ASC Report No. 35/2010

Adaptive Coupling of FEM and BEM: Simple Error Estimators and Convergence

Markus Aurada, Michael Feischl, Michael Karkulik, Dirk Praetorius

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-03-2

Most recent ASC Reports

34/2010 *Michael Feischl, Marcus Page, Dirk Praetorius*
Convergence and Quasi-Optimality of Adaptive FEM with Inhomogeneous Dirichlet Data

33/2010 *Michael Feischl, Marcus Page, Dirk Praetorius*
Convergence of Adaptive FEM for some Elliptic Obstacle Problem with Inhomogeneous Dirichlet Data

32/2010 *Othmar Koch, Friedrich Kupka, Bernhard Löw-Baselli, A. Mayrhofer, F. Zaussinger*
SDIRK Methods for the ANTARES Code

31/2010 *Roland Donninger, Birgit Schörkhuber, Peter C. Aichelburg*
On Stable Self-similar Blow Up for Equivalent Wave Maps: The Linearized Problem

30/2010 *Matthias Langer, Harald Woracek*
The Exponential Type of the Fundamental Solution of an Indefinite Hamiltonian System

29/2010 *Harald Woracek*
An Addendum to M.G. Krein's Inverse Spectral Theorem for Strings

28/2010 *Philipp Dörsek, Josef Teichmann*
A Semigroup Point of View On Splitting Schemes For Stochastic (Partial) Differential Equations

27/2010 *Ansgar Jüngel*
Dissipative quantum fluid models

26/2010 *Sabine Hittmeir, Ansgar Jüngel*
Cross Diffusion Preventing Blow Up in the Two-dimensional Keller-Segel Model

25/2010 *Anton Arnold, Irene M. Gamba, Maria Pia Gualdani, Stéphane Mischler, Clément Mouhot, Christof Sparber*
The Wigner-Fokker-Planck Equation: Stationary States and Large Time Behavior

Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

ISBN 978-3-902627-03-2

ADAPTIVE COUPLING OF FEM AND BEM: SIMPLE ERROR ESTIMATORS AND CONVERGENCE

MARKUS AURADA, MICHAEL FEISCHL, MICHAEL KARKULIK, AND DIRK PRAETORIUS

ABSTRACT. A posteriori error estimators and adaptive mesh-refinement have themselves proven to be important tools for scientific computing. For error control in finite element methods (FEM), there is a broad variety of a posteriori error estimators available, and convergence as well as optimality of adaptive FEM is well-studied in the literature. This is, however, in sharp contrast to the boundary element method (BEM) and to the coupling of FEM and BEM. In our contribution, we present an easy-to-implement error estimator for some FEM-BEM coupling which, to the best of our knowledge, has not been proposed in the literature before. The derived mesh-refining algorithm provides the first adaptive coupling procedure which is mathematically proven to converge.

1. SYMMETRIC FEM-BEM COUPLING

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ with $\Gamma = \partial\Omega$, we consider the nonlinear interface problem

$$(1)\qquad \begin{cases} -\mathrm{div}(A\nabla u^{\mathrm{int}}) = f & \text{in } \Omega^{\mathrm{int}} := \Omega, \\ -\Delta u^{\mathrm{ext}} = 0 & \text{in } \Omega^{\mathrm{ext}} := \mathbb{R}^2 \backslash \overline{\Omega}, \\ u^{\mathrm{int}} - u^{\mathrm{ext}} = u_0 & \text{on } \Gamma, \\ (A\nabla u^{\mathrm{int}} - \nabla u^{\mathrm{ext}}) \cdot n = \phi_0 & \text{on } \Gamma, \\ u^{\mathrm{ext}}(x) = a \log|x| + \mathcal{O}(1/|x|) & \text{as } |x| \to \infty, \end{cases}$$

where n denotes the outer unit normal vector. The given data satisfy $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, and $\phi_0 \in H^{-1/2}(\Gamma)$, and the (possibly nonlinear) operator $A : L^2(\Omega)^2 \to L^2(\Omega)^2$ is assumed to be strongly monotone and Lipschitz continuous.

