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# Stability of the Trace of the Polynomial $L^2$-projection an Triangles

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Bakkalaureatsarbeit

# Stability of the Trace of the Polynomial $L^2$-projection on Triangles

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**Abstract**

This bachelor thesis deals with the stability and approximation of the trace of the polynomial $L^2$-projection on triangles. We consider the $L^2$-projection $\Pi_N^{2D} : L^2(T) \to \mathcal{P}_N(T)$ onto $\mathcal{P}_N(T)$, where $T$ is the reference triangle $\{(x, y) : -1 < x < 1, -1 < y < -x\}$ and show the following result

$$\|\Pi_N^{2D} u\|_{L^2(\Gamma)}^2 \leq C\|u\|_{L^2(T)}\|u\|_{H^1(T)}, \quad \forall u \in H^1(T),$$

where we denote by $\Gamma$ one edge of $\partial T$.
At the end we will present a method to compute numerically the stability constant $C$ in the estimate above and show the computational results. We will also compute two related stability constants, namely, the stability constant for the corresponding one-dimensional statement

$$|(\Pi_N^{1D} u)(\pm 1)|^2 \leq C\|u\|_{L^2(-1,1)}\|u\|_{H^1(-1,1)}, \quad \forall u \in H^1(-1, 1),$$

where $\Pi_N^{1D} : L^2(-1, 1) \to \mathcal{P}_N(-1, 1)$ is the $L^2$-projection onto the space of polynomials of degree $N$, and the stability constant $C_N$ in the two-dimensional bound

$$\|\Pi_N^{2D} u\|_{L^2(\Gamma)}^2 \leq C_N\|u\|_{H^1(T)}^2, \quad \forall u \in H^1(T).$$

Here, $C_N$ is seen to be $O(N)$.

# Contents

# 1 Introduction

Polynomial approximation plays an important role in high order numerical methods such as spectral-, *hp*- and *hp*-Discontinuous Galerkin Finite Element Methods. Basic building blocks of these numerical methods and polynomial approximation in general are projection operators that map into spaces of polynomials. In this work, we study in more detail a specific projection operator, namely, the $L^2$-projection $\Pi_N^{2D} : L^2(T) \to \mathcal{P}_N(T)$ onto the space of polynomials $\mathcal{P}_N(T)$ of degree $N$ on a triangle $T \subset \mathbb{R}^2$. For this operator, we show in Theorem 2.17 the following estimate, where $\Gamma$ denotes an edge of the triangle $T$:

$$\|\Pi_N^{2D} u\|_{L^2(\Gamma)}^2 \leq C\|u\|_{L^2(T)}\|u\|_{H^1(T)}, \tag{1.0.1}$$

As an application of this result we show that the restriction to the edge $\Gamma$ of the polynomial approximation $\Pi_N^{2D} u$ of a function $u$ is an optimal order approximation in $L^2(\Gamma)$:

**Theorem 1.1** *For $N \in \mathbb{N}_0$ denote by $\Pi_N^{2D}$ the $L^2(T)$-projection onto $\mathcal{P}_N(T)$. Then there exists a constant $C > 0$ independent of $N$ and $u$ such that*

$$\|u - \Pi_N^{2D} u\|_{L^2(\Gamma)} \leq CN^{-s+1/2}\|u\|_{H^s(T)} \qquad \forall u \in H^s(T), s \in \mathbb{N}.$$

*Proof.* Before proving Theorem 1.1, we mention that relevant notation is introduced below in Section 1.1.

Since $\Pi_N^{2D} : L^2(T) \to \mathcal{P}_N(T)$ is a projection onto $\mathcal{P}_N$, there holds for any polynomial $p \in \mathcal{P}_N$ that $\Pi_N^{2D} p = p$. Using this and the triangle inequality we have

$$\|u - \Pi_N^{2D} u\|_{L^2(\Gamma)} \leq \|u - p\|_{L^2(\Gamma)} + \|\Pi_N^{2D}(u - p)\|_{L^2(\Gamma)}.$$

By applying (1.0.1) we get

$$\|u - p\|_{L^2(\Gamma)} + \|\Pi_N^{2D}(u - p)\|_{L^2(\Gamma)} \lesssim \|u - p\|_{L^2(\Gamma)} + \|u - p\|_{L^2(T)}^{1/2}\|u - p\|_{H^1(T)}^{1/2}.$$

Now the estimate $\|u\|_{L^2(\Gamma)}^2 \lesssim \|u\|_{L^2(T)}\|u\|_{H^1(T)}$ (see [3, Thm. 1.6.6]) yields

$$\|u - \Pi_N^{2D} u\|_{L^2(\Gamma)} \lesssim \|u - p\|_{L^2(T)}^{1/2}\|u - p\|_{H^1(T)}^{1/2}.$$

Further, for $u \in H^s(T)$ we have the following two estimates taken from [11, Thm. B.4]

$$\|u - \Pi_N^{H^1} u\|_{H^1(T)} \lesssim N^{-(s-1)}\|u\|_{H^s(T)}$$
$$\|u - \Pi_N^{H^1} u\|_{L^2(T)} \lesssim N^{-s}\|u\|_{H^s(T)},$$

where we denote by $\Pi_N^{H^1} : H^1(T) \to H^1(T) \cap \mathcal{P}_N(T)$ the $H^1$-projection. Since $p$ was arbitrary we conclude

$$\begin{aligned}\|u - \Pi_N^{2D} u\|_{L^2(\Gamma)} &\lesssim N^{-(s-1)/2}N^{-s/2}\|u\|_{H^s(T)} \\ &\lesssim N^{-s+1/2}\|u\|_{H^s(T)}\end{aligned}$$

□

We close this introduction by mentioning that the analog of (1.0.1) for tensor product domains such as intervals, squares, and cubes has been established in [7]. Likewise, the analog of Theorem 1.1 for these geometries can be found there. We mention that for tensor product domains, Theorem 1.1 was established earlier in [12] for special case $s \geq 1$. The novel aspect of the present work is the non-trivial generalization to triangles.

## 1.1 General Notation

We will denote points in $\mathbb{R}^n$, $n \in \mathbb{N}$, by underlined letters, i.e., $\underline{x} = (x_1, x_2, \ldots, x_n)$.

An $n$-dimensional multi-index is a $n$-tuple $s = (s_1, \ldots, s_n) \in \mathbb{N}_0^n$. For a sufficiently smooth function $u$ we define

$$D^s u := \frac{\partial^{|s|} u}{\partial x_1^{s_1} \cdots \partial x_n^{s_n}},$$

where $|s| := \sum_{i=1}^n s_i$ is called the order of the multi-index.

Furthermore, we introduce the reference square $S := (-1,1)^2$, the reference triangle $T := \{(\xi_1, \xi_2) : -1 < \xi_1 < 1, -1 < \xi_2 < -\xi_1\}$ and denote one edge of $T$ by $\Gamma := (-1,1) \times \{-1\}$.

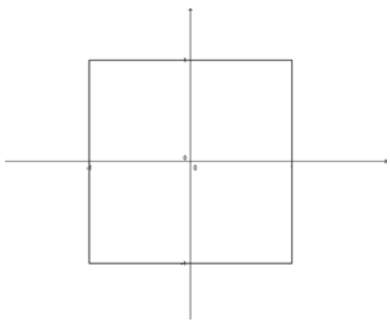

Figure 1: reference square

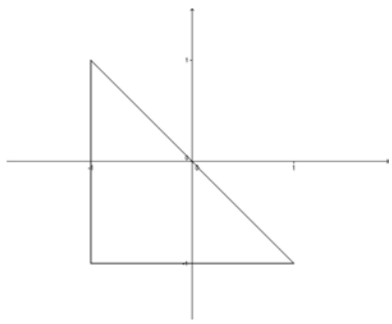

Figure 2: reference triangle

In the following estimates, there will also occur constants, which we will denote by a capital $C$. We do not consider $C$ as a fixed constant that has the same value every time it appears in a proof. In fact, we will regard $C$ as an expression that absorbs all the constants that arise in the current step. However, $C$ will be always independent of critical parameters and functions involved. Sometimes we will also abbreviate notations like $a \leq Cb$ by writing $a \lesssim b$.

Furthermore, we define the space of polynomials of degree $N$ by

$$\mathcal{P}_N := \{x^i : i \leq N, i \in \mathbb{N}_0\},$$

and the restriction to a set D by

$$\mathcal{P}_N(D) := \{p \in \mathcal{P}_N : p : D \to \mathbb{R}\}.$$

## 1.2 Sobolev Spaces

A standard reference for Sobolev spaces is [1]. We also refer to [2, p.62, 1.25].

Let $\Omega$ be an open set in $\mathbb{R}^n$ and $k \in \mathbb{N}_0$. We define an inner product on $C^\infty(\Omega)$, the space of all infinitely differentiable functions on $\Omega$:

$$\langle u, v\rangle_{H^k(\Omega)} := \sum_{|s|\leq k} \int_\Omega D^s u D^s v dx,$$

where $s$ is a multi-index. The expression

$$\|u\|_{H^k(\Omega)} := \sqrt{\langle u, v\rangle_{H^k(\Omega)}}$$

turns the space

$$\{u \in C^\infty(\Omega) \colon \|u\|_{H^k(\Omega)} < \infty\}$$

into a normed space. Now we can define Sobolev spaces.

**Definition 1.2** *Let $\Omega \subset \mathbb{R}^n$ open. The Sobolev space $H^k(\Omega)$ is the closure of the set $\{u \in C^\infty(\Omega) : \|u\|_{H^k(\Omega)} < \infty\}$ with respect to the norm $\|.\|_{H^k(\Omega)}$. Hence,*

$$H^k(\Omega) := \overline{\{u \in C^\infty(\Omega) : \|u\|_{H^k(\Omega)} < \infty\}}^{\|.\|_{H^k(\Omega)}}.$$

## 1.3 Jacobi Polynomials

For the estimate on triangles it will be convenient to use Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which form a family of polynomial solutions of appropriate Sturm-Liouville problems. Furthermore, they form a class of orthogonal polynomials on the interval $(-1, 1)$ with respect to the weight functions $(1-x)^\alpha, (1-x)^\beta$ $(\alpha, \beta > -1)$. We have the following orthogonality property (see, e.g. [9, p. 351])

$$\int_{-1}^{1} (1-x)^\alpha (1-x)^\beta P_p^{(\alpha,\beta)} P_q^{(\alpha,\beta)} dx = \begin{cases} 0 & , p \neq q \\ \frac{2^{\alpha+\beta+1}}{2p+\alpha+\beta+1} \frac{\Gamma(p+\alpha+1)\Gamma(p+\beta+1)}{p!\Gamma(p+\alpha+\beta+1)} & , p = q \end{cases} \tag{1.3.1}$$

where $p, q \in \mathbb{N}_0$. We abbreviate

$$\gamma_p^{(\alpha,\beta)} := \frac{2^{\alpha+\beta+1}}{2p+\alpha+\beta+1} \frac{\Gamma(p+\alpha+1)\Gamma(p+\beta+1)}{p!\Gamma(p+\alpha+\beta+1)}. \tag{1.3.2}$$

In further proofs we will need several properties of Jacobi polynomials. We have the following useful fomulas (see [9, p. 350 f]):

**Recursion Relations**

$$a_n^1 P_{n+1}^{(\alpha,\beta)}(x) = (a_n^2 + a_n^3 x) P_n^{(\alpha,\beta)}(x) - a_n^4 P_{n-1}^{(\alpha,\beta)}(x) \tag{1.3.3}$$

with

$$\begin{aligned} a_n^1 &:= 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta) \\ a_n^2 &:= (2n+\alpha+\beta+1)(\alpha^2-\beta^2) \\ a_n^3 &:= (2n+\alpha+\beta)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2) \\ a_n^4 &:= 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2) \end{aligned}$$

$$b_n^1(x) \frac{d}{dx} P_n^{(\alpha,\beta)}(x) = b_n^2(x) P_n^{(\alpha,\beta)}(x) + b_n^3(x) P_{n-1}^{(\alpha,\beta)}(x) \tag{1.3.4}$$

with

$$\begin{aligned} b_n^1(x) &:= (2n+\alpha+\beta)(1-x^2) \\ b_n^2(x) &:= n\,(\alpha-\beta-(2n+\alpha+\beta)x) \\ b_n^3(x) &:= 2(n+\alpha)(n+\beta) \end{aligned}$$

**Special Values**

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} \tag{1.3.5}$$

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x) \tag{1.3.6}$$

**Special Cases** For the Legendre Polynomial $L_n(x)$ there holds

$$L_n(x) = P_n^{(0,0)}(x) \tag{1.3.7}$$

**Miscellaneous Relations**

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+n+1) P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{1.3.8}$$

$$2n \int_{-1}^{x} (1-t)^{\alpha}(1+t)^{\beta} P_n^{(\alpha,\beta)}(t)\, dt = -(1-x)^{\alpha+1}(1+x)^{\beta+1} P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{1.3.9}$$

## 1.4 Orthogonal Polynomials on Triangles

To introduce orthogonal polynomials on the triangle we need the transformation

$$D : \begin{cases} S \to T \\ (\eta_1, \eta_2) \mapsto (\xi_1, \xi_2) = \left( \frac{(1+\eta_1)(1-\eta_2)}{2} - 1, \eta_2 \right), \end{cases} \tag{1.4.1}$$

sometimes referred to as Duffy transformation (see e.g. [5]), which maps the reference square $S$ onto the reference triangle $T$.
Hence, the inverse is

$$D^{-1} : \begin{cases} T \to S \\ (\xi_1, \xi_2) \mapsto (\eta_1, \eta_2) = \left( 2\frac{(1+\xi_1)}{1-\xi_2} - 1, \xi_2 \right). \end{cases} \tag{1.4.2}$$

Using Jacobi polynomials we define the following polynomials on S:

$$\widetilde{\psi}_{pq}(\eta_1, \eta_2) := P_p^{(0,0)}(\eta_1) \left( \frac{1-\eta_2}{2} \right)^p P_q^{(2p+1,0)}(\eta_2) \tag{1.4.3}$$

Applying definition (1.4.3) we define the functions

$$\psi_{pq} := \tilde{\psi}_{pq} \circ D^{-1}. \tag{1.4.4}$$

The subsequent lemma taken from [10] now shows that $\psi_{pq}$ are orthogonal polynomials of degree $p+q$ on the reference triangle $T$.

**Lemma 1.3** (orthogonal polynomials on the triangle) *The functions $\psi_{pq}$ defined in (1.4.4) satisfy $\psi_{pq} \in \mathcal{P}_{p+q}(T)$, they are $L^2(T)$-orthogonal, and they fulfill*

$$\int_T \psi_{pq}(\xi_1, \xi_2) \psi_{kl}(\xi_1, \xi_2) d\xi_1 d\xi_2 = \delta_{pk} \delta_{ql} \frac{2}{2p+1} \frac{1}{p+q+1}.$$

*Proof.* We start with the assertion that $\psi_{pq}$ is a polynomial of degree $p+q$. With $D^{-1}$ defined in (1.4.2), we get

$$\psi_{pq}(\xi_1, \xi_2) = \widetilde{\psi}_{pq}(2\frac{1+\xi_1}{1-\xi_2}, \xi_2) = P_p^{(0,0)} \left( 2\frac{1+\xi_1}{1-\xi_2}, \xi_2 \right) \left( \frac{1-\xi_2}{2} \right)^p P_q^{(2p+1,0)}(\xi_2).$$

Expanding the Legendre polynomial $P_p^{(0,0)}$ as $P_p^{(0,0)}(x-1) = \sum_{k=0}^p c_k x^k$, we get

$$\psi_{pq}(\xi_1, \xi_2) = \sum_{k=0}^p c_k 2^{k-p} (1+\xi_1)^k (1-\xi_2)^{p-k} P_q^{(2p+1,0)}(\xi_2).$$

Since $P_q^{(2p+1,0)}$ is a polynomial of degree $q$, we get $\psi_{pq} \in \mathcal{P}_{p+q}(T)$.
Next we demonstrate the orthogonality. By transforming to S we get using (1.3.1) twice

$$\int_T \psi_{pq}(\xi_1,\xi_2)\psi_{kl}(\xi_1,\xi_2)d\xi_1 d\xi_2 = \int_S \widetilde{\psi}_{pq}(\eta_1,\eta_2)\widetilde{\psi}_{kl}(\eta_1,\eta_2)\frac{1-\eta_2}{2}d\eta_1 d\eta_2$$
$$= \int_{-1}^{1}\int_{-1}^{1} P_p^{(0,0)}(\eta_1)P_k^{(0,0)}(\eta_1)\left(\frac{1-\eta_2}{2}\right)^{p+k+1} P_q^{(2p+1,0)}(\eta_2)P_l^{(2k+1,0)}(\eta_2)d\eta_1 d\eta_2$$
$$= \frac{2}{2p+1}\delta_{pk}2^{-(2p+1)}\int_{-1}^{1}(1-\eta_2)^{2p+1}P_q^{(2p+1,0)}(\eta_2)P_l^{(2k+1,0)}(\eta_2)d\eta_2$$
$$= \frac{2}{2p+1}\delta_{pk}\delta_{ql}\frac{2}{2p+2q+2}.$$

□

# 2 Trace Stability of $L^2$-Projection

The main result in this section will be Theorem 2.17, where we will bound the $L^2$-norm of the $L^2$-projection onto $\mathcal{P}_N(T)$ of a function $u \in H^1(T)$ on the edge $\Gamma$. To do so, we will first present some additional properties Jacobi polynomials have in connection with special functions $h_1, h_2$ and $h_3$ and list useful information gathered from the features of the Duffy transformation. These properties are used in the proofs of several lemmata, which finally lead the proof of the main theorem.

