the same manner. ❑

3.10 Remark. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be aPs, and $c : \mathcal{A}_1 \to \mathcal{A}_2^*$ be conjugate linear. The embeddings $\iota_{c,j} : \mathcal{A}_j \to \mathcal{A}_1 \ltimes_c \mathcal{A}_2$ are continuous and map closed subsets of $\mathcal{A}_j$ to closed subsets of $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$. Hence, whenever $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an almost Pontryagin space, then $\iota_{c,j}$ will be morphisms. //

The analogs of Proposition 3.3 and Corollary 3.4 in the aPs-setting read as follows.

3.11 Proposition. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be aPs.*

- *(i) Let $\mathcal{A}$ be an aPs together with morphisms $\iota_j' : \mathcal{A}_j \to \mathcal{A}$, $j = 1, 2$, and let the conjugate linear map $c : \mathcal{A}_1 \to \mathcal{A}_2^*$ and isometry $\phi : \mathcal{A}_1 \ltimes_c \mathcal{A}_2 \to \mathcal{A}$ be as in Proposition 3.3. Then $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an aPs and ϕ is a morphism if and only if*

$$\dim\big(\operatorname{ran}\iota_1' \cap \operatorname{ran}\iota_2'\big) < \infty \quad \text{and} \quad \operatorname{ran}\iota_1' + \operatorname{ran}\iota_2' \text{ closed in } \mathcal{A}\,.$$

- *(ii) Let $c : \mathcal{A}_1 \to \mathcal{A}_2^*$ be a conjugate linear map, and let $\hat{c} : \mathcal{A}_2 \to \mathcal{A}_1^*$ and $\phi : \mathcal{A}_2 \ltimes_{\hat{c}} \mathcal{A}_1 \to \mathcal{A}_1 \ltimes_c \mathcal{A}_2$ be as in Corollary 3.4. Then $\mathcal{A}_2 \ltimes_{\hat{c}} \mathcal{A}_1$ is an aPs if and only if $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is, and in this case ϕ is an isomorphism between these aPs.*

Proof. For the proof of (i) let $\mathcal{A}$ and $\mathcal{A}_j$, ι_j', $j = 1, 2$, be given. Since ι_1' and ι_2' are continuous, the map c is explicitly given by (3.5), it maps $\mathcal{A}_1$ into $\mathcal{A}_2'$ and is $\mathcal{T}_1$-to-τ_{w^*}–continuous. Thus $[.,.]_c$ is continuous. In order to get hands on geometric properties, we make a preliminary observation: Namely that

$$\dim \ker \phi < \infty \iff \dim\big(\operatorname{ran}\iota_1' \cap \operatorname{ran}\iota_2'\big) < \infty \tag{3.11}$$

To see this, let $\pi_1 : \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ denote the projection onto the first component, and consider the map $\mu := \pi_1\circ(\iota_1'\times(-\iota_2')) : \mathcal{A}_1 \ltimes_c \mathcal{A}_2 \to \mathcal{A}$. By (3.7), $\mu(\ker\phi) = \operatorname{ran}\iota_1' \cap \operatorname{ran}\iota_2'$. Moreover, $\ker(\mu|_{\ker\phi}) = \ker\iota_1' \times \ker\iota_2'$. Since $\ker\iota_j' \subseteq \mathcal{A}_j^\circ$, and hence $\dim\ker(\mu|_{\ker\phi}) < \infty$, (3.11) follows.

Assume that $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ is an aPs and $\phi : \mathcal{A}_1 \ltimes_c \mathcal{A}_2 \to \mathcal{A}$ is a morphism. Then $\operatorname{ran}\iota_1' + \operatorname{ran}\iota_2' = \operatorname{ran}\phi$ is closed in $\mathcal{A}$ since ϕ maps closed subspaces to closed subspaces. Moreover, since $\ker\phi \subseteq (\mathcal{A}_1 \ltimes_c \mathcal{A}_2)^\circ$, we must have $\dim\ker\phi < \infty$, and (3.11) gives $\dim(\operatorname{ran}\iota_1' \cap \operatorname{ran}\iota_2') < \infty$.

Conversely, assume that $\dim(\operatorname{ran}\iota_1' \cap \operatorname{ran}\iota_2') < \infty$ and $\operatorname{ran}\iota_1' + \operatorname{ran}\iota_2'$ is closed in $\mathcal{A}$. Then, by (3.11), $\ker\phi$ is finite dimensional. Moreover, since $[\operatorname{ran}\phi]^\circ$ is a neutral subspace of $\mathcal{A}$, $\dim([\operatorname{ran}\phi]^\circ) \leq \operatorname{ind}_-\mathcal{A} + \operatorname{ind}_0\mathcal{A}$. Since $\phi^{-1}([\operatorname{ran}\phi]^\circ) = (\mathcal{A}_1 \ltimes_c \mathcal{A}_2)^\circ$, it follows that

$$\dim(\mathcal{A}_1 \ltimes_c \mathcal{A}_2)^\circ < \infty .$$

The map ϕ is isometric, and hence clearly $\operatorname{ind}_-(\mathcal{A}_1 \ltimes_c \mathcal{A}_2) \leq \operatorname{ind}_-\mathcal{A} < \infty$.

Since $\dim\ker\phi < \infty$, the space $\ker\phi$ is complemented in the Banach space $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$, i.e. we may choose a closed subspace $\mathcal{M}_1$ of $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$ with $\mathcal{M}_1 \dot{+} \ker\phi = \mathcal{A}_1 \ltimes_c \mathcal{A}_2$. Then $\phi|_{\mathcal{M}_1}$ is a continuous bijections between the Banach spaces $\mathcal{M}_1$ and $\operatorname{ran}\phi$, and hence a homeomorphism. Let $\mathcal{N}$ be a closed subspace of $\operatorname{ran}\phi$ with finite codimension which is a Hilbert space with respect to the inner product of $\mathcal{A}$. Then $\mathcal{M} := (\phi|_{\mathcal{M}_1})^{-1}(\mathcal{N})$ is a closed subspace of $\mathcal{M}_1$ with finite codimension and, since ϕ is isometric, is a Hilbert space with respect to the inner product of $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$. Since $\mathcal{M}_1$ itself is closed and has finite codimension in $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$, $\mathcal{M}$ is a subspace with the properties required in (aPs3). Let $\mathcal{L}$ be a closed subspace of $\mathcal{A}_1 \ltimes_c \mathcal{A}_2$, then $\phi(\mathcal{L}) = \phi|_{\mathcal{M}_1}(\mathcal{L}\cap\mathcal{M}_1)$, hence is closed in $\operatorname{ran}\phi$ and thus also in $\mathcal{A}$. As a closed subspace of an aPs, the space $\operatorname{ran}\phi$ is itself an aPs.

The second item is immediate, since ϕ is, besides being bijective and isometric, in any case a homeomorphism. ❑

4 Orthogonal coupling of inner product spaces

Let $\langle\mathcal{L}_1, [.,.]_1\rangle$ and $\langle\mathcal{L}_2, [.,.]_2\rangle$ be inner product spaces. Their direct and orthogonal sum $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ is defined as the linear space $\mathcal{L}_1 \times \mathcal{L}_2$ endowed with the inner product

$$\big[(x_1,x_2),(y_1,y_2)\big] := [x_1,y_1] + [x_2,y_2], \quad (x_1,y_1),(x_2,y_2) \in \mathcal{L}_1[\dot{+}]\mathcal{L}_2 .$$

Properties of $\mathcal{L}_1$ and $\mathcal{L}_2$ immediately transfer to $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$, for example we have

$$\operatorname{ind}_-\mathcal{L}_1[\dot{+}]\mathcal{L}_2 = \operatorname{ind}_-\mathcal{L}_1 + \operatorname{ind}_-\mathcal{L}_2,\ \operatorname{ind}_0\mathcal{L}_1[\dot{+}]\mathcal{L}_2 = \operatorname{ind}_0\mathcal{L}_1 + \operatorname{ind}_0\mathcal{L}_2 .$$

In fact, $(\mathcal{L}_1[\dot{+}]\mathcal{L}_2)^\circ = \mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$. Moreover, remember that, with the notation of the previous section, we can write $\mathcal{L}_1[\dot{+}]\mathcal{L}_2 = \mathcal{L}_1 \ltimes_0 \mathcal{L}_2$, where $0 : \mathcal{L}_1 \to \mathcal{L}_2^*$ denotes the zero map.

The following observation is the starting point for our present considerations.

4.1 Remark. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are nondegenerated inner product spaces, then the direct and orthogonal sum $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ is (up to isomorphisms) the unique inner product space containing $\mathcal{L}_1$ and $\mathcal{L}_2$ isometrically as orthogonal subspaces which together span the whole space. //

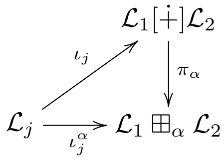
If we move from the nondegenerated to the degenerated situation, then a space with this property will not be unique anymore.