Problem (1) is equivalently stated via the well-known symmetric FEM-BEM coupling: Find $(u, \phi) \in \mathcal{H} := H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that, for all $(v, \psi) \in \mathcal{H}$,

$$(2)\qquad \begin{cases} \langle A\nabla u\,,\, \nabla v\rangle_\Omega + \langle Wu + (K' - \frac{1}{2})\phi\,,\, v\rangle_\Gamma = \langle f\,,\, v\rangle_\Omega + \langle \phi_0 + Wu_0\,,\, v\rangle_\Gamma, \\ \langle \psi\,,\, V\phi - (K - \frac{1}{2})u\rangle_\Gamma = -\langle \psi\,,\, (K - \frac{1}{2})u_0\rangle_\Gamma. \end{cases}$$

Here, V denotes the simple-layer potential, K denotes the double-layer potential with adjoint K', and W denotes the hypersingular integral operator. With the fundamental solution $G(z) = -\frac{1}{2\pi}\log|z|$ of the 2D Laplacian, these boundary integral operators formally read, for $x \in \Gamma$,

$$(3)\qquad (V\psi)(x) = \int_\Gamma G(x-y)\,\psi(y)\,d\Gamma(y),$$

$$(4)\qquad (Kv)(x) = \int_\Gamma \partial_{n(y)}G(x-y)\,v(y)\,d\Gamma(y),$$

$$(5)\qquad (Wv)(x) = -\partial_{n(x)}\int_\Gamma \partial_{n(y)}G(x-y)\,v(y)\,d\Gamma(y).$$

Then, (2) has a unique solution (u, ϕ) which depends continuously on the given data, see e.g. [4]. Moreover, (1) and (2) are linked through $(u, \phi) = (u^{\mathrm{int}}, \partial_n u^{\mathrm{ext}})$ and $u^{\mathrm{ext}} = K(u - u_0) - V\phi$.

Key words and phrases. finite element method (FEM), boundary element method (BEM), FEM-BEM coupling, a posteriori error control, adaptive mesh-refining algorithm, convergence.

2. Galerkin Discretization

For the Galerkin discretization, let $\mathcal{T}_\ell$ be a regular triangulation of Ω into triangles $T_j \in \mathcal{T}_\ell$ and $\mathcal{E}_\ell = \mathcal{T}_\ell|_\Gamma$ be the induced partition of the coupling boundary Γ into piecewise affine line segments $E_j \in \mathcal{E}_\ell$. We then use P1-finite elements $u_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ to approximate u and piecewise constants $\phi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell)$ to approximate ϕ, i.e. the discrete space is defined by $\mathcal{X}_\ell := \mathcal{S}^1(\mathcal{T}_\ell) \times \mathcal{P}^0(\mathcal{E}_\ell) \subset \mathcal{H}$. Now, the Galerkin formulation reads: Find $(u_\ell, \phi_\ell) \in \mathcal{X}_\ell$ such that, for all $(v_\ell, \psi_\ell) \in \mathcal{X}_\ell$,

$$\left\{ \begin{array}{rcl} \langle A\nabla u_\ell\,,\,\nabla v_\ell\rangle_\Omega + \langle W u_\ell + (K' - \frac{1}{2})\phi_\ell\,,\,v_\ell\rangle_\Gamma &=& \langle f\,,\,v_\ell\rangle_\Omega + \langle \phi_0 + W u_0\,,\,v_\ell\rangle_\Gamma, \\ \langle \psi_\ell\,,\,V\phi_\ell - (K - \frac{1}{2})u_\ell\rangle_\Gamma &=& -\langle \psi_\ell\,,\,(K - \frac{1}{2})u_0\rangle_\Gamma. \end{array} \right. \tag{6}$$

Again, we refer to [4] for the fact that the discretization (6) has a unique solution $(u_\ell, \phi_\ell) \in \mathcal{X}_\ell$.

