ASC Report No. 9/2010

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Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-03-2

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ISBN 978-3-902627-03-2

ESTIMATOR REDUCTION AND CONVERGENCE OF ADAPTIVE BEM

M. AURADA, S. FERRAZ-LEITE, AND D. PRAETORIUS

Abstract. A posteriori error estimation and related adaptive mesh-refining algorithms have themselves proven to be powerful tools in nowadays scientific computing. Contrary to adaptive finite element methods, convergence of adaptive boundary element schemes is, however, widely open. We propose a relaxed notion of convergence of adaptive boundary element schemes. Instead of asking for convergence of the error to zero, we only aim to prove estimator convergence in the sense that the adaptive algorithm drives the underlying error estimator to zero. We observe that certain error estimators satisfy an estimator reduction property which is sufficient for estimator convergence. The elementary analysis is only based on Dörfler marking and inverse estimates, but not on reliability and efficiency of the error estimator at hand. In particular, our approach gives a first mathematical justification for the proposed steering of anisotropic mesh-refinements, which is mandatory for optimal convergence behaviour in 3D boundary element computations.

Dedicated to Professor George Hsiao on the occasion of his 75th birthday

1. Introduction

1.1. Convergence of Adaptive Algorithms. In many applications, numerical simulations are based on a triangulation $\mathcal{T}_\ell := \{T_1, \dots, T_N\}$ of the simulation domain. Let the (unknown) exact solution u belong to a certain Hilbert space $\mathcal{H}$ with norm $|||\cdot|||$. Then, for some discrete subspace X_ℓ of $\mathcal{H}$ associated with $\mathcal{T}_\ell$, a numerical approximation $u_\ell \in \mathcal{H}$ is computed. Refinement of $\mathcal{T}_\ell$ yields an improved approximation. Usually, in the context of boundary integral equations, u has certain singularities so that uniform mesh-refinement leads to a poor convergence behaviour for the error $|||u - u_\ell|||$. Contrary, adaptive algorithms have themselves proven to provide an effective means to improve the accuracy of u_ℓ. Based on the local contributions $\rho_\ell(T_j)$ of an a posteriori error estimator ρ_ℓ, these algorithms only refine certain elements $T_j \in \mathcal{T}_\ell$, where the error appears to be large. Starting from an initial mesh $\mathcal{T}_0$, this procedure generates a sequence of triangulations $\mathcal{T}_\ell$ and corresponding discrete solutions $u_\ell \in X_\ell$. However, since adaptive mesh-refinement does not guarantee that

$$\max_{T_j \in \mathcal{T}_\ell} \mathrm{diam}(T_j) \xrightarrow{\ell \to \infty} 0, \tag{1.1}$$

in general, the verification of *convergence*

$$|||u - u_\ell||| \xrightarrow{\ell \to \infty} 0 \tag{1.2}$$

is a nontrivial issue. Whereas (1.2) is well-studied for adaptive finite element methods (AFEM), see [16, 27, 33] and the references therein, this question is essentially open for adaptive boundary element methods (ABEM), where only preliminary convergence results [15, 22] are available.

Date: March 22, 2010.

1.2. Concept of Estimator Reduction. We aim at contributing to the mathematical understanding of h-adaptive BEM. To that end, we primarily ask for *estimator convergence*

$$\rho_\ell \xrightarrow{\ell\to\infty} 0. \tag{1.3}$$

For certain estimators from the BEM literature, we prove that the marking criterion from [17] guarantees some *estimator reduction*

$$\rho_{\ell+1} \le q\,\rho_\ell + C\,|||u_{\ell+1} - u_\ell||| \quad \text{for all } \ell \in \mathbb{N} \tag{1.4}$$

with ℓ-independent constants $0 < q < 1$ and $C > 0$. If the sequence u_ℓ of discrete solutions is convergent to some (unknown) limit

$$u_\infty := \lim_{\ell\to\infty} u_\ell \in \mathcal{H}, \tag{1.5}$$

as is the case in usual adaptive Galerkin schemes, the estimator reduction (1.4) already implies the estimator convergence (1.3). If furthermore the estimator ρ_ℓ provides some upper bound for the error $|||u - u_\ell|||$, this yields convergence (1.2) and, in particular, $u = u_\infty$.

1.3. Main Results & Outline. In Section 2, we state our version of the adaptive algorithm (Algorithm 2.1), observe that adaptive Galerkin BEM always guarantees the a priori convergence (1.5), and prove that the estimator reduction (1.4) thus implies the estimator convergence (1.3). In the remainder of this work, the weakly-singular integral equation for the 2D and 3D Laplacian serves as model problem. This and the lowest-order Galerkin BEM are stated in Section 3. We then focus on $(h - h/2)$-type estimators from [23] and averaging estimators from [12, 14]. In Section 4, we observe that isotropic mesh-refinement steered by these estimators guarantees (1.4), see Theorem 4.1 and 4.3. In 3D BEM, however, anisotropic mesh-refinement is, in general, necessary to resolve edge singularities effectively. Even in the context of FEM, there are —to the best of our knowledge— no rigorous convergence results for adaptive Galerkin schemes with anisotropic mesh-refinement. In Section 5, we consider a heuristics from [23] to steer anisotropic adaptive mesh-refinement. First, this idea is generalized from the $(h - h/2)$-error estimator to the averaging error estimators. Second, we prove that the proposed adaptive schemes again guarantee the estimator reduction (1.4), see Theorem 5.1 and 5.2. This means that the concept of estimator reduction gives a mathematical justification for the anisotropic refinement criterion used and allows for a first convergence result of adaptive anisotropic 3D BEM. Numerical experiments included in Section 4 and Section 5 underline our theoretical findings.

1.4. Some Remarks. We stress that the verification of estimator reduction (1.4) in Theorem 4.1, 4.3, 5.1, and 5.2 depends only on the definition of the local mesh-refinement and on a local inverse estimate. Moreover, our analysis applies to a quite general class of local mesh-refinement rules, e.g., any rule based on newest-vertex bisection or even anisotropic mesh-refinement with rectangular elements. In particular, the proof of estimator convergence (1.3) does neither use reliability, nor efficiency of the error estimator ρ_ℓ at hand, i.e., estimator convergence is independent of whether ρ_ℓ provides a lower or upper bound for $|||u - u_\ell|||$.

It came as a surprise to us that a convergence result (1.2) for adaptive Galerkin schemes can thus be obtained by an elementary and straight-forward analysis. In particular, we see that convergence (1.2) relies only on the reliability of ρ_ℓ, but not on (discrete local) efficiency as used e.g. in [26, 27] in the context of AFEM.

1.5. Possible Generalizations & Extensions. The results of this paper, although stated for the weakly-singular integral equation, also apply to other elliptic integral equations and the corresponding $(h-h/2)$ or averaging error estimators; we refer to [20] for $(h-h/2)$-based error estimators and to [13, 14] for averaging error estimators for some hypersingular integral equation in 2D. Moreover, the concept of estimator reduction provides a general framework for convergence analysis. Further applications read as follows:

First, in BEM computations, it is usually necessary to discretize the given data to work with discrete integral operators only. If the data are discretized by projecting them appropriately, this provides some a priori convergence of the data. Although the discrete solutions u_ℓ are then computed with respect to different right-hand sides, this allows to prove a priori convergence (1.5). This concept is followed in [3, 4] to generalize this work for a 2D integral equation with approximate right-hand side, where given Dirichlet and Neumann data as well as the given volume force are appropriately discretized.

Second, the same ideas are followed to analyze an adaptive FEM-BEM coupling for some nonlinear 2D transmission problem [2], which is proven to converge.

Third, although in the context of AFEM for linear problems, even optimality results are available [7, 16, 33], it is worth noting that our approach also covers the residual-type error estimators. The current convergence (and optimality) analysis includes the proof of (2.5), although not explicitly stated, see [16, Corollary 3.4]. By use of (2.5) and the Galerkin orthogonality, one then proves a contraction property for the weighted sum $\Delta_\ell := \|\!|u - u_\ell|\!\|^2 + \gamma\,\rho_\ell^2$ of Galerkin error and residual error estimator, where $\gamma \sim C^{-2}$ with the constant C from (1.4), see [16, Proof of Theorem 4.1] or [22, Proof of Theorem 4].

Fourth, for elliptic obstacle problems, it is shown in [29] that the reliable error estimator from [5] satisfies (2.5) without a strong restriction on the local mesh-refinement like the *interior node property*, which has been introduced in [26] and used, e.g., in [5, 6, 9, 10, 11]. Moreover, Galerkin schemes for elliptic obstacle problems also guarantee the a priori convergence (1.5) so that our concept also applies [28].

Fifth, for adaptive finite volume methods (AFVM), the a priori convergence (1.5) is, in general, open. With minor modifications of the proof of [16], it is, however, easily seen that the residual error estimator ρ_ℓ from [21] satisfies the estimator reduction (1.4). Since ρ_ℓ provides an upper bound on the FVM error, our theoretical observation reads as follows: If AFVM leads to some convergent sequence of discrete solutions u_ℓ, its limit u_∞ is necessarily the exact solution u.

2. Some Abstract Observations

2.1. Adaptive Mesh-Refining Algorithm. Usually, the a posteriori error estimator ρ_ℓ can be written in the form

$$\rho_\ell := \Big(\sum_{T\in\mathcal{T}_\ell} \rho_\ell(T)^2\Big)^{1/2}, \tag{2.1}$$

where $\rho_\ell(T)$ is a *computable* quantity which measures —at least heuristically— the local contribution of the error $|||u - u_\ell|||_T$ on $T \in \mathcal{T}_\ell$. We consider the usual adaptive algorithm

$$\boxed{\texttt{solve}} \to \boxed{\texttt{estimate}} \to \boxed{\texttt{mark}} \to \boxed{\texttt{refine}} \tag{2.2}$$

which formally reads as follows:

Algorithm 2.1. *Fix $0 < \theta < 1$ and let $\mathcal{T}_\ell$ with $\ell = 0$ be the initial triangulation. For each $\ell = 0, 1, 2, \dots$ do:*

(i) *Compute discrete solution u_ℓ and error estimator ρ_ℓ.*
(ii) *Determine a set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that*

$$\theta \sum_{T \in \mathcal{T}_\ell} \rho_\ell(T)^2 \le \sum_{T \in \mathcal{M}_\ell} \rho_\ell(T)^2. \tag{2.3}$$

(iii) *Refine at least marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$.*
(iv) *Increase counter $\ell \mapsto \ell + 1$ an iterate.* □

The marking criterion (2.3) was introduced by Dörfler [17] to analyze convergence (1.2) of AFEM for the Poisson problem with ρ_ℓ being the residual error estimator. He proved convergence up to some tolerance $\tau > 0$ prescribed by the a priori resolution of the given data. His convergence result was improved by [26] who included the resolution and convergence of the data in terms of the so-called *data oscillation*. Optimality of this AFEM was first proved by [33], where the set $\mathcal{M}_\ell$ has to be chosen with minimal cardinality.