## 2.1 Preliminaries

### 2.1.1 Properties of $h_1, h_2$ and $h_3$

We define

$$\begin{aligned} h_1(q,\alpha) &:= -\frac{2(q+1)}{(2q+\alpha+1)(2q+\alpha+2)} \\ h_2(q,\alpha) &:= \frac{2\alpha}{(2q+\alpha+2)(2q+\alpha)} \\ h_3(q,\alpha) &:= \frac{2(q+\alpha)}{(2q+\alpha+1)(2q+\alpha)}, \end{aligned} \tag{2.1.1}$$

where $\alpha > -1$ and $q \in \mathbb{N}_0$. The following lemma establishes a connection between the integral of weighted Jacobi polynomials and the terms in (2.1.1).

**Lemma 2.1** *Let $\alpha > -1$ and $q \geq 1$. With the terms $h_1, h_2$ and $h_3$ defined in (2.1.1) there holds*

$$\int_{-1}^{x} (1-t)^\alpha P_q^{(\alpha,0)}(t)\,dt = -(1-x)^\alpha \Big( h_1(q,\alpha) P_{q+1}^{(\alpha,0)}(x) + h_2(q,\alpha) P_q^{(\alpha,0)(x)} + h_3(q,\alpha) P_{q-1}^{(\alpha,0)}(x) \Big)$$

*Proof.* The proof relies on the relations satisfied by Jacobi polynomials explained in Section 1.3. Using rearranged versions of (1.3.4), (1.3.8) and (1.3.9) we obtain

$$\begin{aligned} \int_{-1}^{x} (1-t)^\alpha P_q^{(\alpha,0)}(t)\ dt &\overset{(1.3.9)}{=} -\frac{1}{2q}(1+x)(1-x)^{\alpha+1} P_{q-1}^{(\alpha+1,1)}(x) \\ &= -\frac{1}{2q}(1-x^2)(1-x)^\alpha P_{q-1}^{(\alpha+1,1)}(x) \\ &\overset{(1.3.8)}{=} -(1-x)^\alpha \frac{1}{2q}(1-x^2)\frac{2}{q+\alpha+1}\frac{d}{dx}P_q^{(\alpha,0)}(x) \\ &\overset{(1.3.4)}{=} -(1-x)^\alpha \frac{1}{q}\frac{1}{q+\alpha+1}\frac{q(\alpha-(2q+\alpha)x)P_q^{(\alpha,0)}(x) + 2q(q+\alpha)P_{q-1}^{(\alpha,0)}(x)}{2q+\alpha} \\ &= -(1-x)^\alpha \frac{\alpha P_q^{(\alpha,0)}(x) + 2(q+\alpha)P_{q-1}^{(\alpha,0)}(x) - (2q+\alpha)xP_q^{(\alpha,0)}(x)}{(q+\alpha+1)(2q+\alpha)} \end{aligned}$$

(1.3.3) allows us now to replace the term $xP_q^{(\alpha,0)}(x)$ by terms involving $P_{q+1}^{(\alpha,0)}(x)$, $P_q^{(\alpha,0)}(x)$ and $P_{q-1}^{(\alpha,0)}(x)$. Hence, we get

$$\begin{aligned}\int_{-1}^{x}(1-t)^\alpha P_q^{(\alpha,0)}(t)\;dt = -(1-x)^\alpha \frac{1}{(q+\alpha+1)(2q+\alpha)}\Bigg\{&\alpha P_q^{(\alpha,0)}(x) + 2(q+\alpha)P_{q-1}^{(\alpha,0)}(x)\\ &- \frac{1}{(2q+\alpha+1)(2q+\alpha+2)}\Big(2(q+1)(q+\alpha+1)(2q+\alpha)P_{q+1}^{(\alpha,0)}(x)\\ &+ 2q(q+\alpha)(2q+\alpha+2)P_{q-1}^{(\alpha,0)}(x) - (2q+\alpha+1)\alpha^2 P_q^{(\alpha,0)}(x)\Big)\Bigg\}\end{aligned}$$

Rearranging terms gives

$$\begin{aligned}&\int_{-1}^{x}(1-t)^\alpha P_q^{(\alpha,0)}(t)\,dt\\ &= -(1-x)^\alpha \frac{1}{(q+\alpha+1)(2q+\alpha)}\Bigg\{-\frac{2(q+1)(q+\alpha+1)(2q+\alpha)}{(2q+\alpha+1)(2q+\alpha+2)}P_{q+1}^{(\alpha,0)}(x)\\ &\quad + \alpha\frac{2q+2\alpha+2}{2q+\alpha+2}P_q^{(\alpha,0)}(x) + 2(q+\alpha)\frac{q+\alpha+1}{2q+\alpha+1}P_{q-1}^{(\alpha,0)}(x)\Bigg\}\\ &= -(1-x)^\alpha\Bigg\{-\underbrace{\frac{2(q+1)}{(2q+\alpha+1)(2q+\alpha+2)}}_{h_1(q,\alpha)}P_{q+1}^{(\alpha,0)}(x)\\ &\quad + \underbrace{\frac{2\alpha}{(2q+\alpha+2)(2q+\alpha)}}_{h_2(q,\alpha)}P_q^{(\alpha,0)}(x) + \underbrace{\frac{2(q+\alpha)}{(2q+\alpha+1)(2q+\alpha)}}_{h_3(q,\alpha)}P_{q-1}^{(\alpha,0)}(x)\Bigg\}\end{aligned}$$

□

Essential in further proofs is also the following observation.

**Lemma 2.2** (magic cancellation) *Let the functions $h_1, h_2$ and $h_3$ be defined as in (2.1.1) and let $\alpha > -1$. Then there holds for $q \geq 0$*

$$(-1)^q\frac{1}{\gamma_q^{(\alpha,0)}}h_1(q,\alpha) + (-1)^{q+1}\frac{1}{\gamma_{q+1}^{(\alpha,0)}}h_2(q+1,\alpha) + (-1)^{q+2}\frac{1}{\gamma_{q+2}^{(\alpha,0)}}h_3(q+2,\alpha) = 0$$

*and*

$$h_2(q,a) - h_1(q,a) = h_3(q,a).$$

*Proof.* We recall the definition of $\gamma_p^{(\alpha,\beta)}$ in (1.3.2) and obtain in particular

$$\gamma_q^{(\alpha,0)} = \frac{2^{\alpha+1}}{2q+\alpha+1},$$

which leads together with the definition of $h_1, h_2$ and $h_3$ to

$$\begin{aligned}
&(-1)^q \frac{1}{\gamma_q^{(\alpha,0)}} h_1(q,\alpha) + (-1)^{q+1} \frac{1}{\gamma_{q+1}^{(\alpha,0)}} h_2(q+1,\alpha) + (-1)^{q+2} \frac{1}{\gamma_{q+2}^{(\alpha,0)}} h_3(q+2,\alpha) \\
&\quad = (-1)^q \frac{2q+\alpha+1}{2^{\alpha+1}} (-1) \frac{2q+2}{(2q+\alpha+1)(2q+\alpha+2)} \\
&\qquad + (-1)^{q+1} \frac{2q+\alpha+3}{2^{\alpha+1}} \frac{2\alpha}{(2q+\alpha+4)(2q+\alpha+2)} \\
&\qquad + (-1)^{q+2} \frac{2q+\alpha+5}{2^{\alpha+1}} \frac{2(q+\alpha+2)}{(2q+\alpha+5)(2q+\alpha+4)} \\
&\quad = \frac{(-1)^{q+1}}{2^\alpha} \left( \frac{q+1}{2q+\alpha+2} + \frac{(2q+\alpha+3)\alpha}{(2q+\alpha+4)(2q+\alpha+2)} - \frac{q+\alpha+2}{2q+\alpha+4} \right) \\
&\quad = \frac{(-1)^{q+1}}{2^\alpha} \left( \frac{(q+1)(2q+\alpha+4) + (2q+\alpha+3)\alpha - (q+\alpha+2)(2q+\alpha+2)}{(2q+\alpha+4)(2q+\alpha+2)} \right)
\end{aligned}$$

Simply multiplying out the numerator concludes the proof regarding the first equation.

Inserting the definition of $h_1, h_2$ and $h_3$ also leads in the case of the second equation to the conclusion

$$\begin{aligned}
h_2(q,\alpha) - h_1(q,\alpha) &= \frac{2\alpha(2q+\alpha+1) + 2(q+1)(2q+\alpha)}{(2q+\alpha)(2q+\alpha+1)(2q+\alpha+2)} \\
&= \frac{4q^2 + 4q + 6q\alpha + 2\alpha^2 + 4\alpha}{(2q+\alpha)(2q+\alpha+1)(2q+\alpha+2)} \\
&= \frac{(2q+\alpha+2)(2q+2\alpha)}{(2q+\alpha)(2q+\alpha+1)(2q+\alpha+2)} = h_3(q,\alpha)
\end{aligned}$$

□

### 2.1.2 Properties of the Duffy Transformation

Regarding the use of the Duffy transformation when integrating over T we have the following basic information.
The Jacobian matrix is constituted by

$$J_D = \frac{1}{2} \begin{pmatrix} (1-\eta_2) & -(1+\eta_1) \\ 0 & 2 \end{pmatrix} \tag{2.1.2}$$

and therefore we have the Jacobian determinant

$$\det(J_D) = \frac{1-\eta_2}{2}. \tag{2.1.3}$$

Furthermore for sufficiently smooth functions $u$ on $T$ we define the transformed function by

$$\tilde{u} := u \circ D \tag{2.1.4}$$

we have

$$\partial_{\eta_1} \tilde{u}(\eta_1, \eta_2) = \frac{1-\eta_2}{2} (\partial_1 u) \circ D \tag{2.1.5}$$

$$\partial_{\eta_2} \tilde{u}(\eta_1, \eta_2) = -\frac{1+\eta_1}{2} (\partial_1 u) \circ D + (\partial_2 u) \circ D, \tag{2.1.6}$$

where $\partial_1$ is the partial derivative in the first argument.
In particular, we have according to the theorem of integration by substitution for multiple variables

$$\int_S |\tilde{u}(\eta_1, \eta_2)|^2 \frac{1-\eta_2}{2} d\eta_1 d\eta_2 = \|u\|^2_{L^2(T)} \tag{2.1.7}$$

$$\begin{aligned}\int_S |\partial_{\eta_1} \tilde{u}(\eta_1, \eta_2)|^2 \frac{2}{(1-\eta_2)} d\eta_1 d\eta_2 &\overset{(2.1.5)}{=} \int_S |(\partial_1 u) \circ D(\eta_1, \eta_2)|^2 \frac{1-\eta_2}{2} d\eta_1 d\eta_2 \\ &= \|\partial_{\xi_1} u\|^2_{L^2(T)} \leq \|\nabla u\|^2_{L^2(T)} \end{aligned} \tag{2.1.8}$$

$$\begin{aligned}\int_S |\partial_{\eta_2} \tilde{u}(\eta_1, \eta_2)|^2 \frac{1-\eta_2}{2} d\eta_1 d\eta_2 &\overset{(2.1.6)}{\lesssim} \int_S |(\partial_2 u) \circ D(\eta_1, \eta_2)|^2 \frac{1-\eta_2}{2} \qquad (2.1.9) \\ &\quad + |(\partial_1 u) \circ D(\eta_1, \eta_2)|^2 \frac{(1+\eta_1)^2(1-\eta_2)}{2} d\eta_1 d\eta_2 \\ &\lesssim \|\nabla u\|^2_{L^2(T)}. \end{aligned} \tag{2.1.10}$$

Next we want to present a lemma concerning the properties of the Duffy transformation on an edge of the reference triangle $T$.

**Lemma 2.3** *Let $D$ be the transformation defined in (1.4.1) and $\Gamma = (-1, 1) \times \{-1\}$. Then $D(\Gamma) = \Gamma$ and $D$ is an isometric isomorphism with respect to the $L^2(\Gamma)$-norm.*

*Proof.* Obviously $D$ is an isomorphism, so we will only show the isometry property.
Let u be a quadratic integrable function on T and reconsider the transformed function $\tilde{u}$ as defined in (2.1.4). We have

$$\begin{aligned}\|\tilde{u}\|^2_{L^2(\Gamma)} &= \int_\Gamma |\tilde{u}(\eta_1, \eta_2)|^2 \, d\eta_1 d\eta_2 = \int_{-1}^{1} \left|u\big(D(\eta_1, -1)\big)\right|^2 d\eta_1 \\ &= \int_{-1}^{1} \left|u\left(\frac{(1+\eta_1)\,2}{2} - 1, -1\right)\right|^2 d\eta_1 = \int_{-1}^{1} |u(\eta_1, -1)|^2 \, d\eta_1 = \|u\|^2_{L^2(\Gamma)}\end{aligned}$$

□

## 2.2 Expansion in terms of $\psi_{pq}$

It is essential to expand functions $u \in L^2(T)$ in terms of orthogonal polynomials on the triangle as introduced in the Section 1.4. This will be the basis for further calculations.

At first we have to introduce orthogonal systems.

**Definition 2.4** *Let $\mathcal{H}$ be an inner product space with an inner product $\langle .,. \rangle$. A sequence $(v_n) \subset \mathcal{H}$ is called orthogonal system, if any two elements of $(v_n)$ are orthogonal to each other, i.e. $\langle v_i, v_j \rangle = 0, \forall i \neq j$.*

In view of this defnition we have the following well-known fact:

**Lemma 2.5** *Let $\mathcal{H}$ be an inner product space and let $(v_n) \subset \mathcal{H}$ be an orthogonal system with respect to the inner product $\langle .,. \rangle$. Furthermore, let $\{v_n : n \in \mathbb{N}_0\}$ be dense in $\mathcal{H}$ and let $\|.\|$ be the norm induced by the inner product. Then any $u \in \mathcal{H}$ can be expanded as*

$$u = \sum_{n=0}^{\infty} \frac{\langle u, v_n \rangle}{\langle v_n, v_n \rangle} v_n$$

*and we have the following equality also known as Parseval's identity*

$$\|u\|^2 = \langle u, u \rangle = \sum_{n=0}^{\infty} \frac{1}{\langle v_n, v_n \rangle} | \langle u, v_n \rangle |^2.$$

*Proof.* see [6, p. 236ff]

□

Applying Lemma 2.5 to the set of orthogonal polynomials on the triangle $(\psi_{pq})_{p,q \in \mathbb{N}_0}$ we obtain that any $u \in L^2(T)$ can be expanded as

$$u = \sum_{p,q=0}^{\infty} \frac{1}{\langle \psi_{pq}, \psi_{pq} \rangle} u_{pq} \psi_{pq}, \tag{2.2.1}$$

where the coefficients $u_{pq}$ are given by

$$u_{pq} := \int_T u(\xi_1, \xi_2) \psi_{pq}(\xi_1, \xi_2) d\xi_1 d\xi_2. \tag{2.2.2}$$

Using Lemma 1.3 we have

$$\langle \psi_{pq}, \psi_{pq} \rangle = \int_T |\psi_{pq}(\xi_1, \xi_2)|^2 \, d\xi_1 d\xi_2 = \frac{2}{2p+1} \frac{1}{p+q+1} = \gamma_p^{(0,0)} \gamma_q^{(2p+1,0)} 2^{-(2p+1)}$$

and therefore

$$u = \sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}} u_{pq} \psi_{pq}. \tag{2.2.3}$$

Furthermore, we want to rearrange the coefficients $u_{pq}$. If we define

$$U_p(\eta_2) := \int_{-1}^{1} \tilde{u}(\eta_1, \eta_2) P_p^{(0,0)}(\eta_1) d\eta_1, \tag{2.2.4}$$

we have

$$\begin{aligned} u_{pq} &= \int_S \tilde{u}(\eta_1, \eta_2) P_p^{(0,0)}(\eta_1) \left(\frac{1-\eta_2}{2}\right)^{p+1} P_q^{(2p+1,0)}(\eta_2) \\ &= \int_{-1}^{1} \left(\frac{1-\eta_2}{2}\right)^{p+1} U_p(\eta_2) P_q^{(2p+1,0)}(\eta_2) d\eta_2. \end{aligned}$$

Introducing

$$\widetilde{U}_p(\eta_2) := \frac{U_p(\eta_2)}{(1-\eta_2)^p}, \tag{2.2.5}$$

we arrive at

$$u_{pq} = 2^{-(p+1)} \int_{-1}^{1} (1-\eta_2)^{2p+1} \widetilde{U}_p(\eta_2) P_q^{(2p+1,0)}(\eta_2) d\eta_2. \tag{2.2.6}$$

Next we will try to extract information about the properties of the above defined $U_p$ and $\widetilde{U}_p$ in terms of the $L^2$ and $H^1$-norm.

