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Saddle-node bifurcation of viscous profiles

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Abstract

Traveling wave solutions of viscous conservation laws, that are associated to Lax shocks of the inviscid equation, have generically a transversal viscous profile. In the case of a non-transversal viscous profile we show by using Melnikov theory that a parametrized perturbation of the profile equation leads generically to a saddle-node bifurcation of these solutions. An example of this bifurcation in the context of magnetohydrodynamics is given. The spectral stability of the traveling waves generated in the saddle-node bifurcation is studied via an Evans function approach. It is shown that generically one real eigenvalue of the linearization of the viscous conservation law around the parametrized family of traveling waves changes its sign at the bifurcation point. Hence this bifurcation describes the basic mechanism of a stable traveling wave which becomes unstable in a saddle-node bifurcation.

Keywords:
viscous conservation law, traveling wave, bifurcation, spectral stability, Evans function
2010 MSC: 34C37, 34D15, 35L65, 76W05

1. Introduction

We consider viscous conservation laws in one space dimension, which are partial differential equations (PDEs) of the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with a spatial variable $x \in \mathbb{R}$ and a time variable $t \in \mathbb{R}_+^0$. The unknown function $u(x,t)$ takes its values in an open convex set $U \subseteq \mathbb{R}^n$ and the given non-linear flux function $f : U \to \mathbb{R}^n$ is smooth. We assume that the inviscid system

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \tag{2}$$

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is hyperbolic, i.e. the Jacobian matrix of the flux function, $\frac{df}{du}(u)$, is diagonalizable with real eigenvalues, $\lambda_1(u) \leq \ldots \leq \lambda_n(u)$, for all $u \in U$.

A viscous shock wave is a traveling wave solution of (1),

$$u(x,t) := \bar{u}(\xi) \quad \text{with} \quad \xi := x - s \cdot t,$$

whose viscous profile $\bar{u}(\xi)$ travels with speed $s \in \mathbb{R}$ and approaches distinct constant endstates, $\lim_{\xi \to \pm\infty} \bar{u}(\xi) = u^{\pm}$. The viscous profile $\bar{u}(\xi)$ is governed by the autonomous system of ordinary differential equations (ODEs),

$$\frac{du}{d\xi} = f(u) - su - c =: F(u), \tag{3}$$

with a constant vector

$$c := f(u^+) - su^+ = f(u^-) - su^-. \tag{4}$$

Geometrically, a viscous profile corresponds to a heteroclinic orbit connecting the stationary points u^- and u^+. We assume that the endstates $u^{\pm}$ and the speed s correspond to a Lax k-shock of the inviscid conservation law (2), i.e. the Rankine-Hugoniot condition (4) and the Lax inequalities

$$\lambda_{k-1}(u^-) < s < \lambda_k(u^-) \tag{5}$$

and

$$\lambda_k(u^+) < s < \lambda_{k+1}(u^+) \tag{6}$$

hold. Therefore, the endstates $u^{\pm}$ are hyperbolic stationary points of the profile equation (3). The stable manifold theorem [1] implies the existence of an $n-k+1$-dimensional unstable manifold $W^u(u^-)$ and a k-dimensional stable manifold $W^s(u^+)$. Thus a viscous profile $\bar{u}(\xi)$ lies in the intersection of the unstable manifold $W^u(u^-)$ and the stable manifold $W^s(u^+)$.

The viscous profile $\bar{u}(\xi)$ is called *transversal*, if for all points p of its orbit $\{\bar{u}(\xi) \mid \xi \in \mathbb{R}\}$ the union of the tangent spaces $T_pW^u(u^-) \cup T_pW^s(u^+)$ is equal to $\mathbb{R}^n$. Generically, a viscous profile associated to a Lax k-shock is transversal and hence is persistent under small perturbations of the profile equation (3).

In this work, we will mainly consider non-transversal viscous profiles and their behavior under perturbations. In Section 2, we use Melnikov theory to analyze the existence and bifurcation of their heteroclinic orbits under perturbations of the profile equation. We verify that generically a saddle-node bifurcation of viscous profiles occurs. In Section 3, we present an example for this bifurcation scenario related to planar waves in magnetohydrodynamics (MHD).

Finally we study the spectral stability of viscous profiles generated in a saddle-node bifurcation. As usual the viscous conservation law is considered in a moving coordinate frame $(x,t) \mapsto (\xi := x - st, t)$, such that (1) becomes an evolutionary system

$$\frac{du}{dt} = \frac{d^2u}{d\xi^2} - \frac{d}{d\xi}f(u) + s\frac{du}{d\xi} =: \mathcal{F}(u) \tag{7}$$

and viscous profiles of (3) are stationary solutions of (7). It is a natural idea, to study the stability of a viscous shock wave $u(\xi,t)=\bar{u}(\xi)$ by analyzing the spectrum $\sigma(L)$ of the linearized operator,

$$Lp := \frac{d\mathcal{F}}{du}(\bar{u})p = \frac{d}{d\xi}\left(\frac{dp}{d\xi} - \frac{dF}{du}(\bar{u})p\right). \tag{8}$$

The linear operator L has an eigenvalue $\kappa \in \mathbb{C}$, if there exists an eigenfunction $p \neq 0$ such that $Lp(\xi) = \kappa p(\xi)$. In particular, zero is an eigenvalue of L with associated eigenfunction $\frac{d\bar{u}}{d\xi}(\xi)$.

Definition 1.1. Let l denote the dimension of the manifold of viscous profiles that connect the endstates $u^{\pm}$. A viscous shock wave $u(x,t)=\bar{u}(\xi)$ is *spectrally stable*, if the linear operator $L = \frac{d\mathcal{F}}{du}(\bar{u})$ has no spectrum in the closed right half-plane $\overline{\mathbb{C}_+}$ except for an eigenvalue zero with multiplicity l.

Remark. Zumbrun and collaborators [2, 3, 4] proved that a spectrally stable viscous shock wave is indeed non-linearly stable.

We distinguish between point spectrum $\sigma_p(L)$, that consists of all isolated eigenvalues with finite multiplicity, and the essential spectrum $\sigma_{ess}(L) = \sigma(L) \setminus \sigma_p(L)$, see [5]. Since the coefficients of the linear operator L approach constants as $\xi \to \pm\infty$, the essential spectrum lies in the left half-plane and is bounded to the right by a curve which touches the imaginary axis in the origin [6]. Thus the point spectrum will decide upon spectral stability of a viscous shock wave.

In Section 4, we present an Evans function approach to locate the point spectrum and give an alternative proof for the analytic continuation of the Evans function. In Section 5, we will use a parametrization of the family of viscous profiles and the associated family of Evans functions to locate the point spectrum in a small neighborhood of the origin. In the main result, we prove that generically a transcritical bifurcation will occur in the zero set equation of the Evans function. This allows to identify scenarios in which the perturbed viscous profiles will either be or not be spectrally stable in the sense of Definition 1.1. In Section 6, we will investigate the existence and multiplicity of eigenfunctions to the eigenvalue zero. In the Appendix A, we present the generalized cross product of vectors and will construct bounded solutions of a linear system of ordinary differential equations.

2. Bifurcation of viscous profiles

We study the bifurcation of viscous profiles as a system parameter varies. A natural parameter to look at is the shock speed s, however we allow more general perturbations by considering a smooth flux function $f(u,\mu)$ depending on a parameter $\mu \in \mathbb{R}$. The associated profile equation is

$$\frac{du}{d\xi} = f(u,\mu) - s\ u - c =: F(u,\mu). \tag{9}$$

In the following, we assume that

(A1) For $\mu = 0$, there exists a viscous shock wave $u(x,t,0) = \bar{u}(\xi,0)$ with shock speed s_0, distinct constant endstates $u^\pm$, and relative flux c_0.

(A2) The viscous shock wave is associated to a Lax k-shock, i.e. the Jacobians $\frac{df}{du}(u^\pm)$ have real eigenvalues $\lambda_j(u^\pm)$ with associated eigenvectors $r_j(u^\pm)$ for $j = 1,\dots,n$, which are given in increasing order of magnitude and satisfy the inequalities (5) and (6).

(A3) The viscous profile $\bar{u}(\xi,0)$ is non-transversal in the least degenerate way, i.e. for all points p on its orbit $\{\bar{u}(\xi,0) \mid \xi \in \mathbb{R}\}$ the tangent spaces of the stable and unstable manifolds, $W^s(u^-)$ and $W^u(u^+)$, satisfy,

$$\dim\left(T_pW^u(u^-) + T_pW^s(u^+)\right) = n-1, \tag{10}$$

or equivalently

$$\dim\left(T_pW^u(u^-) \cap T_pW^s(u^+)\right) = 2. \tag{11}$$

Remark 2.1. To simplify our notation, we will omit for $(\mu, s, c) = (0, s_0, c_0)$ the dependence on the parameters; for example, we will write $\bar{u}(\xi)$ instead of $\bar{u}(\xi,0)$, $u^\pm$ instead of $u^\pm(0)$, etc.

We analyze the existence and bifurcation of heteroclinic orbits of system (9) for (μ, s, c) close to $(0, s_0, c_0)$ via Melnikov theory as outlined in [7]. Let $p = \bar{u}(0)$ be a point on the unperturbed heteroclinic orbit and consider a $(n-1)$-dimensional space Y centered at p and orthogonal to the heteroclinic orbit. We decompose the space Y as

$$Y = U \oplus V_u \oplus V_s \oplus W,$$

with the one-dimensional space $U = Y \cap T_pW^u(u^-) \cap T_pW^s(u^+)$, the one-dimensional space W orthogonal to $T_pW^u(u^-) \cup T_pW^s(u^+)$ and the remaining spaces V^u and V^s within $Y \cap T_pW^u(u^-)$ and $Y \cap T_pW^s(u^+)$, respectively. Let $\phi(\xi)$ be a solution of the linearized profile equation

$$\frac{dz}{d\xi} = \frac{dF}{du}(\bar{u})z \tag{12}$$

with $\phi(0) \in U$ and let ν be a coordinate in the direction of U. It is well known that the orthogonal space W is spanned by $\psi(0)$, where $\psi(\xi)$ is the unique (up to a multiplicative constant) globally bounded solution of the linear differential equation,

$$\frac{d\psi}{d\xi} = -\left(\frac{dF}{du}(\bar{u})\right)^T \psi, \tag{13}$$

with $\psi(0) \in W$, see also Theorem Appendix A.3. The Melnikov function $M(\nu, \mu, s, c)$ measures the distance between $W^u(u^-) \cap Y$ and $W^s(u^+) \cap Y$ in the direction W. In particular, the Melnikov function $M(\nu, \mu, s, c)$ vanishes if and only if there exists a viscous profile for the parameter values (μ, s, c). The results in [7] imply the following theorem.

Theorem 2.2. *Suppose the assumptions (A1)–(A3) hold. Then a Melnikov function $M(\nu, \mu, s, c)$ exists, that is smooth in a small neighborhood of $(0, 0, s_0, c_0)$. Moreover, it satisfies the identities*

$$M(0, 0, s_0, c_0) = 0, \quad \frac{\partial M}{\partial \nu}(0, 0, s_0, c_0) = 0, \tag{14}$$

$$\frac{\partial M}{\partial \mu}(0, 0, s_0, c_0) = \int_{\mathbb{R}} < \psi, \frac{\partial F}{\partial \mu}(\bar{u}) > ds \tag{15}$$

as well as

$$\frac{\partial^2 M}{\partial \nu^2}(0, 0, s_0, c_0) = \int_{\mathbb{R}} < \psi, \frac{\partial^2 F}{\partial u^2}(\bar{u})(\phi, \phi) > ds\,, \tag{16}$$

where $\phi(\xi)$ is a solution of (12) with $\phi(0) \in U$ and $\psi(\xi)$ is the unique (up to a multiplicative constant) globally bounded solution of (13) given in Theorem Appendix A.3.

The Melnikov function $M(\nu, \mu, s, c)$ has a singularity at $(0, 0, s_0, c_0)$, whose nature is determined by its higher order derivatives. We will focus on the least degenerate situation and assume that

(A4) The derivatives (15) and (16) of the Melnikov function $M(\nu, \mu, s, c)$ are non-zero.