4.2 Definition. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces, and let α be a linear subspace of $\mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$. Define

$$\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2 := \big(\mathcal{L}_1[\dot{+}]\mathcal{L}_2\big)/_{\alpha}\,.$$

We refer to $\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$ as the orthogonal coupling of $\mathcal{L}_1$ and $\mathcal{L}_2$ with overlapping relation α.

Moreover, if ι_j be the canonical embedding of $\mathcal{L}_j$ into $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$, and π_α the canonical projection of $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ onto $(\mathcal{L}_1[\dot{+}]\mathcal{L}_2)/_\alpha$, we set $\iota_1^\alpha := \pi_\alpha \circ \iota_1$, $\iota_2^\alpha := \pi_\alpha \circ \iota_2$, that is



//

4.3 Remark. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces, and let α be a linear subspace of $\mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$.

(*i*) Since $\mathcal{L}_1^\circ \times \mathcal{L}_2^\circ = (\mathcal{L}_1[\dot{+}]\mathcal{L}_2)^\circ$, the mappings $\iota_1^\alpha : \mathcal{L}_1 \to \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$ and $\iota_2^\alpha : \mathcal{L}_2 \to \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$ are both isometric. Moreover,

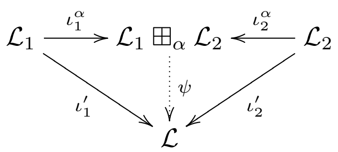
$$\iota_1^\alpha(\mathcal{L}_1) \perp \iota_2^\alpha(\mathcal{L}_2) \quad \text{and} \quad \mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2 = \operatorname{ran} \iota_1^\alpha + \operatorname{ran} \iota_2^\alpha\,.$$

(*ii*) The mappings ι_1^α and ι_2^α are both injective if and only if the linear subspace α is the graph of a bijective map $\alpha : \operatorname{dom}\alpha \to \operatorname{ran}\alpha$ between some linear subspaces $\operatorname{dom}\alpha \subseteq \mathcal{L}_1^\circ$ and $\operatorname{ran}\alpha \subseteq \mathcal{L}_2^\circ$. In order to see this, note that

$$(0, x_2) \in \alpha \iff \iota_2^\alpha(x_2) = 0, \quad (x_1, 0) \in \alpha \iff \iota_1^\alpha(x_1) = 0\,.$$

//

4.4 Proposition. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces. Moreover, let $\mathcal{L}$ be an inner product space together with isometric maps $\iota_j' : \mathcal{L}_j \to \mathcal{L}$, $j = 1, 2$, such that $\iota_1'(\mathcal{L}_1) \perp \iota_2'(\mathcal{L}_2)$. Then there exists a unique linear subspace $\alpha \subseteq \mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$, such that*



(4.1)

with some injective and isometric linear map ψ. Explicitly, α is given as

$$\alpha = \big\{(x_1, x_2) \in \mathcal{L}_1 \times \mathcal{L}_2 : \iota_1'(x_1) = -\iota_2'(x_2)\big\}\,.$$

The map ψ in the diagram (4.1) *is uniquely determined. Explicitly, ψ is given as*

$$\psi\big((x_1, x_2)/_\alpha\big) = \iota_1'(x_1) + \iota_2'(x_2)\,.$$

The map ι_j^α is injective if and only if ι_j' has this property. Moreover, if $\operatorname{ran}\iota_1' + \operatorname{ran}\iota_2' = \mathcal{L}$*, then ψ is bijective.*

Proof. The map $\phi(x) := \iota_1'(x) + \iota_2'(x)$ is an isometry of $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ into $\mathcal{L}$. It satisfies

$$\begin{array}{ccccc} \mathcal{L}_1 & \xrightarrow{\iota_1} & \mathcal{L}_1[\dot{+}]\mathcal{L}_2 & \xleftarrow{\iota_2} & \mathcal{L}_2 \\ & \searrow{\scriptstyle \iota_1'} & \downarrow{\scriptstyle \phi} & \swarrow{\scriptstyle \iota_2'} & \\ & & \mathcal{L} & & \end{array} \tag{4.2}$$

and $\ker\phi = \{(x_1, x_2) \in \mathcal{L}_1[\dot{+}]\mathcal{L}_2 : \iota_1'(x) = -\iota_2'(x)\}$. We are going to show that $\ker\phi \subseteq (\mathcal{L}_1[\dot{+}]\mathcal{L}_2)^\circ$. To this end, let $(x_1, x_2) \in \ker\phi$ be given. If $y_1 \in \mathcal{L}_1$, then

$$\big[(x_1, x_2), (y_1, 0)\big]_{\mathcal{L}_1[\dot{+}]\mathcal{L}_2} = [x_1, y_1]_{\mathcal{L}_1} = \big[\iota_1'(x_1), \iota_1'(y_1)\big] = \big[-\iota_2'(x_2), \iota_1'(y_1)\big] = 0\,.$$

An analogous computation will show that $[(x_1, x_2), (0, y_2)] = 0$ for all $y_2 \in \mathcal{L}_2$. Hence, the linear subspace $\alpha := \ker\phi$ qualifies as being used to define $\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$.

Let ψ be the isometry which makes the diagram

$$\begin{array}{ccc} \mathcal{L}_1[\dot{+}]\mathcal{L}_2 & \xrightarrow{\phi} & \mathcal{L} \\ \downarrow{\scriptstyle \pi_\alpha} & \nearrow{\scriptstyle \psi} & \\ (\mathcal{L}_1[\dot{+}]\mathcal{L}_2)/\alpha & & \end{array}$$

commute.

Clearly, the map ψ is injective and the diagram (4.1) commutes. Moreover,

$$\operatorname{ran}\psi = \operatorname{ran}\phi = \operatorname{ran}\iota_1' + \operatorname{ran}\iota_2'\,.$$

From the injectivity of ψ it also follows that ι_j^α is injective if and only if ι_j' has this property.

In order to show uniqueness, assume that (4.1) holds with some $\alpha' \subseteq \mathcal{L}_1^\circ \times \mathcal{L}_2^\circ$ and $\psi' : \mathcal{L}_1 \boxplus_{\alpha'} \mathcal{L}_2 \to \mathcal{L}$. Then we have

$$\begin{array}{ccccc} & & \mathcal{L}_1[\dot{+}]\mathcal{L}_2 & & \\ & \nearrow{\scriptstyle \iota_1} & \downarrow{\scriptstyle \pi_{\alpha'}} & \nwarrow{\scriptstyle \iota_2} & \\ \mathcal{L}_1 & \xrightarrow{\iota_1^{\alpha'}} & \mathcal{L}_1 \boxplus_{\alpha'} \mathcal{L}_2 & \xleftarrow{\iota_2^{\alpha'}} & \mathcal{L}_2 \\ & \searrow{\scriptstyle \iota_1'} & \downarrow{\scriptstyle \psi'} & \swarrow{\scriptstyle \iota_2'} & \\ & & \mathcal{L} & & \end{array}$$

By uniqueness in Proposition 3.3, recall that $\mathcal{L}_1[\dot{+}]\mathcal{L}_2$ can be viewed as $\mathcal{L}_1 \ltimes_0 \mathcal{L}_2$, we must have $\psi' \circ \pi_{\alpha'} = \phi$. Since ψ' is injective, this implies

$$\alpha' = \ker\pi_{\alpha'} = \ker\big(\psi' \circ \pi_{\alpha'}\big) = \ker\phi = \alpha\,.$$

The map ψ is uniquely determined by (4.1), since $\operatorname{ran}\iota_1^\alpha$ and $\operatorname{ran}\iota_2^\alpha$ together span $\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$. ❑

Combining Proposition 4.4 with Remark 4.3, (ii), we obtain the following corollary.

4.5 Corollary. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be inner product spaces. An inner product space contains isomorphic copies of $\mathcal{L}_1$ and $\mathcal{L}_2$ as orthogonal subspaces which span the whole space, if and only if it is isomorphic to $\mathcal{L}_1 \boxplus_\alpha \mathcal{L}_2$ with some bijective map α between subspaces of $\mathcal{L}_1^\circ$ and $\mathcal{L}_2^\circ$.* ❑

4.6 Remark. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be aPs, and let α be a linear subspace of $\mathcal{A}_1^\circ \times \mathcal{A}_2^\circ$, then also $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ is an aPs. Moreover, we have

$$\operatorname{ind}_-\big(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2\big) = \operatorname{ind}_- \mathcal{A}_1 + \operatorname{ind}_- \mathcal{A}_2\,,$$

$$\operatorname{ind}_0\big(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2\big) = \operatorname{ind}_0 \mathcal{A}_1 + \operatorname{ind}_0 \mathcal{A}_2 - \dim\alpha\,.$$

//

The aPs-version of Proposition 4.4 now reads as follows.