Note that the set $\mathcal{R}_\ell := \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}$ of elements which are eventually refined in (iii) is most likely a superset of $\mathcal{M}_\ell$ due to certain mesh properties which have to be conserved, e.g., regularity of the mesh, avoidance of high-order hanging nodes, uniform boundedness of the K-mesh constant etc.

2.2. A Priori Convergence of Adaptive Galerkin Schemes. If X_ℓ consists of $\mathcal{T}_\ell$-piecewise polynomials, Algorithm 2.1 provides strictly nested discrete subspaces, i.e. $X_\ell \subset X_{\ell+1}$ for all $\ell \in \mathbb{N}$. The first lemma now proves that nestedness of the discrete spaces implies the a priori convergence (1.5) of Galerkin schemes. Although this result, even in a more general formulation, is found e.g. in [27, Lemma 4.2] or [15, Lemma 1.1], we include a proof for the convenience of the reader. By others, it will become clear why the a priori limit u_∞ does not necessarily coincide with the continuous solution u.

Lemma 2.2. *Suppose that $\mathcal{H}$ is a Hilbert space with norm $|||\cdot|||$ and X_ℓ is a sequence of nested closed subspaces, i.e. $X_\ell \subseteq X_{\ell+1}$. For fixed $u \in \mathcal{H}$, let $u_\ell \in X_\ell$ be the best approximation with respect to X_ℓ, i.e.*

$$|||u - u_\ell||| = \min_{v_\ell \in X_\ell} |||u - v_\ell|||. \tag{2.4}$$

Then, the limit $\lim_{\ell\to\infty} u_\ell \in \mathcal{H}$ exists. In particular, there holds $\lim_{\ell\to\infty} |||u_{\ell+1} - u_\ell||| = 0$.

Proof. Let X_∞ be the closure of $\bigcup_{\ell=0}^\infty X_\ell$ in $\mathcal{H}$. Then, X_∞ is a closed subspace of $\mathcal{H}$, and the best approximation $u_\infty \in X_\infty$ of u with respect to X_∞ exists. Best approximation in Hilbert spaces is realized in terms of the orthogonal projection so that the Pythagoras theorem reads

$$|||u - u_\ell|||^2 = |||u - u_\infty|||^2 + |||u_\infty - u_\ell|||^2.$$

In particular, u_ℓ is even the best approximation of u_∞ with respect to X_ℓ. Let $\varepsilon > 0$. Since $\bigcup_{\ell=0}^\infty X_\ell$ is dense in X_∞ and since the spaces X_ℓ are nested, we may choose some index ℓ_0 and some element $v_{\ell_0} \in X_{\ell_0}$ such that $|||u_\infty - v_{\ell_0}||| \le \varepsilon$. For $\ell \ge \ell_0$, the inclusion $X_{\ell_0} \subseteq X_\ell$ thus concludes $|||u_\infty - u_\ell||| = \min_{v_\ell \in X_\ell} |||u_\infty - v_\ell||| \le |||u_\infty - v_{\ell_0}||| \le \varepsilon$. □

2.3. Estimator Reduction Implies Estimator Convergence. Finally, we include the elementary proof that estimator reduction (1.4) implies estimator convergence (1.3). For ρ_ℓ being the error and α_ℓ being the data oscillations, the following result can also be found in [26], where vanishing data oscillations imply the convergence of AFEM.

Lemma 2.3. *Suppose that the sequence of error estimators $(\rho_\ell)_{\ell\in\mathbb{N}}$ satisfies some estimator reduction property*

$$\rho_{\ell+1} \le q\,\rho_\ell + \alpha_\ell \quad \textit{for all } \ell \in \mathbb{N}_0 \tag{2.5}$$

with some fixed constant $0 < q < 1$ and some non-negative sequence $(\alpha_\ell)_{\ell\in\mathbb{N}}$ which satisfies $\lim_{\ell\to\infty} \alpha_\ell = 0$. Then, there holds the estimator convergence $\lim_{\ell\to\infty} \rho_\ell = 0$.

Proof. By induction on ℓ, the estimator reduction (2.5) implies

$$\rho_{\ell+1} \le q^{\ell+1}\rho_0 + \sum_{j=0}^{\ell} q^{\ell-j}\alpha_j \le q^{\ell+1}\rho_0 + \|(\alpha_n)\|_\infty \sum_{k=0}^{\ell} q^k \le \rho_0 + \frac{\|(\alpha_n)\|_\infty}{1-q}$$

with $\|(\alpha_n)\|_\infty$ the supremum norm of the bounded sequence (α_n). In particular, the sequence (ρ_n) is bounded and $0 \le M := \limsup_{\ell\to\infty} \rho_\ell < \infty$ exists. Again, we apply (2.5) to see

$$M = \limsup_{\ell\to\infty} \rho_{\ell+1} \le q \limsup_{\ell\to\infty} \rho_\ell + \limsup_{\ell\to\infty} \alpha_\ell = q\,M.$$

With $0 < q < 1$, this yields $0 \le \liminf_{\ell\to\infty} \rho_\ell \le \limsup_{\ell\to\infty} \rho_\ell = 0$ and thus convergence to zero. □

3. Model Problem

3.1. Weakly-Singular Integral Equation. Throughout, we consider the first-kind integral equation

$$(Vu)(x) := \int_\Gamma G(x,y)u(y)\,d\Gamma(y) = f(x) \quad \text{for } x \in \Gamma \tag{3.1}$$

with weakly-singular integral kernel

$$G(x,y) = -\frac{1}{2\pi} \log|x-y| \text{ for } d = 2 \quad \text{and} \quad G(x,y) = +\frac{1}{4\pi}\frac{1}{|x-y|} \text{ for } d = 3. \tag{3.2}$$

Here, Γ is an open piece of the boundary $\partial\Omega$ of a Lipschitz domain Ω in $\mathbb{R}^d$, and $d\Gamma$ denotes the integration along the arc or on the manifold for $d = 2, 3$, respectively. We recall the definition of the fractional order Sobolev space $H^{1/2}(\Gamma)$ as trace space

$$H^{1/2}(\Gamma) := \{F|_\Gamma \,:\, F \in H^1(\Omega)\}. \tag{3.3}$$

The negative order Sobolev space $\mathcal{H} := \widetilde{H}^{-1/2}(\Gamma)$ is the algebraic topological dual with respect to the extended L^2-scalar product $\langle \cdot , \cdot \rangle$. For $d = 2$, we additionally assume $\mathrm{diam}(\Omega) < 1$. Then, $V : \mathcal{H} \to \mathcal{H}^*$ is a symmetric and elliptic isomorphism, see e.g. [25, 32, 30]. Thus,

$$\langle\!\langle u , v \rangle\!\rangle := \langle V u , v \rangle \quad \text{for all } u, v \in \mathcal{H} \tag{3.4}$$

defines a scalar product, and the induced energy norm $|\!|\!| v |\!|\!| := \langle\!\langle v , v \rangle\!\rangle^{1/2}$ is an equivalent norm on $\mathcal{H}$. For $f \in H^{1/2}(\Gamma)$, the model problem (3.1) is equivalently stated in the variational form

$$\langle\!\langle u , v \rangle\!\rangle = \langle f , v \rangle \quad \text{for all } v \in \mathcal{H}, \tag{3.5}$$

and the Riesz theorem guarantees solvability and uniqueness of the solution $u \in \mathcal{H}$.

3.2. Discrete Spaces & Galerkin Formulation. We consider the lowest-order Galerkin scheme: Let $\mathcal{T}_\ell = \{T_1, \dots, T_N\}$ be an *almost-regular* triangulation of Γ, i.e. there hold

- $\bigcup_{i=1}^N \overline{T}_i = \overline{\Gamma}$,
- each T_i is closed and non-degenerate, i.e. $|T_i| > 0$,
- $|T_i \cap T_j| = 0$ for $i \neq j$,

where $|\cdot|$ denotes the $(d-1)$-dimensional surface measure. In 2D, we restrict to affine line segments T_i. In 3D, we assume that the elements $T_i \in \mathcal{T}_\ell$ are flat triangles or rectangles, and the triangulation is called *regular* in the sense of Ciarlet, if additionally, for all elements $T_i, T_j \in \mathcal{T}_\ell$ with $T_i \neq T_j$, the intersection $T_i \cap T_j$ is either empty, a vertex of both T_i and T_j, or a common edge.

By $X_\ell := \mathcal{P}^0(\mathcal{T}_\ell)$, we denote the space of all $\mathcal{T}_\ell$-piecewise constant functions on Γ. The Galerkin solution $u_\ell \in X_\ell$ is the uniquely determined solution of the variational form

$$\langle\!\langle u_\ell , v_\ell \rangle\!\rangle = \langle f , v_\ell \rangle \quad \text{for all } v_\ell \in X_\ell. \tag{3.6}$$

We stress the Galerkin-orthogonality

$$\langle\!\langle u - u_\ell , v_\ell \rangle\!\rangle = 0 \quad \text{for all } v_\ell \in X_\ell. \tag{3.7}$$

In particular, the Galerkin solution u_ℓ is the best approximation of u with respect to X_ℓ and the energy norm, cf. Lemma 2.2.