This implies the occurrence of a saddle-node bifurcation of viscous profiles and is proved like a saddle-node bifurcation of fixed points of a vector field, see e.g. [8].

Theorem 2.3. *Suppose the assumptions (A1)–(A4) hold. Then a saddle-node bifurcation of viscous profiles occurs at $(\mu, s, c) = (0, s_0, c_0)$. More precisely, there exists a smooth function $\mu(\nu, s, c)$ defined in a sufficiently small neighborhood of $(0, s_0, c_0)$, such that $\mu(0, s_0, c_0) = 0$ and $M(\nu, \mu(\nu, s_0, c_0), s_0, c_0) = 0$ for all sufficiently small ν. In addition, the identities*

$$\frac{d\mu}{d\nu}(0, s_0, c_0) = 0 \quad \textit{and} \quad \frac{d^2\mu}{d\nu^2}(0, s_0, c_0) \neq 0 \tag{17}$$

hold. Hence, there exists a family of viscous profiles $\bar{u}(\xi, \nu) := \bar{u}(\xi, \mu(\nu))$ that is parametrized by ν.

Again, we will suppress the dependence of the function μ on the (constant) parameters s_0 and c_0. The corresponding bifurcation diagram is shown in Figure 1. On one side of the bifurcation point $\mu = 0$ two viscous profiles exist, which coalesce into a single one as μ reaches zero and cease to exist as the parameter μ moves beyond zero. The simplest situation where this bifurcation scenario can be realized is associated to a Lax 2-shock in $\mathbb{R}^3$, see Figure 2. Using the parametrization of the viscous profiles in Theorem 2.3, we obtain a basis of the intersection of the tangent spaces.

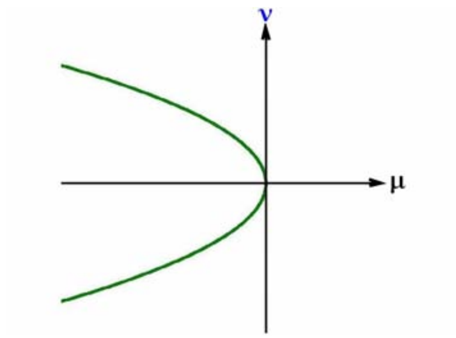

Figure 1: Bifurcation diagram of a saddle-node bifurcation.

Lemma 2.4. *Suppose the assumptions (A1)–(A4) hold. Then the function* $\frac{\partial \bar{u}}{\partial \nu}(\xi, \nu) \mid_{\nu=0}$ *is a solution of*

$$\frac{dz}{d\xi} = \frac{dF}{du}(\bar{u})z,$$

which decays to zero at an exponential rate as $\xi \to \pm\infty$*. Moreover,*

$$T_{\bar{u}(\xi)}W^u(u^-) \cap T_{\bar{u}(\xi)}W^s(u^+) = \operatorname{span}\left\{\frac{\partial \bar{u}}{\partial \xi}(\xi), \frac{\partial \bar{u}}{\partial \nu}(\xi)\right\}. \tag{18}$$

3. Viscous shock waves in MHD

Planar waves in magnetohydrodynamics (MHD) are governed by a system of hyperbolic–parabolic conservation laws. Freistühler and Szmolyan proved that all magnetohydrodynamic shocks have viscous profiles in a certain range of the dissipation coefficients [9, Theorem 1.1]. Moreover, they show that the viscous profiles with the same relative flux are generated in a global bifurcation [9, Theorem 1.3]. Following their conjecture, we will prove via Melnikov theory that a saddle-node bifurcation of viscous profiles occurs.

An application of geometric singular perturbation theory [10, 11] leads to the study of the reduced system,

$$\left.\begin{aligned} \frac{db}{d\xi} &= (\tau - d^2)b - c, \\ \frac{db_*}{d\xi} &= (\tau - d^2)b_*, \\ \frac{d\tau}{d\xi} &= \mu\biggl(\frac{1}{2}\|\mathbf{b}\|^2 + \tau - j + \\ &\qquad + \frac{1}{k\tau}\Bigl(-\frac{\tau^2}{2} - \frac{d^2}{2}\|\mathbf{b}\|^2 - bc + e\Bigr)\biggr), \end{aligned}\right\} \tag{19}$$

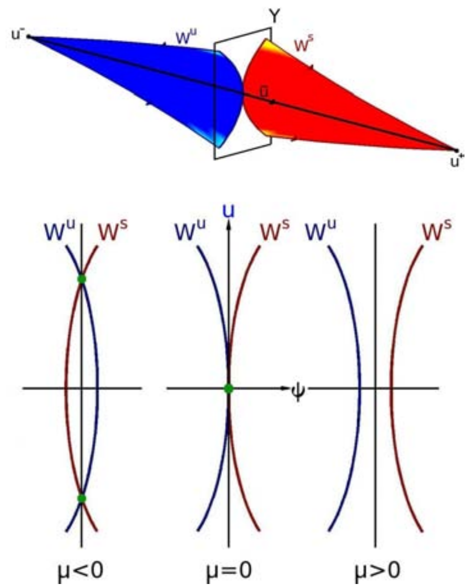

Figure 2: Non-transversal viscous profile $\bar{u}(\xi)$ with unstable and stable manifolds, $W^u(u^-)$ and $W^s(u^+)$; position of $W^u(u^-)$ and $W^s(u^+)$ in the section Y as μ passes through zero.

within the domain $\overline{U}^3 := \mathbb{R}^2 \times \mathbb{R}_+$. The dissipation ratio $\mu = \frac{\nu}{\lambda}$ is the parameter of interest, all other parameters are constant and $k > 1$. We will refer to the function and the right hand side of system (19) as $u(\xi) := (b, b_*, \tau)^T(\xi)$ and $F(u, \mu) := (F_1, F_2, F_3)^T(u, \mu)$, respectively.

Lemma 3.1. (i) *For $\mu = \mu_0$, there exists a profile $\bar{u}(\xi)$ that connects two saddle points u_1 and u_2. Moreover, the profile has the form*

$$\bar{u}(\xi) = (\bar{b}, 0, \bar{\tau})^T(\xi) \tag{20}$$

for some scalar functions $\bar{b}, \bar{\tau} : \mathbb{R} \to \mathbb{R}$, which are strict monotonically decreasing with respect to ξ.

(ii) *The profile $\bar{u}(\xi)$ exists by a non-transversal intersection of the invariant manifolds $W^u(u_1)$ and $W^s(u_2)$. In particular, for any point p on the orbit $\{\bar{u}(\xi) \,|\, \xi \in \mathbb{R}\}$ the identity*

$$T_p W^u(u_1) = T_p W^s(u_2) = \operatorname{span}\left\{\frac{\partial \bar{u}}{\partial \xi}(\xi), v(\xi)\right\} \tag{21}$$

holds, where the derivative of the profile $\frac{\partial \bar{u}}{\partial \xi}(\xi)$ and the function

$$v(\xi) = \begin{pmatrix} 0 \\ \exp\left(\int_0^\xi (\bar{\tau}(s) - d^2) ds\right) \\ 0 \end{pmatrix} \tag{22}$$

are two linearly independent, bounded solutions of the linearized profile equation

$$\frac{dp}{d\xi} = \frac{dF}{du}(\bar{u}, \mu_0)p. \tag{23}$$

Proof. (*i*) The system (19) has four stationary points (which are independent of μ). Due to [9, Lemma 4.7], two stationary points labeled u_1 and u_2 are saddle points, since the associated eigenvalues of the linearized vector field satisfy (with appropriate numbering) the inequalities $\lambda_1(u_1, \mu_0) < 0 < \lambda_2(u_1, \mu_0) < \lambda_3(u_1, \mu_0)$ and $\lambda_1(u_2, \mu_0) < \lambda_2(u_2, \mu_0) < 0 < \lambda_3(u_2, \mu_0)$, respectively. By the result of [9, Lemma 5.1], there exists a profile $\bar{u}(\xi)$ connecting u_1 with u_2 that lies in the plane $\{(b, b_*, \tau)^T \in \bar{U}^3 \mid b_* = 0\}$. Thus the profile has the form (20) for some scalar functions $\overline{b}, \overline{\tau} : \mathbb{R} \to \mathbb{R}$. In the proof of [9, Lemma 5.1], it is observed that the functions $g(b, \tau)$ and $h(b, \tau)$, whence $F_1(b, 0, \tau)$ and $F_3(b, 0, \tau)$, are negative along the profile $\bar{u}(\xi)$ and vanish only in the stationary points u_1 and u_2. Hence the scalar functions $\overline{b}(\xi)$ and $\overline{\tau}(\xi)$ decrease strict monotonically with respect to ξ.

(*ii*) The intersection of the invariant manifolds $W^u(u_1)$ and $W^s(u_2)$ is non-transversal at the parameter value μ_0, due to the reflectional symmetry of system (19), $(b, b_*, \tau) \mapsto (b, -b_*, \tau)$. Moreover, the second ODE of the linearized profile equation,

$$\begin{aligned}
\frac{dp_1}{d\xi} &= (\overline{\tau} - d^2)p_1 + \overline{b}p_3, \\
\frac{dp_2}{d\xi} &= (\overline{\tau} - d^2)p_2, \\
\frac{dp_3}{d\xi} &= \frac{\partial F_3}{\partial b}(\bar{u}, \mu_0)p_1 + \frac{\partial F_3}{\partial \tau}(\bar{u}, \mu_0)p_3,
\end{aligned}$$

decouples. Hence the function (22) and the derivative of the profile are solutions of the linearized profile equation and linearly independent. In addition, the profile $\bar{u}(\xi)$ tends to endstates, which satisfy by the results of [9, Lemma 4.4] the inequalities

$$\lim_{\xi \to -\infty} \overline{\tau}(\xi) = \tau_1 > d^2 \quad \text{and} \quad \lim_{\xi \to +\infty} \overline{\tau}(\xi) = \tau_2 < d^2.$$

Thus the integral

$$\int_0^\xi (\overline{\tau}(s) - d^2)ds \to -\infty$$

diverges in both limits $\xi \to \pm\infty$ to $-\infty$ and we conclude that $v(\xi)$ is globally bounded on $\mathbb{R}$. Since the invariant manifolds $W^u(u_1)$ and $W^s(u_2)$ are two-dimensional, we obtain that for any point p on the orbit $\{\bar{u}(\xi) \,|\, \xi \in \mathbb{R}\}$ the identity (21) holds. □

Thus the assumptions (A1)–(A3) of Section 2 hold and we conclude from Theorem 2.2 the following result.