4.7 Proposition. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be aPs. Moreover, let $\mathcal{A}$ be an aPs together with morphisms $\iota_j' : \mathcal{A}_j \to \mathcal{A}$, $j = 1, 2$, such that $\iota_1'(\mathcal{A}_1) \perp \iota_2'(\mathcal{A}_2)$. Then the isometry $\psi : \mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2 \to \mathcal{A}$ in Proposition 4.4 is a morphism.*

Proof. We wish to apply Proposition 3.11, (i), with the presently given data $\mathcal{A}, \mathcal{A}_j, \iota_j'$, $c := 0$, and the map ϕ in (4.2). To this end note first that $\operatorname{ran}\iota_1' \cap \operatorname{ran}\iota_2'$ is, as a neutral subspace of $\mathcal{A}$, finite dimensional. Since $\operatorname{ran}\iota_1'$ and $\operatorname{ran}\iota_2'$ are, as closed subspaces of the aPs $\mathcal{A}$, themselves aPs, we may choose closed subspaces $\mathcal{M}_j$ of $\operatorname{ran}\iota_j'$, $j = 1, 2$, which are closed, have finite codimension in $\operatorname{ran}\iota_j'$, and are Hilbert spaces with respect to the inner product inherited from $\mathcal{A}$. Clearly, they are orthogonal to each other. This also implies that $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$. Their sum $\mathcal{M} := \mathcal{M}_1[\dot{+}]\mathcal{M}_2$ is thus also a Hilbert space in the inner product of $\mathcal{A}$. Since moreover $\mathcal{M}$ is, as the orthogonal sum of two uniformly positive subspaces, itself uniformly positive, $\mathcal{M}$ is closed in the norm of $\mathcal{A}$. Clearly, $\mathcal{M}$ has finite codimension in $\operatorname{ran}\iota_1' + \operatorname{ran}\iota_2'$, and we conclude that $\operatorname{ran}\iota_1' + \operatorname{ran}\iota_2'$ is closed in the norm of $\mathcal{A}$.

Proposition 3.11 implies that the map ϕ in (4.2) is an aPs-morphism. Hence, also ψ is such. ❑

4.8. Concrete realization of $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$: Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces, and let α be a bijective map between some subspaces $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ of $\mathcal{A}_1^\circ$ and $\mathcal{A}_2^\circ$, respectively. The space $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ can also be described explicitly. To this end choose closed subspaces $\mathcal{A}_{1,r}$ and $\mathcal{A}_{2,r}$ such that

$$\mathcal{A}_1 = \mathcal{A}_{1,r}[\dot{+}]\mathcal{A}_1^\circ,\ \mathcal{A}_2 = \mathcal{A}_{2,r}[\dot{+}]\mathcal{A}_2^\circ\,,$$

choose D_1 and D_2 such that

$$\mathcal{A}_1^\circ = D_1 \dot{+} \operatorname{dom}\alpha,\ \mathcal{A}_2^\circ = D_2 \dot{+} \operatorname{ran}\alpha\,,$$

and set $D := \operatorname{ran}\alpha$. Consider the almost Pontryagin space

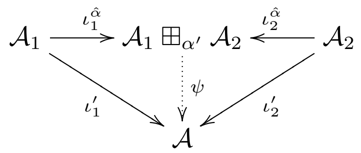
$$\mathcal{A} := \mathcal{A}_{1,r}[\dot{+}]\big(D_1 \dot{+} D \dot{+} D_2\big)[\dot{+}]\mathcal{A}_{2,r} \tag{4.3}$$

where the inner product and topology on $\mathcal{A}_{1,r}$ and $\mathcal{A}_{2,r}$ is the one inherited from $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, and where $D_1 \dot{+} D \dot{+} D_2$ is neutral and endowed with the euclidean topology. Moreover, define $\iota_1' : \mathcal{A}_1 \to \mathcal{A}$ by

$$\iota_1'|_{\mathcal{A}_{1,r}\dot{+}D_1} := \mathrm{id},\ \iota_1'|_{\operatorname{dom}\alpha} := -\alpha\,,$$

and let $\iota_2' : \mathcal{A}_2 \to \mathcal{A}$ be the identity map. Then ι_1' and ι_2' are morphisms. Moreover, it is apparent from their definition that $\iota_1'(\mathcal{A}_1) \perp \iota_2'(\mathcal{A}_2)$ and $\iota_1'(\mathcal{A}_1) + \iota_2'(\mathcal{A}_2) = \mathcal{A}$.

By Proposition 4.4 there exist $\hat{\alpha} \subseteq \mathcal{A}_1^\circ \times \mathcal{A}_2^\circ$ and an isomorphism $\psi : \mathcal{A}_1 \boxplus_{\hat{\alpha}} \mathcal{A}_2 \to \mathcal{A}$ with



Thereby the linear subspace $\hat{\alpha}$ is given as $\hat{\alpha} = \{(x_1, x_2) \in \mathcal{A}_1^\circ \times \mathcal{A}_2^\circ : \iota_1'(x_1) = \iota_2'(x_2)\}$. Write $x_1 = a_1 + b_1$ according to the decomposition $\mathcal{A}_1^\circ = D_1 \dot{+} \operatorname{dom} \alpha$, and let $x_2 = a_2 + b_2$ according to $\mathcal{A}_2^\circ = D_2 \dot{+} \operatorname{ran} \alpha$. Then $\iota_1'(x_1) = a_1 - \alpha(b_1)$ and $\iota_2'(x_2) = a_2 + b_2$. Hence we have $\iota_1'(x_1) = \iota_2'(x_2)$ if and only if $a_1 = a_2 = 0$ and $b_2 = \alpha(b_1)$. This, in turn, is equivalent to $(x_1, x_2) \in \alpha$.

We see that $\hat{\alpha} = \alpha$, and hence ψ is actually an isomorphism between $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ and $\mathcal{A}$, i.e. $\mathcal{A}$ can be regarded as a concrete realization of $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$. //

5 The canonical Pontryagin space extension of an almost Pontryagin space

There is a natural way to associate with a given almost Pontryagin space a Pontryagin space by means of a factorization process. Namely, for an almost Pontryagin space $\mathcal{A}$ we define

$$\mathfrak{P}(\mathcal{A}) := \mathcal{A}/\mathcal{A}^\circ$$

There is also another natural way to associate with a given almost Pontryagin space a Pontryagin space by means of an extension process; and this construction has turned out important.

5.1 Definition. Let $\mathcal{A}$ be an aPs. A pair $(\iota, \mathcal{P})$ is called a canonical Pontryagin space extension of $\mathcal{A}$, if $\mathcal{P}$ is a Pontryagin space, $\iota : \mathcal{A} \to \mathcal{P}$ is an injective morphism, and

$$\dim \mathcal{P}/\iota(\mathcal{A}) = \operatorname{ind}_0 \mathcal{A}\,.$$

We also sometimes say that $\mathcal{P}$ is a canonical Pontryagin space extension of $\mathcal{A}$ with extension embedding ι. //

Let us note that, for each canonical Pontryagin space extension $\mathcal{P}$ of $\mathcal{A}$,

$$\operatorname{ind}_- \mathcal{P} = \operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A}\,.$$

Canonical Pontryagin space extensions are in some sense minimal among all Pontryagin spaces which contain $\mathcal{A}$ as an isometric subspace: If $\mathcal{P}$ is a Pontryagin spaces which contains $\mathcal{A}$ as an isometric subspace, then certainly $\dim \mathcal{P}/\mathcal{A} \geq \operatorname{ind}_0 \mathcal{A}$ and $\operatorname{ind}_- \mathcal{P} \geq \operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A}$.

5.2. Existence of canonical Pontryagin space extensions: Let $\mathcal{A}$ be an almost Pontryagin space. Choose a closed subspace $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{B}[\dot{+}]\mathcal{A}^\circ$, and let C be a linear space with $\dim C = \dim \mathcal{A}^\circ$. Consider the linear space

$$\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}) := \mathcal{A} \dot{+} C = \mathcal{B} \dot{+} \mathcal{A}^\circ \dot{+} C\,,$$

and define on this linear space an inner product $[.,.]$ by the requirements

$$[.,.]|_{\mathcal{A}\times\mathcal{A}} = [.,.]_{\mathcal{A}}, \quad \mathcal{B} \perp C, \quad \mathcal{A}^\circ \# C\,.$$

Here we use the notation $A\#B$ to express that A and B are skewly linked, i.e. that A and B are neutral, $\dim A = \dim B$, and $A\dot{+}B$ is nondegenerated, cf. [B, §I.10] or [IKL, §I.3].