3.3. Shape-Regularity and K-Mesh Constant. Let $h_\ell, \varrho_\ell \in L^\infty(\Gamma)$ denote the associated mesh-size functions of $\mathcal{T}_\ell$, where $h_\ell|_T := \mathrm{diam}(T)$ is the diameter of an element $T \in \mathcal{T}_\ell$ and where $\varrho_\ell|_T$ denotes the diameter of the largest inscribed circle in T. In 3D, the *shape-regularity constant*

$$\sigma(\mathcal{T}_\ell) := \max_{T \in \mathcal{T}_\ell} \frac{h_\ell|_T}{\varrho_\ell|_T} = \|h_\ell / \varrho_\ell\|_{L^\infty(\Gamma)}.$$

explicitly enters the estimates (3.13) and (3.18) in the analysis of the error estimators considered below, whereas $h_\ell = \varrho_\ell$ and hence $\sigma(\mathcal{T}_\ell) = 1$ for 2D. A mesh-refining strategy and the corresponding sequence $\mathcal{T}_\ell$ of generated meshes is called *isotropic* if

$$\sup_{\ell \in \mathbb{N}} \sigma(\mathcal{T}_\ell) < \infty. \tag{3.8}$$

In Section 4 we focus on such mesh-refinements, whereas in Section 5 our analysis is extended to some anisotropic adaptive algorithm, where possibly $\sup_{\ell \in \mathbb{N}} \sigma(\mathcal{T}_\ell) = \infty$.

The inverse estimate (3.10) used in the analysis of the error estimators under consideration depends on the uniform boundedness

$$\sup_{\ell\in\mathbb{N}} \kappa(\mathcal{T}_\ell) < \infty, \tag{3.9}$$

where the K-mesh constant $\kappa(\mathcal{T}_\ell) \geq 1$ is the smallest constant satisfying the following conditions:

- For any $T_j, T_k \in \mathcal{T}_\ell$ with $T_j \cap T_k \neq \emptyset$ holds $h_\ell|_{T_j}/h_\ell|_{T_k} \leq \kappa(\mathcal{T}_\ell)$ as well as $\varrho_\ell|_{T_j}/\varrho_\ell|_{T_k} \leq \kappa(\mathcal{T}_\ell)$, i.e. the local mesh-widths of neighbouring elements do not vary too rapidly.
- For any node $z \in \overline{\Gamma}$ of $\mathcal{T}_\ell$ holds $\#\{T \in \mathcal{T}_\ell \,:\, z \in T\} \leq \kappa(\mathcal{T}_\ell)$, i.e. each node does not belong to too many elements of $\mathcal{T}_\ell$.

We stress that the constant $C_{\rm inv}$ in the estimates (3.13) and (3.18) depend on (an upper bound of) the K-mesh constant $\kappa(\mathcal{T}_\ell)$. Therefore, any mesh-refining algorithm used must ensure the uniform bound (3.9).

3.4. $(h - h/2)$-Type and Averaging-Based Error Estimators. In the following, let $\widehat{\mathcal{T}}_\ell$ be the uniform refinement of $\mathcal{T}_\ell$. We denote by $\widehat{u}_\ell \in \widehat{X}_\ell := \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$ the corresponding Galerkin solution.

For the analysis below, we recall the inverse estimate

$$\|\varrho_\ell^{1/2} v_\ell\|_{L^2(\Gamma)} \leq C_{\rm inv}\, |||v_\ell||| \quad \text{for all } v_\ell \in X_\ell \tag{3.10}$$

from [24, Theorem 3.6], where the constant $C_{\rm inv} > 0$ depends only on the K-mesh constant $\kappa(\mathcal{T}_\ell)$. Moreover, the L^2-orthogonal projection Π_ℓ onto $X_\ell = \mathcal{P}^0(\mathcal{T}_\ell)$ satisfies the approximation estimate

$$|||v - \Pi_\ell v||| \leq C_{\rm apx} \|h_\ell^{1/2}(v - \Pi_\ell v)\|_{L^2(\Gamma)} \leq C_{\rm apx} \|h_\ell^{1/2} v\|_{L^2(\Gamma)}, \tag{3.11}$$

where the constant $C_{\rm apx}$ depends only on Γ, cf. [12, Theorem 4.1, Lemma 4.3].

With these estimates (3.10)–(3.11), we can state the following main result from [23].

Proposition 3.1 ($(h - h/2)$-Type Error Estimators). *The a posteriori error estimators*

$$\begin{array}{rclrcl} \eta_\ell &=& |||\widehat{u}_\ell - u_\ell||| & \widetilde{\eta}_\ell &=& |||(1 - \Pi_\ell)\widehat{u}_\ell||| \\ \mu_\ell &=& \|\varrho_\ell^{1/2}(\widehat{u}_\ell - u_\ell)\|_{L^2(\Gamma)} & \widetilde{\mu}_\ell &=& \|\varrho_\ell^{1/2}(1 - \Pi_\ell)\widehat{u}_\ell\|_{L^2(\Gamma)} \end{array} \tag{3.12}$$

satisfy the estimates

$$\widetilde{\mu}_\ell \leq \mu_\ell \leq \sqrt{2}\, C_{\rm inv}\, \eta_\ell \quad \text{and} \quad \eta_\ell \leq \widetilde{\eta}_\ell \leq C_{\rm apx}\, \sigma(\mathcal{T}_\ell)^{1/2}\, \widetilde{\mu}_\ell. \tag{3.13}$$

Moreover, η_ℓ, μ_ℓ, and $\widetilde{\mu}_\ell$ are always efficient in the sense that

$$\eta_\ell \leq C_{\rm eff}\, |||u - u_\ell||| \tag{3.14}$$

with known efficiency constant $C_{\rm eff} = 1$. Finally, reliability of η_ℓ in the sense that

$$|||u - u_\ell||| \leq C_{\rm rel}\, \eta_\ell \tag{3.15}$$

with some constant $C_{\rm rel} > 0$ is equivalent to the saturation assumption

$$|||u - \widehat{u}_\ell||| \leq C_{\rm sat}\, |||u - u_\ell||| \tag{3.16}$$

with some constant $0 < C_{\rm sat} < 1$. □

Next, we additionally consider the space $X_\ell^{(1)} := \mathcal{P}^1(\mathcal{T}_\ell)$ of all $\mathcal{T}_\ell$-piecewise affine, but not necessarily continuous functions. Let $\mathbb{G}_\ell^{(1)}$ and $\Pi_\ell^{(1)}$ denote the Galerkin and L^2-projections onto $X_\ell^{(1)}$. The work [12] proposes to use averaging on large patches for a posteriori error estimation. We recall the following main result from [12, 14].

Proposition 3.2 (Averaging-Based Error Estimators). *The error estimators*

$$\begin{aligned} \alpha_\ell &= |||(1-\mathbb{G}_\ell^{(1)})\widehat{u}_\ell||| & \widetilde{\alpha}_\ell &= |||(1-\Pi_\ell^{(1)})\widehat{u}_\ell||| \\ \beta_\ell &= \|\varrho_\ell^{1/2}(1-\mathbb{G}_\ell^{(1)})\widehat{u}_\ell\|_{L^2(\Gamma)} & \widetilde{\beta}_\ell &= \|\varrho_\ell^{1/2}(1-\Pi_\ell^{(1)})\widehat{u}_\ell\|_{L^2(\Gamma)} \end{aligned} \tag{3.17}$$

satisfy the estimates

$$\widetilde{\beta}_\ell \le \beta_\ell \le \sqrt{2}\, C_{\rm inv}\, \alpha_\ell, \quad \alpha_\ell \le |||u-u_\ell|||, \quad \text{and} \quad \alpha_\ell \le \widetilde{\alpha}_\ell \le C_{\rm apx}\, \sigma(\mathcal{T}_\ell)^{1/2}\widetilde{\beta}_\ell. \tag{3.18}$$

The error estimators $\widetilde{\beta}_\ell$, β_ℓ, and α_ℓ are, in particular, efficient to estimate $|||u-u_\ell|||$. Moreover, there holds

$$\alpha_\ell \le |||u-\widehat{u}_\ell||| + |||(1-\mathbb{G}_\ell^{(1)})u|||, \tag{3.19}$$

which is understood as efficiency of $\widetilde{\beta}_\ell$, β_ℓ, and α_ℓ with respect to $|||u-\widehat{u}_\ell|||$, up to terms of higher order. Let $\widehat{\mathbb{G}}_\ell$ denote the Galerkin projection onto $\widehat{X}_\ell$. Provided that

$$q_\ell := |||(1-\widehat{\mathbb{G}}_\ell)\mathbb{G}_\ell^{(1)} : \mathcal{H}\to\mathcal{H}||| = \max_{v_\ell^{(1)}\in X_\ell^{(1)}\setminus\{0\}} \min_{\widehat{v}_\ell\in\widehat{X}_\ell} \frac{|||v_\ell^{(1)}-\widehat{v}_\ell|||}{|||v_\ell^{(1)}|||} < 1, \tag{3.20}$$

there even holds

$$|||u-\widehat{u}_\ell||| \le (1-q_\ell^2)^{-1/2}\big(\alpha_\ell + |||(1-\mathbb{G}_\ell^{(1)})u|||\big), \tag{3.21}$$

which is interpreted as reliability of α_ℓ and $\widetilde{\alpha}_\ell$ with respect to $|||u-\widehat{u}_\ell|||$, up to terms of higher order. □

Note that the $(h-h/2)$-based error estimators $\rho_\ell \in \{\mu_\ell, \widetilde{\mu}_\ell\}$ as well as the averaging-based error estimators $\rho_\ell \in \{\beta_\ell\, \widetilde{\beta}_\ell\}$ can be employed to steer Algorithm 2.1 via

$$\begin{aligned} \mu_\ell(T) &:= \|\varrho_\ell^{1/2}(\widehat{u}_\ell-u_\ell)\|_{L^2(T)}, & \widetilde{\mu}_\ell(T) &:= \|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{u}_\ell\|_{L^2(T)}, \\ \beta_\ell(T) &:= \|\varrho_\ell^{1/2}(1-\mathbb{G}_\ell^{(1)})\widehat{u}_\ell\|_{L^2(T)}, & \widetilde{\beta}_\ell(T) &:= \|\varrho_\ell^{1/2}(1-\Pi_\ell^{(1)})\widehat{u}_\ell\|_{L^2(T)}. \end{aligned} \tag{3.22}$$

In the following, we now aim to verify the estimator convergence (1.3) for each choice.