Lemma 3.2. *The Melnikov function $M(\nu,\mu)$ is smooth in a small neighborhood of the point $(\nu,\mu) = (0,\mu_0)$ and satisfies the identities*

$$\frac{\partial M}{\partial \mu}(0,\mu_0) = \int_{\mathbb{R}} a\ w\ F_1(\bar{u},\mu_0)\ \frac{\partial F_3}{\partial \mu}(\bar{u},\mu_0)\ d\xi \tag{24}$$

as well as

$$\frac{\partial^2 M}{\partial \nu^2}(0,\mu_0) = \int_{\mathbb{R}} a\ w^3\ F_1(\bar{u},\mu_0)\ \frac{\partial^2 F_3}{\partial b_*^2}(\bar{u},\mu_0)\ d\xi, \tag{25}$$

where

$$a(\xi) := \exp\Big(- \int_0^\xi \text{trace}\Big(\frac{dF}{du}(\bar{u},\mu_0)\Big) dx\Big)$$

and $v(\xi) = (0, w(\xi), 0)^T$ is taken from Lemma 3.1. Moreover, the derivative of the Melnikov function (24) is non-zero at the point $(0,\mu_0)$. Whereas, (25) is non-zero, if one of the following conditions holds:

(i) *The function $(k\overline{\tau}(\xi) - d^2)$ has a common sign for all $\xi \in \mathbb{R}$.*
(ii) *The expression $k\tau_2 - d^2$ is positive.*

Proof. By the results of Theorem 2.2, the Melnikov function satisfies

$$\frac{\partial M}{\partial \mu}(0,\mu_0) = \int_{\mathbb{R}} < \psi, \frac{\partial F}{\partial \mu}(\bar{u},\mu_0) >\ d\xi,$$

where $\psi(s)$ is the unique (up to a multiplicative constant) bounded solution of the adjoint differential equation of (23). We derive from the results of Theorem Appendix A.3 and Lemma 3.1 the expression

$$\psi(\xi) := a(\xi)\Big(\frac{\partial \bar{u}}{\partial \xi} \times v\Big)(\xi). \tag{26}$$

In addition, the derivative of the vector field $F(u,\mu)$ with respect to μ satisfies

$$\frac{\partial F}{\partial \mu}(\bar{u}(\xi),\mu_0) = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial F_3}{\partial \mu}(\bar{u}(\xi),\mu_0) \end{pmatrix}.$$

Hence, the first order derivative of the Melnikov function is obtained as

$$\begin{aligned}
\frac{\partial M}{\partial \mu}(0,\mu_0) &= \int_{\mathbb{R}} < \psi, \frac{\partial F}{\partial \mu}(\bar{u},\mu_0) >\ d\xi = \\
&= \int_{\mathbb{R}} a\ < \Big(\frac{\partial \bar{u}}{\partial \xi} \times v\Big), \frac{\partial F}{\partial \mu}(\bar{u},\mu_0) >\ d\xi \\
&= \int_{\mathbb{R}} a\ \det \begin{pmatrix} F_1 & 0 & 0 \\ 0 & w & 0 \\ F_3 & 0 & \frac{\partial F_3}{\partial \mu} \end{pmatrix} d\xi \\
&= \int_{\mathbb{R}} a\ w\ F_1(\bar{u},\mu_0)\ \frac{\partial F_3}{\partial \mu}(\bar{u},\mu_0)\ d\xi
\end{aligned}$$

where $a(\xi) := \exp\big(-\int_0^\xi \text{trace}\big(\frac{dF}{du}(\bar{u},\mu_0)\big)dx\big)$. The third equality holds by the results of Lemma Appendix A.2. The integrand

$$a\ w\ F_1(\bar{u},\mu_0)\ \frac{\partial F_3}{\partial \mu}(\bar{u},\mu_0)$$

is the product of scalar and continuous functions, which do not change sign by the equation $\frac{\partial F_3}{\partial \mu}(\bar{u},\mu_0) = \frac{g(\bar{b},\overline{\tau})}{2k\overline{\tau}}$ and the results of Lemma 3.1. Thus the integrand has a common sign and is integrable, which implies that $\frac{\partial M}{\partial \mu}(0,\mu_0)$ does not vanish.

By the results of Theorem 2.2, the Melnikov function also satisfies

$$\frac{\partial^2 M}{\partial \nu^2}(0,\mu_0) = \int_{\mathbb{R}} < \psi, \frac{\partial^2 F}{\partial u^2}(\bar{u},\mu_0)(v,v) > \ d\xi,$$

with $\psi(\xi)$ given by (26). In addition, the special form of the solution $v(\xi) = (0, w(\xi), 0)^T$ implies that

$$\begin{aligned}\frac{\partial^2 F}{\partial u^2}(\bar{u},\mu_0)(v,v) &= \left(0,0,\frac{\partial^2 F_3}{\partial b_*^2}(\bar{u},\mu_0)\ w^2\right)^T \\ &= \left(0,0,\frac{k\overline{\tau}-d^2}{k\overline{\tau}}\ w^2\right)^T,\end{aligned}$$

where k is bigger than one and $\overline{\tau}(\xi)$ is positive for all $\xi \in \mathbb{R}$. We use these expressions to obtain

$$\begin{aligned}\frac{\partial^2 M}{\partial \nu^2}(0,\mu_0) &= \int_{\mathbb{R}} < \psi, \frac{\partial^2 F}{\partial u^2}(\bar{u},\mu_0)(v,v) > \ d\xi \\ &= \int_{\mathbb{R}} a\ < \left(\frac{\partial \bar{u}}{\partial \xi} \times v\right), \frac{\partial^2 F}{\partial u^2}(v,v) > \ d\xi \\ &= \int_{\mathbb{R}} a\ \det\begin{pmatrix} F_1 & 0 & 0 \\ 0 & w & 0 \\ F_3 & 0 & \frac{\partial^2 F_3}{\partial b_*^2} w^2 \end{pmatrix} d\xi \\ &= \int_{\mathbb{R}} a\ w^3\ F_1(\bar{u},\mu_0)\ \frac{\partial^2 F_3}{\partial b_*^2}(\bar{u},\mu_0)\ d\xi,\end{aligned}$$

where the third equality holds by the results of Lemma Appendix A.2. The integrand

$$a\ w^3\ F_1(\bar{u},\mu_0)\ \frac{\partial^2 F_3}{\partial b_*^2}(\bar{u},\mu_0)$$

is the product of scalar factors. The functions $a(\xi)$, $w(\xi)$ and $F_1(\bar{u}(\xi),\mu_0)$ do not change sign by the results of Lemma 3.1. Additionally, the continuous function $\frac{\partial^2 F_3}{\partial b_*^2}(\bar{u}(\xi),\mu_0) = \frac{k\overline{\tau}-d^2}{k\overline{\tau}}(\xi)$ does not vanish by the first assumption. Thus the integrand has a common sign and is integrable, which implies that $\frac{\partial^2 M}{\partial \nu^2}(0,\mu_0)$ is non-zero.

By the results of Lemma 3.1, the coordinate functions of the profile $\bar{u}(\xi) = (\bar{b}, 0, \bar{\tau})^T(\xi)$ decrease strict monotonically to $(b_2, 0, \tau_2)^T$. Thus the second assumption, $k\tau_2 - d^2 > 0$, implies that for all $\xi \in \mathbb{R}$ the inequality $k\bar{\tau}(\xi) - d^2 > 0$ holds and we obtain the statement from the previous result. □

An investigation of the nullclines in [12, 9] leads to a classification (C1)-(C3) of intersection scenarios, see also Lemma 4.5 in [9]. Thus we can identify a parameter regime such that a saddle-node bifurcation of heteroclinic orbits occurs.

Corollary 3.3. *In case (C3) where H^- and H^+ both intersect G_1 a saddle-node bifurcation of heteroclinic orbits will occur.*

Proof. In case (C3), all stationary points of system (19) are elements of the intersection $G_1 \cap H$ and the component G_1 lies entirely above the line $\tau = \frac{d^2}{k}$. Thus the τ coordinates of all stationary points are greater than $\frac{d^2}{k}$, which implies by Lemma 3.2 that the second order derivative of the Melnikov function $\frac{\partial^2 M}{\partial \nu^2}(0, \mu_0)$ is not zero. This verifies the assumptions of Theorem 2.3, which implies the occurrence of a saddle-node bifurcation. □

4. Evans function for viscous shock waves

To locate the point spectrum in the eigenvalue problem associated to a viscous shock wave, we consider the variables $(p, q := \frac{dp}{d\xi} - \frac{dF}{du}(\bar{u})p)^T(\xi)$ and rewrite the eigenvalue equation $Lp = \kappa p$ as a system of first order ODEs

$$\frac{d}{d\xi}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{dF}{du}(\bar{u}) & I_n \\ \kappa I_n & 0_n \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} =: \mathbb{A}(\xi, \kappa)\begin{pmatrix} p \\ q \end{pmatrix}. \tag{27}$$

The matrix $\mathbb{A}(\xi, \kappa)$ is analytic in κ and differentiable in ξ, because $F(u)$ is smooth and the viscous profile $\bar{u}(\xi)$ is differentiable. Due to the Gap Lemma [13, 14], there exist $\beta > 0$ and subspaces $\mathcal{S}(\kappa)$ and $\mathcal{U}(\kappa)$ of $\mathbb{C}^{2n}$ that are analytic in $\kappa \in \mathbb{C}_\beta := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq -\beta\}$ and reduce to the stable space $\mathcal{S}(\kappa)$ and unstable space $\mathcal{U}(\kappa)$ of initial values of the eigenvalue equation (27) for $\kappa \in \overline{\mathbb{C}_+}^{\bullet} := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0,\ z \neq 0\}$. Moreover, these spaces have dimension n and one can choose analytic bases such that $\mathcal{S}(\kappa) = \mathrm{span}\{\eta_j^s(0, \kappa) \mid j = 1, \ldots, n\}$ and $\mathcal{U}(\kappa) = \mathrm{span}\{\eta_j^u(0, \kappa) \mid j = 1, \ldots, n\}$, see also [15]. For $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$, the existence of an eigenfunction is equivalent to a non-trivial intersection of the spaces $\mathcal{S}(\kappa)$ and $\mathcal{U}(\kappa)$, which will be studied via the Evans function.

Theorem 4.1 ([13])**.** *Suppose $\beta > 0$ is sufficiently small. Then the* Evans function*, defined as*

$$\begin{aligned} E :&\mathbb{C}_\beta \to \mathbb{C}, \\ &\kappa \mapsto E(\kappa) := \det\left(\eta_1^u, \ldots, \eta_n^u, \eta_1^s, \ldots, \eta_n^s\right)(0, \kappa), \end{aligned}$$

has the following properties

(i) *$E(\kappa)$ is analytic in κ for $\kappa \in \mathbb{C}_\beta$.*

(ii) *For $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$, $E(\kappa) = 0$ if and only if $\kappa \in \sigma_p(L)$.*

(iii) *The algebraic multiplicity of the eigenvalue $\kappa \in \sigma_p(L)$ equals its order as a root of the Evans function.*

Remark. The Evans function approach was introduced in the setting of reaction-diffusion equations. In this case the properties of the Evans function in a domain of consistent splitting, as stated in Theorem 4.1, have been proved in the article [16].

Remark 4.2. In the article [2], Zumbrun and Howard base their spectral analysis on the resolvent kernel, rather than the resolvent, and define the effective spectrum as the set of poles for the meromorphic continuation of the resolvent kernel into the essential spectrum. Moreover, an effective eigenprojection with respect to a spectral parameter is defined via the residue of the resolvent kernel. The range of an effective eigenprojection is referred to as the effective eigenspace and its elements, the effective eigenfunctions, can be arranged in Jordan chains. Then the effective spectrum coincides with the zero set (of the analytic continuation) of the Evans function and the multiplicity of an effective eigenvalue is equal to the order of the roots of the Evans function. We will refer to effective eigenfunctions that decay exponentially in the limits $\xi \to \pm\infty$ as genuine eigenfunctions.

To investigate the onset of instability, we will study the Evans function in a small neighborhood of the origin. Therefore, we will need the expansion of solutions of the eigenvalue equation (27) in terms of $|\kappa|$, see also [13, Lemma 3.1]. We will give an alternative proof via geometric singular perturbation theory [10, 17, 11].

Lemma 4.3. *Suppose the assumption (A2) holds. Then, for $\kappa \in \mathbb{C}$ with $|\kappa| < \varepsilon$, the solutions $\eta_j^{s/u}(\xi, \kappa)$ of the eigenvalue equation (27) with initial values $\eta_j^{s/u}(0, \kappa)$ satisfy the expansion*

$$\eta_j^s(\xi, \kappa) = \begin{pmatrix} p_{n+j}(\xi) \\ 0 \end{pmatrix} + o(|\kappa|), \quad j = 1, \ldots, k, \tag{28}$$

$$\eta_j^s(\xi, \kappa) = \begin{pmatrix} p_{n+j}(\xi) \\ r_j(u^+) \end{pmatrix} + o(|\kappa|), \quad j = k+1, \ldots, n, \tag{29}$$

with functions $p_{n+j}(\xi)$ that decay exponentially fast to zero as $\xi \to +\infty$, and

$$\eta_j^u(\xi, \kappa) = \begin{pmatrix} p_j(\xi) \\ r_j(u^-) \end{pmatrix} + o(|\kappa|), \quad j = 1, \ldots, k-1, \tag{30}$$

$$\eta_j^u(\xi, \kappa) = \begin{pmatrix} p_j(\xi) \\ 0 \end{pmatrix} + o(|\kappa|), \quad j = k, \ldots, n. \tag{31}$$

with functions $p_j(\xi)$ that decay exponentially fast to zero as $\xi \to -\infty$.