It is easy to see that $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A})$ is a Pontryagin space. Moreover, the set-theoretic inclusion map ι_{ext} of $\mathcal{A}$ into $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A})$ is a morphism. Clearly, ι_{ext} is injective and $\dim \mathcal{P}_{\mathrm{ext}}(\mathcal{A})/\mathcal{A} = \dim \mathcal{A}^\circ$. //

We will see in Corollary 5.6 below, that canonical Pontryagin space extensions are unique up to isomorphism.

a. Extension of morphisms.

It is important to see how morphisms between almost Pontryagin spaces can be extended to morphisms between canonical Pontryagin space extensions. First we deal with concrete extensions as constructed in 5.2.

5.3 Proposition. *Let $\mathcal{A}_1, \mathcal{A}_2$ be almost Pontryagin spaces, and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism. Let spaces $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1/\ker\phi)$ and $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_2)$ be constructed as in 5.2 from some choices of subspaces $\mathcal{B}_1 \subseteq \mathcal{A}_1/\ker\phi$ and $\mathcal{B}_2 \subseteq \mathcal{A}_2$, respectively. Then there exists a morphism $\tilde\phi : \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1/\ker\phi) \to \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_2)$, such that*

$$\begin{array}{ccccc} \mathcal{A}_1 & \xrightarrow{\pi} & \mathcal{A}_1/\ker\phi & \xrightarrow{\iota_{\mathrm{ext}}} & \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1/\ker\phi) \\ {\scriptstyle\phi}\downarrow & & & & \downarrow{\scriptstyle\tilde\phi} \\ \mathcal{A}_2 & & \xrightarrow[\iota_{\mathrm{ext}}]{\qquad\qquad} & & \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_2) \end{array} \tag{5.1}$$

Proof. There exists an injective morphism $\phi' : \mathcal{A}_1/\ker\phi \to \mathcal{A}_2$ such that

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\pi} & \mathcal{A}_1/\ker\phi \\ {\scriptstyle\phi}\downarrow & \swarrow{\scriptstyle\phi'} & \\ \mathcal{A}_2 & & \end{array}$$

cf. Remark 2.5, (vii). Obviously, it is enough to prove the assertion for ϕ'. Hence, we may assume without loss of generality that ϕ is injective.

The subspace $(\iota_{\mathrm{ext}}\circ\phi)(\mathcal{B}_1)$ of $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_2)$ is closed and nondegenerated. Moreover, $(\iota_{\mathrm{ext}}\circ\phi)(\mathcal{A}_1^\circ)$ is a neutral subspace of $(\iota_{\mathrm{ext}}\circ\phi)(\mathcal{B}_1)^\perp$. Hence there exists a subspace C' of $(\iota_{\mathrm{ext}}\circ\phi)(\mathcal{B}_1)^\perp$, such that $(\iota_{\mathrm{ext}}\circ\phi)(\mathcal{A}_1^\circ)\#C'$, cf. [B, §I.10].

The space $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1)$ is defined as $\mathcal{B}_1[\dot{+}](\mathcal{A}_1^\circ\dot{+}C)$ with $\mathcal{A}_1^\circ\#C$. Choose a basis $\{\delta_1,\dots,\delta_n\}$ of $\mathcal{A}_1^\circ$, and let $\{\epsilon_1,\dots,\epsilon_n\}$ be a basis of C with

$$[\delta_j,\epsilon_k] = \begin{cases} 0\,, & j\neq k \\ 1\,, & j=k \end{cases}$$

Since $\iota_{\mathrm{ext}}\circ\phi$ is injective, the set $\{(\iota_{\mathrm{ext}}\circ\phi)(\delta_1),\dots,(\iota_{\mathrm{ext}}\circ\phi)(\delta_n)\}$ is a basis of $(\iota_{\mathrm{ext}}\circ\phi)(\mathcal{A}_1^\circ)$. Hence there exists a basis $\{\epsilon_1',\dots,\epsilon_n'\}$ of C' such that

$$\big[(\iota_{\mathrm{ext}}\circ\phi)(\delta_j),\epsilon_k'\big] = \begin{cases} 0\,, & j\neq k \\ 1\,, & j=k \end{cases}$$

With these notations define $\tilde{\phi} : \mathfrak{P}_{\text{ext}}(\mathcal{A}_1) \to \mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$ by

$$\tilde{\phi}|_{\iota_{\text{ext}}(\mathcal{A}_1)} := \iota_{\text{ext}} \circ \phi \circ \iota_{\text{ext}}^{-1}, \quad \tilde{\phi}(\epsilon_j) := \epsilon_j', \; j = 1, \ldots, n \,.$$

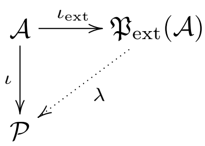
It is straightforward to check that $\tilde{\phi}$ is isometric. Moreover, the fact that (5.1) commutes is built into the definition. ❑

5.4 Remark. The extension $\tilde{\phi}$ in Proposition 5.3 is in general not unique. In fact, whenever $\mathcal{P}$ is a Pontryagin space with

$$(\iota_{\text{ext}} \circ \phi)(\mathcal{A}_1) \subseteq \mathcal{P} \subseteq \mathfrak{P}_{\text{ext}}(\mathcal{A}_2) \,,$$

the choice of $\tilde{\phi}$ can be made such that $\operatorname{ran} \tilde{\phi} \subseteq \mathcal{P}$. //

5.5 Corollary. *Let $\mathcal{A}$ be an almost Pontryagin space, and let $(\iota, \mathcal{P})$ be a canonical Pontryagin space extension of $\mathcal{A}$. Moreover, let $(\iota_{\text{ext}}, \mathfrak{P}_{\text{ext}}(\mathcal{A})$ be the canonical Pontryagin space extension constructed in 5.2 from some subspace $\mathcal{B}$. Then there exists an isomorphism of $\lambda : \mathfrak{P}_{\text{ext}}(\mathcal{A}) \to \mathcal{P}$ such that*



Proof. Since $\mathcal{P}$ is a Pontryagin space, we have $\mathfrak{P}_{\text{ext}}(\mathcal{P}) = \mathcal{P}$ and $\iota_{\text{ext}} = \text{id}$. Proposition 5.3 applied with the map $\iota : \mathcal{A} \to \mathcal{P}$ gives a morphism $\lambda : \mathfrak{P}_{\text{ext}}(\mathcal{A}) \to \mathcal{P}$. Since a morphism between Pontryagin spaces is injective, we conclude from $\lambda(\iota_{\text{ext}}(\mathcal{A})) = \iota(\mathcal{A})$ and

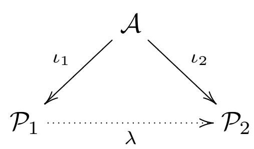
$$\dim \mathcal{P}/\iota(\mathcal{A}) = \dim \mathcal{A}^\circ = \dim \mathfrak{P}_{\text{ext}}(\mathcal{A})/\iota_{\text{ext}}(\mathcal{A}) \,,$$

that λ is an isomorphism. ❑

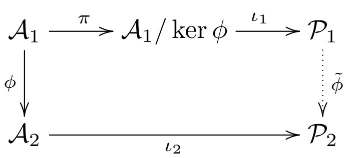
This fact has some immediate, but important, consequences.

5.6 Corollary.

(i) *Let $\mathcal{A}$ be an almost Pontryagin space. If $(\iota_1, \mathcal{P}_1)$ and $(\iota_2, \mathcal{P}_2)$ are canonical Pontryagin space extension of $\mathcal{A}$, then there exists an isomorphism $\lambda : \mathcal{P}_1 \to \mathcal{P}_2$ with*



(ii) *Let $\mathcal{A}_1, \mathcal{A}_2$ be almost Pontryagin spaces, and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism. Let $(\iota_1, \mathcal{P}_1)$ and $(\iota_2, \mathcal{P}_2)$ be canonical Pontryagin space extensions of $\mathcal{A}_1/\ker\phi$ and $\mathcal{A}_2$, respectively. Then there exists a morphism $\tilde{\phi} : \mathcal{P}_1 \to \mathcal{P}_2$, such that*



b. Compatibility with orthogonal coupling.

The following result, although a fairly simple consequence of Proposition 5.3, turns out useful.