**Lemma 2.6** (properties of $U_p$) *Let $u \in H^1(T)$ and $U_p$ be defined in (2.2.4). Then there exists a constant $C > 0$ independent of $p$ and $u$ such that*

$$\sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \int_{-1}^{1} |U_p(\eta_2)|^2 \frac{1-\eta_2}{2} d\eta_2 = \|u\|_{L^2(T)}^2, \tag{2.2.7}$$

$$\sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \int_{-1}^{1} |U_p'(\eta_2)|^2 \frac{1-\eta_2}{2} d\eta_2 \le C \|\nabla u\|_{L^2(T)}^2, \tag{2.2.8}$$

$$\sum_{p=0}^{\infty} \frac{p^2}{\gamma_p^{(0,0)}} \int_{-1}^{1} |U_p(\eta_2)|^2 \frac{2}{1-\eta_2} d\eta_2 \le C \|\nabla u\|_{L^2(T)}^2. \tag{2.2.9}$$

*Furthermore, we have for $\Gamma = (-1,1) \times \{-1\}$*

$$\sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} |U_p(-1)|^2 = \|u\|_{L^2(\Gamma)}^2. \tag{2.2.10}$$

*Proof.* (2.2.7) follows from the definition of $U_p$, since the definition of $U_p$ implies (for fixed $\eta_2$) the representation

$$\tilde{u}(\eta_1, \eta_2) = \sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} U_p(\eta_2) P_p^{(0,0)}(\eta_1),$$

which in turn gives

$$\int_{-1}^{1} |\tilde{u}(\eta_1,\eta_2)|^2 d\eta_1 = \sum_{p=0}^{\infty} \frac{1}{\left(\gamma_p^{(0,0)}\right)^2} |U_p(\eta_2)|^2 \int_{-1}^{1} \left|P_p^{(0,0)}(\eta_1)\right|^2 d\eta_1$$
$$= \sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{2p+1}{2} |U_p(\eta_2)|^2 \frac{2}{2p+1}$$
$$= \sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} |U_p(\eta_2)|^2. \tag{2.2.11}$$

(2.1.7) yields that multiplication with $\frac{1-\eta_2}{2}$ and integration in $\eta_2$ gives (2.2.7).

Likewise, (2.2.8) is a consequence of the fact that $\partial_{\eta_2}\tilde{u}$ has the representation

$$\partial_{\eta_2}\tilde{u}(\eta_1,\eta_2) = \sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} U_p'(\eta_2) P_p^{(0,0)}(\eta_1).$$

Similar argumentations as in case of (2.2.7) with the difference that we have to use (2.1.9) instead of (2.1.7) then yield

$$\sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \int_{-1}^{1} \left|U_p'(\eta_2)\right|^2 \frac{1-\eta_2}{2} d\eta_2 \lesssim \|\partial_{\xi_2} u\|_{L^2(T)}^2,$$

which immediatedly leads to (2.2.8).

According the third estimate, the abbreviation $z_p(\eta_2) := \int_{\eta_1} (\partial_{\eta_1}\tilde{u})(\eta_1,\eta_2) P_p^{(0,0)}(\eta_1) d\eta_1$ leads us to the representation

$$\partial_{\eta_1}\tilde{u}(\eta_1,\eta_2) = \sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} z_p(\eta_2) P_p^{(0,0)}(\eta_1).$$

Similar argumentations as in case of (2.2.7) lead to

$$\int_{-1}^{1} |\partial_{\eta_1}\tilde{u}|^2 \, d\eta_1 = \sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} |z_p(\eta_2)|^2.$$

Multiplication with $\frac{2}{1-\eta_2}$, integration in $\eta_2$ and application of (2.1.8) yields

$$\sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \int_{-1}^{1} \frac{2}{1-\eta_2} |z_p(\eta_2)|^2 d\eta_2 = \int_{-1}^{1}\int_{-1}^{1} \frac{2}{1-\eta_2} |\partial_{\eta_1}\tilde{u}|^2 \, d\eta_1 d\eta_2 \overset{(2.1.8)}{\leq} \|\nabla u\|_{L^2(T)}^2. \tag{2.2.12}$$

We do now use Lemma 2.1 with $\alpha = 0$. For $p \geq 1$ we have

$$\int_{-1}^{x} P_p^{(0,0)}(t)\,dt = \frac{1}{2p+1}\left(P_{p+1}^{(0,0)}(x) - P_{p-1}^{(0,0)}(x)\right)$$

and integrate by parts. Hence, we get (note that $\int_{-1}^{1} P_p^{(0,0)}(t)\,dt = 0$)

$$\begin{aligned}
U_p(\eta_2) &= \int_{-1}^{1} \tilde{u}(\eta_1,\eta_2) P_p^{(0,0)}(\eta_1) d\eta_1 \\
&= -\frac{1}{2p+1}\left(\int_{-1}^{1} \partial_{\eta_1}\tilde{u}(\eta_1,\eta_2)P_{p+1}^{(0,0)}(\eta_1)d\eta_1 - \int_{-1}^{1} \partial_{\eta_1}\tilde{u}(\eta_1,\eta_2)P_{p-1}^{(0,0)}(\eta_1)d\eta_1\right) \\
&= -\frac{1}{2p+1}\left(z_{p+1}(\eta_2) - z_{p-1}(\eta_2)\right)
\end{aligned}$$

Therefore we have $p|U_p(\eta_z)| \lesssim |z_{p-1}(\eta_2)| + |z_{p+1}(\eta_2)|$ and we conclude by inserting into (2.2.12):

$$\sum_{p=1}^{\infty} p^2 \frac{1}{\gamma_p^{(0,0)}} \int_{-1}^{1} |U_p(\eta_2)|^2 \frac{2}{1-\eta_2} d\eta_2 \lesssim \|\nabla u\|_{L^2(T)}^2.$$

Starting at $p = 0$ finally gives (2.2.9).

For the estimate (2.2.10), we use (2.2.11) with $\eta_2 = -1$. Noting Lemma 2.3 we have

$$\|u\|_{L^2(\Gamma)}^2 = \|\tilde{u}(\,.\,,-1)\|_{L^2(-1,1)}^2 = \int_{-1}^{1} |\tilde{u}(\eta_1,-1)|^2\, d\eta_1$$

and therefore the result follows.

□

**Lemma 2.7** (properties of $\widetilde{U}_p$) *Let $u \in H^1(T)$ and $\widetilde{U}_p$ be defined in (2.2.5). Then there exists a constant $C > 0$ independent of $p$ and $u$ such that*

$$\sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \int_{-1}^{1} \frac{(1-\eta_2)^{2p+1}}{2} \left|\widetilde{U}_p(\eta_2)\right|^2 d\eta_2 = \|u\|_{L^2(T)}^2, \tag{2.2.13}$$

$$\sum_{p=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \int_{-1}^{1} \frac{(1-\eta_2)^{2p+1}}{2} \left|\widetilde{U}_p'(\eta_2)\right|^2 d\eta_2 \leq C\|\nabla u\|_{L^2(T)}^2. \tag{2.2.14}$$

*Proof.* We recall

$$\begin{aligned}
\widetilde{U}_p(\eta_2) &= (1-\eta_2)^{-p} U_p(\eta_2), \\
\widetilde{U}_p'(\eta_2) &= (1-\eta_2)^{-p} U_p'(\eta_2) + p(1-\eta_2)^{-(p+1)} U_p(\eta_2)
\end{aligned}$$

Hence,

$$\frac{1}{2}\int_{-1}^{1}(1-\eta_2)^{2p+1}\left|\widetilde{U}_p(\eta_2)\right|^2 d\eta_2 = \int_{-1}^{1}\frac{1-\eta_2}{2}\left|U_p(\eta_2)\right|^2 d\eta_2$$

$$\frac{1}{2}\int_{-1}^{1}(1-\eta_2)^{2p+1}\left|\widetilde{U}_p'(\eta_2)\right|^2 d\eta_2 \lesssim \int_{-1}^{1}\frac{1-\eta_2}{2}\left|U_p'(\eta_2)\right|^2 d\eta_2 + p^2\int_{-1}^{1}\frac{2}{1-\eta_2}\left|U_p(\eta_2)\right|^2 d\eta_2$$

Inserting now the results of Lemma 2.6 concludes the argument.

□

**Corollary 2.8** *Assume the hypotheses of Lemma 2.7. Then there exists a constant $C$ independent of $p$ and $u$ such that, by defining*

$$\tilde{u}_{pq} := \int_{-1}^{1}(1-\eta_2)^{2p+1}\widetilde{U}_p(\eta_2)P_q^{(2p+1,0)}(\eta_2)d\eta_2 = 2^{p+1}u_{pq}, \tag{2.2.15}$$

$$\tilde{u}'_{pq} := \int_{-1}^{1}(1-\eta_2)^{2p+1}\widetilde{U}_p'(\eta_2)P_q^{(2p+1,0)}(\eta_2)d\eta_2, \tag{2.2.16}$$

*we have*

$$\sum_{p,q=0}^{\infty}\frac{1}{\gamma_p^{(0,0)}}\frac{1}{\gamma_q^{(2p+1,0)}}|\tilde{u}_{pq}|^2 = 2\|u\|^2_{L^2(T)}, \tag{2.2.17}$$

$$\sum_{p,q=0}^{\infty}\frac{1}{\gamma_p^{(0,0)}}\frac{1}{\gamma_q^{(2p+1,0)}}|\tilde{u}'_{pq}|^2 \le C\|\nabla u\|^2_{L^2(T)}. \tag{2.2.18}$$

*Proof.* Expanding $\widetilde{U}_p$ in terms of orthogonal polynomials $P_q^{(2p+1,0)}$ yields the representation

$$\widetilde{U}_p(\eta_2) = \sum_{q=0}^{\infty}\frac{1}{\gamma_q^{(2p+1,0)}}\tilde{u}_{pq}P_q^{(2p+1,0)}(\eta_2).$$

Since we have from Lemma 2.5

$$\int_{-1}^{1}(1-\eta_2)^{2p+1}|\widetilde{U}_p(\eta_2)|^2 d\eta_2 = \sum_{q=0}^{\infty}\frac{1}{\gamma_q^{(2p+1,0)}}|\tilde{u}_{pq}|^2$$

the statement (2.2.17) follows directly from Lemma 2.7. Analogously, we deal with (2.2.18), where we expand $\widetilde{U}_p'$ and again conclude with (2.2.14) of Lemma 2.7.

□

## 2.3 Connections between $\tilde{u}_{pq}$ and $\tilde{u}'_{pq}$

A key ingredient of the proof of Theorem 2.17 are connections between $\tilde{u}_{pq}$ and $\tilde{u}'_{pq}$. We start with a one-dimensional situation:

**Lemma 2.9** *Let $U \in C^1(-1,1)$ and let $(1-x)^\alpha U(x)$ be integrable. Furthermore, let*

$$\lim_{x\to 1}(1-x)^{1+\alpha}U(x) = 0 \quad \text{and} \quad \lim_{x\to -1}(1+x)U(x) = 0.$$

*Let $h_1$, $h_2$ and $h_3$ be defined in (2.1.1). We define*

$$u_q := \int_{-1}^{1} (1-x)^\alpha U(x) P_q^{(\alpha,0)}(x)dx,$$
$$b_q := \int_{-1}^{1} (1-x)^\alpha U'(x) P_q^{(\alpha,0)}(x)dx.$$

*Then for $q \geq 1$ and $\alpha > -1$ the following relationship holds:*

$$u_q = h_1(q,\alpha)b_{q+1} + h_2(q,\alpha)b_q + h_3(q,\alpha)b_{q-1}$$

*Proof.* From (1.3.9) we have for $x \to -1$

$$\int_{-1}^{x} (1-t)^\alpha P_q^{(\alpha,0)}(t)dt = O(1+x)$$

and for $x \to 1$

$$\int_{-1}^{x} (1-t)^\alpha P_q^{(\alpha,0)}(t)dt = -\int_{x}^{1} (1-t)^\alpha P_q^{(\alpha,0)}(t)dt$$
$$= -\left(\frac{(1-t)^{\alpha+1}}{\alpha+1} P_q^{(\alpha,0)}(t)\right)\Bigg|_x^1 + \int_x^1 \frac{(1-t)^{\alpha+1}}{\alpha+1}\frac{d}{dt}P_q^{(\alpha,0)}(t)dt = O\left((1-x)^{\alpha+1}\right).$$

Hence, using the stipulated behavior of $U$ at the endpoints, the following integration by parts can be justified:

$$u_q = \int_{-1}^{1} (1-x)^\alpha U(x) P_q^{(\alpha,0)}(x)dx$$
$$= \left(U(x)\int_{-1}^{x}(1-t)^\alpha P_q^{(\alpha,0)}(x)\right)\Bigg|_{-1}^{1} - \int_{-1}^{1} U'(x)\int_{-1}^{x}(1-t)^\alpha P_q^{(\alpha,0)}(t)dtdx.$$

In particular, we note that $b_q$ is well-defined. Furthermore,

$$u_q = -\int_{-1}^{1} U'(x)\int_{-1}^{x}(1-t)^\alpha P_q^{(\alpha,0)}(t)dtdx$$
$$= \int_{-1}^{1}(1-x)^\alpha U'(x)\left(h_1(q,\alpha)P_{q+1}^{(\alpha,0)}(x) + h_2(q,\alpha)P_q^{(\alpha,0)}(x) + h_3(q,\alpha)P_{q-1}^{(\alpha,0)}(x)\right)dx$$
$$= h_1(q,\alpha)b_{q+1} + h_2(q,\alpha)b_q + h_3(q,\alpha)b_{q-1},$$

where in the third equation we appealed to Lemma 2.1.

□

According to Lemma 2.9 the following corollary makes a connection between $\tilde{u}_{pq}$ and $\tilde{u}'_{pq}$:

**Corollary 2.10** *Let $\tilde{u}_{pq}$ and $\tilde{u}'_{pq}$ be defined in (2.2.15) and (2.2.16), and let $h_1$, $h_2$ and $h_3$ be defined in (2.1.1). Then for $q \geq 1$ and $p \geq 0$ there holds*

$$\tilde{u}_{pq} = h_1(q, 2p+1)\tilde{u}'_{p,q+1} + h_2(q, 2p+1)\tilde{u}'_{p,q} + h_3(q, 2p+1)\tilde{u}'_{p,q-1}.$$

*Proof.* To prove this corollary we want to make use of Lemma 2.9. Therefore, we have to clarify that the conditions in the lemma are satisfied. We proceed in two steps. First, we require $u \in C^\infty(\mathbb{R}^2)$ and show the statement in this case and then we argue by density to achieve results in $H^1(T)$.

Step 1: By assuming that $u \in C^\infty(\mathbb{R}^2)$ we get $\tilde{u} \in C^1([-1,1]^2)$. Hence, for fixed $p$, if we recall the definition of $U_p$ in (2.2.4), we see that the map $\eta_2 \mapsto U_p(\eta_2)$ is smooth on $[-1,1]$. Considering the definition of $\widetilde{U}_p$

$$\widetilde{U}_p(\eta_2) = \frac{U_p(\eta_2)}{(1-\eta_2)^p},$$

we see that $\widetilde{U}_p \in C^1([-1,1))$ and that $\widetilde{U}_p$ has at most one pole of maximal order $p$ at the point $\eta_2 = 1$. In view of these preliminary considerations we conclude that the following limits exist and that the conditions in Lemma 2.9 are satisfied:

$$\lim_{\eta_2 \to 1} (1-\eta_2)^{2p+2}\widetilde{U}_p(\eta_2) = \lim_{\eta_2 \to 1} (1-\eta_2)^{p+2}U_p(\eta_2) = 0.$$

and

$$\lim_{\eta_2 \to -1} (1+\eta_2)\widetilde{U}_p(\eta_2) = 0.$$

Now the statement follows directly from Lemma 2.9 when looking at the definition of $\tilde{u}_{pq}$ and $\tilde{u}'_{pq}$ and consequently replacing $U$ with $\widetilde{U}_p$ and $\alpha$ with $2p+1$.