Proof. We augment the profile equation (3) and the eigenvalue equation (27) to obtain the autonomous system

$$\begin{aligned}\frac{du}{d\xi} &= F(u),\\ \frac{dp}{d\xi} &= \frac{dF}{du}(u)p + q,\\ \frac{dq}{d\xi} &= \kappa p,\end{aligned} \tag{32}$$

which is singularly perturbed at $\kappa = 0$ and has stationary points $U^\pm = (u^\pm, 0, 0)$. We will construct the invariant manifolds for κ in a small neighborhood of the origin and use the parametrization $\kappa = \rho \exp(i\phi)$ with $\rho \in [0, \varepsilon]$ and $\phi = [0, 2\pi[$. The manifold of equilibria for $\kappa = 0$ is given by $M_0 = M_0^- \cup M_0^+$ with

$$M_0^\pm := \left\{ (u, p, q)^T \in \mathbb{C}^{3n} \;\middle|\; u = u^\pm, \right.$$
$$\left. p = -\left(\frac{dF}{du}(u^\pm)\right)^{-1} q, \quad q \in \mathbb{C}^n \right\}.$$

For $\kappa = 0$, the linearization of the augmented system (32) at any point in the critical manifold $M_0^\pm$ has exactly $n = \dim(M_0^\pm)$ eigenvalues with zero real-part. Hence, the critical manifolds $M_0^\pm$ are normally hyperbolic and geometric singular perturbation theory [10, 17, 11] is applicable. At first, we will construct the invariant manifold $W^s(U^+)$ for $\rho = 0$ and note that its the total space of a fiber bundle with an invariant base space $W^{s,slow}(U^+)$ within M_0^+ and the fiber $W^{s,fast}(U^+)$. The equations on the slow time scale $\tau := \rho\xi$ are

$$\begin{aligned}\rho\frac{du}{d\tau} &= F(u),\\ \rho\frac{dp}{d\tau} &= \frac{dF}{du}(u)p + q,\\ \frac{dq}{d\tau} &= \exp(i\phi)p.\end{aligned}$$

The reduced problem $\rho = 0$ is only defined on M_0 and the slow flow on M_0^+ is governed by

$$\frac{dq}{d\tau} = -\exp(i\phi)\left(\frac{dF}{du}(u^+)\right)^{-1} q.$$

Any subspace spanned by eigenvectors of $\left(\frac{dF}{du}(u^+)\right)^{-1}$ will remain invariant. However, for κ in the domain $\overline{\mathbb{C}_+}^{\bullet}$ the invariant manifold $W^s(U^+)$ should be the stable manifold of the stationary point U^+. By the assumptions, the eigenvalues $-\exp(i\phi)(\lambda_j(u^+) - s)^{-1}$ with associated eigenvectors $r_j(u^+)$ for $j = k+1, \ldots, n$

have negative real part as long as $\phi \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. Thus we obtain the invariant manifold $W^{s,slow}(U^+)$ within the critical manifold M_0^+ as

$$\begin{aligned} W^{s,slow}(U^+) := \Bigg\{ (u,p,q)^T \in \mathbb{C}^{3n} \,\Bigg|\, & u = u^+, \\ & p = -\left(\frac{dF}{du}(u^+)\right)^{-1} q, \\ & q \in span\{r_{k+1}(u^+), \dots, r_n(u^+)\} \Bigg\}. \end{aligned}$$

The fibers emanating from the critical manifold M_0^+ are described by the equations on the fast time scale ξ. The augmented system reduces for $\rho = 0$ to

$$\begin{aligned} \frac{du}{d\xi} &= F(u), \\ \frac{dp}{d\xi} &= \frac{dF}{du}(u)p + q, \\ \frac{dq}{d\xi} &= 0. \end{aligned}$$

We consider without loss of generality the fiber with base point U^+, i.e. solutions satisfying the boundary condition $\lim_{\xi \to \infty}(u,p,q)^T(\xi) = (u^+, 0, 0)^T$. The constant solution $u(\xi) \equiv u^+$ solves the first equation and the q coordinates are identically zero. Thus the invariant manifold $W^{s,fast}(U^+)$ in the fast directions has at the stationary point U^+ the tangent space

$$\begin{aligned} T_{U^+}W^{s,fast}(U^+) = \{ (u,p,q)^T \in \mathbb{C}^{3n} \mid & q = 0, \\ & u,p \in span\{r_1(u^+), \dots, r_k(u^+)\} \ \}. \end{aligned}$$

To sum up, the invariant manifold $W^s(U^+)$ is the total space of a fiber bundle with base space $W^{s,slow}(U^+) \subset M_0^+$ and the fiber $W^{s,fast}(U^+)$. Since the critical manifold M_0^+ is normally hyperbolic it perturbs smoothly to an invariant manifold M_ρ^+ for $\rho \in [0, \varepsilon]$ small. This implies that the construction of the $W^s(U^+)$ persists for small ρ. The solutions of the eigenvalue equation can be extracted from the invariant manifold $W^s(U^+)$ of the augmented system (32), whereas the fibration of $W^s(U^+)$ explains the different behavior of the solutions of (27). For example, the solutions $\eta_j^s(\xi, \kappa)$ for $j = 1, \dots, k$ and $j = k+1, \dots, n$ are related to the fast manifold $W^{s,fast}(U^+)$ and the slow manifold $W^{s,slow}(U^+)$, respectively. In this way, we obtain the expansions (28) and (29). Similarly, we are able to prove the (validity of) the expansions (30) and (31). □

In the following, we study the Evans function $E(\kappa)$ in a neighborhood of the origin by calculating its derivatives at $\kappa = 0$. We restrict our calculations to the real half-line $\kappa \in \mathbb{R}_+$, such that the Evans function and the involved vectors can be chosen to be real valued. For transversal viscous profiles corresponding to Lax shocks the following result was proven in [13]. As a preparation for our analysis of non-transversal viscous profiles we give the proof in our setup.

Theorem 4.4 ([13])**.** *Suppose the assumption (A2) holds. The first derivative of the Evans function satisfies*

$$\frac{dE}{d\kappa}(0) = c\ \det(p_k, \ldots, p_n, p_{n+2}, \ldots, p_{n+k})(0) \cdot \Delta,$$

with a non-zero constant $c \in \mathbb{R}$*, the vectors* $p_j(0)$*,* $j = 1, \ldots, 2n$*, in Lemma 4.3 and the Liu-Majda determinant* Δ *in* (33)*.*

Remark 4.5. The Liu-Majda determinant

$$\Delta := \det(r_1(u^-), \ldots, r_{k-1}(u^-), u^+ - u^-, r_{k+1}(u^+), \ldots, r_n(u^+)) \tag{33}$$

is spanned by certain eigenvectors of the Jacobian of the flux function at the endstates and the vector $u^+ - u^-$. The associated Liu-Majda condition asserts that the Liu-Majda determinant does not vanish,

$$\Delta \neq 0\,, \tag{34}$$

which is necessary for dynamical stability of the Lax shock as a solution of the inviscid conservation law [18].

Proof. We consider the Evans function in Theorem 4.1,

$$E(\kappa) = \det(\eta_1^u, \ldots, \eta_n^u, \eta_1^s, \ldots, \eta_n^s)(0, \kappa).$$

Since the derivative of the profile $\frac{d\bar{u}}{d\xi}(\xi)$ is a genuine eigenfunction for $\kappa = 0$, we can assume without loss of generality that the vectors $\eta_k^u(0, \kappa)$ and $\eta_1^s(0, \kappa)$ satisfy the identity

$$\eta_k^u(0,0) = \eta_1^s(0,0) = \begin{pmatrix} \frac{d\bar{u}}{d\xi} \\ 0 \end{pmatrix}(0), \tag{35}$$

which implies $E(0) = 0$. We differentiate the Evans function with respect to κ by the Leibniz rule, evaluate the derivative at $\kappa = 0$ and obtain

$$\frac{dE}{d\kappa}(0) = \det\left(\eta_1^u, \ldots, \eta_n^u, \frac{\partial \eta_1^s}{\partial \kappa} - \frac{\partial \eta_k^u}{\partial \kappa}, \eta_2^s, \ldots, \eta_n^s\right)(0). \tag{36}$$

All other summands vanish, since they contain the pair of linearly dependent vectors (35).

The solutions of the eigenvalue problem (27) with initial values $\eta_1^s(0, \kappa)$ and $\eta_k^u(0, \kappa)$ at $\xi = 0$ will be denoted as $\eta_1^s(\xi, \kappa)$ and $\eta_k^u(\xi, \kappa)$, respectively. Their derivative with respect to κ is governed by the differential equations,

$$\frac{\partial}{\partial \xi}\begin{pmatrix} \frac{\partial p}{\partial \kappa} \\ \frac{\partial q}{\partial \kappa} \end{pmatrix} = \begin{pmatrix} \frac{dF}{du}(\bar{u}(\xi)) & I_n \\ \kappa I_n & 0_n \end{pmatrix}\begin{pmatrix} \frac{\partial p}{\partial \kappa} \\ \frac{\partial q}{\partial \kappa} \end{pmatrix} + \begin{pmatrix} 0 \\ p \end{pmatrix}.$$

For $\kappa = 0$, we observe that

$$\frac{d}{d\xi}\frac{\partial}{\partial\kappa}q(\xi,0) = p(\xi,0).$$

Thus we obtain from integration with respect to ξ and (35) the identities

$$\frac{\partial \eta_k^u}{\partial\kappa}(\xi,0) = \begin{pmatrix} \frac{\partial p_k}{\partial\kappa}(\xi) \\ \bar{u}(\xi) - u^- \end{pmatrix} \tag{37}$$

and

$$\frac{\partial \eta_1^s}{\partial\kappa}(\xi,0) = \begin{pmatrix} \frac{\partial p_{n+1}}{\partial\kappa}(\xi) \\ \bar{u}(\xi) - u^+ \end{pmatrix}. \tag{38}$$

We insert the vectors (37) and (38) in identity (36), change the order of the vectors with k_p permutations and obtain the matrix in block diagonal form

$$\frac{dE}{d\kappa}(0) = (-1)^{k_p} \det \begin{pmatrix} A & B \\ 0_n & C \end{pmatrix},$$

with quadratic matrices $A := (p_k, \ldots, p_n, p_{n+2}, \ldots, p_{n+k})(0) \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, the null matrix $0_n \in \mathbb{R}^{n\times n}$ and

$$C := (r_1(u^-), \ldots, r_{k-1}(u^-), -(u^+ - u^-), r_{k+1}(u^+), \ldots, r_n(u^+)).$$

Thus we derive from the identity

$$\det \begin{pmatrix} A & B \\ 0_n & C \end{pmatrix} = \det(A)\det(C)$$

the stated result. □

Corollary 4.6 ([13])**.** *Suppose the assumption (A2) and the Liu-Majda condition* (34) *hold. If the viscous profile* $\bar{u}(\xi)$ *is transversal, then* $\kappa = 0$ *is a simple root of the Evans function.*

Proof. The matrix $A = (p_k, \ldots, p_n, p_{n+2}, \ldots, p_{n+k})(0)$ is spanned by the tangent vectors of the invariant manifolds $W^u(u^-)$ and $W^s(u^+)$ of the profile equation (3). The assumption of a transversal intersection along the viscous profile implies that the tangent vectors are linearly independent. Hence the factor $\det(A)$ will be non-zero. Together with the Liu-Majda condition (34), we obtain that the first derivative of the Evans function at $\kappa = 0$ does not vanish. Thus the order of the root $\kappa = 0$ is one. □

In case of a non-transversal viscous profile corresponding to a Lax shock, the first derivative of the Evans function at $\kappa = 0$ vanishes, which may signal the onset of instability. We will study this situation in the remainder of this work.