5.7 Proposition. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be almost Pontryagin spaces and let α be a bijective function between subspaces of $\mathcal{A}_1^\circ$ and $\mathcal{A}_2^\circ$. Then there exist morphisms $\tilde\iota_1^\alpha$ and $\tilde\iota_2^\alpha$, such that*

$$\begin{array}{ccccc} \mathcal{A}_1 & \xrightarrow{\ \iota_1^\alpha\ } & \mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2 & \xleftarrow{\ \iota_2^\alpha\ } & \mathcal{A}_2 \\ \downarrow{\scriptstyle \iota_{\text{ext}}} & & \downarrow{\scriptstyle \iota_{\text{ext}}} & & \downarrow{\scriptstyle \iota_{\text{ext}}} \\ \mathfrak{P}_{\text{ext}}(\mathcal{A}_1) & \xrightarrow[\tilde\iota_1^\alpha]{} & \mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2) & \xleftarrow[\tilde\iota_2^\alpha]{} & \mathfrak{P}_{\text{ext}}(\mathcal{A}_2) \end{array} \tag{5.2}$$

The choice of $\tilde\iota_1^\alpha$ and $\tilde\iota_2^\alpha$ can be made such that $\operatorname{ran}\tilde\iota_1^\alpha \cap \operatorname{ran}\tilde\iota_2^\alpha$ *is a nondegenerated subspace of* $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)$ *with dimension* $2\dim(\operatorname{dom}\alpha)$ *which contains* $(\iota_{\text{ext}} \circ \iota_1^\alpha)(\operatorname{dom}(\alpha))$.

Proof. By Remark 4.3, (*ii*), the maps ι_1^α and ι_2^α are injective. Hence Proposition 5.3 guarantees existence of $\tilde\iota_1^\alpha$ and $\tilde\iota_2^\alpha$ which satisfy (5.2). We have to show that they can be chosen so to satisfy the stated additional requirement. To this end we use the concrete realization of orthogonal couplings given in 4.8, the concrete form of canonical Pontryagin space extensions given in 5.2, and trace the construction of $\tilde\iota_1^\alpha$ and $\tilde\iota_2^\alpha$ in the proof of Proposition 5.3.

Choose closed subspaces $\mathcal{A}_{j,r}$ of $\mathcal{A}_j$ with $\mathcal{A}_j = \mathcal{A}_{j,r}[\dot+]\mathcal{A}_j^\circ$, $j = 1, 2$, choose D_j with $\mathcal{A}_1^\circ = D_1 \dot+ \operatorname{dom}\alpha$ and $\mathcal{A}_2^\circ = D_2 \dot+ \operatorname{ran}\alpha$, and set $D := \operatorname{ran}\alpha$. We then have

$$\mathcal{A}_1 = \mathcal{A}_{1,r}[\dot+](D_1 \dot+ \operatorname{dom}\alpha), \quad \mathcal{A}_2 = \mathcal{A}_{2,r}[\dot+](D_2 \dot+ D)\,.$$

Then we can identify

$$\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2 \cong \mathcal{A}_{1,r}[\dot+](D_1 \dot+ D \dot+ D_2)[\dot+]\mathcal{A}_{2,r}\,.$$

In this identification, the embeddings ι_1^α and ι_2^α act as

$$\iota_1^\alpha(x_r + x_1 + x_d) = x_r + x_1 + \alpha(x_d), \quad x_r \in \mathcal{A}_{1,r},\ x_1 \in D_1,\ x_d \in \operatorname{dom}\alpha\,,$$

$$\iota_2^\alpha(x) = x, \quad x \in \mathcal{A}_2\,,$$

and the isotropic part of $\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2$ is given as

$$\big(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2\big)^\circ = D_1 \dot+ D \dot+ D_2\,,$$

For the construction of $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1)$, $\mathfrak{P}_{\text{ext}}(\mathcal{A}_2)$, and $\mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)$, we use the closed nondegenerated subspaces $\mathcal{A}_{1,r}$, $\mathcal{A}_{2,r}$, and $\mathcal{A}_{1,r}[\dot+]\mathcal{A}_{2,r}$, respectively. Then we can write (note that $\dim\operatorname{dom}\alpha = \dim\operatorname{ran}\alpha$)

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}_1) = \mathcal{A}_{1,r}\,[\dot+]\,\Big((D_1 \dot+ C_1)[\dot+](\operatorname{dom}\alpha \dot+ C)\Big)$$

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}_2) = \mathcal{A}_{2,r}\,[\dot+]\,\Big((D_2 \dot+ C_2)[\dot+](D \dot+ C)\Big)$$

$$\mathfrak{P}_{\text{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2) = \mathcal{A}_{1,r}\,[\dot+]\,\Big((D_1 \dot+ C_1)[\dot+](D \dot+ C)[\dot+](D_2 \dot+ C_2)\Big)\,[\dot+]\,\mathcal{A}_{2,r}$$

with neutral spaces C_1, C_d, C, C_2 satisfying $C_1 \# D_1$, $C \# \operatorname{dom}\alpha$, $C_2 \# D_2$, $C \# D$, and the extension embeddings are the respective set-theoretic inclusion maps. The maps constructed in Proposition 5.3 act as

$$\tilde{\iota}_1^\alpha\big(x_r + (x_1 + y_1) + (x_d + y)\big) = x_r + (x_1 + y_1) + (\alpha(x_d) + y),$$
$$x_r \in \mathcal{A}_{1,r}, x_1 \in D_1, y_1 \in C_1, x_d \in \operatorname{dom}\alpha, y \in C\,,$$

$$\tilde{\iota}_2^\alpha(x) = x, \quad x \in \mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_2)\,.$$

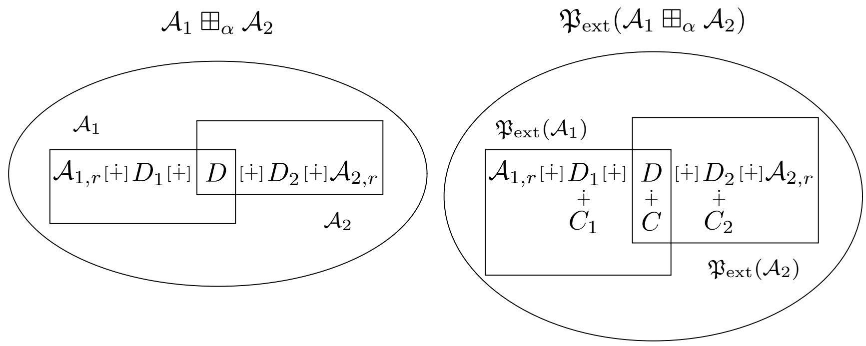
From this we see that

$$\operatorname{ran}\tilde{\iota}_1^\alpha \cap \operatorname{ran}\tilde{\iota}_2^\alpha = D \dotplus C\,.$$

❑

5.8 Remark. In the situation of Proposition 5.7, the mappings ι_1^α and ι_2^α are both injective, all extension embeddings ι_{ext} are by definition injective, and $\tilde{\iota}_1^\alpha, \tilde{\iota}_1^\alpha$ are morphisms with nondegenerated domain and are thus also injective. Hence, we can think of $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)$ as the biggest of the six spaces in (5.2) which contains the other ones.

If we supress the distinction between $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, and think of them both as being equal to the space D, we can picture the situation as follows:



//

We will in our later discussions encounter the situation that $\operatorname{dom}\alpha = \mathcal{A}_1^\circ$ and $\operatorname{ran}\alpha = \mathcal{A}_2^\circ$ in Proposition 5.7. In fact, the computation of inner products given in Lemma 5.9 below plays an important role in [SW3]. Hence, we shall now discuss this case in some more detail.

Note that, due to $\operatorname{dom}\alpha = \mathcal{A}_1^\circ$ and $\operatorname{ran}\alpha = \mathcal{A}_2^\circ$, we have $D_1 = D_2 = C_1 = C_2 = 0$ and $(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)^\circ = D = \mathcal{A}_1^\circ = \mathcal{A}_2^\circ$. Denote by P_D, P_C, $P_{\mathcal{A}_{1,r}[\dotplus]\mathcal{A}_{2,r}}$ and $P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2}$ the projections of the space $\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2)$ onto the space denoted as index according to the above pictured direct sum decomposition. Thus, e.g., we have $\operatorname{ran} P_D = D$ and $\ker P_D = \mathcal{A}_{1,r}[\dotplus]C[\dotplus]\mathcal{A}_{2,r}$.

5.9 Lemma. *Assume that in the situation of Proposition 5.7 we have* $\operatorname{dom}\alpha = \mathcal{A}_1^\circ$ *and* $\operatorname{ran}\alpha = \mathcal{A}_2^\circ$. *Then the following hold:*

(*i*) *We have*

$$P_{\mathcal{A}_{1,r}[\dotplus]\mathcal{A}_{2,r}} + P_D + P_C = I, \quad P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} + P_C = I$$

$$P_{\mathcal{A}_{1,r}[\dotplus]\mathcal{A}_{2,r}}\big(\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_j)\big) = \mathcal{A}_{j,r},\ P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2}\big(\mathfrak{P}_{\mathrm{ext}}(\mathcal{A}_j)\big) = \mathcal{A}_j, \quad j = 1, 2$$

Let elements $x_1 \in \mathfrak{P}_{\rm ext}(\mathcal{A}_1)$ *and* $x_2 \in \mathfrak{P}_{\rm ext}(\mathcal{A}_2)$ *be given.*

(*ii*) $[x_1, x_2] = [P_D x_1, P_C x_2] + [P_C x_1, P_D x_2]$.