Step 2: Let $u \in H^1(T)$. Since $C^\infty(\mathbb{R}^2)$ is dense in $H^1(T)$, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C^\infty(\mathbb{R}^2)$ such that $u_n \to u$ in $H^1(T)$ for $n \to \infty$. Because we have already proved that $u_n$, $n \in \mathbb{N}$ satisfies our statement, ensuring that the sequences of coefficients $\tilde{u}_{pq}$ and $\tilde{u}'_{pq}$ converge for fixed $p$ and $q$ will conclude the proof:

We recall that

$$\tilde{u}_{pq} = 2^{p+1}u_{pq} = 2^{p+1}\int_T u(\xi_1, \xi_2)\psi_{pq}(\xi_1, \xi_2)d\xi_1 d\xi_2.$$

Since $(\psi_{pq})_{p,q\in\mathbb{N}_0}$ forms an orthogonal basis for $L^2(T)$ and since $H^1(T) \subset L^2(T)$, the maps $F : u \mapsto \tilde{u}_{pq}$ are continuous linear functionals on $H^1(T)$ and thus $\lim_{n\to\infty} F(u_n) = F(u)$. In case of $\tilde{u}'_{pq}$ we study the functionals $\widetilde{F} : u \mapsto \tilde{u}'_{pq}$ that map $C^\infty(\mathbb{R}^2)$ into $\mathbb{R}$. Since $\widetilde{F}$ is a linear functional that is continuous with respect to the $H^1(T)$-norm we see by density of $C^\infty(\mathbb{R}^2)$ in $H^1(T)$ that it is indeed a well-defined continuous linear functional on $H^1(T)$ and thus again $\lim_{n\to\infty} \widetilde{F}(u_n) = \widetilde{F}(u)$.

□

Next, we show a short lemma that will be useful in the proof of Lemma 2.12.

**Lemma 2.11** *Let $\alpha \geq 0$ and $q \geq 1$. Then there exists a constant $C > 0$ independent of $q$ and $\alpha$ such that*

$$\alpha \sum_{j=q+\alpha}^{N} \frac{1}{j^2} \leq C \frac{\alpha}{q+\alpha}, \qquad \forall N = q+\alpha, q+\alpha+1, \ldots$$

*Proof.* For $n \in \mathbb{N}$ we have, since $x \mapsto \frac{1}{x^2}$ is monotone decreasing

$$\sum_{j=n}^{\infty} \frac{1}{j^2} = \frac{1}{n^2} + \sum_{j=n+1}^{\infty} \frac{1}{j^2} \leq \frac{1}{n^2} + \int_n^{\infty} \frac{1}{x^2} dx = \frac{1}{n^2} + \frac{1}{n} \leq \frac{2}{n},$$

where, in the last step, we used $n \geq 1$. Hence we conclude since $q + \alpha \geq 1$

$$\alpha \sum_{j=q+\alpha}^{\infty} \frac{1}{j^2} \leq 2 \frac{\alpha}{q+\alpha}.$$

□

The following lemma is very technical, but it will lead to Corollary 2.13 which will yield, in combination with Lemma 2.8, the tool to the conclusion in Theorem 2.17.

**Lemma 2.12** *Assume the hypotheses of Lemma 2.9. Let $\alpha \geq 0$. Let $u_q$ and $b_q$ be defined as in Lemma 2.9. Then for $q \geq 1$ there exists a constant $C$ independent of $q$ and $\alpha$ such that*

$$|b_{q-1}|^2 + |b_q|^2 \leq C 2^{\alpha+1} \left( \sum_{j \geq q} \frac{1}{\gamma_j^{(\alpha,0)}} u_j^2 \right)^{1/2} \left( \sum_{j \geq q-1} \frac{1}{\gamma_j^{(\alpha,0)}} b_j^2 \right)^{1/2}.$$

*Proof.* We may assume that the right-hand side of the estimate in the lemma is finite.

In view of the sign properties of $h_1$, $h_2$, $h_3$ and Lemma 2.2 we have

$$|h_1(q,\alpha)| + |h_2(q,\alpha)| = |h_3(q,\alpha)|. \tag{2.3.1}$$

We introduce the abbreviation

$$\alpha_q := \frac{h_2(q,\alpha)}{h_3(q,\alpha)} = \frac{\alpha(2q+\alpha+1)}{(2q+\alpha+2)(q+\alpha)},$$
$$\varepsilon_q := \alpha_q(1-\alpha_{q+1}) = \frac{\alpha(q+2)(2q+\alpha+1)}{(2q+\alpha+4)(q+1+\alpha)(q+\alpha)}.$$

By rearranging terms in Lemma 2.9 and using the triangle inequality we get

$$|h_3(q,\alpha)|\,|b_{q-1}| \leq |u_q| + |h_2(q,\alpha)|\,|b_q| + |h_1(q,\alpha)|\,|b_{q+1}|.$$

We set

$$z_q := \frac{|u_q|}{|h_3(q,\alpha)|} \tag{2.3.2}$$

and by applying (2.3.1) we arrive at

$$|b_{q-1}| \le z_q + \alpha_q|b_q| + (1-\alpha_q)|b_{q+1}|. \tag{2.3.3}$$

Iterating (2.3.3) once gives

$$\begin{aligned}
|b_{q-1}| &\le z_q + \alpha_q\big(z_{q+1} + \alpha_{q+1}|b_{q+1}| + (1-\alpha_{q+1})|b_{q+2}|\big) + (1-\alpha_q)|b_{q+1}| \\
&\le z_q + \alpha_q z_{q+1} + \big(1-\alpha_q(1-\alpha_{q+1})\big)|b_{q+1}| + \alpha_q(1-\alpha_{q+1})|b_{q+2}| \\
&= z_q + \alpha_q z_{q+1} + (1-\varepsilon_q)|b_{q+1}| + \varepsilon_q|b_{q+2}|
\end{aligned}$$

Squaring and Cauchy-Schwarz yields

$$\begin{aligned}
b_{q-1}^2 \le{}& (z_q + \alpha_q z_{q+1})^2 + 2(z_q + \alpha_q z_{q+1})\big((1-\varepsilon_q)|b_{q+1}| + \varepsilon_q|b_{q+2}|\big) \\
&+ (1-\varepsilon_q)^2 b_{q+1}^2 + \varepsilon_q^2 b_{q+2}^2 + 2\varepsilon_q(1-\varepsilon_q)|b_{q+1}|\,|b_{q+2}| \\
\le{}& (z_q + \alpha_q z_{q+1})^2 + 2(z_q + \alpha_q z_{q+1})\big((1-\varepsilon_q)|b_{q+1}| + \varepsilon_q|b_{q+2}|\big) \\
&+ \big((1-\varepsilon_q)^2 + \varepsilon_q(1-\varepsilon_q)\big)b_{q+1}^2 + \big(\varepsilon_q^2 + \varepsilon_q(1-\varepsilon_q)\big)b_{q+2}^2.
\end{aligned}$$

If we abbreviate for the first two addends

$$f_q := (z_q + \alpha_q z_{q+1})^2 + 2(z_q + \alpha_q z_{q+1})\big((1-\varepsilon_q)|b_{q+1}| + \varepsilon_q|b_{q+2}|\big) \tag{2.3.4}$$

we obtain

$$b_{q-1}^2 \le f_q + (1-\varepsilon_q)b_{q+1}^2 + \varepsilon_q b_{q+2}^2,$$

which we rewrite as

$$b_{q-1}^2 - b_{q+1}^2 \le f_q + \varepsilon_q\left(b_{q+2}^2 - b_{q+1}^2\right). \tag{2.3.5}$$

Next, we want to employ a telescoping sum. Since we assume that the sums in the right side of the statement of this lemma are finite, i.e.

$$\sum_j \frac{1}{\gamma_j^{(\alpha,0)}} u_j^2 < \infty, \qquad \sum_j \frac{1}{\gamma_j^{(\alpha,0)}} b_j^2 < \infty, \tag{2.3.6}$$

and since $\frac{1}{\gamma_j^{(\alpha,0)}} \lesssim (j+\alpha)2^{-\alpha}$ we have

$$\sqrt{q}\;|b_q| \to 0 \quad \text{for} \quad q \to \infty.$$

Hence, we can write

$$
\begin{aligned}
b_{q-1}^2 + b_q^2 &= \sum_{j=0}^{\infty} b_{q-1+2j}^2 - b_{q-1+2j+2}^2 + b_{q+2j}^2 - b_{q+2j+2}^2 \\
&\le \sum_{j=0}^{\infty} f_{q+2j} + \varepsilon_{q+2j}\left(b_{q+2+2j}^2 - b_{q+1+2j}^2\right) + f_{q+1+2j} + \varepsilon_{q+1+2j}\left(b_{q+3+2j}^2 - b_{q+2+2j}^2\right) \\
&= \sum_{j=0}^{\infty} f_{q+j} - \sum_{j=0}^{\infty} \varepsilon_{q+2j} b_{q+1+2j}^2 + \sum_{j=0}^{\infty} \left(\varepsilon_{q+2j} - \varepsilon_{q+2j+1}\right) b_{q+2+2j}^2 + \sum_{j=0}^{\infty} \varepsilon_{q+1+2j} b_{q+3+2j}^2 \\
&= \sum_{j=0}^{\infty} f_{q+j} - \varepsilon_q b_{q+1}^2 - \sum_{j=0}^{\infty} \varepsilon_{q+2+2j} b_{q+3+2j}^2 \\
&\quad + \sum_{j=0}^{\infty} \left(\varepsilon_{q+2j} - \varepsilon_{q+2j+1}\right) b_{q+2+2j}^2 + \sum_{j=0}^{\infty} \varepsilon_{q+1+2j} b_{q+3+2j}^2 \\
&= \sum_{j=0}^{\infty} f_{q+j} - \varepsilon_q b_{q+1}^2 + \sum_{j=0}^{\infty} \left(\varepsilon_{q+1+2j} - \varepsilon_{q+2+2j}\right) b_{q+3+2j}^2 + \sum_{j=0}^{\infty} \left(\varepsilon_{q+2j} - \varepsilon_{q+2j+1}\right) b_{q+2+2j}^2 \\
&= \sum_{j=0}^{\infty} f_{q+j} - \varepsilon_q b_{q+1}^2 + \sum_{j=0}^{\infty} \left(\varepsilon_{q+j} - \varepsilon_{q+j+1}\right) b_{q+2+j}^2.
\end{aligned}
$$

We conclude, noting that $\varepsilon_q \ge 0$,

$$
b_{q-1}^2 + b_q^2 \le b_{q-1}^2 + b_q^2 + \varepsilon_q b_{q+1}^2 \le F_q + S_{q+2}, \tag{2.3.7}
$$

where

$$
F_q := \sum_{j \ge q} f_j, \tag{2.3.8}
$$

$$
S_q := \sum_{j \ge q} \varepsilon_j' b_j^2 \quad \text{with} \quad \varepsilon_j' := |\varepsilon_{j-2} - \varepsilon_{j-1}|. \tag{2.3.9}
$$

By positivity of $\varepsilon_j'$ and $f_j$ we have $S_{q+1} \le S_q$ as well as $F_{q+1} \le F_q$. Therefore, we get from (2.3.7) and the definition of $S_q$

$$
\begin{aligned}
S_q &= \varepsilon_q' b_q^2 + \varepsilon_{q+1}' b_{q+1}^2 + S_{q+2} \\
&\le S_{q+2} + \max\{\varepsilon_q', \varepsilon_{q+1}'\} S_{q+3} + \max\{\varepsilon_q', \varepsilon_{q+1}'\} F_{q+1} \\
&\le (1 + \max\{\varepsilon_q', \varepsilon_{q+1}'\}) S_{q+2} + \max\{\varepsilon_q', \varepsilon_{q+1}'\} F_q.
\end{aligned}
$$

Applying the notation

$$
\varepsilon_q'' := \max\{\varepsilon_q', \varepsilon_{q+1}'\}
$$

we have

$$S_q \leq (1 + \varepsilon''_q) S_{q+2} + \varepsilon''_q F_q. \tag{2.3.10}$$

Iterating (2.3.10) $N$ times leads to

$$S_q \leq S_{q+2N+2} \prod_{j=0}^{N} (1 + \varepsilon''_{q+2j}) + \sum_{j=0}^{N} \varepsilon''_{q+2j} F_{q+2j} \prod_{i=0}^{j-1} (1 + \varepsilon''_{q+2i}). \tag{2.3.11}$$

A calculation shows

$$\varepsilon'_j \lesssim \frac{\alpha(\alpha + j)^3}{(\alpha + j)^5} = \frac{\alpha}{(\alpha + j)^2} \tag{2.3.12}$$

From the definition of $S_q$ in (2.3.9), (2.3.6), and (2.3.12) it follows that $\lim_{q\to\infty} S_q = 0$. Furthermore, we can bound the product uniformly in $N$:

$$\prod_{j=0}^{N} (1 + \varepsilon''_{q+2j}) = \exp\left( \sum_{j=0}^{N} \ln(1 + \varepsilon''_{q+2j}) \right) \leq \exp\left( \sum_{j=0}^{N} \varepsilon''_{q+2j} \right), \tag{2.3.13}$$

where in the last estimate we used the fact that $\ln(1 + x) \leq x$ for $x \geq 0$. From (2.3.12) we get

$$\sum_{j=0}^{N} \varepsilon''_{q+2j} \lesssim \sum_{j=0}^{N} \frac{\alpha}{(\alpha + q + 2j)^2} \lesssim \alpha \sum_{j=q}^{N} \frac{1}{(\alpha + j)^2} \lesssim \frac{\alpha}{\alpha + q}, \quad \forall N = q, q + 1, \ldots, \tag{2.3.14}$$

where we have used Lemma 2.11 in the last step. Since $\frac{\alpha}{\alpha+q} < 1$, inserting (2.3.14) in (2.3.13) gives

$$\prod_{j=0}^{N} (1 + \varepsilon''_{q+2j}) \leq C. \tag{2.3.15}$$

Now, by passing to the limit $N \to \infty$ in (2.3.11), we conclude a closed form bound for $S_q$:

$$S_q \leq \sum_{j=0}^{\infty} \varepsilon''_{q+2j} F_{q+2j} \prod_{i=0}^{j-1} (1 + \varepsilon''_{q+2i}).$$

Applying (2.3.14), (2.3.15), (2.3.12) and the definition of $F_q$ we can simplify

$$S_q \lesssim \sum_{j=0}^{\infty} \varepsilon''_{q+2j} F_{q+2j} \lesssim \sum_{j\geq q} \sum_{i\geq j} f_i \frac{\alpha}{(\alpha + j)^2} = \sum_{i\geq q} f_i \sum_{j=q}^{i} \frac{\alpha}{(\alpha + j)^2} \lesssim \frac{\alpha}{\alpha + q} F_q.$$

Inserting this estimate in (2.3.7) and using $\frac{\alpha}{\alpha+q+2} < 1$, we arrive at

$$b_{q-1}^2 + b_q^2 \lesssim F_q + \frac{\alpha}{\alpha+q+2} F_{q+2} \lesssim F_q + F_{q+2} \lesssim F_q. \tag{2.3.16}$$

We are left with estimating $F_q$. By the definition of $F_q$ in (2.3.8) and the definition of $f_q$ in (2.3.4) we have

$$F_q = \sum_{j \geq q} (z_j + \alpha_j z_{j+1})^2 + 2 \sum_{j \geq q} (z_j + \alpha_j z_{j+1}) \big( (1 - \varepsilon_j)|b_{j+1}| + \varepsilon_j |b_{j+2}| \big). \tag{2.3.17}$$

Now we estimate both sums separately starting with the first one:

$$\sum_{j \geq q} (z_j + \alpha_j z_{j+1})^2 \lesssim \sum_{j \geq q} z_j^2 + \underbrace{\alpha_j^2}_{\leq 1} z_{j+1}^2 \lesssim \sum_{j \geq q} z_j^2. \tag{2.3.18}$$

Next, we use the relation between $u_q$ and $b_q$ from Lemma 2.9. Furthermore, we note that $h_3(q, \alpha) \gtrsim 2^{-(\alpha+1)} \gamma_q^{(\alpha,0)}$. Consequently we obtain

$$\begin{aligned} z_q^2 = \frac{|u_q|^2}{|h_3(q,\alpha)|^2} &\lesssim 2^{\alpha+1} \frac{|u_q|}{\gamma_q^{(\alpha,0)}} \frac{|u_q|}{h_3(q,\alpha)} \\ &= 2^{\alpha+1} \frac{|u_q|}{\gamma_q^{(\alpha,0)}} \frac{1}{h_3(q,\alpha)} \, |h_1(q,\alpha) b_{q+1} + h_2(q,\alpha) b_q + h_3(q,\alpha) b_{q-1}| \\ &\lesssim 2^{\alpha+1} \frac{|u_q|}{\gamma_q^{(\alpha,0)}} \big( (1-\alpha_q)|b_{q+1}| + \alpha_q |b_q| + |b_{q-1}| \big). \end{aligned}$$

Inserting this in the bound (2.3.18), we get by applying the Cauchy-Schwarz inequality for sums

$$\begin{aligned} \sum_{j \geq q} (z_j + \alpha_j z_{j+1})^2 &\lesssim 2^{\alpha+1} \sum_{j \geq q} \frac{1}{\gamma_j^{(\alpha,0)}} |u_j| \big( (1-\alpha_j)|b_{j+1}| + \alpha_j |b_j| + |b_{j-1}| \big) \\ &\lesssim 2^{\alpha+1} \left( \sum_{j \geq q} \frac{1}{\gamma_j^{(\alpha,0)}} |u_j|^2 \right)^{1/2} \left( \sum_{j \geq q-1} \frac{1}{\gamma_j^{(\alpha,0)}} |b_j|^2 \right)^{1/2}. \end{aligned}$$

We continue by estimating the second sum in (2.3.17). Using again $z_q^2 \lesssim 2^{\alpha+1} |u_q| / \gamma_q^{(\alpha,0)}$

we get

$$\begin{aligned}
&\sum_{j\geq q}(z_j+\alpha_j z_{j+1})\big((1-\varepsilon_j)|b_{j+1}|+\varepsilon_j|b_{j+2}|\big)\\
&\quad\lesssim 2^{\alpha+1}\sum_{j\geq q}\frac{1}{\gamma_j^{(\alpha,0)}}\big(|u_j|+\underbrace{\alpha_j}_{\leq 1}|u_{j+1}|\big)\big(\underbrace{(1-\varepsilon_j)}_{\leq 1}|b_{j+1}|+\underbrace{\varepsilon_j}_{\leq 1}|b_{j+2}|\big)\\
&\quad\lesssim 2^{\alpha+1}\sum_{j\geq q}\frac{1}{\gamma_j^{(\alpha,0)}}\left(|u_j|+|u_{j+1}|\right)\left(|b_{j+1}|+|b_{j+2}|\right)\\
&\quad\lesssim 2^{\alpha+1}\left(\sum_{j\geq q}\frac{1}{\gamma_j^{(\alpha,0)}}|u_j|^2\right)^{1/2}\left(\sum_{j\geq q}\frac{1}{\gamma_{j+1}^{(\alpha,0)}}|b_{j+1}|^2\right)^{1/2}\\
&\quad\lesssim 2^{\alpha+1}\left(\sum_{j\geq q}\frac{1}{\gamma_j^{(\alpha,0)}}|u_j|^2\right)^{1/2}\left(\sum_{j\geq q-1}\frac{1}{\gamma_j^{(\alpha,0)}}|b_j|^2\right)^{1/2}.
\end{aligned}$$

In view of (2.3.16) the last two estimates conclude the proof.