5. Bifurcation analysis of $E(\kappa,\nu)=0$

Following the analysis in Section 2, we consider a family of viscous conservation laws

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}f(u,\mu)=\frac{\partial^2 u}{\partial x^2} \tag{39}$$

with associated profile equation

$$\frac{du}{d\xi}=f(u,\mu)-s\,u-c=:F(u,\mu). \tag{40}$$

Under the assumptions (A1)–(A4), Theorem 2.3 implies the existence of a family of viscous profiles $\bar{u}(\xi,\nu)=\bar{u}(\xi,\mu(\nu))$, which is generated in a saddle-node bifurcation. In the next step, we study the spectral stability of this family of viscous profiles $\bar{u}(\xi,\nu)$. The evolutionary equation (7) is given by

$$\frac{du}{dt}=\frac{d^2u}{d\xi^2}-\frac{d}{d\xi}f(u,\mu)+s\frac{du}{d\xi}=:\mathcal{F}(u,\mu) \tag{41}$$

and the operator $\mathcal{F}(u,\mu)$ linearized along the viscous profiles $\bar{u}(\xi,\nu)$ is

$$Lp:=\frac{d\mathcal{F}}{du}(\bar{u},\mu(\nu))p=\frac{d}{d\xi}\left(\frac{dp}{d\xi}-\frac{dF}{du}(\bar{u},\mu(\nu))p\right). \tag{42}$$

Theorem 5.1. *Suppose the assumptions (A1)–(A2) hold. For sufficiently small positive constants β and δ, a family of Evans functions,*

$$E:\mathbb{C}_\beta\times(-\delta,\delta)\to\mathbb{C},\quad (\kappa,\nu)\mapsto E(\kappa,\nu)\,, \tag{43}$$

that is analytic in κ and smooth in ν, can be defined like in Theorem 4.1.

Proof. For a fixed $\nu\in(-\delta,\delta)$, define the Evans function $E(\kappa,\nu)$ as in Theorem 4.1. Thus $E(\kappa,\nu)$ is analytic in κ. Moreover, the parameter ν enters the eigenvalue problem through its dependence on the viscous profile. Therefore the family of Evans functions inherits the smooth dependence on ν from the family of viscous profiles $\bar{u}(\xi,\nu)$. □

Lemma 5.2. *Suppose the assumptions (A1)–(A4) hold. Then the genuine eigenfunctions for $\kappa=0$ of the eigenvalue problem* (27) *form a two-dimensional linear space which is spanned by the functions*

$$\begin{pmatrix}\frac{\partial\bar{u}}{\partial\xi}\\ 0\end{pmatrix}(\xi,0)\quad \textit{and}\quad \begin{pmatrix}\frac{\partial\bar{u}}{\partial\nu}\\ 0\end{pmatrix}(\xi,0). \tag{44}$$

Moreover, the Evans function $E(\kappa,\nu)$ and its derivatives satisfy the identities

$$E(0,0)=0,\quad \frac{\partial E}{\partial\kappa}(0,0)=0\quad \textit{and}\quad \frac{\partial E}{\partial\nu}(0,0)=0. \tag{45}$$

Proof. Any tangent vector of the invariant manifolds $W^u(u^-)$ and $W^s(u^+)$ satisfies the linearized profile equation

$$\frac{dp}{d\xi} = \frac{dF}{du}(\bar{u}, 0)p$$

and therefore also the eigenvalue equation

$$\kappa p = \frac{d}{d\xi}\left(\frac{dp}{d\xi} - \frac{dF}{du}(\bar{u}(\xi,\nu), \mu(\nu))p\right) \tag{46}$$

for $(\kappa, \nu) = (0, 0)$. However, only a tangent vector, that is lying in the intersection of the tangent spaces $T_{\bar{u}(\xi)}W^u(u^-)$ and $T_{\bar{u}(\xi)}W^s(u^+)$, is a bounded function. By assumption (A3), the intersection is two-dimensional and, by Lemma 2.4, it is spanned by the functions $\frac{\partial \bar{u}}{\partial \xi}(\xi, 0)$ and $\frac{\partial \bar{u}}{\partial \nu}(\xi, 0)$. Thus we conclude the first statement. Moreover, the identities (45) are derived by a similar reasoning as in the proof of Theorem 4.4. □

Lemma 5.2 implies that the non-transversal viscous profile $\bar{u}(\xi, 0)$ is not spectrally stable in the sense of Definition 1.1.

Remark 5.3. Due to Lemma 2.4 and for sufficiently small ν, the derivative of the viscous profile $\bar{u}(\xi, \nu)$ with respect to ξ is a genuine eigenfunction to $\kappa = 0$. Therefore the Evans function $E(\kappa, \nu)$ and its derivatives with respect to the parameter ν vanish identically for $\kappa = 0$ and sufficiently small ν.

The identities $\frac{\partial E}{\partial \kappa}(0, 0) = 0$ and $E(0, \nu) = 0$ for all sufficiently small ν indicate a bifurcation in the equation $E(\kappa, \nu) = 0$ defining the zero set of the Evans function. The nature of the singularity of the Evans function at the origin is studied via its higher order derivatives.

The following preliminary result is a direct consequence of Lemma 2.4.

Lemma 5.4. *Suppose the assumptions (A1)–(A4) hold. Then the function*

$$\bar{v}(\xi) := \int_{-\infty}^{\xi} \frac{\partial \bar{u}}{\partial \nu}(x, 0)dx$$

is continuous and bounded on $\mathbb{R}$*. In addition,* $\bar{v}(\xi)$ *approaches constant endstates* $v^{\pm} := \lim_{\xi \to \pm\infty} \bar{v}(\xi)$*.*

First, we establish the connection between (a derivative of) the Evans function and (a derivative of) the Melnikov function.

Theorem 5.5. *Suppose the assumptions (A1)–(A4) hold. Then the derivative of the Evans function* (43) *satisfies*

$$\begin{aligned} &\frac{\partial^2 E}{\partial \nu \partial \kappa}(0, 0) = \\ &= c \cdot \Delta \cdot \int_{\mathbb{R}} < \psi, \frac{d^2 F}{du^2}(\bar{u}, 0)\left(\frac{\partial \bar{u}}{\partial \nu}, \frac{\partial \bar{u}}{\partial \nu}\right)(\xi, 0) > \, d\xi \end{aligned} \tag{47}$$

with a non-zero real constant c, the Liu-Majda determinant Δ *in* (33)*, and the function* $\psi(\xi)$ *in Theorem 2.2.*

Proof. We consider the Evans function $E(\kappa,\nu)$ in Theorem 5.1. Due to Lemma 5.2 and Remark 5.3, we can assume without loss of generality that for sufficiently small ν solutions of the eigenvalue equation are given by

$$\eta_k^u(\xi,0,\nu) = \eta_1^s(\xi,0,\nu) = \begin{pmatrix} \frac{\partial \bar{u}}{\partial \xi} \\ 0 \end{pmatrix}(\xi,\nu) \tag{48}$$

and

$$\eta_{k+1}^u(\xi,0,0) = \eta_2^s(\xi,0,0) = \begin{pmatrix} \frac{\partial \bar{u}}{\partial \nu} \\ 0 \end{pmatrix}(\xi,0). \tag{49}$$

We rewrite the Evans function,

$$E(\kappa,\nu) = \det\left(\eta_1^u,\ldots,\eta_n^u,\eta_1^s-\eta_k^u,\eta_2^s-\eta_{k+1}^u,\eta_3^s,\ldots,\eta_n^s\right)(0,\kappa,\nu),$$

differentiate with respect to κ and ν by the Leibniz rule and evaluate the derivative at $(\kappa,\nu)=(0,0)$ to obtain

$$\begin{aligned}
&\frac{\partial^2 E}{\partial\nu\partial\kappa}(0,0) = \\
&= \det\left(\eta_1^u,\ldots,\eta_n^u,\frac{\partial(\eta_1^s-\eta_k^u)}{\partial\kappa},\frac{\partial(\eta_2^s-\eta_{k+1}^u)}{\partial\nu},\eta_3^s,\ldots,\eta_n^s\right)(0)+ \\
&+\det\left(\eta_1^u,\ldots,\eta_n^u,\frac{\partial(\eta_1^s-\eta_k^u)}{\partial\nu},\frac{\partial(\eta_2^s-\eta_{k+1}^u)}{\partial\kappa},\eta_3^s,\ldots,\eta_n^s\right)(0).
\end{aligned}$$

All other summands vanish at $(\kappa,\nu)=(0,0)$, since they contain a vector $(\eta_1^s - \eta_k^u)(0,0,0)$ and/or $(\eta_2^s-\eta_{k+1}^u)(0,0,0)$ which coincide with the null vector.

We consider the solutions of the eigenvalue equation

$$\frac{d}{d\xi}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{dF}{du}(\bar{u}(\xi,\nu),\mu(\nu)) & I_n \\ \kappa I_n & 0_n \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}, \tag{50}$$

satisfying the identities (48) and (49). Their derivatives with respect to the spectral parameter κ are governed by the system of differential equations

$$\frac{\partial}{\partial\xi}\begin{pmatrix} \frac{\partial p}{\partial\kappa} \\ \frac{\partial q}{\partial\kappa} \end{pmatrix} = \begin{pmatrix} \frac{dF}{du}(\bar{u}(\xi,\nu),\mu(\nu)) & I_n \\ \kappa I_n & 0_n \end{pmatrix}\begin{pmatrix} \frac{\partial p}{\partial\kappa} \\ \frac{\partial q}{\partial\kappa} \end{pmatrix} + \begin{pmatrix} 0 \\ p \end{pmatrix}.$$

In the proof of Theorem 4.4, we obtained the expressions

$$\frac{\partial\eta_k^u}{\partial\kappa}(\xi,0,0) = \begin{pmatrix} z_k(\xi) \\ \bar{u}(\xi)-u^- \end{pmatrix} \tag{51}$$

and

$$\frac{\partial\eta_1^s}{\partial\kappa}(\xi,0,0) = \begin{pmatrix} z_{n+1}(\xi) \\ \bar{u}(\xi)-u^+ \end{pmatrix}, \tag{52}$$

where the functions $z_i(\xi)$ are defined as $z_i(\xi) := \frac{\partial p_i}{\partial \kappa}(\xi, 0, 0)$ for $i = k, n+1$. In a similar way, we derive

$$\frac{\partial \eta^u_{k+1}}{\partial \kappa}(\xi, 0, 0) = \begin{pmatrix} z_{k+1}(\xi) \\ \bar{v}(\xi) - v^- \end{pmatrix} \tag{53}$$

and

$$\frac{\partial \eta^s_2}{\partial \kappa}(\xi, 0, 0) = \begin{pmatrix} z_{n+2}(\xi) \\ \bar{v}(\xi) - v^+ \end{pmatrix}, \tag{54}$$

where the continuous and bounded function $\bar{v}(\xi) := \int_{-\infty}^{\xi} \frac{\partial \bar{u}}{\partial \nu}(x, 0)\ dx$ with asymptotic endstates $v^{\pm} := \lim_{\xi \to \pm\infty} \bar{v}(\xi)$ is taken from Lemma 5.4 and the functions $z_i(\xi)$ are defined as $z_i(\xi) := \frac{\partial p_i}{\partial \kappa}(\xi, 0, 0)$ for $i = k+1, n+2$.