(*iii*) *We have* $P_C x_1 = P_C x_2$ *if and only if*

$$[x_1, h] = [x_2, h], \ \ h \in D\,.$$

In this case $x_1 + P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2 = P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_1 + x_2$.

Let, moreover, elements $y_1 \in \mathfrak{P}_{\rm ext}(\mathcal{A}_1)$ *and* $y_2 \in \mathfrak{P}_{\rm ext}(\mathcal{A}_2)$ *be given.*

(*iv*) *If* $P_C x_1 = P_C x_2$ *and* $P_C y_1 = P_C y_2$, *then*

$$[x_1 + P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, y_1 + P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] = [x_1, y_1] + [x_2, y_2]\,.$$

Proof. The formulas in (*i*) are immediate from the definitions of the corresponding projections. In order to see the equality asserted in (*ii*) we compute

$$[x_1, x_2] = \big[(P_{\mathcal{A}_{1,r}[\dot+]\mathcal{A}_{2,r}} + P_D + P_C)x_1, (P_{\mathcal{A}_{1,r}[\dot+]\mathcal{A}_{2,r}} + P_D + P_C)x_2\big] =$$
$$= \big[P_{\mathcal{A}_{1,r}[\dot+]\mathcal{A}_{2,r}} x_1, P_{\mathcal{A}_{1,r}[\dot+]\mathcal{A}_{2,r}} x_2\big] + \big[(P_D + P_C)x_1, (P_D + P_C)x_2\big] =$$
$$= [P_D x_1, P_C x_2] + [P_C x_1, P_D x_2]$$

We come to the proof of (*iii*). We have, for each $h \in D$,

$$[x_1, h] = \big[(P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} + P_C)x_1, h\big] = [P_C x_1, h]$$
$$[x_2, h] = \big[(P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} + P_C)x_2, h\big] = [P_C x_2, h]$$

Since $D\#C$, the asserted equivalence follows. Moreover, in case that $P_C x_1 = P_C x_2$, we have

$$x_1 + P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2 = P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_1 + P_C x_1 + P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2 =$$
$$= P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_1 + P_C x_2 + P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2 = P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_1 + x_2$$

Finally, assume that we are in the situation given in (*iv*). We first compute

$$[x_2, y_2] = \big[(P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} + P_C)x_2, (P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} + P_C)y_2\big] =$$
$$= [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] + [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_C y_2] + [P_C x_2, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] =$$
$$= [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] + [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_C y_1] + [P_C x_1, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2]$$

Hence we obtain that

$$[x_1 + P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, y_1 + P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] =$$
$$= [x_1, y_1] + [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, y_1] + [x_1, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] + [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] =$$
$$= [x_1, y_1] + [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_C y_1] + [P_C x_1, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] + [P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} x_2, P_{\mathcal{A}_1 \boxplus_\alpha \mathcal{A}_2} y_2] =$$
$$= [x_1, y_1] + [x_2, y_2]$$

❑

6 Almost Pontryagin space completions

In the context of almost Pontryagin spaces, completions have been investigated in [KWW]; some basic ideas going back to [JLT]. In these papers existence of completions was shown, and it was seen that completions are related to linear functionals.

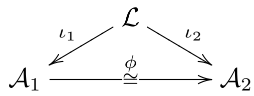
In this section we give a much more complete treatment of this topic. As a byproduct we also obtain an alternative proof of the uniqueness part in [KWW, Proposition 4.4], where a more 'basis dependent' approach was used. Recall the definition of almost Pontryagin space completions.

6.1 Definition. Let $\langle \mathcal{L}, [.,.] \rangle$ be an inner product space. A pair $(\iota, \mathcal{A})$ is called an aPs-completion of $\mathcal{L}$, if $\mathcal{A}$ is an aPs, and ι is an isometric map of $\mathcal{L}$ onto a dense subspace of $\mathcal{A}$. //

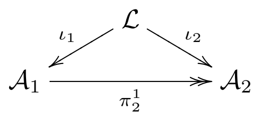
Two aPs-completions of an inner product space $\mathcal{L}$ might be 'the same', or one might be 'larger' than the other. This is made precise by the following notions.

6.2 Definition. Let $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ be two aPs-completions of an inner product space $\mathcal{L}$.

(*i*) We call $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ isomorphic, and write $(\iota_1, \mathcal{A}_1) \cong (\iota_2, \mathcal{A}_2)$, if there exists an isomorphism ϕ of $\mathcal{A}_1$ onto $\mathcal{A}_2$, such that $\phi \circ \iota_1 = \iota_2$, i.e.



(*ii*) We write $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$, if there exists a surjective morphism π^1_2 of $\mathcal{A}_1$ onto $\mathcal{A}_2$, such that $\pi^1_2 \circ \iota_1 = \iota_2$, i.e.



//

Obviously, isomorphism is an equivalence relation on the set of all aPs-completions of $\mathcal{L}$, and the relation $\succeq$ is reflexive and transitive. Moreover, a short argument will show that

$$\Big((\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2) \wedge (\iota_2, \mathcal{A}_2) \succeq (\iota_1, \mathcal{A}_1)\Big) \iff (\iota_1, \mathcal{A}_1) \cong (\iota_2, \mathcal{A}_2)$$

We conclude that $\succeq$ induces a partial order on the set of all aPs-completions of $\mathcal{L}$ modulo isomorphism.

6.3 Remark. If $(\iota_1, \mathcal{A}_1)$ is an aPs-completion of $\mathcal{L}$, $\mathcal{A}_2$ is an aPs, and π is a surjective morphism of $\mathcal{A}_1$ onto $\mathcal{A}_2$, then $(\pi \circ \iota_1, \mathcal{A}_2)$ is an aPs-completion of $\mathcal{L}$ and $(\iota_1, \mathcal{A}_1) \succeq (\pi \circ \iota_1, \mathcal{A}_2)$. //

Let $\mathcal{L}$ be an inner product space. If in some aPs-completion $(\iota, \mathcal{A})$ of $\mathcal{L}$ the space $\mathcal{A}$ is nondegenerated, i.e. a Pontryagin space, we will speak of $(\iota, \mathcal{A})$ as a Pontryagin space completion of $\mathcal{L}$.

6.4 Remark. It is well-known, see e.g. [B, §V.2,§I.11], that $\mathcal{L}$ admits a Pontryagin space completion if and only if $\operatorname{ind}_- \mathcal{L} < \infty$. Moreover, in this case, each two Pontryagin space completions are isomorphic. Since $\operatorname{ind}_- \mathcal{L} < \infty$ is obviously a necessary condition for existence of an aPs-completion, we conclude that $\mathcal{L}$ admits an aPs-completion if and only if $\operatorname{ind}_- \mathcal{L} < \infty$. //

Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$, and consider the map $\mathfrak{L}$ which assigns to each aPs-completion $(\iota, \mathcal{A})$ of $\mathcal{L}$ the linear subspace

$$\mathfrak{L}(\iota, \mathcal{A}) := \iota^* \mathcal{A}'$$

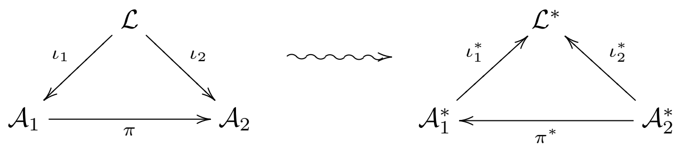
of the algebraic dual $\mathcal{L}^*$ of $\mathcal{L}$. Here ι^* denotes the (algebraic) adjoint of ι, that is $\iota^* : \mathcal{A}^* \to \mathcal{L}^*$ and $\iota^* f = f \circ \iota$.

The next statement already contains a good portion of our description of aPs-completions.