□

**Corollary 2.13** *Assume the same hypotheses as in Lemma 2.12. Then for every $p\geq 0$ there exists a constant $C>0$ independent of $p$ and $\alpha$ such that*

$$|b_p|^2\leq C2^{\alpha+1}\left(\sum_{j\geq p+1}\frac{1}{\gamma_j^{(\alpha,0)}}u_j^2\right)^{1/2}\left(\sum_{j\geq p}\frac{1}{\gamma_j^{(\alpha,0)}}b_j^2\right)^{1/2}.$$

*Proof.* The proof follows directly from Lemma 2.12. For $p=0$ we apply Lemma 2.12 with $q=1$. Then we have

$$|b_0|^2\leq|b_0|^2+|b_1|^2\lesssim 2^{\alpha+1}\left(\sum_{j\geq 1}\frac{1}{\gamma_j^{(\alpha,0)}}u_j^2\right)^{1/2}\left(\sum_{j\geq 0}\frac{1}{\gamma_j^{(\alpha,0)}}b_j^2\right)^{1/2}.$$

Analogously, for $p\geq 1$ we apply Lemma 2.12 correspondingly with $q\geq 2$.

□

## 2.4 Trace Results for Triangles

In this section we will provide the final spadework to Theorem 2.17, especially regarding results on the edge $\Gamma$. Then, at last, we will write down Theorem 2.17 and present the proof.

First, we want to show a representation for the transformed function $\tilde{u}$ on the edge $\Gamma=(-1,1)\times\{-1\}$, where $u\in L^2(T)$. By (1.3.5) and (1.3.6) we note

$$P_q^{(2p+1,0)}(-1)=(-1)^q P_q^{(0,2p+1)}(1)=(-1)^q.$$

Since the Duffy transformation reduces on $\Gamma$ to the identity we obtain:

$\psi_{pq}(\xi_1,-1)=\widetilde{\psi}_{pq}\left(D^{-1}(\xi_1,-1)\right)=\widetilde{\psi}_{pq}(\eta_1,-1)=P_p^{(0,0)}(\eta_1)P_q^{(2p+1,0)}(-1)=(-1)^q P_p^{(0,0)}(\eta_1).$

By applying the representation of $u$ in (2.2.3) we arrive at

$$\tilde{u}(\eta_1,-1)=u(\xi_1,-1)=\sum_{p=0}^{\infty}P_p^{(0,0)}(\eta_1)\frac{2^{2p+1}}{\gamma_p^{(0,0)}}\left(\sum_q(-1)^q u_{pq}\frac{1}{\gamma_q^{(2p+1,0)}}\right). \tag{2.4.1}$$

In particular, we have

$$\|u\|^2_{L^2(\Gamma)}=\|\tilde{u}\|^2_{L^2(\Gamma)}=\sum_{p=0}^{\infty}\frac{1}{\gamma_p^{(0,0)}}\left|\sum_{q=0}^{\infty}(-1)^q u_{pq}\frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}}\right|^2. \tag{2.4.2}$$

Next, we will see that the infinite sum over $q$ in (2.4.1) and (2.4.2) can be expressed as a finite sum:

**Lemma 2.14** *Let $N\geq 1$. There holds*

$$\sum_{q=N}^{\infty}(-1)^q\frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}}u_{pq}=(-1)^N h_2(N,2p+1)\frac{2^p}{\gamma_N^{(2p+1,0)}}\tilde{u}'_{p,N}+\sum_{q=N-1}^{N}(-1)^{q+1}h_3(q+1,2p+1)\frac{2^p}{\gamma_{q+1}^{(2p+1,0)}}\tilde{u}'_{p,q}.$$

*Proof.* We have in view of Corollary 2.10

$$\begin{aligned}
\sum_{q=N}^{\infty}(-1)^q\frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}}u_{pq} &\overset{(2.2.15)}{=}\sum_{q=N}^{\infty}(-1)^q\frac{2^p}{\gamma_q^{(2p+1,0)}}\tilde{u}_{pq}\\
&=\sum_{q=N}^{\infty}(-1)^q\frac{2^p}{\gamma_q^{(2p+1,0)}}\left(h_1(q,2p+1)\tilde{u}'_{p,q+1}+h_2(q,2p+1)\tilde{u}'_{p,q}+h_3(q,2p+1)\tilde{u}'_{p,q-1}\right)\\
&=\sum_{q=N+1}^{\infty}(-1)^{q-1}h_1(q-1,2p+1)\frac{2^p}{\gamma_{q-1}^{(2p+1,0)}}\tilde{u}'_{p,q}\\
&\quad+\sum_{q=N}^{\infty}(-1)^q h_2(q,2p+1)\frac{2^p}{\gamma_q^{(2p+1,0)}}\tilde{u}'_{p,q}+\sum_{q=N-1}^{\infty}(-1)^{q+1}h_3(q+1,2p+1)\frac{2^p}{\gamma_{q+1}^{(2p+1,0)}}\tilde{u}'_{p,q}\\
&=\sum_{q=N+1}^{\infty}(-1)^q\tilde{u}'_{p,q}2^p\left[-\frac{h_1(q-1,2p+1)}{\gamma_{q-1}^{(2p+1,0)}}+\frac{h_2(q,2p+1)}{\gamma_q^{(2p+1,0)}}-\frac{h_3(q+1,2p+1)}{\gamma_{q+1}^{(2p+1,0)}}\right]\\
&\quad+(-1)^N h_2(N,2p+1)\frac{2^p}{\gamma_N^{(2p+1,0)}}\tilde{u}'_{p,N}+\sum_{q=N-1}^{N}(-1)^{q+1}h_3(q+1,2p+1)\frac{2^p}{\gamma_{q+1}^{(2p+1,0)}}\tilde{u}'_{p,q}
\end{aligned}$$

By Lemma 2.2, the expression in brackets vanishes and that concludes the proof. □

Since Lemma 2.14 assumes $N \geq 1$, the terms corresponding to $q = 0$ in (2.4.2) are not included. We study this case in Lemma 2.16 below. But first, we have to provide the following short lemma.

**Lemma 2.15** *For $a, b \in \mathbb{R}$ there holds*

$$\min\{a^2, b^2\} \leq |a| \cdot |b|$$

*Proof.* W.l.o.g. we assume that $|a| \leq |b|$. Hence, we have

$$\min\{a^2, b^2\} = a^2 \leq |a| \cdot |b|.$$

$\square$

**Lemma 2.16** *Let $u \in H^1(T)$ and consider the representation of the norms in (2.4.2). For $p \geq 1$ and $q = 0$ there exists a constant $C > 0$ independent of $p$ and $u$ such that*

$$\sum_{p=1}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \left| u_{p0} \frac{2^{2p+1}}{\gamma_0^{(2p+1,0)}} \right|^2 \leq C \|u\|_{L^2(T)} \|u\|_{H^1(T)}$$

*Proof.* Since $\gamma_0^{(2p+1,0)} = \frac{2^{2p+1}}{p+1}$ and $\gamma_p^{(0,0)} = \frac{2}{2p+1} \lesssim \frac{1}{p+1}$, we get

$$\sum_{p=1}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \left| u_{p0} \frac{2^{2p+1}}{\gamma_0^{(2p+1,0)}} \right|^2 \lesssim \sum_{p=1}^{\infty} (p+1)^3 \left| u_{p0} \right|^2.$$

To bound the sum on the right-hand side, we note that an integration by parts gives

$$\begin{aligned} 2^{p+1} u_{p0} &= \int_{-1}^{1} (1-\eta_2)^{p+1} U_p(\eta_2) d\eta_2 \\ &= \frac{1}{p+2} \left( 2^{p+2} U_p(-1) + \int_{-1}^{1} (1-\eta_2)^{p+2} U_p'(\eta_2) d\eta_2 \right), \end{aligned}$$

where the first equation is due to (2.2.6). These two equations yield two representations for $u_{p0}$. Considering the first one, we get by employing the Cauchy-Schwarz inequality

$$\begin{aligned} |2^{p+1} u_{p0}|^2 &= \left| \int_{-1}^{1} (1-\eta_2)^{p+1} U_p(\eta_2) d\eta_2 \right|^2 \\ &\leq \left( \int_{-1}^{1} \left( (1-\eta_2)^{p+\frac{1}{2}} \right)^2 d\eta_2 \right) \left( \int_{-1}^{1} (1-\eta_2) |U_p(\eta_2)|^2 d\eta_2 \right) \\ &= \frac{2^{2p+2}}{p+1} \int_{-1}^{1} \frac{1-\eta_2}{2} |U_p(\eta_2)|^2 d\eta_2 \end{aligned} \qquad (2.4.3)$$

Once again we make use of the Cauchy-Schwarz inequality and obtain for the second representation

$$|2^{p+1}u_{p0}|^2 \leq \frac{2}{(p+2)^2}\left(2^{2p+4}|U_p(-1)|^2 + \left|\int_{-1}^{1}(1-\eta_2)^{p+2}U_p'(\eta_2)d\eta_2\right|^2\right),$$

where

$$\begin{aligned}\left|\int_{-1}^{1}(1-\eta_2)^{p+2}U_p'(\eta_2)d\eta_2\right|^2 &\leq \left(\int_{-1}^{1}\left((1-\eta_2)^{p+\frac{3}{2}}\right)^2 d\eta_2\right)\left(\int_{-1}^{1}(1-\eta_2)|U_p'(\eta_2)|^2 d\eta_2\right)\\ &= \frac{2^{2p+4}}{p+2}\int_{-1}^{1}\frac{1-\eta_2}{2}|U_p'(\eta_2)|^2.\end{aligned}$$

Inserting this in the bound before yields

$$\begin{aligned}|2^{p+1}u_{p0}|^2 &\leq 2\frac{2^{2p+4}}{(p+2)^2}\left(|U_p(-1)|^2 + \frac{1}{p+2}\int_{-1}^{1}\frac{1-\eta_2}{2}|U_p'(\eta_2)|^2 d\eta_2\right)\\ &\leq 2\frac{2^{2p+4}}{(p+1)^2}\left(|U_p(-1)|^2 + \frac{1}{p+1}\int_{-1}^{1}\frac{1-\eta_2}{2}|U_p'(\eta_2)|^2 d\eta_2\right).\end{aligned} \tag{2.4.4}$$

Next, we abbreviate

$$\begin{aligned}\sigma_p^2 &:= \int_{-1}^{1}\frac{1-\eta_2}{2}|U_p(\eta_2)|^2 d\eta_2,\\ \tau_p^2 &:= \int_{-1}^{1}\frac{1-\eta_2}{2}|U_p'(\eta_2)|^2 d\eta_2.\end{aligned}$$

Hence, applying (2.4.3) and (2.4.4) we have

$$\begin{aligned}|2^{p+1}u_{p0}|^2 &\leq \min\left\{\frac{2^{2p+2}}{p+1}\sigma_p^2\ ,\ 2\frac{2^{2p+4}}{(p+1)^2}\left(|U_p(-1)|^2 + \frac{1}{p+1}\tau_p^2\right)\right\}\\ &\leq 2\frac{2^{2p+4}}{(p+1)^2}|U_p(-1)|^2 + 2\min\left\{\frac{2^{2p+2}}{p+1}\sigma_p^2\ ,\ \frac{2^{2p+4}}{(p+1)^3}\tau_p^2\right\}\\ &\leq 2\frac{2^{2p+4}}{(p+1)^2}|U_p(-1)|^2 + 2\frac{2^{2p+3}}{(p+1)^2}\sigma_p\tau_p,\end{aligned}$$

where we used Lemma 2.15 in the last step. This leads us to the following:

$$|u_{p0}|^2 \lesssim \frac{1}{(p+1)^2}|U_p(-1)|^2 + \frac{1}{(p+1)^2}\sigma_p\tau_p. \tag{2.4.5}$$

Hence, we conclude

$$\begin{aligned}\sum_{p=1}^{\infty}(p+1)^3|u_{p0}|^2 &\lesssim \sum_{p=1}^{\infty}(p+1)|U_p(-1)|^2+\sum_{p=1}^{\infty}(p+1)\sigma_p\tau_p \\ &\lesssim \sum_{p=1}^{\infty}(p+1)|U_p(-1)|^2+\left(\sum_{p=1}^{\infty}(p+1)\sigma_p^2\right)^{1/2}\left(\sum_{p=1}^{\infty}(p+1)\tau_p^2\right)^{1/2} \\ &\lesssim \|u\|^2_{L^2(\Gamma)}+\|u\|_{L^2(T)}\|\nabla u\|_{L^2(T)}\end{aligned}$$

where, in the last inequality, we appealed to Lemma 2.6. Since there is the non-trivial estimate $\|u\|^2_{L^2(\Gamma)} \lesssim \|u\|_{L^2(T)}\|u\|_{H^1(T)}$ of [3, Thm. 1.6.6] and $\|\nabla u\|_{L^2(T)} \leq \|u\|_{H^1(T)}$ the statement follows.

□

Now at the end of this section we finally arrive at the main theorem as we have all tools ready to prove it.