4. Convergence for Isotropic Mesh-Refinement

In this section, we prove some estimator convergence (1.3) in case of isotropic mesh-refinement (3.8). As has been stated above, the analysis of the error estimators needs uniform boundedness (3.9) of the K-mesh constant. Moreover, the convergence analysis below needs the inclusions

$$X_\ell \subseteq X_{\ell+1} \quad \text{as well as} \quad \widehat{X}_\ell \subseteq \widehat{X}_{\ell+1} \tag{4.1}$$

to employ the a priori convergence (1.5) from Lemma 2.2 for both sequences of spaces. Whereas the first inclusion is guaranteed for any mesh-refinement rule, the second inclusion for the uniformly refined meshes $\widehat{\mathcal{T}}_\ell$ is crucial. We stress that the subsequently introduced mesh-refinement rules guarantee (4.1) even in the stronger form $X_\ell \subseteq X_{\ell+1} \subseteq \widehat{X}_\ell \subseteq \widehat{X}_{\ell+1}$.

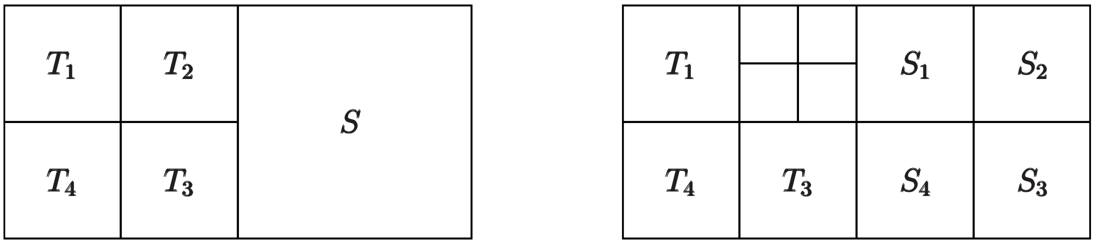

FIGURE 1. For isotropic mesh-refinement with rectangular elements, a marked element T is always refined uniformly into four new elements T_j. This isotropic refinement obviously yields $h_{\ell+1}|_T = \frac{1}{2}\,h_\ell|_T$ and $\varrho_{\ell+1}|_T = \frac{1}{2}\varrho_\ell|_T$ for the refined mesh-sizes. Moreover, one hanging node per edge is allowed (left). If, in the left configuration, element T_2 is marked for refinement, we mark element S for refinement as well (right).

To verify some estimator reduction (1.4), it is essential to observe that (local) mesh-refinement leads to a uniform decay

$$\varrho_{\ell+1}|_T \le q_{\rm refine}\,\varrho_\ell|_T \quad \text{for all } T \in \mathcal{M}_\ell, \tag{4.2}$$

for all marked elements as well as to boundedness

$$\varrho_{\ell+1}|_T \le \varrho_\ell|_T \le C_{\rm refine}\,\varrho_{\ell+1}|_T \quad \text{for all } T \in \mathcal{T}_\ell \tag{4.3}$$

for all elements. Here, the constants $0 < q_{\rm refine} < 1$ and $C_{\rm refine} > 1$ may depend only on the chosen mesh-refinement. Throughout this section, we now assume that the mesh-refinement satisfies (3.9) and (4.1)–(4.3). Possible choices for 2D and isotropic 3D BEM are discussed in the following two sections.

4.1. Mesh-Refinement in 2D BEM. For $d = 2$, we assume that the elements $T \in \mathcal{T}_\ell$ are affine line segments so that $\varrho_\ell = h_\ell$, i.e. $\sigma(\mathcal{T}_\ell) = 1$. When refined, an element T is bisected into two elements of half length. Note that this guarantees (4.1)–(4.3) with $q_{\rm refine} = 1/2$ and $C_{\rm refine} = 2$.

In order to ensure the uniform boundedness (3.9) of the K-mesh constant, we use an algorithm from [4, Section 2.2]: If $T_i \in \mathcal{T}_\ell$ is marked for refinement, any neighbour T_j with

$$h_\ell|_{T_j}/h_\ell|_{T_i} > \kappa(\mathcal{T}_0) \tag{4.4}$$

is recursively marked for refinement as well. This guarantees $\kappa(\mathcal{T}_\ell) \le 2\,\kappa(\mathcal{T}_0)$ for all generated meshes $\mathcal{T}_\ell$. Note that this extends the set of elements $\mathcal{R}_\ell := \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1} \supseteq \mathcal{M}_\ell$ which are eventually refined in step (iii) of Algorithm 2.1. However, one can prove that this mesh-refinement is optimal in the sense that the number of elements in $\mathcal{T}_\ell$ is essentially bounded by the number of marked elements, i.e.

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C(\mathcal{T}_0) \sum_{j=0}^{\ell-1} \#\mathcal{M}_j, \tag{4.5}$$

where $C(\mathcal{T}_0) \ge 1$ depends only on $\mathcal{T}_0$, see [4, Theorem 2.5].

4.2. Isotropic Mesh-Refinement in 3D BEM. For $d = 3$, we restrict to the particular cases that the triangulation $\mathcal{T}_\ell$ is either a regular triangulation consisting of flat triangles or an almost-regular triangulation consisting of flat rectangles with hanging nodes of order at most 1, cf. Figure 1. Extensions to more general triangulations can easily be included into our analysis.

First, if $\mathcal{T}_\ell$ is a regular triangulation into triangles, we note that all refinement rules based on newest vertex bisection (NVB) satisfy the additional property (4.1), whereas the popular red-green-blue refinement can lead to $\widehat{X}_\ell \not\subseteq \widehat{X}_{\ell+1}$. See [36, Chapter 5] for an overview on mesh-refinement rules. We refer to [31] for the fact that NVB based mesh-refinement only leads to finitely many similarity classes of triangles. In particular, $\sigma(\mathcal{T}_\ell)$ can be bounded uniformly by a constant that only depends on the initial mesh $\mathcal{T}_0$. Note that regularity and uniform shape regularity of $\mathcal{T}_\ell$, in particular, imply the uniform K-mesh property (3.9). Moreover, an elementary calculation proves that any NVB based refinement rule guarantees (4.2)–(4.3), where $0 < q_{\rm refine} < 1$ and $C_{\rm refine} > 1$ again depend only on $\mathcal{T}_0$. Finally, it has been proven by [7] and more generally in [34] that any mesh-refinement based on NVB is optimal in the sense of (4.5).

Second, if $\mathcal{T}_\ell$ consists of rectangular elements, a marked element T is split uniformly into four similar sons. To allow local mesh refinement, we admit hanging nodes of first order, see Figure 1. In particular, the shape-regularity constant $\sigma(\mathcal{T}_\ell) = \sigma(\mathcal{T}_0)$ does not change, and there holds $\kappa(\mathcal{T}_\ell) \le 4\,\kappa(\mathcal{T}_0)$. Moreover, (4.1) is clearly satisfied. Finally, as in 2D, the estimates (4.2)–(4.3) hold with $q_{\rm refine} = 1/2$ and $C_{\rm refine} = 2$.

4.3. Estimator Reduction for $(h - h/2)$-Type Error Estimators. We now consider the $(h-h/2)$-error estimators μ_ℓ and $\widetilde{\mu}_\ell$ from Proposition 3.1 and use its local contributions from (3.22) to steer Algorithm 2.1. For both choices, the following theorem states some estimator reduction (1.4) as well as the estimator convergence (1.3).

Theorem 4.1. *Let $0 < \theta < 1$ be a fixed constant and let $\mu_\ell(T)$ and $\widetilde{\mu}_\ell(T)$ be the indicators defined in* (3.22). *Let $0 < q_{\rm refine} < 1$ be the constant from* (4.2).

(i) *Suppose that we use the indicators $\rho_\ell(T) := \mu_\ell(T)$ in Algorithm 2.1. Then,*

$$\mu_{\ell+1} \le \big(1 - (1 - q_{\rm refine})\,\theta\big)^{1/2}\,\mu_\ell + C_{\rm mesh}\,\big(|||\widehat{u}_{\ell+1} - \widehat{u}_\ell||| + |||u_{\ell+1} - u_\ell|||\big) \tag{4.6}$$

for all $\ell \in \mathbb{N}_0$.

(ii) *Suppose that we use the indicators $\rho_\ell(T) := \widetilde{\mu}_\ell(T)$ in Algorithm 2.1. Then,*

$$\widetilde{\mu}_{\ell+1} \le (1-\theta)^{1/2}\,\widetilde{\mu}_\ell + C_{\rm mesh}\,|||\widehat{u}_{\ell+1} - \widehat{u}_\ell||| \tag{4.7}$$

for all $\ell \in \mathbb{N}_0$.