In addition, we calculate the derivatives of the solutions of (50) with respect to ν, which satisfy the system of differential equations

$$\begin{aligned} \frac{\partial}{\partial \xi} \begin{pmatrix} \frac{\partial p}{\partial \nu} \\ \frac{\partial q}{\partial \nu} \end{pmatrix} &= \begin{pmatrix} \frac{dF}{du}(\bar{u}(\xi, \nu), \mu(\nu)) & I_n \\ \kappa I_n & 0_n \end{pmatrix} \begin{pmatrix} \frac{\partial p}{\partial \nu} \\ \frac{\partial q}{\partial \nu} \end{pmatrix} + \\ &+ \begin{pmatrix} \frac{d^2F}{du^2}(\bar{u}(\xi, \nu), \mu(\nu))(\frac{\partial \bar{u}}{\partial \nu}, p) \\ 0 \end{pmatrix} + \\ &+ \begin{pmatrix} \frac{\partial^2 F}{\partial \mu \partial u}(\bar{u}(\xi, \nu), \mu(\nu))\ p\ \frac{d\mu}{d\nu}(\nu) \\ 0 \end{pmatrix}. \end{aligned} \tag{55}$$

The functions $\eta^u_k(\xi, \kappa, \nu)$ and $\eta^s_1(\xi, \kappa, \nu)$ satisfy the identities (48). Hence, the difference vector $(\eta^s_1 - \eta^u_k)(\xi, \kappa, \nu)$ vanishes identically for $\kappa = 0$ and sufficiently small ν, which implies

$$\frac{\partial}{\partial \nu}\left(\eta^s_1 - \eta^u_k\right)(\xi, 0, \nu) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{56}$$

The solutions $\eta^u_{k+1}(\xi, \kappa, \nu)$ and $\eta^s_2(\xi, \kappa, \nu)$ are chosen such that the identities (49) hold and are part of the fast manifold. By Lemma 4.3, their q-coordinates demonstrate for $\kappa = 0$ and sufficiently small ν the asymptotic behavior

$$\lim_{\xi \to -\infty} q_{k+1}(\xi, 0, \nu) = 0$$

and

$$\lim_{\xi \to +\infty} q_{n+2}(\xi, 0, \nu) = 0.$$

In addition, the order of taking the limit and the derivative, respectively, can be interchanged for these functions and their derivatives satisfy for $\kappa = 0$ and sufficiently small ν the asymptotic behavior

$$\lim_{\xi \to -\infty} \frac{\partial q_{k+1}}{\partial \nu}(\xi, 0, \nu) = 0 \tag{57}$$

and

$$\lim_{\xi \to +\infty} \frac{\partial q_{n+2}}{\partial \nu}(\xi, 0, \nu) = 0. \tag{58}$$

The derivatives of the solutions $\eta^u_{k+1}(\xi, \kappa, \nu)$ and $\eta^s_2(\xi, \kappa, \nu)$ are governed by the differential equations (55). In particular, the q-vectors satisfy for $\kappa = 0$ and sufficiently small ν the equations $\frac{\partial}{\partial \xi} \frac{\partial q_i}{\partial \nu}(\xi, 0, \nu) = 0 \in \mathbb{R}^n$. Thus we conclude that the q-vectors are constant and equal the null vector due to the limits (57) and (58). Hence, we obtain the expression

$$\frac{\partial}{\partial \nu}(\eta^s_2 - \eta^u_{k+1})(\xi, 0, 0) = \begin{pmatrix} y_{n+2} - y_{k+1} \\ 0 \end{pmatrix}(\xi, 0) \tag{59}$$

with functions $y_i(\xi)$ defined as $y_i(\xi) := \frac{\partial p_i}{\partial \nu}(\xi, 0, 0)$ for $i = k+1, n+2$.

We insert the vectors (51)–(54), (56) and (59) into the derivative of the Evans function $E(\kappa, \nu)$ at $(\kappa, \nu) = (0, 0)$ and obtain

$$\begin{aligned} &\frac{\partial^2 E}{\partial \nu \partial \kappa}(0,0) = \\ &= \det \begin{pmatrix} U^s_p & U^f_p & * & y_{n+2} - y_{k+1} & \tilde{S}^f_p & S^s_p \\ U^s_q & 0 & -[\bar{u}] & 0 & 0 & S^s_q \end{pmatrix}(0) + \\ &\quad + \det \begin{pmatrix} U^s_p & U^f_p & 0 & z_{n+2} - z_{k+1} & \tilde{S}^f_p & S^s_p \\ U^s_q & 0 & 0 & -[\bar{v}] & 0 & S^s_q \end{pmatrix}(0) \end{aligned}$$

with matrices

$$\begin{aligned} U^s_p(0) &:= (p_1, \ldots, p_{k-1})(0) \in \mathbb{R}^{n \times (k-1)}, \\ U^s_q(0) &:= (r_1(u^-), \ldots, r_{k-1}(u^-)) \in \mathbb{R}^{n \times (k-1)}, \\ U^f_p(0) &:= (p_k, \ldots, p_n)(0) \in \mathbb{R}^{n \times (n-k+1)}, \\ \tilde{S}^f_p(0) &:= (p_{n+3}, \ldots, p_{n+k})(0) \in \mathbb{R}^{n \times (k-2)}, \\ S^s_p(0) &:= (p_{n+k+1}, \ldots, p_{2n})(0) \in \mathbb{R}^{n \times (n-k)} \end{aligned}$$

and

$$S^s_q(0) := (r_{k+1}(u^+), \ldots, r_n(u^+)) \in \mathbb{R}^{n \times (n-k)}.$$

The second determinant vanishes, since it contains a null vector. However, in the first determinant we change the order of the vectors by k_p permutations to obtain a matrix in block diagonal form,

$$\begin{aligned} &\frac{\partial^2 E}{\partial \nu \partial \kappa}(0,0) = \\ &= (-1)^{k_p} \cdot \det \begin{pmatrix} U^f_p & \tilde{S}^f_p & y_{n+2} - y_{k+1} & U^s_p & * & S^s_p \\ 0 & 0 & 0 & U^s_q & -[\bar{u}] & S^s_q \end{pmatrix}, \end{aligned} \tag{60}$$

and factorize the expression into the product

$$\frac{\partial^2 E}{\partial\nu\partial\kappa}(0,0) = \\ = (-1)^{k_p+1}\cdot\Delta\cdot\det(U_p^f,\tilde{S}_p^f,y_{n+2}-y_{k+1})(0) \tag{61}$$

with the Liu-Majda determinant $\Delta = \det(U_q^s,[\bar{u}],S_q^s)$, see also (33). We write the second determinant as a sum of determinants and evaluate each summand in turn. The functions that span the matrices $U^f(\xi,0,0)$ and $\tilde{S}_p^f(\xi,0,0)$ are solutions of the linearized profile equation, which decay in at least one limit. In addition, the function $y_{n+2}(\xi) = \frac{\partial p_{n+2}}{\partial\nu}(\xi,0,0)$ is governed by the system of differential equations (55), which simplifies for $(\kappa,\nu)=(0,0)$ to

$$\frac{\partial y_{n+2}}{\partial\xi} = \frac{dF}{du}(\bar{u},0)y_{n+2} + \frac{d^2F}{du^2}(\bar{u},0)\left(\frac{\partial\bar{u}}{\partial\nu},\frac{\partial\bar{u}}{\partial\nu}\right)(\xi,0),$$

since the identities $p_{n+2}(\xi,0,0) = \frac{\partial\bar{u}}{\partial\nu}(\xi,0)$, $\frac{\partial q_{n+2}}{\partial\nu}(\xi,0,0)\equiv 0$, $\mu(0)=0$ and $\frac{d\mu}{d\nu}(0) = 0$ hold. The function $y_{n+2}(\xi)$ and the inhomogeneity of its differential equation are bounded on $\mathbb{R}_+$. Thus we can apply Lemma Appendix A.4 and obtain

$$\det(U_p^f,\tilde{S}_p^f,y_{n+2})(0) = \\ = -\int_0^{+\infty} < \psi, \frac{d^2F}{du^2}(\bar{u},0)\left(\frac{\partial\bar{u}}{\partial\nu},\frac{\partial\bar{u}}{\partial\nu}\right)(\xi,0) > d\xi.$$

In a similar way we derive

$$\det(U_p^f,\tilde{S}_p^f,y_{k+1})(0) = \\ = \int_{-\infty}^{0} < \psi, \frac{d^2F}{du^2}(\bar{u},0)\left(\frac{\partial\bar{u}}{\partial\nu},\frac{\partial\bar{u}}{\partial\nu}\right)(\xi,0) > d\xi.$$

Hence, the second determinant in (61) satisfies

$$\det(U_p^f,\tilde{S}_p^f,y_{n+2}-y_{k+1})(0) = \\ = -\int_{\mathbb{R}} < \psi, \frac{d^2F}{du^2}(\bar{u},0)\left(\frac{\partial\bar{u}}{\partial\nu},\frac{\partial\bar{u}}{\partial\nu}\right)(\xi,0) > d\xi.$$

We combine this expression with (61) and obtain the stated result. □

Corollary 5.6. *Suppose the assumptions (A1)–(A4) and the Liu-Majda condition* (34) *hold. Then the mixed derivative,* $\frac{\partial^2 E}{\partial\nu\partial\kappa}(\kappa,\nu)$*, of the Evans function* (43) *is non-zero at the point* $(\kappa,\nu)=(0,0)$*.*

Proof. We obtained in Theorem 5.5 the second order derivative of the Evans function as

$$\frac{\partial^2 E}{\partial\nu\partial\kappa}(0,0) =$$
$$= c\cdot\Delta\cdot\int_{\mathbb{R}} < \psi, \frac{d^2F}{du^2}(\bar{u},0)\left(\frac{\partial\bar{u}}{\partial\nu},\frac{\partial\bar{u}}{\partial\nu}\right)(\xi,0) > d\xi\cdot$$

with a non-zero constant c and the Liu-Majda determinant Δ in (33). By the results of Theorem 2.2 and Lemma 2.4, the integral expression equals the second order derivative of the Melnikov function at $(\nu,\mu)=(0,0)$,

$$\int_{\mathbb{R}} < \psi, \frac{d^2F}{du^2}(\bar{u},0)\left(\frac{\partial\bar{u}}{\partial\nu},\frac{\partial\bar{u}}{\partial\nu}\right)(\xi,0) > d\xi = \frac{\partial^2 M}{\partial\nu^2}(0,0),$$

which is non-zero by assumption (A4). In addition, we assumed that the Liu-Majda determinant does not vanish. Hence, the derivative of the Evans function is the product of non-zero factors, which proves the assertion. □

Theorem 5.7. *Suppose the assumptions (A1)–(A4) and the Liu-Majda condition* (34) *hold as well as the derivative of Evans function* $\frac{\partial^2 E}{\partial\kappa^2}(0,0)$ *is non-zero. Then a transcritical bifurcation occurs in the equation* $E(\kappa,\nu)=0$ *at the bifurcation point* $(\kappa,\nu)=(0,0)$*. In particular, the zero set of the Evans function consists close to the origin of two curves*

$$\{(\kappa,\nu)\in\mathbb{R}^2 \mid \kappa\equiv 0, \nu\in(-\varepsilon,\varepsilon)\} \tag{62}$$

and

$$\left\{(\kappa,\nu)\in\mathbb{R}^2 \mid \nu=\nu(\kappa), \kappa\in(-\varepsilon,\varepsilon)\right\}, \tag{63}$$

where ε *is a sufficiently small positive constant, and* $\nu:(-\varepsilon,\varepsilon)\to\mathbb{R}$, $\kappa\mapsto\nu(\kappa)$ *is a differentiable function such that* $\nu(0)=0$ *and*

$$\frac{d\nu}{d\kappa}(0) = -\frac{1}{2}\frac{\frac{\partial^2 E}{\partial\kappa^2}}{\frac{\partial^2 E}{\partial\nu\partial\kappa}}(0,0)\neq 0. \tag{64}$$

Moreover, the curves intersect transversally at the point $(\kappa,\nu)=(0,0)$.

Proof. It follows from equations (45), Corollary 5.6 and the assumption $\frac{\partial^2 E}{\partial\kappa^2}(0,0)\neq 0$, that a transcritical bifurcation as described in the theorem occurs, see e.g. [8]. □

Remark 5.8. If the Evans function satisfies instead $\frac{\partial^2 E}{\partial\kappa^2}(0,0)=0$, then other bifurcation scenarios such as a pitchfork bifurcation are possible, see Figure 3. In any case the curves of the zero set of the Evans function will intersect transversally at the bifurcation point.

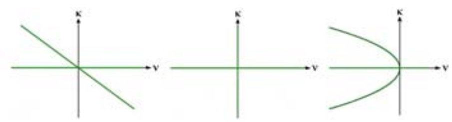

Figure 3: Bifurcation diagram of a transcritical, a degenerate and a pitchfork bifurcation.

In the next step, we identify viscous shock waves that are not spectrally stable.