**Theorem 2.17** (trace stability of $L^2$-projection) *For $N \in \mathbb{N}_0$ denote by $\Pi_N$ the $L^2(T)$-projection onto $\mathcal{P}_N(T)$. There exists a constant $C > 0$ independent of $N$ and $u$ such that*

$$\|\Pi_N u\|^2_{L^2(\Gamma)} \leq C\|u\|_{L^2(T)}\|u\|_{H^1(T)} \qquad \forall u \in H^1(T).$$

*Proof.* Since $\|u\|^2_{L^2(\Gamma)}\lesssim\|u\|_{L^2(T)}\|u\|_{H^1(T)}$ (see [3, Thm. 1.6.6]) we will show instead the statement $\|u-\Pi_N u\|^2_{L^2(\Gamma)}\lesssim\|u\|_{L^2(T)}\|u\|_{H^1(T)}$. By (2.4.2), we have to bound

$$\begin{aligned}\|u-\Pi_N u\|^2_{L^2(\Gamma)} &= \sum_{p=0}^{\infty}\frac{1}{\gamma_p^{(0,0)}}\left|\sum_{q=\max\{0,N+1-p\}}^{\infty}(-1)^q\frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}}u_{pq}\right|^2 \\ &= \sum_{p=0}^{N}\frac{1}{\gamma_p^{(0,0)}}\left|\sum_{q=N+1-p}^{\infty}(-1)^q\frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}}u_{pq}\right|^2+\sum_{p=N+1}^{\infty}\frac{1}{\gamma_p^{(0,0)}}\left|\sum_{q=0}^{\infty}(-1)^q\frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}}u_{pq}\right|^2 \\ &\lesssim \underbrace{\sum_{p=0}^{N}\frac{1}{\gamma_p^{(0,0)}}\left|\sum_{q=N+1-p}^{\infty}(-1)^q\frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}}u_{pq}\right|^2}_{=:S_1}+\underbrace{\sum_{p=N+1}^{\infty}\frac{1}{\gamma_p^{(0,0)}}\left|\sum_{q=1}^{\infty}(-1)^q\frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}}u_{pq}\right|^2}_{=:S_2} \\ &\quad+\underbrace{\sum_{p=N+1}^{\infty}\frac{1}{\gamma_p^{(0,0)}}\left|\frac{2^{2p+1}}{\gamma_0^{(2p+1,0)}}u_{p0}\right|^2}_{=:S_3}\end{aligned}$$

Lemma 2.16 immediately gives $S_3 \lesssim \|u\|_{L^2(T)}\|u\|_{H^1(T)}$. From Lemma 2.14, the estimates

$$\begin{aligned}&h_2(N+1-p,2p+1) \lesssim \frac{p}{N^2} \lesssim \frac{1}{N}, \qquad \forall p=0,\ldots,N \\ &h_3(N+1-p,2p+1), \quad h_3(N+2-p,2p+1) \lesssim \frac{N+p}{N^2} \lesssim \frac{1}{N} \qquad \forall p=0,\ldots,N\end{aligned}$$

and

$$\frac{1}{\gamma_{N+1-p}^{(2p+1,0)}} = \frac{N+2}{2^{2p+1}}, \qquad \frac{1}{\gamma_{N+2-p}^{(2p+1,0)}} = \frac{N+3}{2^{2p+1}}$$

we obtain for $S_1$

$$S_1 \lesssim \sum_{p=0}^{N} \frac{1}{\gamma_p^{(0,0)}} \left( \left|2^{-(p+1)} \tilde{u}'_{p,N+1-p}\right|^2 + \left|2^{-(p+1)} \tilde{u}'_{p,N-p}\right|^2 \right)$$

Analogously, we get for $S_2$

$$S_2 \lesssim \sum_{p=N+1}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \left( \left|2^{-(p+1)} \tilde{u}'_{p,1}\right|^2 + \left|2^{-(p+1)} \tilde{u}'_{p,0}\right|^2 \right)$$

Applying Corollary 2.13 the powers of two in the estimates above and in the corollary annihilate each other. Hence, $S_1 + S_2$ gives

$$\begin{aligned} S_1 + S_2 \lesssim & \sum_{p=0}^{N} \frac{1}{\gamma_p^{(0,0)}} \left( \sum_{q \geq N+1-p} \frac{1}{\gamma_q^{(2p+1,0)}} |\tilde{u}_{pq}|^2 \right)^{1/2} \left( \sum_{q \geq N-p} \frac{1}{\gamma_q^{(2p+1,0)}} |\tilde{u}'_{pq}|^2 \right)^{1/2} \\ & + \sum_{p=N+1}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \left( \sum_{q \geq 1} \frac{1}{\gamma_q^{(2p+1,0)}} |\tilde{u}_{pq}|^2 \right)^{1/2} \left( \sum_{q \geq 0} \frac{1}{\gamma_q^{(2p+1,0)}} |\tilde{u}'_{pq}|^2 \right)^{1/2} \\ \lesssim & \|u\|_{L^2(T)} \|\nabla u\|_{L^2(T)}, \end{aligned}$$

where in the last estimate, we have used the Cauchy-Schwarz inequality for sums and Corollary 2.8. Since $\|\nabla u\|_{L^2(T)} \leq \|u\|_{H^1(T)}$ this concludes the proof.

□

# 3 Numerical computations

## 3.1 Description of the numerical method

In this section, we test numerically the stability properties of the $L^2$-projection onto the space of polynomials of degree $N$. To do so, we want to compute the constant $C$ in the estimate $\|\Pi_N u\|^2_{L^2(\Gamma)} \leq C\|u\|_{L^2(T)}\|u\|_{H^1(T)}$ of Theorem 2.17. Our discretization of this problem takes the form of a maximization problem given by

$$\sup_{u\in\mathcal{P}_{kN}} \frac{\|\Pi_N u\|^2_{L^2(\Gamma)}}{\|u\|_{L^2(T)}\|u\|_{H^1(T)}} \tag{3.1.1}$$

or, equivalently,

$$\sup_{u\in\mathcal{P}_{kN}} \{\|\Pi_N u\|^2_{L^2(\Gamma)} : \|u\|_{L^2(T)}\|u\|_{H^1(T)} = 1\}, \tag{3.1.2}$$

where $k \in \mathbb{N}$, but usually we will use $k \in \{1, 2, 3\}$.
As we see in (3.1.2), we want to solve a maximization problem with respect to the side condition $\|u\|_{L^2(T)}\|u\|_{H^1(T)} = 1$. The method of Lagrange multipliers provides a strategy to tackle this task.

**Theorem 3.1** (method of Lagrange multipliers) *Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and $g : D \subset \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$. If $f$ has in $\underline{y}$ a local constrained extremum with the side condition $g = 0$ and the differential $\nabla_{\underline{y}} g(\underline{y}) \in \mathbb{R}^{m\times n}$ has rank $m$, then there exists $\underline{\lambda} \in \mathbb{R}^m$ such that $(\underline{\lambda}, \underline{y})$ is a stationary point for the function $\mathcal{L} : D \times \mathbb{R}^m \to \mathbb{R}$*

$$\mathcal{L}(\underline{x}, \underline{\mu}) := f(\underline{x}) + \underline{\mu}^T g(\underline{x}).$$

*I.e. we have*

$$\nabla_{\underline{y}}\mathcal{L}(\underline{y}, \underline{\lambda}) = 0$$
$$\nabla_{\underline{\lambda}}\mathcal{L}(\underline{y}, \underline{\lambda}) = 0.$$

*Proof.* See [8, Thm. 10.6.1].

□

We define the Lagrange function $\mathcal{L}(u, \lambda)$ with Lagrange multiplier $\lambda$ as follows:

$$\mathcal{L}(u, \lambda) := \|\Pi_N u\|^2_{L^2(\Gamma)} - \lambda(\|u\|_{L^2(T)}\|u\|_{H^1(T)} - 1). \tag{3.1.3}$$

To handle the norms, we want to employ the expansion of u in (2.2.3):

$$u = \sum_{p,q=0}^{\infty} \frac{1}{\gamma_p^{(0,0)}} \frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}} u_{pq}\psi_{pq}.$$

With the abbreviation

$$c_{pq} = \left( \int_T |\psi_{pq}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \right)^{-1} = \frac{1}{\gamma_p^{(0,0)}} \frac{2^{2p+1}}{\gamma_q^{(2p+1,0)}} = \frac{1}{2}(2p+1)(p+q+1),$$

we arrive at

$$u = \sum_{p,q=0}^{\infty} c_{pq} u_{pq} \psi_{pq},$$

Since we take the supremum of $u \in \mathcal{P}_{kN}$, we have

$$u = \sum_{p,q=0}^{kN} c_{pq} u_{pq} \psi_{pq}. \tag{3.1.4}$$

We can now approach the computation of the norms. We want to express every norm as a vector-matrix-vector multiplication, i.e. $\|u\|^2 = \underline{u}^T A \underline{u}$, with a vector $\underline{u}$ that contains the coefficients $u_{pq}$. Therefore, we need to convert double indices $(p, q)$ into single indices. Defining a map

$$num : \begin{cases} \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 \\ (p,q) \mapsto num(p,q) = \frac{1}{2}(p+q+2)(p+q+1) - (1-p), \end{cases} \tag{3.1.5}$$

where $p+q \leq kN$, leads us to the triangular look-up matrix $I$, which gives the corresponding single index. We define the matrix by

$$I_{pq} := num(p,q). \tag{3.1.6}$$

In particular, we have

$$I = \begin{pmatrix} num(0,0) & num(0,1) & num(0,2) & \cdots & num(0,kN) \\ num(1,0) & num(1,1) & \cdots & & \\ num(2,0) & \cdots & & & \\ \vdots & & & & \\ num(kN,0) & & & & \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3 & \cdots \\ 2 & 4 & \cdots & \\ 5 & \cdots & & \\ \vdots & & & \\ & & & \end{pmatrix}$$

Hence, we get the conversion of the double index $(p, q)$ to a single index by looking up the entry $I_{pq} = num(p, q)$.

In order to set up the Lagrangian $\mathcal{L}$, we need to realize the expressions $\|u\|^2_{L^2(T)}$, $\|\nabla u\|^2_{L^2(T)}$, and $\|\Pi_N u\|^2_{L^2(\Gamma)}$, for $u \in \mathcal{P}_{kN}$.

**Matrix representation of $\|u\|^2_{L^2(T)}$**

We have

$$\begin{aligned}\|u\|^2_{L^2(T)} &= \langle u, u\rangle_{L^2(T)} \\ &= \sum_{p,q,k,l=0}^{kN} c_{pq}c_{kl}u_{pq}u_{kl}\,\langle \psi_{pq}, \psi_{kl}\rangle_{L^2(T)} \\ &= \sum_{p,q,k,l=0}^{kN} c_{pq}c_{kl}u_{pq}u_{kl}\delta_{pk}\delta_{ql}\frac{2}{2p+1}\frac{1}{p+q+1}, \end{aligned} \tag{3.1.7}$$

where in the last step we applied to Lemma 1.3. We get

$$\|u\|^2_{L^2(T)} = \underline{u}^T M \underline{u}, \tag{3.1.8}$$

with the vector $\underline{u}$ that satisfies $u_{I_{pq}} = u_{pq}$ and the matrix $M$, whose entries are given by

$$M_{I_{pq}I_{kl}} = c_{pq}c_{kl}\delta_{pk}\delta_{ql}\frac{2}{2p+1}\frac{1}{p+q+1}, \quad p+q, k+l \leq kN.$$

**Matrix representation of $\|\Pi_N u\|^2_{L^2(\Gamma)}$**

Analogous to the procedure above, we obtain for $\|\Pi_N u\|^2_{L^2(\Gamma)}$:

$$\begin{aligned}\|\Pi_N u\|^2_{L^2(\Gamma)} &= \langle\, \Pi_N u, \Pi_N u\rangle_{L^2(T)} \\ &= \sum_{\substack{p,q,k,l=0 \\ p+q\leq N,\, k+l\leq N}}^{kN} c_{pq}c_{kl}u_{pq}u_{kl}\,\langle \psi_{pq}, \psi_{kl}\rangle_{L^2(\Gamma)}\end{aligned}$$

We use (1.3.5), (1.3.6) and the fact that the Legendre polynomials $L_n(x) = P_n^{(0,0)}$ satisfy $\int_{-1}^{1} L_i(x)L_j(x)dx = \frac{2}{2i+1}\delta_{ij}$ to obtain

$$\begin{aligned}\langle \psi_{pq}, \psi_{kl}\rangle_{L^2(\Gamma)} &= \int_\Gamma \psi_{pq}(\xi_1,\xi_2)\psi_{kl}(\xi_1,\xi_2)d\xi_1 d\xi_2 \\ &= \int_{-1}^{1} \psi_{pq}(\xi_1,-1)\psi_{kl}(\xi_1,-1)d\xi_1 \\ &= \int_{-1}^{1} \psi_{pq}(2\frac{(1+\xi_1)}{1-(-1)} - 1, -1)\psi_{kl}(2\frac{(1+\xi_1)}{1-(-1)} - 1, -1)d\xi_1 \\ &= \int_{-1}^{1} \widetilde{\psi}_{pq}(\xi_1,-1)\widetilde{\psi}_{kl}(\xi_1,-1)d\xi_1\end{aligned}$$

$$= \int_{-1}^{1} P_p^{(0,0)}(\xi_1) P_q^{(2p+1,0)}(-1) P_k^{(0,0)}(\xi_1) P_l^{(2k+1,0)}(-1) d\xi_1$$

$$= (-1)^q \begin{pmatrix} q \\ q \end{pmatrix} (-1)^l \begin{pmatrix} l \\ l \end{pmatrix} \int_{-1}^{1} L_p(\xi_1) L_k(\xi_1) d\xi_1$$

$$= (-1)^{q+l} \frac{2}{2p+1} \delta_{pk}$$

Hence,

$$\|\Pi_N u\|^2_{L^2(\Gamma)} = \sum_{\substack{p,q,k,l=0 \\ p+q \le N,\, k+l \le N}}^{kN} c_{pq} c_{kl} u_{pq} u_{kl} (-1)^{q+l} \frac{2}{2p+1} \delta_{pk}. \tag{3.1.9}$$

In particular, we have

$$\|\Pi_N u\|^2_{L^2(\Gamma)} = \underline{u}^T T \underline{u}, \tag{3.1.10}$$

where the entries of the matrix $T$ are given by

$$T_{I_{pq} I_{kl}} = c_{pq} c_{kl} (-1)^{q+l} \frac{2}{2p+1} \delta_{pk}, \quad p+q, k+l \le N.$$

**Matrix representation of $\|\nabla u\|^2_{L^2(T)}$**

To examine $\|\nabla u\|^2_{L^2(T)}$ we introduce the transformed function $\tilde{u} = u \circ D$. We have

$$\nabla \tilde{u}(\eta_1, \eta_2) = \nabla(u \circ D)(\eta_1, \eta_2) = (\nabla u) \circ D(\eta_1, \eta_2) \cdot J_D,$$

where $J_D$ is the Jacobian matrix defined in (2.1.2) and $\nabla u$ is thought of as a row vector. Therefore

$$\nabla \tilde{u}(\eta_1, \eta_2) \cdot J_D^{-1} = (\nabla u) \circ D(\eta_1, \eta_2), \tag{3.1.11}$$

with $J_D^{-1} = \frac{1}{\det J_D} \begin{pmatrix} 1 & \frac{1+\eta_1}{2} \\ 0 & \frac{1-\eta_2}{2} \end{pmatrix}$. Using (3.1.11) we achieve

$$\begin{aligned}
\|\nabla u\|^2_{L^2(T)} &= \int_T |\nabla u(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \\
&= \int_S |(\nabla u) \circ D(\eta_1, \eta_2)|^2 |\det J_D| d\eta_1 d\eta_2 \\
&= \int_S \left[(\nabla u) \circ D(\eta_1, \eta_2)\right] \left[(\nabla u) \circ D(\eta_1, \eta_2)\right]^T |\det J_D| d\eta_1 d\eta_2 \\
&= \int_S \left[\nabla \tilde{u}(\eta_1, \eta_2) \cdot J_D^{-1}\right] \left[\nabla \tilde{u}(\eta_1, \eta_2) \cdot J_D^{-1}\right]^T |\det J_D| d\eta_1 d\eta_2 \\
&= \int_S \nabla \tilde{u}(\eta_1, \eta_2) J_D^{-1} J_D^{-T} \nabla \tilde{u}(\eta_1, \eta_2)^T |\det J_D| d\eta_1 d\eta_2.
\end{aligned}$$

For the matrix product we have

$$\begin{aligned}
\widetilde{D} := |\det J_D| J_D^{-1} J_D^{-T} &= \frac{1}{|\det J_D|} \begin{pmatrix} 1 & \frac{1+\eta_1}{2} \\ 0 & \frac{1-\eta_2}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1+\eta_1}{2} & \frac{1-\eta_2}{2} \end{pmatrix} \\
&= \frac{2}{1-\eta_2} \begin{pmatrix} 1 + \frac{(1+\eta_1)^2}{4} & \frac{(1+\eta_1)(1-\eta_2)}{4} \\ \frac{(1+\eta_1)(1-\eta_2)}{4} & \frac{(1-\eta_2)^2}{4} \end{pmatrix} \\
&= \begin{pmatrix} \frac{4+(1+\eta_1)^2}{2(1-\eta_2)} & \frac{1+\eta_1}{2} \\ \frac{1+\eta_1}{2} & \frac{1-\eta_2}{2} \end{pmatrix}.
\end{aligned}$$

Since $\tilde{u}(\eta_1, \eta_2) = \sum_{p,q=0}^{kN} c_{pq} u_{pq} \widetilde{\psi}_{pq}(\eta_1, \eta_2)$ we arrive at

$$\begin{aligned}
\|\nabla u\|_{L^2(T)}^2 &= \int_S \nabla \tilde{u}(\eta_1, \eta_2) \widetilde{D} \nabla \tilde{u}(\eta_1, \eta_2)^T d\eta_1 d\eta_2 \\
&= \sum_{p,q,k,l=0}^{kN} c_{pq} c_{kl} u_{pq} u_{kl} \int_S \partial_{\eta_1} \widetilde{\psi}_{pq} \partial_{\eta_1} \widetilde{\psi}_{kl} \frac{4+(1+\eta_1)^2}{2(1-\eta_2)} \; + \; \partial_{\eta_2} \widetilde{\psi}_{pq} \partial_{\eta_1} \widetilde{\psi}_{kl} \frac{1+\eta_1}{2} \\
&\quad + \; \partial_{\eta_1} \widetilde{\psi}_{pq} \partial_{\eta_2} \widetilde{\psi}_{kl} \frac{1+\eta_1}{2} \; + \; \partial_{\eta_2} \widetilde{\psi}_{pq} \partial_{\eta_2} \widetilde{\psi}_{kl} \frac{1-\eta_2}{2} d\eta_1 d\eta_2.
\end{aligned} \tag{3.1.12}$$