(iii) *The constant $C_{\rm mesh} > 0$ depends only on the chosen mesh-refinement and the initial mesh $\mathcal{T}_0$. The last two terms on the right-hand side of* (4.6) *as well as the last term on the right-hand side of* (4.7) *vanish as $\ell \to \infty$. In particular, Lemma 2.3 applies and proves $\lim\limits_{\ell\to\infty} \mu_\ell = 0 = \lim\limits_{\ell\to\infty} \widetilde{\mu}_\ell$.*

Proof. The triangle inequality proves

$$\mu_{\ell+1} \le \|\varrho_{\ell+1}^{1/2}(\widehat{u}_\ell - u_\ell)\|_{L^2(\Gamma)} + \|\varrho_{\ell+1}^{1/2}((\widehat{u}_{\ell+1} - u_{\ell+1}) - (\widehat{u}_\ell - u_\ell))\|_{L^2(\Gamma)}$$

Note that (4.3) also applies to uniform mesh-refinement. With this and $(\widehat{u}_{\ell+1}-u_{\ell+1})-(\widehat{u}_\ell - u_\ell) \in \widehat{X}_\ell$, the inverse estimate (3.10) gives

$$\begin{aligned}\|\varrho_{\ell+1}^{1/2}((\widehat{u}_{\ell+1}-u_{\ell+1})-(\widehat{u}_\ell-u_\ell))\|_{L^2(\Gamma)} &\le C_{\rm refine}\,\|\widehat{\varrho}_{\ell+1}^{1/2}((\widehat{u}_{\ell+1}-\widehat{u}_\ell)-(u_{\ell+1}-u_\ell))\|_{L^2(\Gamma)}\\ &\le C_{\rm refine}\,C_{\rm inv}\,|||(\widehat{u}_{\ell+1}-\widehat{u}_\ell)-(u_{\ell+1}-u_\ell)|||.\end{aligned}$$

We next use (4.2)–(4.3) in the form

$$\varrho_{\ell+1}|_T \le q_{\rm refine}\varrho_\ell|_T \quad \text{for all } T\in\mathcal{M}_\ell \quad \text{as well as} \quad \varrho_{\ell+1}|_T \le \varrho_\ell|_T \quad \text{for all } T\in\mathcal{T}_\ell\backslash\mathcal{M}_\ell.$$

The marking strategy (2.3) gives

$$\begin{aligned}\|\varrho_{\ell+1}^{1/2}(\widehat{u}_\ell-u_\ell)\|_{L^2(\Gamma)}^2 &= \sum_{T\in\mathcal{M}_\ell}\|\varrho_{\ell+1}^{1/2}(\widehat{u}_\ell-u_\ell)\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{M}_\ell}\|\varrho_{\ell+1}^{1/2}(\widehat{u}_\ell-u_\ell)\|_{L^2(T)}^2\\ &\le q_{\rm refine}\sum_{T\in\mathcal{M}_\ell}\|\varrho_\ell^{1/2}(\widehat{u}_\ell-u_\ell)\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{M}_\ell}\|\varrho_\ell^{1/2}(\widehat{u}_\ell-u_\ell)\|_{L^2(T)}^2\\ &= (q_{\rm refine}-1)\sum_{T\in\mathcal{M}_\ell}\mu_\ell(T)^2 + \sum_{T\in\mathcal{T}_\ell}\mu_\ell(T)^2\\ &\le \big(1-(1-q_{\rm refine})\,\theta\big)\,\mu_\ell^2.\end{aligned}$$

This concludes the proof of (4.6) with $C_{\rm mesh} = C_{\rm refine}C_{\rm inv}$. According to the inclusions (4.1), Lemma 2.2 proves that $\widehat{u}_\ell$ and u_ℓ converge to certain limits $\widehat{u}_\infty$ and u_∞, respectively. Consequently, the terms $|||\widehat{u}_{\ell+1}-\widehat{u}_\ell|||$ and $|||u_{\ell+1}-u_\ell|||$ vanish as $\ell\to\infty$.

The proof of (4.7) follows along the same lines, but we use $\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)} = 0$ for $T\in\mathcal{M}_\ell$ instead. □

Under the saturation assumption (3.16), we can now prove that the adaptive algorithm leads to convergence $u_\infty = u$.

Corollary 4.2. *Let $0<\theta<1$ be a fixed constant and suppose that we use either μ_ℓ or $\widetilde{\mu}_\ell$ for marking in Algorithm 2.1. Assume that the saturation assumption* (3.16) *is valid, at least for infinitely many steps ℓ of the adaptive algorithm. Then, there holds*

$$\lim_{\ell\to\infty}\mu_\ell = \lim_{\ell\to\infty}\widetilde{\mu}_\ell = \lim_{\ell\to\infty}|||u-u_\ell||| = 0. \tag{4.8}$$

Proof. The saturation assumption (3.16) is equivalent to the reliability (3.15) of η_ℓ, cf. Theorem 3.1. Moreover, the mesh-refining strategy in this section is isotropic. Therefore, μ_ℓ as well as $\widetilde{\mu}_\ell$ are equivalent to η_ℓ, cf. (3.13). Finally, convergence of the estimator, e.g., $\lim_{\ell\to\infty}\mu_\ell = 0$, implies $|||u-u_\ell||| \le C_{\rm rel}\,\eta_\ell \to 0$ as $\ell\to\infty$. □

Remark 1. *In* [22, Theorem 8]*, we prove the following result: Suppose that we use the indicators $\mu_\ell(T)$ for marking in Algorithm 2.1. Under the saturation assumption* (3.16)*, there are constants $\gamma,\kappa\in(0,1)$ such that $\Delta_\ell := |||u-u_\ell|||^2 + |||u-\widehat{u}_\ell|||^2 + \gamma\,\mu_\ell(T)^2 \ge 0$ satisfies $\Delta_{\ell+1}\le\kappa\,\Delta_\ell$. In particular, one obtains convergence $\Delta_\ell\to 0$ as $\ell\to\infty$.— The same result holds for μ_ℓ replaced by $\widetilde{\mu}_\ell$, cf.* [22, Theorem 7]. *Although the results in* [22] *are stronger, so are the assumptions, i.e., uniform saturation assumption* (3.16) *for all steps $\ell = 0,1,2,\dots$.*

Contrary to those results, we now have decoupled the convergence of the error estimator in Theorem 4.1 from the convergence of the error $\lim_{\ell\to\infty} |||u - u_\ell||| = 0$. □

Remark 2. *Independently of the saturation assumption* (3.16)*, Theorem 4.1 provides additional information on the a priori limits* $u_\infty := \lim_\ell u_\ell$ *and* $\widehat{u}_\infty := \lim_\ell \widehat{u}_\ell$*. With estimator convergence* $\lim_\ell \mu_\ell = 0 = \lim_\ell \widetilde{\mu}_\ell$ *and equivalence* (3.13) *to* η_ℓ*, we observe* $0 = \lim_\ell \eta_\ell = |||\widehat{u}_\infty - u_\infty|||$*, whence* $u_\infty = \widehat{u}_\infty$*.* □

4.4. Estimator Reduction for Averaging Error Estimators. We consider the averaging error estimators β_ℓ and $\widetilde{\beta}_\ell$ from Proposition 3.2 and its local contributions from (3.22) to steer Algorithm 2.1. Following the lines of proof of Theorem 4.1, we obtain the following estimator reductions (4.9)–(4.10) for β_ℓ and $\widetilde{\beta}_\ell$, respectively.

Theorem 4.3. *Let* $0 < \theta < 1$ *be a fixed constant and let* $\beta_\ell(T)$ *and* $\widetilde{\beta}_\ell(T)$ *be the indicators defined in* (3.17)*. Let* $0 < q_{\rm refine} < 1$ *be the constant from* (4.2)*.*

(i) *Suppose that we use the indicators* $\rho_\ell(T) := \beta_\ell(T)$ *in Algorithm 2.1. Then,*

$$\beta_{\ell+1} \le \big(1 - (1 - q_{\rm refine})\,\theta\big)^{1/2}\,\beta_\ell + C_{\rm mesh}\,\big(|||\widehat{u}_{\ell+1} - \widehat{u}_\ell||| + |||\mathbb{G}^{(1)}_{\ell+1}\widehat{u}_{\ell+1} - \mathbb{G}^{(1)}_\ell\widehat{u}_\ell|||\big) \tag{4.9}$$

for all $\ell \in \mathbb{N}_0$*.*

(ii) *Suppose that we use the indicators* $\rho_\ell(T) := \widetilde{\beta}_\ell(T)$ *in Algorithm 2.1. Then,*

$$\widetilde{\beta}_{\ell+1} \le (1 - \theta)^{1/2}\,\widetilde{\beta}_\ell + C_{\rm mesh}\,|||\widehat{u}_{\ell+1} - \widehat{u}_\ell||| \tag{4.10}$$

for all $\ell \in \mathbb{N}_0$*.*

(iii) *The constant* $C_{\rm mesh} > 0$ *depends only on the chosen mesh-refinement and the initial mesh* $\mathcal{T}_0$*. The last two terms on the right-hand side of* (4.9) *as well as the last term on the right-hand side of* (4.10) *vanish as* $\ell \to \infty$*. In particular, Lemma 2.3 applies and proves* $\lim_{\ell\to\infty} \beta_\ell = 0 = \lim_{\ell\to\infty} \widetilde{\beta}_\ell$*.*

Proof. The proofs of (i) and (ii) follow along the same lines as in Theorem 4.1. To verify (iii), note that Lemma 2.2 proves convergence $\widehat{u}_\infty := \lim_\ell \widehat{u}_\ell$, whence $|||\widehat{u}_{\ell+1} - \widehat{u}_\ell||| \to 0$ as $\ell \to \infty$. Therefore, Lemma 2.3 applies to $\widetilde{\beta}_\ell$.

Moreover, $\widehat{u}^{(1)}_\ell := \mathbb{G}^{(1)}_\ell \widehat{u}_\infty \in X^{(1)}_\ell$ is the best approximation of $\widehat{u}_\infty$ with respect to $X^{(1)}_\ell$. Therefore, Lemma 2.2 applies and proves that the limit $\widehat{u}^{(1)}_\infty := \lim_\ell \widehat{u}^{(1)}_\ell$ exists. According to the triangle inequality and stability of the Galerkin projection $\mathbb{G}^{(1)}_\ell$, there holds

$$\begin{aligned} |||\widehat{u}^{(1)}_\infty - \mathbb{G}^{(1)}_\ell\widehat{u}_\ell||| &\le |||\widehat{u}^{(1)}_\infty - \mathbb{G}^{(1)}_\ell\widehat{u}_\infty||| + |||\mathbb{G}^{(1)}_\ell\widehat{u}_\infty - \mathbb{G}^{(1)}_\ell\widehat{u}_\ell||| \\ &\le |||\widehat{u}^{(1)}_\infty - \mathbb{G}^{(1)}_\ell\widehat{u}_\infty||| + |||\widehat{u}_\infty - \widehat{u}_\ell||| \xrightarrow{\ell\to\infty} 0. \end{aligned}$$

This proves $\widehat{u}^{(1)}_\infty = \lim_\ell \mathbb{G}^{(1)}_\ell \widehat{u}_\ell$. Consequently, Lemma 2.3 also applies to β_ℓ. □

