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A local inverse spectral theorem for Hamiltonian systems

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Abstract

We consider 2×2–Hamiltonian systems of the form $y'(t) = zJH(t)y(t)$, $t \in [s_-, s_+)$. If a system of this form is in the limit point case, an analytic function is associated with it, namely its Titchmarsh–Weyl coefficient q_H. The (global) uniqueness theorem due to L. de Branges says that the Hamiltonian H is (up to reparameterization) uniquely determined by the function q_H. In the present paper we give a local uniqueness theorem: if the Titchmarsh–Weyl coefficients q_{H_1} and q_{H_2} corresponding to two Hamiltonian systems are exponentially close, then the Hamiltonians H_1 and H_2 coincide (up to reparameterization) up to a certain point of their domain, which depends on the quantitative degree of exponential closeness of the Titchmarsh–Weyl coefficients.

AMS Classification Numbers: 34B05, 34A55; 46E22, 42A82
Keywords: Hamiltonian system, Titchmarsh–Weyl coefficient, local uniqueness theorem, inverse spectral theorem

1 Introduction and main result

We consider a 2×2-Hamiltonian system without potential, i.e. a 2×2-system of the form

$$y'(t) = zJH(t)y(t), \quad t \in [s_-, s_+), \tag{1.1}$$

with a function $H(t)\colon [s_-, s_+) \to \mathbb{C}^{2\times 2}$ which is locally integrable on $[s_-, s_+)$, takes non-negative matrices as values, and does not vanish identically on any set of positive measure. Moreover, z is a complex parameter, and J denotes the signature matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The function $H(t)$ is also called the Hamiltonian of the system (1.1). Equations of this form frequently occur in analysis and natural sciences; for example in Hamiltonian mechanics, cf. [Arn], [Fl], as generalizations of Sturm–Liouville equations, cf. [R], [GKM], or in the study of strings, cf. [At], [KK2], [Ka2].

Equation (1.1) generates in a natural way a differential operator (actually, in some cases it is a linear relation, i.e. a multi-valued operator). It acts in a certain weighted L^2-space $L^2(H)$ whose elements are 2-vector valued functions; see, e.g. [Ka1], [O], [HSW]. The spectral theory of this operator changes tremendously depending whether the integral $\int_{s_-}^{s_+} \operatorname{tr} H(t)\,dt$ is finite or infinite; in the first case one says that the Hamiltonian H (or the system (1.1)) is in Weyl's limit circle case, in the latter one speaks of limit point case. In the limit point case the operator mentioned above has deficiency indices $(1,1)$; in the limit circle case it has deficiency indices $(2,2)$.

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Of course, 'changes of scale' in the equation (1.1) will not affect the spectral theory of the associated differential operator. The notion of 'changes of scale', however, needs to be defined rigorously. Two Hamiltonians H_1 and H_2 defined on respective intervals (s_-^1, s_+^1) and (s_-^2, s_+^2) are called *reparameterizations* of each other if there exists an absolutely continuous, increasing bijection $\varphi\colon (s_-^2, s_+^2) \to (s_-^1, s_+^1)$ such that φ^{-1} is also absolutely continuous and $H_2(t) = H_1(\varphi(t)) \cdot \varphi'(t)$, $t \in (s_-^2, s_+^2)$. In this case, we write $H_1 \sim H_2$. We have the following connection between solutions in this situation: if y_1 is a solution of (1.1) with $H = H_1$, then $y_2(t) = y_1(\varphi(t))$ is a solution of (1.1) with $H = H_2$ and vice versa.