Applying (1.4.3) and the fact that $\frac{d^k}{dx^k} P_n^{(\alpha,\beta)} = \frac{\Gamma(\alpha+\beta+n+1+k)}{2^k \Gamma(\alpha+\beta+n+1)} P_{n-k}^{(\alpha+k,\beta+k)}$ we get

$$\begin{aligned}
&\partial_{\eta_1} \widetilde{\psi}_{pq}(\eta_1, \eta_2) \\
&\quad = \frac{d}{d\eta_1} P_p^{(0,0)}(\eta_1) \left( \frac{1-\eta_2}{2} \right)^p P_q^{(2p+1,0)}(\eta_2) \\
&\quad = \frac{p+1}{2} P_{p-1}^{(1,1)}(\eta_1) \left( \frac{1-\eta_2}{2} \right)^p P_q^{(2p+1,0)}(\eta_2)
\end{aligned}$$

$$\begin{aligned}
&\partial_{\eta_2} \widetilde{\psi}_{pq}(\eta_1, \eta_2) \\
&\quad = P_p^{(0,0)}(\eta_1) \left[ \left( -\frac{p}{2} \right) \left( \frac{1-\eta_2}{2} \right)^{p-1} P_q^{(2p+1,0)}(\eta_2) + \left( \frac{1-\eta_2}{2} \right)^p \frac{d}{d\eta_2} P_q^{(2p+1,0)}(\eta_2) \right] \\
&\quad = \left( -\frac{p}{2} \right) P_p^{(0,0)}(\eta_1) \left( \frac{1-\eta_2}{2} \right)^{p-1} P_q^{(2p+1,0)}(\eta_2) \\
&\qquad + \frac{2p+q+2}{2} P_p^{(0,0)}(\eta_1) \left( \frac{1-\eta_2}{2} \right)^p P_{q-1}^{(2p+2,1)}(\eta_2).
\end{aligned}$$

Therefore

$$\partial_{\eta_1}\widetilde{\psi}_{pq}\partial_{\eta_1}\widetilde{\psi}_{kl}\frac{4+(1+\eta_1)^2}{2(1-\eta_2)}$$
$$= \frac{(p+1)(k+1)}{2^{p+k+3}}\ (4+(1+\eta_1)^2)P_{p-1}^{(1,1)}(\eta_1)P_{k-1}^{(1,1)}(\eta_1)\ (1-\eta_2)^{p+k-1}P_q^{(2p+1,0)}(\eta_2)P_l^{(2k+1,0)}(\eta_2)$$

$$\partial_{\eta_2}\widetilde{\psi}_{pq}\partial_{\eta_1}\widetilde{\psi}_{kl}\frac{1+\eta_1}{2}$$
$$= -\frac{p(k+1)}{2^{p+k+2}}\ (1+\eta_1)P_p^{(0,0)}(\eta_1)P_{k-1}^{(1,1)}(\eta_1)\ (1-\eta_2)^{p+k-1}P_q^{(2p+1,0)}(\eta_2)P_l^{(2k+1,0)}(\eta_2)$$
$$+ \frac{(2p+q+2)(k+1)}{2^{p+k+3}}\ (1+\eta_1)P_p^{(0,0)}(\eta_1)P_{k-1}^{(1,1)}(\eta_1)\ (1-\eta_2)^{p+k}P_{q-1}^{(2p+2,1)}(\eta_2)P_l^{(2k+1,0)}(\eta_2)$$

$$\partial_{\eta_1}\widetilde{\psi}_{pq}\partial_{\eta_2}\widetilde{\psi}_{kl}\frac{1+\eta_1}{2}$$
$$= -\frac{(p+1)k}{2^{p+k+2}}\ (1+\eta_1)P_{p-1}^{(1,1)}(\eta_1)P_k^{(0,0)}(\eta_1)\ (1-\eta_2)^{p+k-1}P_q^{(2p+1,0)}(\eta_2)P_l^{(2k+1,0)}(\eta_2)$$
$$+ \frac{(p+1)(2k+l+2)}{2^{p+k+3}}\ (1+\eta_1)P_{p-1}^{(1,1)}(\eta_1)P_k^{(0,0)}(\eta_1)\ (1-\eta_2)^{p+k}P_q^{(2p+1,0)}(\eta_2)P_{l-1}^{(2k+2,1)}(\eta_2)$$

$$\partial_{\eta_2}\widetilde{\psi}_{pq}\partial_{\eta_2}\widetilde{\psi}_{kl}\frac{1-\eta_2}{2}$$
$$= \frac{p\,k}{2^{p+k+1}}\ P_p^{(0,0)}(\eta_1)P_k^{(0,0)}(\eta_1)\ (1-\eta_2)^{p+k-1}P_q^{(2p+1,0)}(\eta_2)P_l^{(2k+1,0)}(\eta_2)$$
$$- \frac{p(2k+l+2)}{2^{p+k+2}}\ P_p^{(0,0)}(\eta_1)P_k^{(0,0)}(\eta_1)\ (1-\eta_2)^{p+k}P_q^{(2p+1,0)}(\eta_2)P_{l-1}^{(2k+2,1)}(\eta_2)$$
$$- \frac{(2p+q+2)k}{2^{p+k+2}}\ P_p^{(0,0)}(\eta_1)P_k^{(0,0)}(\eta_1)\ (1-\eta_2)^{p+k}P_{q-1}^{(2p+2,1)}(\eta_2)P_l^{(2k+1,0)}(\eta_2)$$
$$+ \frac{(2p+q+2)(2k+l+2)}{2^{p+k+3}}\ P_p^{(0,0)}(\eta_1)P_k^{(0,0)}(\eta_1)\ (1-\eta_2)^{p+k+1}P_{q-1}^{(2p+2,1)}(\eta_2)P_{l-1}^{(2k+2,1)}(\eta_2).$$

Thus, we have a total of 9 addends in the integral of (3.1.12). Furthermore, we can see, all the terms on the right side of the equations do have coordinatewise product structure and are polynomials.

Defining

$$\begin{aligned}\widetilde{g}_1(\eta_1) &:= (4+(1+\eta_1)^2)P_{p-1}^{(1,1)}(\eta_1)P_{k-1}^{(1,1)}(\eta_1)\\ \widetilde{h}_1(\eta_2) &:= (1-\eta_2)^{p+k-1}P_q^{(2p+1,0)}(\eta_2)P_l^{(2k+1,0)}(\eta_2)\end{aligned}$$

the first equation, for instance, can be decribed as follows:

$$\partial_{\eta_1}\widetilde{\psi}_{pq}\partial_{\eta_1}\widetilde{\psi}_{kl}\frac{4+(1+\eta_1)^2}{2(1-\eta_2)} = \frac{(p+1)(k+1)}{2^{p+k+3}}\widetilde{g}_1(\eta_1)\widetilde{h}_1(\eta_2). \tag{3.1.13}$$

The other representations follow similarily.
To tackle the integral in (3.1.12), we will use two-dimensional Gauss-Legendre quadrature with $kN+1$ points for every addend. Let $\underline{w}$ be the vector of the weights and let $\underline{x} = \underline{y}$ be the vectors of the nodes on the x resp. y-axis. Using (3.1.13) the integral of the first addend in (3.1.12) can be realized through

$$C_m\ \underline{w}^T\left(\widetilde{g}_m(\underline{x})\widetilde{h}_m(\underline{y})^T\right)\underline{w} = C_m\sum_{i=0}^{kN}\sum_{j=0}^{kN} w_i w_j \widetilde{g}_k(x_i)\widetilde{h}_k(y_j), \quad m=1,\dots,9, \tag{3.1.14}$$

where $C_m$ is the constant in (3.1.13) which generally depends on $p, q, k$ and $l$. Proceeding analogously for the other addends, we arrive at

$$\|\nabla u\|^2_{L^2(T)} = \sum_{p,q,k,l=0}^{kN}\left(c_{pq}c_{kl}u_{pq}u_{kl}\sum_{m=1}^{9} C_m\ \underline{w}^T\left(\widetilde{g}_m(\underline{x})\widetilde{h}_m(\underline{y})^T\right)\underline{w}\right). \tag{3.1.15}$$

In particular, we have

$$\|\nabla u\|^2_{L^2(T)} = \underline{u}^T H\underline{u}, \tag{3.1.16}$$

where the entries of the matrix $H$ are given by

$$H_{I_{pq}I_{kl}} = c_{pq}c_{kl}\sum_{m=1}^{9} C_m\ \underline{w}^T\left(\widetilde{g}_m(\underline{x})\widetilde{h}_m(\underline{y})^T\right)\underline{w}.$$

We can now combine all the matrix representations (3.1.8), (3.1.10) and (3.1.16) and recall the Lagrange function $\mathcal{L}(u,\lambda)$ to obtain

$$\mathcal{L}(\underline{u},\lambda) = \underline{u}^T T\underline{u} - \lambda\left(\sqrt{(\underline{u}^T M\underline{u})(\underline{u}^T(H+M)\underline{u})} - 1\right). \tag{3.1.17}$$

To compute the stationary points of the Lagrangian $\mathcal{L}$, we have to differentiate the function with respect to $\lambda$ and $\underline{u}$ and solve $\nabla_{\underline{u},\lambda}\mathcal{L} = 0$.

**Lemma 3.2** *Let $A \in \mathbb{R}^{n\times n}$ be symmetric and $\underline{u} \in \mathbb{R}^n$. Define the function*

$$f : \begin{cases} \mathbb{R}^n \to \mathbb{R} \\ \underline{u} \mapsto \underline{u}^T A \underline{u}. \end{cases}$$

*Then we have*

$$\nabla_{\underline{u}} f(\underline{u}) = 2\,\underline{u}^T A. \tag{3.1.18}$$

*Proof.* We have

$$\frac{\partial f}{\partial u_k}(\underline{u}) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial u_k}(u_i u_j A_{ij}) = \sum_{j=1}^n u_j A_{kj} + \sum_{j=1}^n u_j A_{jk} = 2 \sum_{j=1}^n u_j A_{jk}.$$

According to the definition of the gradient (3.1.18) is obvious.

□

Considering the symmetry of $T$,$H$ and $M$ and defining $\widetilde{H} := H + M$, Lemma 3.2 yields

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda}(\underline{u}, \lambda) &= \sqrt{(\underline{u}^T M \underline{u})(\underline{u}^T (\widetilde{H}) \underline{u})} - 1 \\ \nabla_{\underline{u}} \mathcal{L}(\underline{u}, \lambda) &= 2\,\underline{u}^T T - \lambda \left( \frac{2\,\underline{u}^T M (\underline{u}^T \widetilde{H} \underline{u}) + 2\,\underline{u}^T \widetilde{H} (\underline{u}^T M \underline{u})}{2\sqrt{(\underline{u}^T M \underline{u})(\underline{u}^T \widetilde{H} \underline{u})}} \right) \\ &= 2\,\underline{u}^T T - \lambda \left( \sqrt{\frac{\underline{u}^T \widetilde{H} \underline{u}}{\underline{u}^T M \underline{u}}}\,\underline{u}^T M + \sqrt{\frac{\underline{u}^T M \underline{u}}{\underline{u}^T \widetilde{H} \underline{u}}}\,\underline{u}^T \widetilde{H} \right) \end{aligned}$$

To solve the system of equations $\nabla_{\underline{u},\lambda} \mathcal{L} = 0$ we apply the `MATLAB` function `fsolve`. `fsolve` requires an initial guess $\begin{pmatrix} \underline{u}_0 \\ \lambda_0 \end{pmatrix}$ that should be sufficiently close to a stationary point of the Lagrangian $\mathcal{L}$. Our basic strategy is:

1. Use the solution for polynomial degree $N - 1$ as the initial guess for the case of polynomial degree $N$.
2. For small values of $N$ (e.g. $N = 1$ or $N = 2$) use the solution of the following maximization problem as the initial guess:

   $$\sup_{u \in \mathcal{P}_{kN}} \frac{\|\Pi_N u\|^2_{L^2(\Gamma)}}{\|u\|^2_{H^1(T)}} = \lambda. \tag{3.1.19}$$

   This maximization problem can be solved with `Matlab` since it can be recast as an eigenvalue problem. Let

   $$f(\underline{u}) := \|\Pi_N u\|^2_{L^2(\Gamma)} - \lambda \|u\|^2_{H^1(T)}.$$

>From (3.1.10),(3.1.16) and Lemma 3.2 we get

$$
\begin{aligned}
f(\underline{u}) &= \underline{u}^T T \underline{u} - \lambda\, \underline{u}^T \widetilde{H} \underline{u} \\
\nabla_{\underline{u}} f(\underline{u}) &= 2\, \underline{u}^T T - 2\lambda\, \underline{u}^T \widetilde{H}.
\end{aligned}
$$

To search for maxima of $f$ we set $\nabla_{\underline{u}} f = 0$. Rearranging terms and considering the symmetry of $T$ and $\widetilde{H}$ gives the generalized eigenvalue problem

$$T\underline{u} = \lambda\, \widetilde{H} \underline{u},$$

which can be solved with the `MATLAB` function `eig`. The maximum eigenvalue and the corresponding eigenvector then give the starting vector as required.

### 3.1.1 Pseudocode

For a better understanding of the algorithm described above we will write it down in pseudocode. Algorithm 3.3 needs as input the current polynomial degree N, the starting vector $\underline{x}_0 = (\underline{u}_0, \lambda_0)$ and the factor k that is multiplied with N. For the initial polynomial degree we will set $u_0 = 0$. The output is the stability constant $\lambda$ that satisfies

$$\sup_{u \in \mathcal{P}_{kN}} \frac{\|\Pi_N u\|^2_{L^2(\Gamma)}}{\|u\|_{L^2(T)} \|u\|_{H^1(T)}} = \lambda.$$

**Algorithm 3.3** ( l2proj2D_lagrange)

$h = kN, \quad T, M, H = 0 \in \mathbb{R}^{\frac{1}{2}(h+1)(h+2)}$

generate matrix I as defined in (3.1.6)

generate Gauss-Legendre nodes and weights $\underline{x}, \underline{y}, \underline{w} \in \mathbb{R}^{h+1}$

for $p \in \{0, \dots, h\}$ do {

    for $q \in \{0, \dots, h-p\}$ do {

        for $k \in \{0, \dots, h\}$ do {

            for $l \in \{0, \dots, h-k\}$ do {

                if $(p+q \le N)$ & $(k+l \le N)$ {

                    $T_{I_{pq}I_{kl}} = c_{pq}c_{kl}(-1)^{q+l}\frac{2}{2p+1}\delta_{pk}$

                }

                $M_{I_{pq}I_{kl}} = c_{pq}c_{kl}\delta_{pk}\delta_{ql}\frac{2}{2p+1}\frac{1}{p+q+1}$

$$H_{I_{pq}I_{kl}} = c_{pq}c_{kl} \sum_{m=1}^{9} C_m \; \underline{w}^T \left( \widetilde{g}_m(\underline{x}) \widetilde{h}_m(\underline{y})^T \right) \underline{w}$$

$\}$

$\}$

$\}$

$\}$

$H = H + M$

if $\underline{u}_0 = 0$ do $\{$

solve $T\underline{u} = \lambda\, H\underline{u}$ with Matlab function eig $\big($ i.e. $\big[(\mu_i)_i, (\underline{v}_i)_i\big] = \mathrm{eig}(T,\, H)$ $\big)$

set $\lambda_0 = \mu_{i_0} = \max_i \mu_i$ and $\underline{u}_0 = \underline{v}_{i_0}$

$\}$

$\underline{x}_0 = (\underline{u}_0, \lambda_0)$

solve $\nabla_{\underline{u},\lambda}\mathcal{L} = 0$ with Matlab function fsolve with initial value $\underline{x}_0$ $\big($ i.e. $\underline{x} = \mathrm{fsolve}(\nabla_{\underline{u},\lambda}\mathcal{L},\, \underline{x}_0)$ $\big)$

$\lambda = \underline{x}_{\,end}$

## 3.2 Computational Results

In this section we will show our computational results regarding the computed stability constants of several similar one-dimensional and two-dimensional bounds.