Remark 3. *Note that the last term in* (4.9) *reads* $|||(\mathbb{G}^{(1)}_{\ell+1}\widehat{\mathbb{G}}_{\ell+1} - \mathbb{G}^{(1)}_\ell\widehat{\mathbb{G}}_\ell)u|||$*. Since the spaces* $X^{(1)}_\ell$ *and* $\widehat{X}_\ell$ *are not nested, the operator* $\mathbb{G}^{(1)}_\ell\widehat{\mathbb{G}}_\ell$ *is* not *a Galerkin projection. This prevents to use the arguments of* [22] *to prove some contraction property for the (weighted)*

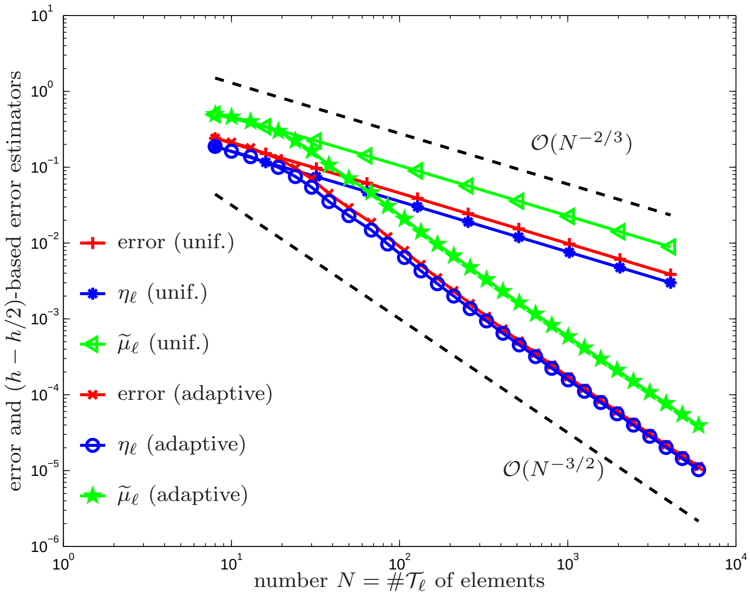

FIGURE 2. Error $|||u - u_\ell|||$ (red) as well as error estimators η_ℓ (blue) and $\widetilde{\mu}_\ell$ (green) for uniform and adaptive mesh-refinement. For comparison, we plot the slopes of the experimental convergence rates $\mathcal{O}(N^{-\alpha})$ for $\alpha \in \{2/3, 3/2\}$.

sum of error and β_ℓ — provided that β_ℓ is reliable. Instead, our new argument applies directly and proves $\lim_{\ell\to\infty} \beta_\ell = 0$. $\square$

Remark 4. *As before, Theorem 4.3 provides additional knowledge on the a priori limits $u_\infty := \lim_\ell u_\ell$, $\widehat{u}_\infty := \lim_\ell \widehat{u}_\ell$, $\widehat{u}_\infty^{(1)} := \lim_\ell \mathbb{G}_\ell^{(1)} \widehat{u}_\ell$. ¿From estimator convergence $\lim_\ell \beta_\ell = 0 = \lim_\ell \widetilde{\beta}_\ell$ and equivalence* (3.18)*, we obtain $\lim_\ell \alpha_\ell = 0$, whence $\widehat{u}_\infty = \widehat{u}_\infty^{(1)}$. Moreover,* [19, Theorem 5.3] *states the equivalence of α_ℓ and η_ℓ. The proof, given only for 2D BEM, can be extended to isotropic 3D BEM as well. Altogether, we thus obtain $\lim_\ell \eta_\ell = 0$ and conclude $u_\infty = \widehat{u}_\infty = \widehat{u}_\infty^{(1)}$.* $\square$

4.5. Numerical Experiment for 2D BEM. We consider the numerical solution of Symm's integral equation

$$Vu = 1 \tag{4.11}$$

on the boundary $\Gamma = \partial\Omega$ of the square $\Omega = (0, 0.5)^2$. Note that $\mathrm{diam}(\Omega) < 1$ to ensure the ellipticity of the simple-layer potential V. The uniform initial mesh $\mathcal{T}_0$ consisted of 8 line segments with length 0.125.

In Algorithm 2.1, we use the $(h-h/2)$-based error estimator $\widetilde{\mu}_\ell$ and $\theta = 0.25$ to steer the adaptive mesh-refinement. The exact solution $u \in H^{-1/2}(\Gamma)$ of (4.11) is unknown. To compute the error $|||u - u_\ell|||$, we used the Galerkin orthonality

$$|||u - u_\ell|||^2 = |||u|||^2 - |||u_\ell|||^2. \tag{4.12}$$

The discrete energy $|||u_\ell|||^2$ is computed by use of the Galerkin matrix. The exact energy $|||u|||^2$ is obtained by Aitken's Δ^2-extrapolation applied to a sequence of discrete energies with

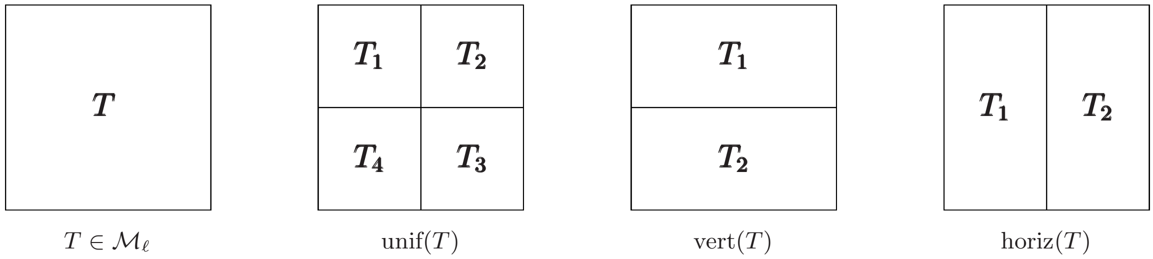

FIGURE 3. The extended Algorithm 2.1 in Section 5 gives a criterion whether a marked rectangle $T \in \mathcal{M}_\ell$ (left) is refined isotropically into four elements $T_1, \dots, T_4$ or anisotropically into two elements T_1 and T_2. In the latter case, the algorithm decides whether vertical or horizontal refinement seems to be more appropriate.

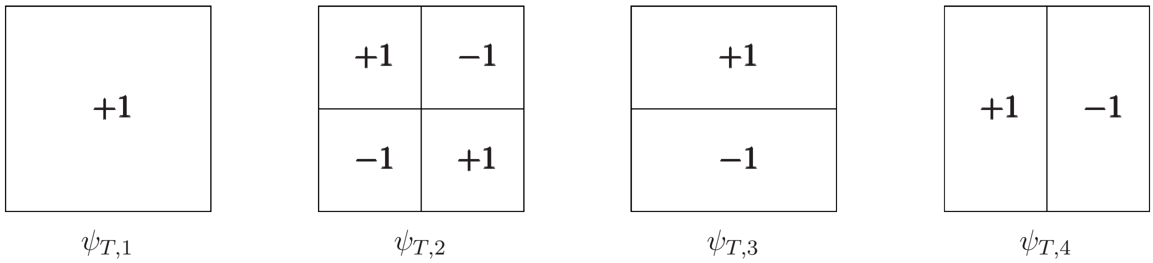

FIGURE 4. For each rectangle $T \in \mathcal{T}_\ell$, we introduce four $\widehat{\mathcal{T}}_\ell$-piecewise constant functions $\psi_{T,j} \in \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$, which are extended by zero to $\Gamma \setminus T$.

respect to uniform mesh-refinement. Throughout, we use the extrapolated value $|||u|||^2 = 5.14807864569$.

Figure 2 shows the numerical results for the error $|||u - u_\ell|||$ as well as the $(h - h/2)$-based error estimators η_ℓ and $\widetilde{\mu}_\ell$, plotted over the number $N = \#\mathcal{T}_\ell$ of elements for uniform and adaptive mesh-refinement. For uniform mesh-refinement, we observe an experimental convergence rate $\mathcal{O}(N^{-2/3})$. This is due to singularities of u at the four corners of Γ. We stress that for a piecewise smooth solution u, theory predicts a convergence behaviour $\mathcal{O}(N^{-3/2})$ instead, cf. [30]. We stress that this optimal order of convergence is regained by use of adaptive mesh-refinement.

In any case, we observe that the curves of the error $|||u - u_\ell|||$ and the curves of the corresponding error estimators η_ℓ and μ_ℓ are parallel within a certain range. This gives experimental evidence that, in this experiment, the saturation assumption (3.16) holds.

5. Convergence for Anisotropic Mesh-Refinement

We consider the model problem of Section 3. Since isotropic mesh-refinement does usually not recover the optimal order of convergence in 3D BEM computations, we extend the refinement strategy in Algorithm 2.1. For the ease of presentation, we restrict to rectangular boundary elements. We use a strategy introduced in [23, Section 4.5] for the $(h - h/2)$-based estimators μ_ℓ and $\widetilde{\mu}_\ell$ to decide whether a marked rectangle $T \in \mathcal{T}_\ell$ is refined isotropically into four rectangles or anisotropically into two rectangles, respectively, cf. Figure 3. For $\widetilde{\mu}_\ell$, we prove that this strategy yields the estimator reduction. Finally, we extend these ideas to prove the estimator reduction for some anisotropic mesh-refinement steered by $\widetilde{\beta}_\ell$.

5.1. Estimator Reduction for $(h-h/2)$-Type Error Estimator $\widetilde{\mu}_\ell$. Let $T_1,\dots,T_4 \in \widehat{\mathcal{T}}_\ell$ denote the four son-elements of a marked coarse-mesh rectangle $T \in \mathcal{T}_\ell$, where we use the same numbering as for the isotropic refinement of Figure 3. We consider the four piecewise constant functions $\psi_{T,j} \in \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$ from Figure 4 and observe that $\{\psi_{T,1},\dots,\psi_{T,4}\}$ is an L^2-orthogonal basis of $\mathcal{P}^0(\{T_1,\dots,T_4\})$. Therefore, the already computed $\widehat{u}_\ell|_T \in \mathcal{P}^0(\{T_1,\dots,T_4\})$ can be written in the form

$$\widehat{u}_\ell|_T = \sum_{j=1}^{4} c_{T,j}\psi_{T,j} \text{ with Fourier coefficients } c_{T,j} = \frac{(\psi_{T,j}\,,\widehat{u}_\ell)_{L^2(T)}}{\|\psi_{T,j}\|^2_{L^2(T)}} = \frac{(\psi_{T,j}\,,\widehat{u}_\ell)_{L^2(T)}}{|T|^2}. \tag{5.1}$$

The decision whether isotropic or anisotropic refinement is more appropriate, is now done as follows: Let $0<\tau<1$ be an additional parameter. We assume that $T \in \mathcal{M}_\ell$ is marked for refinement in Algorithm 2.1.