Corollary 5.9. *Suppose the assumptions of Theorem 5.7 hold. Then the viscous shock waves with viscous profiles* $\bar{u}(\xi,\nu)$ *are not spectrally stable in the sense of Definition 1.1 for* $\nu = 0$ *and for sufficiently small positive (negative) values of* ν *if the constant* (64) *is positive (negative).*

Proof. By the result of Theorem 5.7, the zero set of the Evans function close to the origin consists of two curves (62) and (63), which represent effective eigenvalues. Since the derivative of a viscous profile is always a genuine eigenfunction associated to the effective eigenvalue zero, the curve (62) is present. The other curve (63) has a representation with respect to κ and the function $\nu(\kappa)$ satisfies the identity (64). For $\nu = 0$ the viscous shock wave is not spectrally stable due to the result of Lemma 5.2. Whereas in the other parameter regimes there exist positive real eigenvalues κ and the associated viscous shock wave is again not spectrally stable. □

Theorem 5.10. *In addition to the assumptions of Theorem 5.7, assume that* $\kappa = 0$ *is the only eigenvalue of the eigenvalue problem* (50) *associated to the viscous profile* $\bar{u}(\xi,0)$ *in* $\mathbb{C}_\beta$. *Then the viscous profiles* $\bar{u}(\xi,\nu)$ *are spectrally stable in the sense of Definition 1.1 for sufficiently small* $\nu > 0$ $(\nu < 0)$ *if the constant* (64) *is negative (positive).*

Proof. Eigenvalues close to zero behave as described in Theorem 5.7. Hence there exists a simple eigenvalue $\kappa = 0$ and a simple eigenvalue $\kappa < 0$ for ν in the considered ranges. It is well known that there exists a $R > 0$ such that $|\kappa| > R$ implies $E(\kappa,\nu) \neq 0$ for sufficiently small ν. Since $E(\kappa,0)$ has no other zeros in $\mathbb{C}_\beta$, $E(\kappa,\nu)$ has no other zeros in $\mathbb{C}_\beta$ for sufficiently small ν. □

6. Effective eigenvalue $\kappa = 0$

We consider a viscous profile $\bar{u}(\xi,0)$, which satisfies the assumptions (A1)–(A4) in Section 2 as well as the Liu-Majda condition (34), and study the existence and multiplicity of effective eigenfunctions to the effective eigenvalue zero.

Lemma 6.1. *Suppose the assumptions (A1)–(A4) hold. For* $(\kappa,\nu) = (0,0)$, *the eigenvalue equation* (50) *has* $n-1$ *linearly independent, globally bounded solutions, whose q-vector is constant, but different from the null vector.*

Proof. A solution of the proposed form has to satisfy the differential equation

$$\frac{dp}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}, 0)p(\xi) + q \tag{65}$$

with a constant vector $q \in \mathbb{R}^n \backslash \{0\}$. Since the homogeneous system associated to (65) has exponential dichotomies on $\mathbb{R}_-$ as well as $\mathbb{R}_+$, Palmer's Lemma [19, Lemma 4.2] is applicable. Moreover, we observe from Theorem 2.2 that

$$\frac{d\psi}{d\xi}(\xi) = -\left(\frac{dF}{du}(\bar{u}(\xi), 0)\right)^T \psi(\xi)$$

has a unique (up to a multiplicative factor) bounded solution $\psi(\xi)$. Thus a bounded solution of (65) exists if and only if

$$\int_{\mathbb{R}} < \psi(\xi), q > \; d\xi = 0.$$

For constant vectors q, we obtain a well-defined linear system of equations

$$< \int_{\mathbb{R}} \psi(\xi) \; d\xi, q >= 0.$$

Since the vector $\int_{\mathbb{R}} \psi(\xi) d\xi$ is different from the null vector, the kernel has dimension $n-1$ and we conclude the statement. □

Although the eigenvalue equation (50) for $(\kappa, \nu) = (0, 0)$ has $n-1$ bounded solutions with a constant q-vector different from the null vector, only solutions in the non-trivial intersection of the spaces

$$\text{span}\left\{\eta_j^u(\xi) = \begin{pmatrix} p_j(\xi) \\ r_j(u^-) \end{pmatrix} \middle| \; j = 1, \ldots, k-1\right\} \tag{66}$$

and

$$\text{span}\left\{\eta_j^s(\xi) = \begin{pmatrix} p_{n+j}(\xi) \\ r_j(u^+) \end{pmatrix} \middle| \; j = k+1, \ldots, n\right\} \tag{67}$$

are effective eigenfunctions. However, if the Liu-Majda condition (34) holds, then the intersection of the spaces (66) and (67) is necessarily trivial. In agreement with the modified Fredholm theory in [2], we define

Definition 6.2. A generalized eigenfunction for the eigenvalue $\kappa = 0$ is a non-trivial solution $p(\xi)$ of the generalized eigenvalue equation $L(Lp) = 0$.

Due to Lemma 5.2, the functions $\frac{\partial \bar{u}}{\partial \xi}(\xi)$ and $\frac{\partial \bar{u}}{\partial \nu}(\xi, 0)$ span the linear space of genuine eigenfunctions and are L^1-integrable. Hence, an associated solution $p(\xi)$ of the generalized eigenvalue equation $L(Lp) = 0$ has to satisfy the equation

$$\begin{aligned} Lp(\xi) &= \frac{d}{d\xi}\left(\frac{dp}{d\xi}(\xi) - \frac{dF}{du}(\bar{u}(\xi), 0)p(\xi)\right) \\ &= \gamma_1 \frac{\partial \bar{u}}{\partial \nu}(\xi, 0) + \gamma_2 \frac{\partial \bar{u}}{\partial \xi}(\xi, 0) \end{aligned}$$

for some real constants γ_1 and γ_2. After integrating the last identity with respect to ξ, we obtain the inhomogeneous linear system of ODEs

$$\frac{dp}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}(\xi), 0)p(\xi) + \tilde{b}(\xi) \tag{68}$$

with a continuous and bounded inhomogeneity

$$\tilde{b}(\xi) := \int_{-\infty}^{\xi} \left(\gamma_1 \frac{\partial \bar{u}}{\partial \nu}(x, 0) + \gamma_2 \frac{\partial \bar{u}}{\partial \xi}(x) \right) dx. \tag{69}$$

We will relate the existence of a bounded solution of (68) to the vanishing of the second order derivative of the Evans function, $\frac{\partial^2 E}{\partial \kappa^2}(\kappa, \nu)$, at the origin.

In preparation of Theorem 6.4 we derive an expression for the function $b(\xi)$.

Lemma 6.3. *Suppose the assumptions (A1)–(A4) and the Liu-Majda condition* (34) *hold. Then there exist real constants $\varphi_i \in \mathbb{R}$ for $i = 1, \ldots, n$ such that the identity*

$$\begin{aligned} &-(v^+ - v^-) = \\ &= \sum_{i=1}^{k-1} \varphi_i r_i(u^-) - \varphi_k(u^+ - u^-) + \sum_{i=k+1}^{n} \varphi_i r_i(u^+) \end{aligned} \tag{70}$$

is satisfied. Then the function

$$\begin{aligned} b(\xi) :&= \bar{v}(\xi) - v^+ - \varphi_k(\bar{u}(\xi) - u^+) - \sum_{i=k+1}^{n} \varphi_i r_i(u^+) \\ &= \bar{v}(\xi) - v^- - \varphi_k(\bar{u}(\xi) - u^-) + \sum_{i=1}^{k-1} \varphi_i r_i(u^-) \end{aligned}$$

is well-defined as well as continuous and bounded on $\mathbb{R}$.

Proof. The Liu-Majda condition implies that the set of vectors

$$\{r_i(u^-) \mid i = 1, \ldots, k-1\} \cup \{u^+ - u^-\} \cup \{r_i(u^+) \mid i = k, \ldots, n\}$$

forms a basis of $\mathbb{R}^n$ and the vector $v^+ - v^-$ has a representation (70) with respect to this basis. Thus the function $b(\xi)$ is well-defined and as a linear combination of continuous and bounded functions inherits these properties. ☐

Theorem 6.4. *Suppose the assumptions (A1)–(A4) and the Liu-Majda condition* (34) *hold. Then the second order derivative of the Evans function* (43) *with respect to the spectral parameter κ satisfies*

$$\frac{\partial^2 E}{\partial \kappa^2}(0, 0) = c \cdot \Delta \cdot \int_{\mathbb{R}} < \psi, b > (\xi) d\xi,$$

with a non-zero, real constant c, the Liu-Majda determinant Δ in (33)*, the function $\psi(\xi)$ in Theorem 2.2 and the function $b(\xi)$ in Lemma 6.3.*

Proof. We consider the Evans function $E(\kappa,\nu)$ in Theorem 5.1. Due to Lemma 5.2 and Remark 5.3, we can assume without loss of generality that, for sufficiently small ν, solutions of the eigenvalue equation are given by

$$\eta_k^u(\xi,0,\nu) = \eta_1^s(\xi,0,\nu) = \begin{pmatrix} \frac{\partial \bar{u}}{\partial \xi} \\ 0 \end{pmatrix}(\xi,\nu) \tag{71}$$

and

$$\eta_{k+1}^u(\xi,0,0) = \eta_2^s(\xi,0,0) = \begin{pmatrix} \frac{\partial \bar{u}}{\partial \nu} \\ 0 \end{pmatrix}(\xi,0). \tag{72}$$

We rewrite the Evans function,

$$\begin{aligned} &E(\kappa,\nu) = \\ &= \det\big(\eta_1^u,\ldots,\eta_n^u,\eta_1^s-\eta_k^u,\eta_2^s-\eta_{k+1}^u,\eta_3^s,\ldots,\eta_n^s\big)(0,\kappa,\nu), \end{aligned}$$

differentiate twice with respect to κ by the Leibniz rule and evaluate the derivative at $(\kappa,\nu)=(0,0)$. In this way, we obtain

$$\begin{aligned} &\frac{\partial^2 E}{\partial \kappa^2}(0,0) = \\ &= \det\left(\eta_1^u,\ldots,\eta_n^u,\frac{\partial}{\partial\kappa}(\eta_1^s-\eta_k^u),\frac{\partial}{\partial\kappa}(\eta_2^s-\eta_{k+1}^u),\eta_3^s,\ldots,\eta_n^s\right)(0). \end{aligned}$$

All other summands vanish at $(\kappa,\nu)=(0,0)$, since they contain a vector $(\eta_1^s - \eta_k^u)(0,0,0)$ and/or $(\eta_2^s-\eta_{k+1}^u)(0,0,0)$ which coincide with the null vector.

In the proof of Theorem 5.5 we computed the derivatives of the solutions with respect to the spectral parameter κ and obtained the expressions

$$\frac{\partial}{\partial\kappa}\big(\eta_1^s-\eta_k^u\big)(\xi,0,0) = \begin{pmatrix} z_{n+1}(\xi)-z_k(\xi) \\ -(u^+-u^-) \end{pmatrix}$$

and

$$\frac{\partial}{\partial\kappa}\big(\eta_2^s-\eta_{k+1}^u\big)(\xi,0,0) = \begin{pmatrix} z_{n+2}(\xi)-z_{k+1}(\xi) \\ -(v^+-v^-) \end{pmatrix}$$

with functions $z_i(\xi) := \frac{\partial p_i}{\partial \kappa}(\xi,0,0)$ for $i=k,k+1,n+1,n+2$ and $\bar{v}(\xi) := \int_{-\infty}^{\xi}\frac{\partial\bar{u}}{\partial\nu}(x,0)dx$ with asymptotic limits $v^\pm := \lim_{\xi\to\pm\infty}\bar{v}(\xi)$. We insert these expressions into the derivative of the Evans function $E(\kappa,\nu)$ at $(\kappa,\nu)=(0,0)$ and obtain

$$\begin{aligned} &\frac{\partial^2 E}{\partial \kappa^2}(0,0) = \\ &= \det\begin{pmatrix} U_p^s & U_p^f & z_{n+1}-z_k & z_{n+2}-z_{k+1} & \tilde{S}_p^f & S_p^s \\ U_q^s & 0 & -[\bar{u}] & -[\bar{v}] & 0 & S_q^s \end{pmatrix}(0) \end{aligned} \tag{73}$$

with matrices

$$
\begin{aligned}
U_p^s(0) &:= (p_1, \dots, p_{k-1})(0) \in \mathbb{R}^{n\times(k-1)},\\
U_q^s(0) &:= (r_1(u^-), \dots, r_{k-1}(u^-)) \in \mathbb{R}^{n\times(k-1)},\\
U_p^f(0) &:= (p_k, \dots, p_n)(0) \in \mathbb{R}^{n\times(n-k+1)},\\
\tilde{S}_p^f(0) &:= (p_{n+3}, \dots, p_{n+k})(0) \in \mathbb{R}^{n\times(k-2)},\\
S_p^s(0) &:= (p_{n+k+1}, \dots, p_{2n})(0) \in \mathbb{R}^{n\times(n-k)}
\end{aligned}
$$

and

$$
S_q^s(0) := (r_{k+1}(u^+), \dots, r_n(u^+)) \in \mathbb{R}^{n\times(n-k)}.
$$

In the matrix within the determinant (73) the q-coordinates of $n+1$ vectors are different from the null vector. In addition, the Liu-Majda condition (34) implies that there exist real constants $\varphi_i \in \mathbb{R}$ for $i = 1, \dots, n$ such that the vector $[\bar{v}] = v^+ - v^-$ has a representation

$$
-[\bar{v}] = \sum_{i=1}^{k-1} \varphi_i r_i(u^-) - \varphi_k[\bar{u}] + \sum_{i=k+1}^{n} \varphi_i r_i(u^+). \tag{74}
$$