Consider a Hamiltonian H which is in the limit point case and denote by $W(x,z) = (w_{ij}(x,z))_{i,j=1,2}$ the transpose of the fundamental solution of (1.1), i.e. the solution of the initial value problem

$$\frac{d}{dt}W(t,z)J = zW(t,z)H(t), \quad t \in (s_-, s_+), \qquad W(s_-,z) = I, \tag{1.2}$$

where I denotes the 2×2-identity matrix. Then, for each $\tau \in \mathbb{R}\cup\{\infty\}$, the limit

$$\lim_{t \nearrow s_+} \frac{w_{11}(t,z)\tau + w_{12}(t,z)}{w_{21}(t,z)\tau + w_{22}(t,z)} =: q_H(z)$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ and does not depend on τ. For $\tau = \infty$, the quotient is understood as $w_{11}(t,z)/w_{21}(t,z)$. The function q_H is called the *Titchmarsh–Weyl coefficient* associated with the Hamiltonian H. It belongs to the Nevanlinna class $\mathcal{N}_0$, which means that

$$q_H \text{ is analytic on } \mathbb{C}\setminus\mathbb{R}, \qquad q_H(\overline{z}) = \overline{q_H(z)},\ z \in \mathbb{C}\setminus\mathbb{R},$$
$$\operatorname{Im} q_H(z) \geq 0 \quad \text{for } \operatorname{Im} z > 0.$$

This construction is vital for the spectral theory of the equation (1.1); for example the function q_H can be used to construct a Fourier transform of the space $L^2(H)$ onto a certain L^2-space of scalar valued functions, namely the space $L^2(\mu)$ where μ is the measure in the Herglotz integral representation of the function q_H (appropriately including a possible point mass at infinity). The name Titchmarsh–Weyl *coefficient* comes from the fact that the function

$$y(t) = \begin{pmatrix} w_{11}(t,z) \\ w_{12}(t,z) \end{pmatrix} - q_H(z) \begin{pmatrix} w_{21}(t,z) \\ w_{22}(t,z) \end{pmatrix} \tag{1.3}$$

is the unique (up to scalar multiples) solution of (1.1) that is in $L^2(H)$.

It is a fundamental result due to L. de Branges that for each function $q \in \mathcal{N}_0$ there exists (up to reparameterization) one and only one Hamiltonian H such that $q = q_H$, cf. [dB], [W1]. The uniqueness part of this result may be formulated in the following way.

1.1. *Uniqueness theorem* ([dB]). *Let H_1 and H_2 be Hamiltonians defined on intervals $[s_-^1, s_+^1)$ and $[s_-^2, s_+^2)$, respectively. Assume that H_1 and H_2 are in the limit point case and denote by q_{H_1} and q_{H_2} their respective Titchmarsh–Weyl coefficients. Then the following are equivalent.*

(*i*) $H_1 \sim H_2$.

(ii) $q_{H_1} = q_{H_2}$.

Our aim in this paper is to prove a refinement of this theorem, namely the following local version.

1.2. *Local uniqueness theorem.* *Let H_1 and H_2 be Hamiltonians defined on intervals $[s_-^1, s_+^1)$ and $[s_-^2, s_+^2)$, respectively. Assume that H_1 and H_2 are in the limit point case, and denote by q_{H_1} and q_{H_2} their respective Titchmarsh–Weyl coefficients. Moreover, let $a > 0$ and set*

$$s_a^j := \sup\Big\{t \in [s_-^j, s_+^j) \colon \int_{s_-^j}^t \sqrt{\det H_j(x)}\, dx < a\Big\}, \quad j = 1, 2\,.$$

Then the following are equivalent.

(i) $H_1 \big|_{[s_-^1, s_a^1)} \sim H_2 \big|_{[s_-^2, s_a^2)}$.