- If $c_{T,2}^2 + c_{T,3}^2 \le \frac{\tau}{1-\tau}\, c_{T,4}^2$, we use horizontal refinement to create two sons $T_1, T_2 \in \mathcal{T}_{\ell+1}$.
- If $c_{T,2}^2 + c_{T,4}^2 \le \frac{\tau}{1-\tau}\, c_{T,3}^2$, we use vertical refinement to create two sons $T_1, T_2 \in \mathcal{T}_{\ell+1}$.
- Otherwise, $T \in \mathcal{M}_\ell$ is refined isotropically into four sons $T_1,\dots,T_4 \in \mathcal{T}_{\ell+1}$.

In order to ensure the uniform boundedness of the K-mesh constant $\kappa(\mathcal{T}_\ell)$ we additionally check the mesh-size ratio ϱ_ℓ of neighbouring elements and possibly mark additional elements as it is done in 2D for the mesh-size ratio with respect to h_ℓ. The following theorem states some estimator reduction (1.4) as well as the estimator convergence (1.3) for $\widetilde{\mu}_\ell$.

Theorem 5.1. *Let $0<\theta<1$ and $0<\tau<1$ be fixed constants. Suppose that we use the indicators $\rho_\ell(T) := \widetilde{\mu}_\ell(T)$ defined in (3.22) for marking in Algorithm 2.1 and the above described heuristics to decide the type of refinement. Then,*

$$\widetilde{\mu}_{\ell+1} \le \big(1-\theta(1-\tau)\big)^{1/2}\,\widetilde{\mu}_\ell + \sqrt{2}\,C_{\rm inv}\,|||\widehat{u}_{\ell+1}-\widehat{u}_\ell||| \quad \text{for all } \ell \in \mathbb{N}_0, \tag{5.2}$$

and the second term on the right-hand side vanishes as $\ell\to\infty$. In particular, Proposition 2.3 applies and proves convergence $\lim\limits_{\ell\to\infty}\widetilde{\mu}_\ell = 0$.

Proof. Note that, for rectangular elements, $\widehat{\varrho}_{\ell+1} = \varrho_{\ell+1}/2$. We proceed as in the proof of Theorem 4.1. The triangle inequality and the inverse estimate (3.10) prove

$$\begin{aligned}
\widetilde{\mu}_{\ell+1} &= \|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_{\ell+1}\|_{L^2(\Gamma)}\\
&\le \|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_{\ell}\|_{L^2(\Gamma)} + \|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})(\widehat{u}_{\ell+1}-\widehat{u}_\ell)\|_{L^2(\Gamma)}\\
&\le \|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_{\ell}\|_{L^2(\Gamma)} + \|\varrho_{\ell+1}^{1/2}(\widehat{u}_{\ell+1}-\widehat{u}_\ell)\|_{L^2(\Gamma)}\\
&\le \|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_{\ell}\|_{L^2(\Gamma)} + \sqrt{2}\,C_{\rm inv}\,|||\widehat{u}_{\ell+1}-\widehat{u}_\ell|||,
\end{aligned}$$

where we have additionally used that $\Pi_{\ell+1}$ is even the $\mathcal{T}_{\ell+1}$-elementwise L^2-orthogonal projection. Now, let $T \in \mathcal{M}_\ell$ be a marked element.

- If T is refined isotropically, there holds

$$\|(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)} = 0.$$

- If T is refined by horizontal refinement, there holds
$$\|(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)}^2 = |T|\,\big(c_{T,2}^2 + c_{T,3}^2\big).$$
Moreover, the proposed mesh-refinement yields $c_{T,2}^2 + c_{T,3}^2 \le \frac{\tau}{1-\tau}\,c_{T,4}^2$, which is equivalent to
$$|T|\,\big(c_{T,2}^2 + c_{T,3}^2\big) \le \tau\,|T|\,\big(c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2\big) = \tau\,\|(1-\Pi_\ell)\widehat{u}_\ell\|_{L^2(T)}^2.$$
- If T is refined by vertical refinement, there holds
$$\|(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)}^2 = |T|\,\big(c_{T,2}^2 + c_{T,4}^2\big),$$
and the mesh-refinement strategy yields $c_{T,2}^2 + c_{T,4}^2 \le \frac{\tau}{1-\tau}\,c_{T,3}^2$. This again leads to
$$|T|\,\big(c_{T,2}^2 + c_{T,4}^2\big) \le \tau\,|T|\,\big(c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2\big) = \tau\,\|(1-\Pi_\ell)\widehat{u}_\ell\|_{L^2(T)}^2.$$

Since $\varrho_{\ell+1}|_T \in \mathbb{R}$ is constant, we thus obtain in any case
$$\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)}^2 \le \tau\,\|\varrho_{\ell+1}^{1/2}(1-\Pi_\ell)\widehat{u}_\ell\|_{L^2(T)}^2 \le \tau\,\widetilde{\mu}_\ell(T)^2 \quad \text{for all } T \in \mathcal{M}_\ell.$$
Moreover, there clearly holds
$$\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)}^2 \le \|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{u}_\ell\|_{L^2(T)}^2 = \widetilde{\mu}_\ell(T)^2 \quad \text{for all } T \in \mathcal{T}_\ell\backslash\mathcal{M}_\ell.$$
Together with the marking strategy (2.3), this implies
$$\begin{aligned}
\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(\Gamma)}^2 &= \sum_{T\in\mathcal{M}_\ell} \|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{M}_\ell} \|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)}^2 \\
&\le \tau \sum_{T\in\mathcal{M}_\ell} \widetilde{\mu}_\ell(T)^2 + \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{M}_\ell} \widetilde{\mu}_\ell(T)^2 \\
&= -(1-\tau)\sum_{T\in\mathcal{M}_\ell} \widetilde{\mu}_\ell(T)^2 + \sum_{T\in\mathcal{T}_\ell} \widetilde{\mu}_\ell(T)^2 \\
&\le \big(1-\theta(1-\tau)\big)\,\widetilde{\mu}_\ell^2
\end{aligned}$$
and concludes the proof. □

Remark 5. *In* [23, Section 4.5], *we use the above stated refinement strategy with* $\widetilde{\tau} := \tau/(1-\tau) = 1/2$ *which is equivalent to the choice* $\tau = 1/3$. *In addition to* [23], *we stress the following observation: Suppose that* $\mathcal{M}_\ell$ *is a subset of* $\mathcal{T}_\ell$ *with* (2.3) *and minimal cardinality. For* $\tau < 1/2$, *the proposed criterion cannot mark* $T \in \mathcal{M}_\ell$ *for both, horizontal and vertical refinement. To see this, we argue by contradiction and assume that there holds* $c_{T,2}^2 + c_{T,3}^2 \le \frac{\tau}{1-\tau}\,c_{T,4}^2$ *as well as* $c_{T,2}^2 + c_{T,4}^2 \le \frac{\tau}{1-\tau}\,c_{T,3}^2$ *for some* $T \in \mathcal{M}_\ell$. *Note that this is equivalent to*
$$c_{T,2}^2 + c_{T,3}^2 \le \tau\,(c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2) \quad \textit{and} \quad c_{T,2}^2 + c_{T,4}^2 \le \tau\,(c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2).$$
Now, $\tau < 1/2$ *yields*
$$c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2 \le \tau\,(c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2) + \frac{1}{2}(c_{T,3}^2 + c_{T,4}^2) < c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2,$$
from which we infer $c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2 = 0$. *This however implies* $\widehat{u}_\ell|_T = (\Pi_\ell\widehat{u}_\ell)|_T$. *Then,* $\widetilde{\mu}_\ell(T) = 0$ *contradicts* $T \in \mathcal{M}_\ell$ *according to the minimality of* $\mathcal{M}_\ell$. □

5.2. Estimator Reduction for Averaging-Based Error Estimator $\widetilde{\beta}_\ell$. The ideas of the previous section can be generalized to anisotropic mesh-refinement steered by the averaging estimator $\widetilde{\beta}_\ell$ from (3.17). For $T \in \mathcal{M}_\ell$, let $\Pi^{(1)}_{\text{unif}(T)}$, $\Pi^{(1)}_{\text{vert}(T)}$, and $\Pi^{(1)}_{\text{horiz}(T)}$ denote the L^2-orthogonal projections onto $\mathcal{P}^1(\text{unif}(T))$, $\mathcal{P}^1(\text{vert}(T))$, and $\mathcal{P}^1(\text{horiz}(T))$, respectively, cf. Figure 3. As before, let $0 < \tau < 1$ be an additional parameter and assume that $T \in \mathcal{M}_\ell$ is marked for refinement in Algorithm 2.1.

- If $\|\varrho_\ell^{1/2}(1 - \Pi^{(1)}_{\text{horiz}(T)})\widehat{u}_\ell\|^2_{L^2(T)} \le \tau\widetilde{\beta}_\ell(T)^2$, we use horizontal refinement.
- If $\|\varrho_\ell^{1/2}(1 - \Pi^{(1)}_{\text{vert}(T)})\widehat{u}_\ell\|^2_{L^2(T)} \le \tau\widetilde{\beta}_\ell(T)^2$, we use vertical refinement.
- Otherwise, $T \in \mathcal{M}_\ell$ is refined isotropically into four sons $T_1, \dots, T_4 \in \mathcal{T}_{\ell+1}$.

As above, we check the mesh-size ratio ϱ_ℓ of neighbouring elements and possibly mark them for refinement in order to ensure the uniform boundedness of the K-mesh constant $\kappa(\mathcal{T}_\ell)$. The following theorem states some estimator reduction (1.4) as well as the estimator convergence (1.3) for $\widetilde{\beta}_\ell$.