First of all we present the results to our main estimate from Theorem 2.17

$$\|\Pi_N u\|^2_{L^2(\Gamma)} \le C\|u\|_{L^2(T)}\|u\|_{H^1(T)}, \tag{3.2.1}$$

which we discretized to the maximization problem in (3.1.1). Alongside we also show the results to estimate $C$ in the following bound:

$$\|\Pi_N u\|^2_{L^2(\Gamma)} \le C\|u\|^2_{H^1(T)}. \tag{3.2.2}$$

Both results are collected in Table 1. In the numerical calculations, we have taken the supremum over all $u \in \mathcal{P}_{2N+5}$ instead of simply using $u \in \mathcal{P}_{2N}$. We have done this in order better approximations for low polynomial degrees. In the left coloumn of Table 1 for instance, this modification produced an improvement of approximately 0.05 for $N = 1$ and 0.002 for $N = 2$.

| | $\sup_{u\in\mathcal{P}_{2N+5}} \frac{\|\Pi_N u\|^2_{L^2(\Gamma)}}{\|u\|_{L^2(T)}\|u\|_{H^1(T)}}$ | $\sup_{u\in\mathcal{P}_{2N+5}} \frac{\|\Pi_N u\|^2_{L^2(\Gamma)}}{\|u\|^2_{H^1(T)}}$ |
|---|---|---|
| $N$ | $C$ | $C$ |
| 1 | 1.89127927875177 | 1.51217158397383 |
| 2 | 2.48492724094763 | 1.70653711817479 |
| 3 | 2.84078999271727 | 1.71576980258930 |
| 4 | 3.06972913803173 | 1.69882999609302 |
| 5 | 3.22148699691258 | 1.68145886970881 |
| 6 | 3.32827973990691 | 1.66837798998617 |
| 7 | 3.40793969705832 | 1.65851935763366 |
| 8 | 3.47015236302314 | 1.65088391474287 |
| 9 | 3.52039680138236 | 1.64480277602405 |
| 10 | 3.56198608661588 | 1.63984713663359 |
| 11 | 3.59705680658713 | 1.63573210199244 |
| 12 | 3.62706330324132 | 1.63226133451629 |
| 13 | 3.65303935848581 | 1.62929514033592 |
| 14 | 3.67574554579823 | 1.62673141969948 |
| 15 | 3.69575709815906 | 1.62449382302297 |
| 16 | 3.71351893955889 | 1.62252412030053 |
| 17 | 3.72938166137221 | 1.62077713211213 |
| 18 | 3.74362586798005 | 1.61921727177212 |
| 19 | 3.75647917306558 | 1.61781613110706 |
| 20 | 3.76812835132416 | 1.61655075997341 |
| 21 | 3.77872823341705 | 1.61540241759010 |
| 22 | 3.78840832510274 | 1.61435565138111 |
| 23 | 3.79727782704123 | 1.61339760733066 |
| 24 | 3.80542948466692 | 1.61251750675473 |
| 25 | 3.81294258939904 | 1.61170624444197 |

Table 1: Computed constants $C$ for the 2-dimensional maximization problems above

The constants in the right column show a slightly unexpected behaviour since they rise in the first three polynomial degrees, but then decrease slowly. Still, this behaviour is underpinned by the corresponding one-dimensional results in Table 3.
In the left coloumn of Table 1 we see that the values for the stability constant in (3.2.1) show for higher polynomial degrees a slight convergence behaviour. Since Algorithm 3.3 has complexity $O(N^6)$ we did not go beyond polynomial degree $N = 25$, but it seems that the exact value for the constant lies somewhere around 4.
This is also underpinned by the following graphic (Fig. 3). In a loglog-plot we show the difference of our expected exact value 4 to the stability constants in the left coloumn of Tabel 1 as well as to extrapolated values of these constants denoted by `extrapol`. Furthermore, we assumed the leading order behavior to be $O(N^{-1})$, which is mapped as a green

line. However, we have to qualify the statement of the plot and be cautious when talking about convergence to 4 since the computational errors are still relatively high.

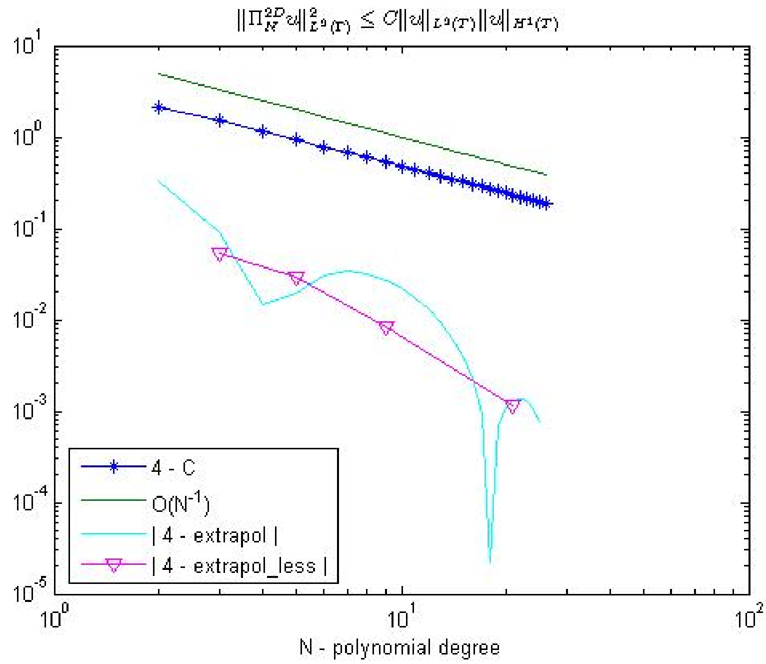


Figure 3: Convergence behaviour of extrapolation for results of Table 1.

The strange behaviour of the error curve for the extrapolated values (cyan line) at higher polynomial degrees is due to the fact that at $N = 17$ the extrapolated values get very close to the exact value 4 and finally exceed it. To counteract this behaviour we implemented the same extrapolation using less points instead of all the data in the left coloumn (magenta line). In particular, we have chosen the values belonging to $N = 2, 4, 8, 20, 25$. We denote the corresponding extrapolated values by `extrapol_less`. In this case, we observe a more regular behavior of the convergence graph.

Next we test the $H^1(T)$-stability of the $L^2$-projection. I.e. we consider the estimate

$$\|\Pi_N u\|^2_{H^1(T)} \leq C_N \|u\|^2_{H^1(T)}, \tag{3.2.3}$$

where $C_N$ will be seen to be $O(N)$. In this case, the computation of the norm $\|\Pi_N u\|^2_{H^1(T)}$ is very simple compared to $\|\Pi_N u\|^2_{L^2(\Gamma)}$ since we just have to cut off the $N \times N$-submatrix from the matrix representation of $\|u\|^2_{H^1(T)}$ and fill the rest of the matrix with zeros. This is due to the fact that from our expansion of $u$ (see (3.1.4))

$$u = \sum_{p,q=0}^{kN+5} c_{pq} u_{pq} \psi_{pq}$$

and from the properties of the $L^2$-projection we obtain

$$\Pi_N u = \sum_{p,q=0}^{N} c_{pq} u_{pq} \psi_{pq}.$$

Similar to the maximization problem in (3.1.19) the dicretization of the estimate (3.2.3) can be recast as an eigenvalue problem. Thus, we can compute the stability constants by applying the Matlab function `eig` as already mentioned in the section subsequent to equation (3.1.19).

| | $\sup_{u\in\mathcal{P}_{2N+5}} \frac{\|\Pi_N u\|^2_{H^1(T)}}{\|u\|^2_{H^1(T)}}$ | $\sup_{u\in\mathcal{P}_{2N+5}} \frac{\|\Pi_N u\|^2_{H^1(T)}}{\|u\|^2_{H^1(T)}(N+1)}$ |
|---|---|---|
| $N$ | $C_N$ | $C_N/(N+1)$ |
| 1 | 1.44342468965527 | 0.72171234482763 |
| 2 | 1.66384487682417 | 0.55461495894139 |
| 3 | 2.01676948808509 | 0.50419237202127 |
| 4 | 2.35066910857151 | 0.47013382171430 |
| 5 | 2.70952348807897 | 0.45158724801316 |
| 6 | 3.10052717355358 | 0.44293245336480 |
| 7 | 3.51997252826326 | 0.43999656603291 |
| 8 | 3.94549105887804 | 0.43838789543089 |
| 9 | 4.37344658206461 | 0.43734465820646 |
| 10 | 4.80329996183314 | 0.43666363289392 |
| 11 | 5.23408675671248 | 0.43617389639271 |
| 12 | 5.66544853736716 | 0.43580373364363 |
| 13 | 6.09706606518657 | 0.43550471894189 |
| 14 | 6.52879509572547 | 0.43525300638169 |
| 15 | 6.96053045736136 | 0.43503315358509 |
| 16 | 7.39223100383616 | 0.43483711787272 |
| 17 | 7.82387326013005 | 0.43465962556279 |
| 18 | 8.25545984396632 | 0.43449788652454 |
| 19 | 8.68699945511594 | 0.43434997275579 |
| 20 | 9.11851181498960 | 0.43421484833284 |
| 21 | 9.55001765514877 | 0.43409171159767 |
| 22 | 9.98154228840591 | 0.43398009949591 |
| 23 | 10.41311006906840 | 0.43387958621119 |
| 24 | 10.84474708933944 | 0.43378988357358 |
| 25 | 11.27647786452010 | 0.43371068709693 |

Table 2: Computed constants $C_N$ for 2-dimensional problem (3.2.3)

The computed values of $C_N$ are listed in the left column of Table 2. These values increase monotonically as $N$ increases.
The right column shows the results of the left one multiplied by a factor $1/(N+1)$. Since the computed values slowly decrease and are bounded by 1, we observe numerically the behaviour $C_N = O(N)$. We mention that for tensor product geometries such as intervals and squares the bound $C_N = O(N)$ has been rigorously established in [4]. These numerical experiments indicate that this result of [4] also holds for triangles.
For better illustration, we emphasize again the dependence on $N$ of $C_N$ in the graphic Fig. 4 below. The blue line simply shows the constants in the left coloumn of Table 2 plotted against $N$. For the green graph we did the following: Applying the Matlab function `polyfit` we tried to fit a first order polynomial $p$ to the data for $C_N$. `polyfit` then yields the polynomial coefficients $a_0$ and $a_1$ that satisfy $p(x) = a_1 x + a_0$. Since $a_1$ gives the slope of $p$ we plot $a_1 N$ against $N$ and see that this graph fits to the graph for the $C_N$. Hence, $a_1$ gives the constant $C$ with $C_N \sim C \cdot N$.

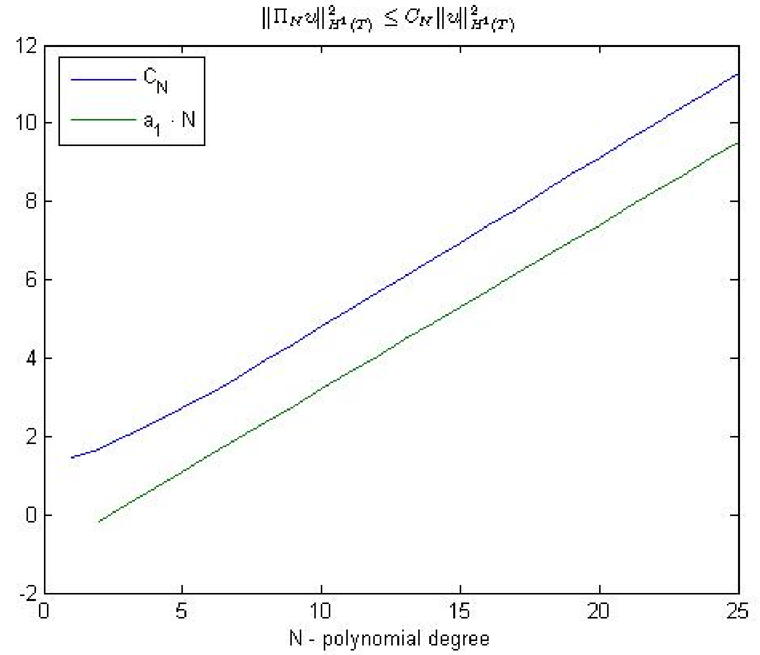


Figure 4: computed value of $C_N$ in (3.2.3).

At last we present one-dimensional results, corresponding to the two-dimensional computations above. The rather difficult computations on the boundary of $T$ do now, in the 1D setting, simplify to a simple evaluation of the 1D $L^2$-projection at the point $x = 1$. Since the 1D calculations have also much less complexity, we can actually go much further concerning the polynomial degree than in the two-dimensional case.

| | $\sup_{u\in\mathcal{P}_{2N+5}} \frac{|(\Pi_N u)(1)|^2}{\|u\|_{L^2}\|u\|_{H^1}}$ | $\sup_{u\in\mathcal{P}_{2N+5}} \frac{|(\Pi_N u)(1)|^2}{\|u\|_{H^1}^2}$ |
|---|---|---|
| $N$ | $C$ | $C$ |
| 1 | 1.23955678848166 | 0.87500000000000 |
| 2 | 1.83060420722495 | 1.14361283167718 |
| 3 | 2.15356301758515 | 1.15072048261852 |
| 4 | 2.34107634418767 | 1.13538600864567 |
| 5 | 2.45950440857071 | 1.11992788388317 |
| 10 | 2.72197669756981 | 1.08267283507986 |
| 15 | 2.82210388094874 | 1.06853381605478 |
| 20 | 2.87406416225027 | 1.06111064764886 |
| 25 | 2.90512455645280 | 1.05653834380496 |
| 30 | 2.92540310256414 | 1.05343954290018 |
| 35 | 2.93948150346736 | 1.05120092371494 |
| 40 | 2.94971129296227 | 1.04950802550192 |
| 45 | 2.95741139670208 | 1.04818299976127 |
| 50 | 2.96337279789142 | 1.04711769879646 |
| 55 | 2.96809547801163 | 1.04624257963827 |
| 60 | 2.97190920471166 | 1.04551089085115 |
| 65 | 2.97503926131727 | 1.04489004319026 |
| 70 | 2.97764417211457 | 1.04435662477469 |
| 75 | 2.97983833781377 | 1.04389338318215 |
| 80 | 2.98170613800169 | 1.04348732490767 |
| 85 | 2.98331100621829 | 1.04312847760604 |
| 90 | 2.98470143645123 | 1.04280906028005 |
| 95 | 2.98591505801295 | 1.04252291272595 |
| 100 | 2.98698146107878 | 1.04226509441461 |
| 105 | 2.98792419448650 | 1.04203159687688 |
| 110 | 2.98876220322098 | 1.04181913380901 |
| 115 | 2.98951087919072 | 1.04162498545355 |
| 120 | 2.99018284042289 | 1.04144688155757 |

Table 3: Computed constants $C$ for 1-dimensional maximization problems above

Recalling the two-dimensional results in Table 1, we do see again convergence in the left column, where here the exact value of the constant seems to be close to 3, as well as the already mentioned unexpected behaviour in the right coloumn.

Also in the one-dimensional case we want to show a graphic (Fig. 5) that indicates the convergence to suspected value 3. The plot is similar to the 2D case but we changed our assumption of the leading order behaviour to $O(N^{-3/2})$ since we realized that $O(N^{-1})$ does not fit properly. Furthermore, to obtain `extrapol_less` we extrapolated with values corresponding to $N = 2, 4, 10, 30, 60, 120$.

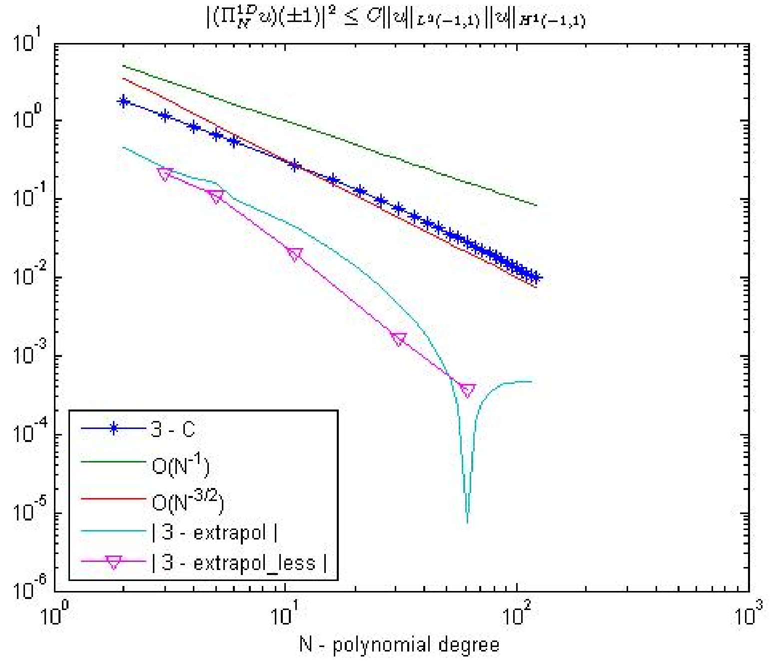


Figure 5: Convergence behaviour of extrapolation for results of Table 3.

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