We take this linear combination into account and transform the determinant,

$$
\begin{aligned}
&\frac{\partial^2 E}{\partial \kappa^2}(0,0) =\\
&= \det \begin{pmatrix} U_p^s & U_p^f & z_{n+1} - z_k & z^+ - z^- & \tilde{S}_p^f & S_p^s \\ U_q^s & 0 & -[\bar{u}] & 0 & 0 & S_q^s \end{pmatrix}(0),
\end{aligned}
$$

where the auxiliary functions are defined as

$$
z^+(\xi) :\,= z_{n+2}(\xi) - \varphi_k z_{n+1}(\xi) - \sum_{i=k+1}^{n} \varphi_i p_{n+i}(\xi)
$$

and

$$
z^-(\xi) :\,= z_{k+1}(\xi) - \varphi_k z_k(\xi) + \sum_{i=1}^{k-1} \varphi_i p_i(\xi),
$$

respectively. In the next step, we change the order of the vectors by k_p permutations to obtain a matrix in block diagonal form,

$$
\begin{aligned}
&\frac{\partial^2 E}{\partial \kappa^2}(0,0) =\\
&= (-1)^{k_p} \cdot \det \begin{pmatrix} U_p^f & \tilde{S}_p^f & z^+ - z^- & U_p^s & z_{n+1} - z_k & S_p^s \\ 0 & 0 & 0 & U_q^s & -[\bar{u}] & S_q^s \end{pmatrix}(0),
\end{aligned}
$$

and factorize the expression into the product

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = (-1)^{k_p+1} \cdot \Delta \cdot \det(U_p^f, \tilde{S}_p^f, z^+ - z^-)(0) \tag{75}$$

with the Liu-Majda determinant Δ in (33). We rewrite the first determinant as a sum of determinants and evaluate each summand in turn. The function $z^+(\xi)$ is governed by a linear differential equation

$$\begin{aligned}
\frac{dz^+}{d\xi}(\xi) =& \frac{dz_{n+2}}{d\xi}(\xi) - \varphi_k \frac{dz_{n+1}}{d\xi}(\xi) - \sum_{i=k+1}^{n} \varphi_i \frac{dp_{n+i}}{d\xi}(\xi) \\
=& \frac{dF}{du}(\bar{u}(\xi)) z_{n+2}(\xi) + (\bar{v}(\xi) - v^+) \\
& - \varphi_k \left[\frac{dF}{du}(\bar{u}(\xi)) z_{n+1}(\xi) + (\bar{u}(\xi) - u^+) \right] \\
& - \sum_{i=k+1}^{n} \varphi_i \left[\frac{dF}{du}(\bar{u}(\xi)) p_{n+i}(\xi) + r_i(u^+) \right] \\
=& \frac{dF}{du}(\bar{u}(\xi)) z^+(\xi) + b^+(\xi)
\end{aligned}$$

with inhomogeneity

$$\begin{aligned}
b^+(\xi) :=& (\bar{v}(\xi) - v^+) - \varphi_k(\bar{u}(\xi) - u^+) - \\
& - \sum_{i=k+1}^{n} \varphi_i r_i(u^+).
\end{aligned}$$

The functions $z^+(\xi)$ and $b^+(\xi)$ are bounded on $\mathbb{R}_+$, since they are linear combinations of bounded functions. Thus, the requirements of Lemma Appendix A.4 are met and we obtain

$$\det(U_p^f, \tilde{S}_p^f, z^+)(0) = -\int_0^{+\infty} < \psi, b^+ > d\xi.$$

In a similar way, we derive the expression

$$\det(U_p^f, \tilde{S}_p^f, z^-)(0) = \int_{-\infty}^{0} < \psi, b^- > d\xi,$$

where the bounded function $b^-(\xi)$ is defined as

$$b^-(\xi) := (\bar{v}(\xi) - v^-) - \varphi_k(\bar{u}(\xi) - u^-) + \sum_{i=1}^{k-1} \varphi_i r_i(u^-).$$

The linear combination (74) implies the identity $b^+(\xi) \equiv b^-(\xi)$ and we define $b(\xi) := b^+(\xi)$. Thus we obtain the expression

$$\det(U_p^f, \tilde{S}_p^f, z^+ - z^-)(0) = -\int_{\mathbb{R}} < \psi, b > d\xi$$

and conclude from (75) the stated result. □

In the following, we will prove the connection between the existence of a bounded solution of the generalized eigenvalue equation for $(\kappa, \nu) = (0, 0)$ and the second order derivative of the Evans function $\frac{\partial^2 E}{\partial \kappa^2}(0, 0)$.

Theorem 6.5. *Suppose the assumptions (A1)–(A4) and the Liu-Majda condition* (34) *hold. Then the second order derivative of the Evans function* (43) *with respect to the spectral parameter,* $\frac{\partial^2 E}{\partial \kappa^2}(\kappa, \nu)$*, vanishes at* $(\kappa, \nu) = (0, 0)$*, if and only if there exists a generalized eigenfunction for the effective eigenvalue zero that is bounded on* $\mathbb{R}$ *and associated to the genuine eigenfunction* $\frac{\partial \bar{u}}{\partial \nu}(\xi, 0) + \varphi_k \frac{\partial \bar{u}}{\partial \xi}(\xi)$ *with the constant* φ_k *from Lemma 6.3.*

Proof. By the assumptions and the result of Theorem 6.4, the second order derivative of the Evans function, $\frac{d^2 E}{d\kappa^2}(0, 0)$, has a factorization into a product of non-zero factors and the definite integral $\int_{\mathbb{R}} < \psi, b > (\xi) d\xi$, where the function $b(\xi)$ is taken from Lemma 6.3 and $\psi(\xi)$ is the unique (up to a multiplicative factor) bounded solution of the adjoint problem in Theorem 2.2. By Palmer's Lemma [19, Lemma 4.2], the condition

$$\int_{\mathbb{R}} < \psi, b > (\xi) d\xi = 0$$

is equivalent to the existence of a bounded solution of the inhomogeneous linear system of differential equations (68). Since the inhomogeneity $b(\xi)$ has the proposed form (69) with $\gamma_1 = 1$ and $\gamma_2 = \varphi_k$, the statement follows. □

Remark 6.6. The results of Sections 5 and 6 are not sufficient to decide spectral stability of the magnetohydrodynamic (MHD) viscous shock waves considered in Section 3. One reason is that the profile equation (19) are a reduced system of the full system governing magnetohydrodynamic viscous shock waves, which is derived by geometric singular perturbation arguments. To study the Evans function near $(0, 0)$ for the full MHD system, a similar slow-fast reduction of the corresponding eigenvalue problem is needed.

Another more severe obstacle is the fact that the intermediate MHD shocks considered in Section 3 do not satisfy the Liu-Majda condition (34). The latter point is addressed in the doctoral thesis [20]. Although the Evans function is more degenerate, similar results on the multiplicity of the effective eigenvalue zero are obtained.

Appendix A. Linear systems of ODEs

We consider a viscous profile $\bar{u}(\xi, 0)$, which satisfies the assumptions (A1)–(A4) in Section 2. Any tangent vector of the associated invariant manifolds $W^u(u^-)$ and $W^s(u^+)$ is a solution of the linearized profile equation

$$\frac{dp}{d\xi} = \frac{dF}{du}(\bar{u}, 0)\, p.$$

In the following, we will construct the unique (up to a multiplicative constant) solution of the adjoint problem (13), that is orthogonal to these tangent vectors, via a generalization of the cross product to higher dimensions.

Definition Appendix A.1 ([21])**.** Let $n \geq 2$ and e_i with $i = 1, \dots, n$ denote the Euclidean basis vectors of the real vector space $\mathbb{R}^n$. For $n-1$ vectors $p_1, \dots, p_{n-1}$ in $\mathbb{R}^n$, we define the *(generalized) cross product* as the vector

$$p_1 \times \cdots \times p_{n-1} = \sum_{j=1}^{n} \det\big(p_1, \dots, p_{n-1}, e_j\big) e_j.$$

We state some properties of the (generalized) cross product.

Lemma Appendix A.2 ([21])**.** *Let $n \geq 2$ and w as well as p_i with $i = 1, \dots, n-1$ be vectors in $\mathbb{R}^n$.*

(i) *The matrix spanned by the given vectors satisfies the identity*

$$\det(p_1, \dots, p_{n-1}, w) = < p_1 \times \cdots \times p_{n-1}, w > . \tag{A.1}$$

(ii) *The cross product $p_1 \times \cdots \times p_{n-1}$ is perpendicular to any vector p_i with $i = 1, \dots, n-1$.*

(iii) *The cross product $p_1 \times \cdots \times p_{n-1}$ is equal to the null vector if and only if the vectors p_i with $i = 1, \dots, n-1$ are linearly dependent.*

(iv) *In addition, let C be a quadratic matrix whose coefficients c_{ij} are defined by $c_{ij} = \frac{<p_i, p_j>}{\|p_i\| \cdot \|p_j\|}$ for $i, j = 1, \dots, n-1$. Then the length of the cross product satisfies*

$$\|p_1 \times \cdots \times p_{n-1}\| = \|p_1\| \cdot \|p_2\| \cdot \ \cdots \ \cdot \|p_{n-1}\| \cdot (\det(C))^{1/2}.$$

The construction of the bounded solution $\psi(\xi)$ in the case of planar [1, 19] and higher dimensional systems [22, Section 4] is well known. However, we use the generalized cross product to obtain an equivalent expression, which will simplify the proofs involving the bounded solution $\psi(\xi)$.

Theorem Appendix A.3. *Let $\{p_i \in C^1(\mathbb{R}; \mathbb{R}^n) \mid i = 1, \dots, n-1\}$ be a basis for the union of the tangent spaces $T_{\bar{u}(\xi)} W^u(u^-)$ and $T_{\bar{u}(\xi)} W^s(u^+)$. Then the tangent vectors $p_i(\xi)$ for $i = 1, \dots, n-1$ are solutions of the linearized profile equation,*

$$\frac{dp}{d\xi} = \frac{dF}{du}(\bar{u}, 0)\, p, \tag{A.2}$$

whose norms decay to zero in the limit $\xi \to -\infty$ and/or $\xi \to +\infty$. In addition, the function $\psi(\xi)$ defined as

$$\psi(\xi) := \exp\left(- \int_0^{\xi} \operatorname{trace}\left(\frac{dF}{du}(\bar{u}, \mu_0) \right) dx \right) (p_1 \times \ldots \times p_{n-1})(\xi) \tag{A.3}$$

is the globally bounded solution of the adjoint problem,

$$\frac{d\psi}{d\xi} = -\left(\frac{dF}{du}(\bar{u}, 0) \right)^T \psi, \tag{A.4}$$

which is unique up to a multiplicative constant and is an element of $C^{\infty}_{\exp}(\mathbb{R};\mathbb{R}^n)$*, i.e. the function* $\psi(\xi)$ *decays exponentially in both limits.*

Later we will need the following technical result, which follows from Palmer's Lemma [19, Lemma 4.2].

Lemma Appendix A.4. *Consider the solutions* $p_i(\xi)$ *for* $i = 1, \ldots, n-1$ *and* $\psi(\xi)$ *of the linearized profile equation and the adjoint differential equation, respectively, in Theorem Appendix A.3. If* $y(\xi)$ *is a bounded solution of the inhomogeneous differential equation*

$$\frac{dy}{d\xi} = \frac{dF}{du}(\bar{u}, 0)\ y + b,$$

where the inhomogeneity $b : \mathbb{R} \to \mathbb{R}^n$ *is a bounded function, then we obtain*

$$\det(p_1, \ldots, p_{n-1}, y)(0) = -\int_0^{+\infty} < \psi, b > \ d\xi$$
$$= \int_{-\infty}^{0} < \psi, b > \ d\xi.$$

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