Theorem 5.2. *Let $0 < \theta < 1$ and $0 < \tau < 1$ be fixed constants. Suppose that we use the indicators $\rho_\ell(T) := \widetilde{\beta}_\ell(T)$ defined in (3.17) in Algorithm 2.1 and the above described heuristics to decide the type of refinement. Then,*

$$\widetilde{\beta}_{\ell+1} \le \big(1 - \theta(1-\tau)\big)^{1/2}\,\widetilde{\beta}_\ell + \sqrt{2}\,C_{\text{inv}}\,|||\widehat{u}_{\ell+1} - \widehat{u}_\ell||| \quad \textit{for all } \ell \in \mathbb{N}_0, \tag{5.3}$$

and the second term on the right-hand side vanishes as $\ell \to \infty$. In particular, Proposition 2.3 applies and proves convergence $\lim_{\ell\to\infty} \widetilde{\beta}_\ell = 0$.

Proof. We follow the lines of the proof of Theorem 5.1. As above, the triangle inequality and the inverse estimate (3.10) prove

$$\widetilde{\beta}_{\ell+1} \le \|\varrho_{\ell+1}^{1/2}(1 - \Pi^{(1)}_{\ell+1})\widehat{u}_\ell\|_{L^2(\Gamma)} + \sqrt{2}\,C_{\text{inv}}\,|||\widehat{u}_{\ell+1} - \widehat{u}_\ell|||.$$

Now, let $T \in \mathcal{M}_\ell$ be a marked element.

- If T is refined isotropically, there holds
$$\|\varrho_{\ell+1}^{1/2}(1 - \Pi^{(1)}_{\ell+1})\widehat{u}_\ell\|_{L^2(T)} = 0.$$
- If T is refined by horizontal refinement, there holds
$$\|\varrho_{\ell+1}^{1/2}(1 - \Pi^{(1)}_{\ell+1})\widehat{u}_\ell\|^2_{L^2(T)} \le \|\varrho_\ell^{1/2}(1 - \Pi^{(1)}_{\text{horiz}(T)})\widehat{u}_\ell\|^2_{L^2(T)} \le \tau\widetilde{\beta}_\ell(T)^2.$$
- If T is refined by vertical refinement, there holds
$$\|\varrho_{\ell+1}^{1/2}(1 - \Pi^{(1)}_{\ell+1})\widehat{u}_\ell\|^2_{L^2(T)} \le \|\varrho_\ell^{1/2}(1 - \Pi^{(1)}_{\text{vert}(T)})\widehat{u}_\ell\|^2_{L^2(T)} \le \tau\widetilde{\beta}_\ell(T)^2.$$

In all cases, we thus obtain

$$\|\varrho_{\ell+1}^{1/2}(1 - \Pi^{(1)}_{\ell+1})\widehat{u}_\ell\|^2_{L^2(T)} \le \tau\,\widetilde{\beta}_\ell(T)^2 \quad \text{for all } T \in \mathcal{M}_\ell.$$

Moreover, there clearly holds

$$\|\varrho_{\ell+1}^{1/2}(1 - \Pi^{(1)}_{\ell+1})\widehat{u}_\ell\|^2_{L^2(T)} \le \widetilde{\beta}_\ell(T)^2 \quad \text{for all } T \in \mathcal{T}_\ell\backslash\mathcal{M}_\ell.$$

Together with the marking strategy (2.3), we again obtain

$$\|\varrho_{\ell+1}^{1/2}(1 - \Pi^{(1)}_{\ell+1})\widehat{u}_\ell\|^2_{L^2(\Gamma)} \le \big(1 - \theta(1-\tau)\big)\,\widetilde{\beta}_\ell^2$$

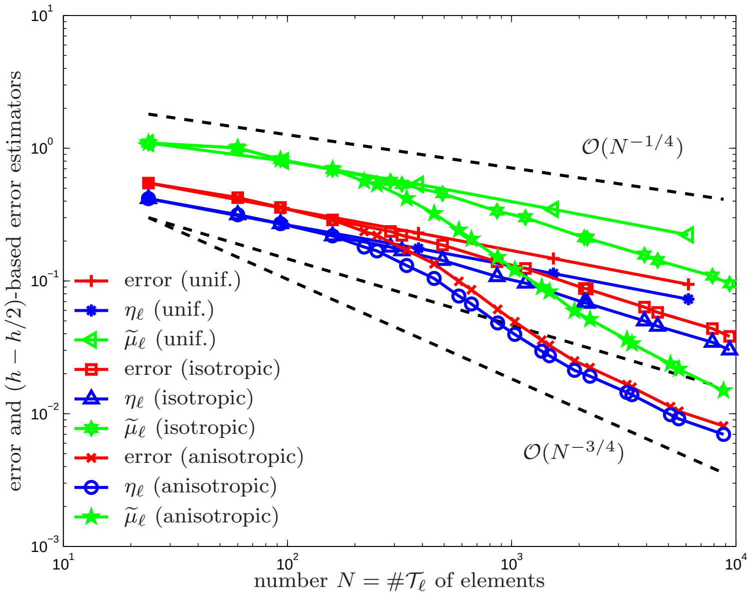

FIGURE 5. Error $|||u - u_\ell|||$ (red) as well as error estimators η_ℓ (blue) and $\widetilde{\mu}_\ell$ (green) for uniform, adaptive isotropic, and adaptive anisotropic mesh-refinement. For comparison, we plot the slopes of the experimental convergence rates $\mathcal{O}(N^{-\alpha})$ for $\alpha \in \{1/4, 1/2, 3/4\}$.

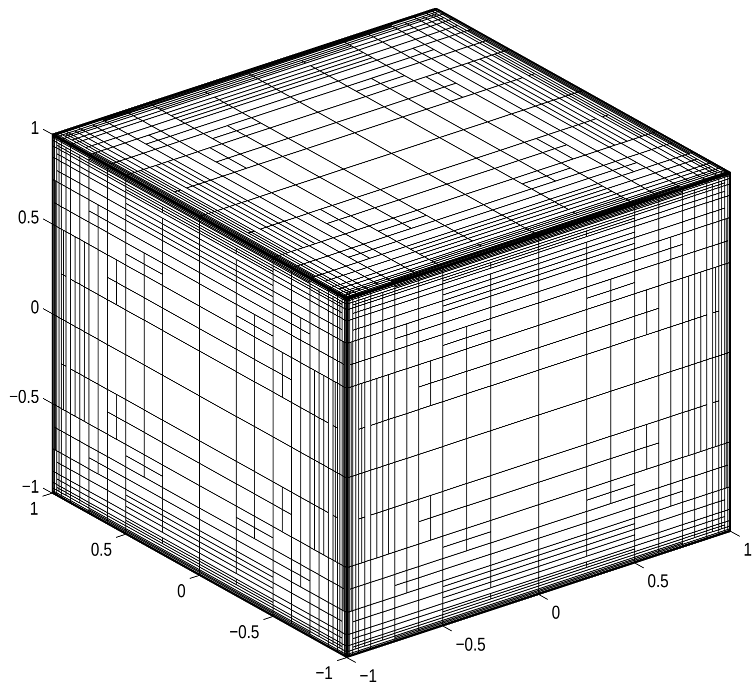

FIGURE 6. Mesh $\mathcal{T}_{19}$ with $N = 5.567$ boundary elements obtained after 19 steps of the adaptive algorithm with anisotropic mesh-refinement.

and conclude the proof. $\square$

5.3. Numerical Experiment for 3D BEM — Isotropic vs. Anisotropic Mesh-Refinement. We consider the numerical solution of Symm's integral equation

$$Vu = 1 \tag{5.4}$$

on the boundary $\Gamma = \partial\Omega$ of the cube $\Omega = (-1, 1)^3$. The uniform initial mesh $\mathcal{T}_0$ consisted of 24 flat square elements with edge length 1.

In Algorithm 2.1, we use the $(h - h/2)$-based error estimator $\widetilde{\mu}_\ell$ and $\theta = 0.25$ to steer the local mesh refinement. In case of anisotropic mesh-refinement, we use the parameter $\tau = 1/3$.

With $N = \#\mathcal{T}_\ell$ being the number of boundary elements, the optimal convergence rate for lowest-order Galerkin BEM is well-known to be $\mathcal{O}(N^{-3/4})$ for a piecewise smooth exact solution u, cf. [30]. The exact solution $u \in H^{-1/2}(\Gamma)$ is, however, unknown, and theory predicts that edge singularities might occur.

Figure 5 visualizes the error $|||u - u_\ell|||$ as well as the $(h - h/2)$-error estimators η_ℓ and $\widetilde{\mu}_\ell$ plotted over the number of elements. As above, the error $|||u - u_\ell|||$ is computed by use of Galerkin orthogonality and the extrapolated value $|||u|||^2 = 16.604658$. All quantities are shown for uniform, adaptive isotropic, and adaptive anisotropic mesh-refining strategies: Uniform mesh-refinement leads to a poor convergence rate $\mathcal{O}(N^{-1/4})$. While adaptive isotropic mesh-refinement accelerates the convergence up to a convergence rate of about $\mathcal{O}(N^{-1/2})$, the observed behavior is still not optimal. Finally, the proposed anisotropic adaptive strategy recovers the optimal order of convergence $\mathcal{O}(N^{-3/4})$. We observe that the anisotropically generated triangulation $\mathcal{T}_{19}$ with $N = 5.567$ elements shown in Figure 6 is highly adapted along the edges, and discrete solutions show some singular behavior along the edges of Γ. Therefore, the anisotropic alignment of the mesh seems to be mandatory for optimal convergence behavior in this example.

Acknowledgement. The second author (SFL) acknowledges a grant of the graduate school *Differential Equations — Models in Science and Engineering*, funded by the Austrian Science Fund (FWF) under grant W800-N05. The research of the third author (DP) is supported through the FWF project *Adaptive Boundary Element Method*, funded by the Austrian Science Fund (FWF) under grant P21732. The results of this paper have been presented at MAFELAP '09 in the BEM minisymposium dedicated to Professor G. Hsiao on the occasion of his 75th birthday.

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