3. A Posteriori Error Control

For a posteriori error estimation, we employ the general concept of $h - h/2$ error estimation: We solve the discrete system (6) twice to obtain Galerkin solutions $(u_\ell, \phi_\ell) \in \mathcal{X}_\ell$ and $(\widehat{u}_\ell, \widehat{\phi}_\ell) \in \widehat{\mathcal{X}}_\ell$, where the enriched space $\widehat{\mathcal{X}}_\ell$ is induced by the uniform refinement $\widehat{\mathcal{T}}_\ell$ of $\mathcal{T}_\ell$ and $\widehat{\mathcal{E}}_\ell = \widehat{\mathcal{T}}_\ell|_\Gamma$. With

$$\eta_\ell = \|(\widehat{u}_\ell, \widehat{\phi}_\ell) - (u_\ell, \phi_\ell)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)}, \tag{7}$$

we observe that, up to some multiplicative constant, we always obtain a lower bound for the error

$$\eta_\ell \lesssim \|(u, \phi) - (u_\ell, \phi_\ell)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)}. \tag{8}$$

Moreover, the converse inequality $\gtrsim$ holds under a saturation assumption

$$\|(u, \phi) - (\widehat{u}_\ell, \widehat{\phi}_\ell)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} \le q\,\|(u, \phi) - (u_\ell, \phi_\ell)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} \tag{9}$$

with some uniform constant $0 < q < 1$. Note that (9) essentially states that the Galerkin scheme has reached some asymptotic regime, i.e. $\|(u, \phi) - (u_\ell, \phi_\ell)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} = \mathcal{O}(h^\alpha)$.

Having defined η_ℓ in (7), we stress that, first, the $H^{-1/2}$-norm can hardly be computed and, second, (u_ℓ, ϕ_ℓ) is hardly ever used in practice since $(\widehat{u}_\ell, \widehat{\phi}_\ell)$ is a better approximation. The remedy for both objectives is given by the $(h - h/2)$-type error estimator

$$\mu_\ell^2 = \|\nabla(\widehat{u}_\ell - I_\ell \widehat{u}_\ell)\|_{L^2(\Omega)}^2 + \|h_\ell^{1/2}(\widehat{\phi}_\ell - \Pi_\ell \widehat{\phi}_\ell)\|_{L^2(\Gamma)}^2, \tag{10}$$

which, up to multiplicative constants, coincides with η_ℓ. Here, $h_\ell|_E = \mathrm{diam}(E)$ is the local mesh-width of $\mathcal{E}_\ell$. Moreover, $I_\ell \widehat{\phi}_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ is the nodal interpolant, and $\Pi_\ell \widehat{\phi}_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$ is the piecewise integral mean, i.e. having computing the improved Galerkin solution $(\widehat{u}_\ell, \widehat{\phi}_\ell)$ it is an elementary and easy-to-implement postprocessing step to compute μ_ℓ.

4. Convergent Adaptive Coupling

For triangles $T \in \mathcal{T}_\ell$ and boundary edges $E \in \mathcal{E}_\ell$, we define

$$\mu_\ell(T) = \|\nabla(\widehat{u}_\ell - I_\ell \widehat{u}_\ell)\|_{L^2(T)} \quad \text{and} \quad \mu_\ell(E) = \mathrm{diam}(E)^{1/2}\|\widehat{\phi}_\ell - \Pi_\ell \widehat{\phi}_\ell\|_{L^2(E)}. \tag{11}$$

Based on these local contributions of μ_ℓ and given some fixed parameter $0 < \theta < 1$ as well as an initial mesh $\mathcal{T}_0$, the usual adaptive algorithm reads as follows:

(i) Refine $\mathcal{T}_\ell$ and $\mathcal{E}_\ell = \mathcal{T}_\ell|_\Gamma$ uniformly to obtain $\widehat{\mathcal{T}}_\ell$ and $\widehat{\mathcal{E}}_\ell = \widehat{\mathcal{T}}_\ell|_\Gamma$.
(ii) Compute Galerkin solution $(\widehat{u}_\ell, \widehat{\phi}_\ell) \in \widehat{\mathcal{X}}_\ell = \mathcal{S}^1(\widehat{\mathcal{T}}_\ell) \times \mathcal{P}^0(\widehat{\mathcal{E}}_\ell)$.
(iii) Find minimal set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \cup \mathcal{E}_\ell$ such that

$$\theta \sum_{\tau \in \mathcal{T}_\ell \cup \mathcal{E}_\ell} \mu_\ell(\tau)^2 \le \sum_{\tau \in \mathcal{M}_\ell} \mu_\ell(\tau)^2. \tag{12}$$

(iv) Refine at least marked elements $T \in \mathcal{T}_\ell \cap \mathcal{M}_\ell$ and edges $E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$.
(v) Increase counter $\ell \mapsto \ell + 1$ and iterate.

In the context of FEM, convergence of such an algorithm has first been proven by [6]. Even optimality is nowadays understood for linear problems [5]. For BEM, convergence of this algorithm has recently been shown by [8]. For the adaptive coupling, the following result from our work [2] is the first convergence result available: One can prove that the adaptive algorithm guarantees

$$\lim_{\ell\to\infty} \mu_\ell = 0, \quad \text{whence} \quad \lim_{\ell\to\infty} (\widehat{u}_\ell, \widehat{\phi}_\ell) = (u, \phi) = \lim_{\ell\to\infty} (u_\ell, \phi_\ell), \tag{13}$$

where only the second convergence hings on the saturation assumption (9). Our proof follows the concept of estimator reduction proposed in [3]: Based on bisection of marked edges and triangles and based on use of the marking strategy (12), one first shows for all $\delta > 0$ and all $\ell \in \mathbb{N}$

$$\mu_{\ell+1}^2 \le (1+\delta)(1-\theta/2)\,\mu_\ell + C\,(1+\delta^{-1})\,\|(\widehat{u}_{\ell+1}, \widehat{\phi}_{\ell+1}) - (\widehat{u}_\ell, \widehat{\phi}_\ell)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)}. \tag{14}$$

Second, one uses the quasioptimality of Galerkin schemes to see that $(\widehat{u}_\ell, \widehat{\phi}_\ell)$ is a Cauchy sequence. With $\delta > 0$ sufficiently small, (14) may be written in Landau small-$\mathcal{O}$-notation

$$\mu_{\ell+1}^2 \le \kappa\,\mu_\ell^2 + \mathcal{O}(1) \tag{15}$$

with $0 < \kappa < 1$. From this, elementary calculus concludes the first limit in (13).

5. Approximation of Given Data

The implementation of the Galerkin scheme (2) involves the terms Wu_0 and Ku_0 which, in general, can hardly be computed analytically. Instead, we assume additional regularity $u_0 \in H^1(\Gamma) \subset C(\Gamma)$ and replace u_0 by its nodal interpolant $u_{0\ell} = I_\ell u_0 \in \mathcal{S}^1(\mathcal{E}_\ell)$. Note that now all occurring integral operators act only on discrete functions and thus correspond to matrices which allow compression schemes like the fast multipole method or hierarchical matrices.

Approximation of u_0 provides a perturbed Galerkin solution $(\widetilde{u}_\ell, \widetilde{\phi}_\ell) \in \mathcal{X}_\ell$ and a perturbed error estimator $\widetilde{\mu}_\ell$. For the overall error estimation, this only leads to an additional data oscillation term

$$\mathrm{osc}_\ell = \|h_\ell^{1/2}(u_0 - u_{0\ell})'\|_{L^2(\Gamma)} \tag{16}$$