6.5 Lemma. *Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$, and let $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ be two aPs-completions of $\mathcal{L}$ with $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$. Then*

$$\mathfrak{L}(\iota_1, \mathcal{A}_1) \supseteq \mathfrak{L}(\iota_2, \mathcal{A}_2) \quad \textit{and} \quad \dim \big(\mathfrak{L}(\iota_1, \mathcal{A}_1) / \mathfrak{L}(\iota_2, \mathcal{A}_2)\big) = \operatorname{ind}_0 \mathcal{A}_1 - \operatorname{ind}_0 \mathcal{A}_2 \,.$$

Proof. Let $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ be a surjective morphism with $\pi \circ \iota_1 = \iota_2$. Passing to adjoints yields



Since π is continuous, we have $\pi^* \mathcal{A}_2' \subseteq \mathcal{A}_1'$. It readily follows that

$$\mathfrak{L}(\iota_2, \mathcal{A}_2) = \iota_2^* \mathcal{A}_2' = \iota_1^* \pi^* \mathcal{A}_2' \subseteq \iota_1^* \mathcal{A}_1' = \mathfrak{L}(\iota_1, \mathcal{A}_1) \,.$$

We need to compute codimension. Since $\operatorname{ran} \iota_1$ is dense in $\mathcal{A}_1$, the restriction of ι_1^* to $\mathcal{A}_1'$ is injective. Thus

$$\dim \big(\mathfrak{L}(\iota_1, \mathcal{A}_1) / \mathfrak{L}(\iota_2, \mathcal{A}_2)\big) = \dim \big(\iota_1^* \mathcal{A}_1' / \iota_1^* \pi^* \mathcal{A}_2'\big) = \dim \big(\mathcal{A}_1' / \pi^* \mathcal{A}_2'\big) \,.$$

Since π is surjective, by the Closed Range Theorem, $\pi^* \mathcal{A}_2'$ is a w^*-closed subspace of $\mathcal{A}_1'$. It follows that

$$\pi^* \mathcal{A}_2' = \overline{\pi^* \mathcal{A}_2'}^{w^*} = (\ker \pi)^\perp \,,$$

and hence

$$\dim \big(\mathcal{A}_1' / \pi^* \mathcal{A}_2'\big) = \dim \big(\mathcal{A}_1' / (\ker \pi)^\perp\big) = \dim (\ker \pi)' \,.$$

Since π is isometric, we have $\ker \pi \subseteq \mathcal{A}_1^\circ$. In particular, $\ker \pi$ is finite dimensional, and hence

$$\dim (\ker \pi)' = \dim \ker \pi \,.$$

The relation $\ker \pi \subseteq \mathcal{A}_1^\circ$ also shows that $\ker \pi = \ker(\pi|_{\mathcal{A}_1^\circ})$. Since π is surjective, we have $\pi^{-1}(\mathcal{A}_2^\circ) = \mathcal{A}_1^\circ$, and hence $\pi|_{\mathcal{A}_1^\circ}$ maps $\mathcal{A}_1^\circ$ surjectively onto $\mathcal{A}_2^\circ$. It follows that

$$\dim \ker \pi = \dim \ker(\pi|_{\mathcal{A}_1^\circ}) = \dim \mathcal{A}_1^\circ - \dim \mathcal{A}_2^\circ \,.$$

Putting together these relations, the desired formula follows. ❑

Lemma 6.5 shows, in particular, that

$$(\iota_1, \mathcal{A}_1) \cong (\iota_2, \mathcal{A}_2) \implies \mathfrak{L}(\iota_1, \mathcal{A}_1) = \mathfrak{L}(\iota_2, \mathcal{A}_2). \tag{6.1}$$

Since each two Pontryagin space completions of $\mathcal{L}$ are isomorphic, the following notion is well-defined.

6.6 Definition. Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$. Choose a Pontryagin space completion $(\iota, \mathcal{P})$ of $\mathcal{L}$, and let a linear subspace $\mathcal{L}'$ of $\mathcal{L}^*$ be defined as

$$\mathcal{L}' := \mathfrak{L}(\iota, \mathcal{P}), \quad (\iota, \mathcal{P}) \text{ Pontryagin space completion of } \mathcal{L}.$$

//

6.7 Remark. The choice of the notation $\mathcal{L}'$ is not accidentially. In fact, using the terminology of [B, §IV.6], the space $\mathfrak{L}(\iota, \mathcal{P})$ is nothing else but the topological dual space of $\mathcal{L}$ with respect to the unique decomposition majorant which $\mathcal{L}$ carries as inner product space with finite negative index. //

The map $\mathfrak{L}$ is defined on the set of all aPs-completions of $\mathcal{L}$, and maps an aPs-completion to a linear subspace of the algebraic dual $\mathcal{L}^*$. Due to (6.1), it induces a map from equivalence classes of aPs-completions modulo isomorphisms to linear subspaces of $\mathcal{L}^*$; we denote this map again by $\mathfrak{L}$. It acts between two partially ordered sets. In the next result we show that it is an injective order homomorphism and determine its range.

6.8 Theorem. *Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$. Then $\mathfrak{L}$ induces an order-isomorphism of the set of all aPs-completions of $\mathcal{L}$ modulo isomorphism onto the set of all linear subspaces of $\mathcal{L}^*$ which contain $\mathcal{L}'$ with finite codimension. Thereby,*

$$\dim \big(\mathfrak{L}(\iota, \mathcal{A}) \big/ \mathcal{L}' \big) = \operatorname{ind}_0 \mathcal{A}. \tag{6.2}$$

Proof.
Step 1: Let $(\iota, \mathcal{A})$ be an aPs-completion of $\mathcal{L}$. Denote by $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ$ the canonical projection, then π is a surjective morphism. Hence, $(\pi \circ \iota, \mathcal{A}/\mathcal{A}^\circ)$ is also an aPs-completion and $(\iota, \mathcal{A}) \succeq (\pi \circ \iota, \mathcal{A}/\mathcal{A}^\circ)$, cf. Remark 6.3. However, since $\mathcal{A}/\mathcal{A}^\circ$ is nondegenerated, $(\pi \circ \iota, \mathcal{A}/\mathcal{A}^\circ)$ actually is a Pontryagin space completion of $\mathcal{L}$. Thus $\mathfrak{L}(\pi \circ \iota, \mathcal{A}/\mathcal{A}^\circ) = \mathcal{L}'$, and we obtain from Lemma 6.5 that $\mathfrak{L}(\iota, \mathcal{A})$ contains $\mathcal{L}'$ with codimension $\operatorname{ind}_0 \mathcal{A}^\circ$.

Step 2: Assume next that $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ are aPs-completions of $\mathcal{L}$ such that $\mathfrak{L}(\iota_1, \mathcal{A}_1) \supseteq \mathfrak{L}(\iota_2, \mathcal{A}_2)$. Therefore, for each given $f \in \mathcal{A}_2'$, there exists $\tilde{f} \in \mathcal{A}_1'$ with $\iota_1^* \tilde{f} = \iota_2^* f$. Since $\iota_1^*|_{\mathcal{A}_1'}$ is injective, this element $\tilde{f}$ is uniquely determined. Hence, a map $\Lambda : \mathcal{A}_2' \to \mathcal{A}_1'$ is well-defined by the requirement

$$\iota_1^*(\Lambda f) = \iota_2^* f, \quad f \in \mathcal{A}_2'.$$

Clearly, Λ is linear.

We are going to apply the Closed Graph Theorem. To check its hypothesis, let a sequence $(f_n)_{n \in \mathbb{N}}$ of functionals $f_n \in \mathcal{A}_2'$ be given, and assume that $f_n \to$

f in $\mathcal{A}_2'$ and $\Lambda f_n \to g$ in $\mathcal{A}_1'$. Since convergence in the norm implies w^*-convergence, we have for each $x \in \mathcal{L}$

$$\begin{array}{l}(\iota_2^* f_n)x = f_n(\iota_2 x) \to f(\iota_2 x) = (\iota_2^* f)x = \iota_1^*(\Lambda f)x \\ \quad \| \\ \iota_1^*(\Lambda f_n)x = (\Lambda f_n)(\iota_1 x) \to g(\iota_1 x) = (\iota_1^* g)x\end{array}$$

Since $\iota_1^*|_{\mathcal{A}_1'}$ is injective, this implies that $\Lambda f = g$. It follows that Λ is bounded.

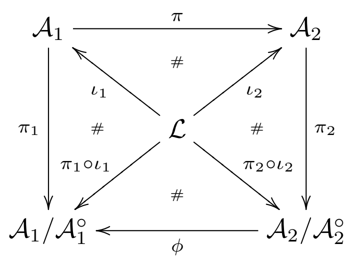
Let $\|.\|_1$ and $\|.\|_2$ be norms on $\mathcal{A}_1$ and $\mathcal{A}_2$ which induce their respective topologies. Moreover, let $\|.\|_1'$ and $\|.\|_2'$ be the corresponding operator norms on $\mathcal{A}_1'$ and $\mathcal{A}_2'$. We compute for $x \in \mathcal{L}$

$$\|\iota_2 x\|_2 = \sup\Big\{ \,|\underbrace{f(\iota_2 x)}_{\substack{\| \\ (\iota_2^* f)x = \iota_1^*(\Lambda f)x = (\Lambda f)(\iota_1 x)}}| : \ f \in \mathcal{A}_2', \|f\|_2' \leq 1\Big\} =$$

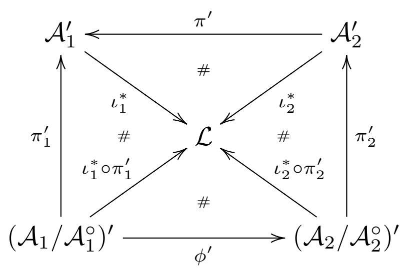
$$= \sup\big\{|\tilde{f}(\iota_1 x)| : \ \tilde{f} \in \underbrace{\Lambda\big(\{f \in \mathcal{A}_2' : \|f\|_2' \leq 1\}\big)}_{\subseteq \{\tilde{f} \in \mathcal{A}_1' : \|\tilde{f}\|_1' \leq \|\Lambda\|\}} \big\} \leq \|\Lambda\| \cdot \|\iota_1 x\|_1 .$$

It follows that $\ker \iota_1 \subseteq \ker \iota_2$, so that $\iota_2 \circ \iota_1^{-1}$ is a well-defined map. Moreover, it follows that $\iota_2 \circ \iota_1^{-1}$ is bounded. Let $\pi : \mathcal{A}_1' \to \mathcal{A}_2'$ be its extension by continuity. Then π is isometric and has dense range in $\mathcal{A}_2'$.