in the definition of the error estimator. More precisely, we have

$$\varrho_\ell^2 = \widetilde{\mu}_\ell^2 + \mathrm{osc}_\ell^2 \lesssim \|(u,\phi) - (\widetilde{u}_\ell, \widetilde{\phi}_\ell)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)}^2 + \mathrm{osc}_\ell^2 \lesssim \varrho_\ell^2, \tag{17}$$

where the upper bound holds under the saturation assumption (9) for the non-perturbed problem. Using the local contributions of ϱ_ℓ for marking in the adaptive algorithm, there still holds

$$\lim_{\ell\to\infty} \varrho_\ell = 0, \quad \text{whence} \quad \lim_{\ell\to\infty} (\widehat{\widetilde{u}}_\ell, \widehat{\widetilde{\phi}}_\ell) = (u, \phi) = \lim_{\ell\to\infty} (\widetilde{u}_\ell, \widetilde{\phi}_\ell), \tag{18}$$

where again only the second convergence hings on the saturation assumption (9). An analogous approach can be used for including the approximation of ϕ_0 by means of local averaging $\phi_{0\ell} = \Pi_\ell \phi_0$. However, there is no necessity here, since no integral operator is applied to ϕ_0.

6. Conclusions

The $h - h/2$ error estimation strategy is a very basic and natural strategy to derive a posteriori error estimators. According to quasioptimality of Galerkin schemes, $(h - h/2)$-based error estimators provide always a lower bound for the unknown error. The upper bound hings on a saturation assumption like (9). Unlike FEM, where the saturation assumption has been verified by [7] for linear model problems, it remains mathematically open for BEM and the FEM-BEM coupling. It is, however, observed empirically in numerical experiments.

In our contribution, we propose a modification of the elementary $h - h/2$ error estimator which is capable to steer an adaptive algorithm. Moreover, the proposed algorithm guarantees estimator convergence (13). Numerical experiments in [2] show that our adaptive algorithm empirically leads to optimal order of convergence with respect to the degrees of freedom. Moreover, if an error accuracy is prescribed, the introduced strategy is more effective than uniform mesh-refinement with respect to computational time and storage requirements.

Finally, the analysis and the adaptive algorithm carry over to other versions of the FEM-BEM coupling as long as Galerkin solutions are known to be quasioptimal [1].

REFERENCES

[1] M. Aurada, M. Feischl, M. Karkulik, D. Praetorius, *A posteriori error estimates for the Johnson-Nédélec FEM-BEM coupling*, work in progress, 2010.

[2] M. Aurada, M. Feischl, D. Praetorius, *Convergence of some adaptive FEM-BEM coupling*, submitted for publication, 2010.

[3] M. Aurada, S. Ferraz-Leite, D. Praetorius, *Estimator reduction and convergence of adaptive BEM*, submitted for publication, 2010.

[4] C. Carstensen and E. Stephan, *Adaptive coupling of boundary elements and finite elements*, RAIRO Modél. Math. Anal. Numér., 24, 779-817, 1995.

[5] J. Cascon, C. Kreuzer, R. Nochetto, and K. Siebert, *Quasi-optimal convergence rate for an adaptive finite element method*, SIAM J. Numer. Anal., 46, 2524-2550, 2008.

[6] W. Dörfler, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal., 33, 1106-1124, 1996.

[7] W. Dörfler and R. Nochetto, *Small data oscillation implies the saturation assumption*, *Numer. Math.*, 91, 1-12, 2002.

[8] S. Ferraz-Leite, C. Ortner, and D. Praetorius, *Convergence of simple adaptive Galerkin schemes based on $h - h/2$ error estimators*, Numer. Math., 116, 291-316, 2010.

VIENNA UNIVERSITY OF TECHNOLOGY, INSTITUTE FOR ANALYSIS AND SCIENTIFIC COMPUTING, WIEDNER HAUPTSTR. 8–10 / E101, 1040 WIEN, AUSTRIA

E-mail address: {Markus.Aurada, Michael.Feischl, Michael Karkulik}@tuwien.ac.at

E-mail address: Dirk.Praetorius@tuwien.ac.at (corresponding author)