Let $\pi_j : \mathcal{A}_j \to \mathcal{A}_j/\mathcal{A}_j^\circ$, $j = 1, 2$ denote the canonical projections. Since $(\pi_1 \circ \iota_1, \mathcal{A}_1/\mathcal{A}_1^\circ)$ and $(\pi_2 \circ \iota_2, \mathcal{A}_2/\mathcal{A}_2^\circ)$ are both Pontryagin space completions of $\mathcal{L}$, there exists an isomorphism ϕ of $\mathcal{A}_2/\mathcal{A}_2^\circ$ onto $\mathcal{A}_1/\mathcal{A}_1^\circ$ with $\phi \circ (\pi_2 \circ \iota_2) = \pi_1 \circ \iota_1$. Thus, in the following diagram, each outer triangle commutes.



Passing to adjoints, gives the outer triangles in



Injectivity of $\iota_1^*|_{\mathcal{A}_1'}$ implies $\pi_1' = \pi' \circ \pi_2' \circ \phi'$. In particular, $\operatorname{ran} \pi_1' \subseteq \operatorname{ran} \pi' \subseteq \mathcal{A}_1'$. However, as we saw in the proof of Lemma 6.5, $\operatorname{ran} \pi_1'$ is a closed subspace of $\mathcal{A}_1'$ with finite codimension. Hence, also $\operatorname{ran} \pi'$ is closed in $\mathcal{A}_1'$. By the Closed

Range Theorem, $\operatorname{ran}\pi$ is closed in $\mathcal{A}_1$, and hence π is surjective. Thus π is a morphism and we have shown that $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$

Step 3: So far, we have seen that $\mathfrak{L}$ maps aPs-completions into the set of all subspaces of $\mathcal{L}^*$ which contain $\mathcal{L}'$ with finite codimension, that (6.2) holds, and that

$$(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2) \iff \mathfrak{L}(\iota_1, \mathcal{A}_1) \supseteq \mathfrak{L}(\iota_2, \mathcal{A}_2)\,.$$

In particular, $\mathfrak{L}(\iota_1, \mathcal{A}_1) = \mathfrak{L}(\iota_2, \mathcal{A}_2)$ if and only if $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ are isomorphic.

It remains to show that, for each given subspace $\mathcal{M}$ with $\mathcal{L}' \subseteq \mathcal{M}$ and $\dim \mathcal{M}/\mathcal{L}' < \infty$, there exists an aPs-completion $(\iota, \mathcal{A})$ of $\mathcal{L}$ with $\mathfrak{L}(\iota, \mathcal{A}) = \mathcal{M}$. The construction of one such completion goes back to [JLT] and was formulated and proved in the aPs-context in [KWW]. Therefore, let us only briefly indicate the method. Put $n := \dim \mathcal{M}/\mathcal{L}'$ and choose $f_1, \ldots, f_n \in \mathcal{L}^*$ such that $\mathcal{M} = \operatorname{span}(\mathcal{L}' \cup \{f_1, \ldots, f_n\})$. Moreover, let $(\iota_\mathcal{P}, \mathcal{P})$ be the Pontryagin space completion of $\mathcal{L}$. Define

- $\rightsquigarrow$ $\mathcal{A} := \mathcal{P}[\dot{+}]\mathbb{C}^n$, and $\mathcal{T}$ the product topology on $\mathcal{A}$,
- $\rightsquigarrow$ $[x + \xi, y + \eta]_\mathcal{A} := [x, y]_\mathcal{P}$, $x, y \in \mathcal{P}, \xi, \eta \in \mathbb{C}^n$,
- $\rightsquigarrow$ $\iota x := x + (f_1(x), \ldots, f_n(x))$, $x \in \mathcal{L}$.

Then one can show that $(\iota, \mathcal{A})$ is an aPs-completion of $\mathcal{L}$ with $\mathfrak{L}(\iota, \mathcal{A}) = \mathcal{M}$. ❑

6.9 Corollary. *Let $(\iota_1, \mathcal{A}_1)$ and $(\iota_2, \mathcal{A}_2)$ be two aPs-completions of an inner product space $\mathcal{L}$. Then $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$ if and only if $\ker \iota_1 \subseteq \ker \iota_2$ and $\iota_2 \circ \iota_1^{-1} : \operatorname{ran} \iota_1 \to \operatorname{ran} \iota_2$ is bounded.*

Proof. If $(\iota_1, \mathcal{A}_1) \succeq (\iota_2, \mathcal{A}_2)$, then the map π_2^1 guaranteed by the definition of $\succeq$ is linear, bounded, and satisfies $\pi_2^1 \circ \iota_1 = \iota_2$. The required properties of ι_1 and ι_2 follow. Conversely, assume that $\ker \iota_1 \subseteq \ker \iota_2$ and $\iota_2 \circ \iota_1^{-1} : \operatorname{ran} \iota_1 \to \operatorname{ran} \iota_2$ is bounded. Let $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ be the extension by continuity of $\iota_2 \circ \iota_1^{-1}$, then $\iota_2^* = \iota_1^* \circ \pi'$ and hence

$$\mathfrak{L}(\iota_2, \mathcal{A}_2) = \iota_2^* \mathcal{A}_2' = \big(\iota_1^* \circ \pi'\big)\mathcal{A}_2' \subseteq \iota_1^* \mathcal{A}_1' = \mathfrak{L}(\iota_1, \mathcal{A}_1)\,.$$

❑

References

[B] J.Bognar: *Indefinite inner product spaces,* Springer Verlag, Berlin 1974.

[IKL] I.S.Iohvidov, M.G.Krein, H.Langer: *Introduction to the spectral theory of operators in spaces with an indefinite metric,* Akademie Verlag, Berlin 1982.

[JLT] P.Jonas, H.Langer, B.Textorius: *Models and unitary equivalence of cyclic selfadjoint operators in Pontryagin spaces,* Oper. Theory Adv. Appl. 59 (1992), 252–284.

[KWW] M.Kaltenbäck, H.Winkler, H.Woracek: *Almost Pontryagin spaces,* Oper. Theory Adv. Appl. 160 (2005), 253–271.

[KW1] M.Kaltenbäck, H.Woracek: *On extensions of hermitian functions with a finite number of negative squares,* J. Operator Theory 40 (1998), 147–183.

[KW2] M.Kaltenbäck, H.Woracek: *The Kreĭn formula for generalized resolvents in degenerated inner product spaces,* Monatsh. Math. 127 (1999), 119–140.

[KW3] M.Kaltenbäck, H.Woracek: *Pontryagin spaces of entire functions III,* Acta Sci. Math. (Szeged) 69 (2003), 241–310.

[PT] F.Philipp, C.Trunk: *G-self-adjoint operators in almost Pontryagin spaces,* Oper. Theory Adv. Appl. 188 (2009), 207–235.

[SW1] H.de Snoo, H.Woracek: *Symmetric relations in almost Pontryagin spaces,* in preparation.

[SW2] H.de Snoo, H.Woracek: *Q-functions and h_0-resolvents in almost Pontryagin spaces,* in preparation.

[SW3] H.de Snoo, H.Woracek: *The Krein formula in almost Pontryagin spaces. A proof via orthogonal coupling,* in preparation.

[W] H.Woracek: *An operator theoretic approach to degenerated Nevanlinna-Pick interpolation,* Math. Nachr. 176 (1995), 335–350.

H.S.V. de Snoo
Johann Bernoulli Institute for Mathematics and Computer Science
University of Groningen
PO Box 407, 9700 AK Groningen
NEDERLAND
email: desnoo@math.rug.nl

H. Woracek
Institut für Analysis und Scientific Computing
Technische Universität Wien
Wiedner Hauptstraße. 8–10/101
A–1040 Wien
AUSTRIA
email: harald.woracek@tuwien.ac.at