(ii) There exists $\theta \in (0, \pi)$ such that for every $\varepsilon > 0$,

$$q_{H_1}(re^{i\theta}) - q_{H_2}(re^{i\theta}) = O\big(e^{(-2a+\varepsilon) r \sin\theta}\big), \quad r \to +\infty\,.$$

(iii) Denote by Γ_α, $\alpha \in (0, \frac{\pi}{2})$, the Stolz angle $\Gamma_\alpha := \{z \in \mathbb{C} \colon \alpha \leq \arg z \leq \pi - \alpha\}$. For every $\alpha \in (0, \frac{\pi}{2})$ we have[1]

$$q_{H_1}(z) - q_{H_2}(z) = O\big((\operatorname{Im} z)^3 e^{-2a \operatorname{Im} z}\big), \quad |z| \to \infty, \ \ z \in \Gamma_\alpha\,.$$

The following result, although a simple consequence of 1.2, is worth being stated separately.

1.3 Corollary. *Let H_1 and H_2 be Hamiltonians defined on intervals $[s_-^1, s_+^1)$ and $[s_-^2, s_+^2)$, respectively. Assume that $\alpha := \int_{s_-^1}^{s_+^1} \sqrt{\det H_1(x)}\, dx < \infty$ and that for some $\theta \in (0, \pi)$ and some $\beta > \alpha$*

$$q_{H_1}(re^{i\theta}) - q_{H_2}(re^{i\theta}) = O\big(e^{-2\beta r \sin\theta}\big), \quad r \to +\infty\,. \tag{1.4}$$

Then $q_{H_1} = q_{H_2}$, i.e. $H_1 \sim H_2$.

The topic of (global) uniqueness theorems in the setting of Hamiltonian systems or Schrödinger operators has been researched for a long time; let us refer for example to [Bor1], [M1], [GL], [Bor2], [dB], [M2], [GS1], [BBW]. Interestingly, local results came up only comparatively recently, starting with [Si] and followed by [GS2], [Be], [CG], [GKM], where Sturm–Liouville equations in Schrödinger form $-y''(t) + q(t)y(t) = \lambda y(t)$ for scalar- and vector-valued cases, and Dirac systems (i.e. Hamiltonian systems of the form $y'(t) = zJy(t) + V(t)y(t)$) were discussed. To our knowledge all these papers deal with cases where the term with the spectral parameter does not depend on t.

The proof of the present local uniqueness theorem 1.2 is carried out by combining several results of de Branges' theory of Hilbert spaces of entire functions, among them the global uniqueness theorem 1.1, with some facts from the theory

[1] Maybe the power $(\operatorname{Im} z)^3$ is not the best growth estimate one can get. However, it is not far from optimal; it is easy to construct examples with $q_{H_1}(z) - q_{H_2}(z) \asymp e^{-2a \operatorname{Im} z}$.

of positive definite functions, cf. [Sa], [AG], [K], and classical results of complex analysis; see, e.g. [Ko]. Let us point out explicitly that we do not obtain a new proof of the (global) uniqueness theorem 1.1 since it enters in our proof of 1.2 [2].

After giving the proofs of 1.2 and Corollary 1.3, which is done in Section 2, we deduce a local Borg-Marchenko uniqueness result for Sturm-Liouville equations without potential[3].

1.4 Remark. The local uniqueness theorem analogous to 1.2 remains true in the setting of indefinite Hamiltonian systems as introduced and studied in [KW/IV]–[KW/VI] [4]. The proof is word by word the same, only in some places one has to refer to Pontryagin space theory instead of classical Hilbert space results. In order to avoid the somewhat tedious introduction of these notions, we decided not to formulate the general 'indefinite' result. We will content ourselves with indicating the proper references in footnotes.

This observation becomes valuable when considering local uniqueness theorems for Sturm–Liouville equations with singular potentials; some examples of such equations have been studied, e.g. in [LLS], [HM], [FL]. However, in the present paper, we will not touch upon this topic. //

2 Proof of the local uniqueness theorem

We need to recall the relations among Hamiltonian systems, de Branges spaces and positive definite functions in some detail.

2.1. Some classes of functions. Besides Hamiltonians and the class $\mathcal{N}_0$ of Nevanlinna functions, some other classes of functions have to be specified.

(i) A continuous function $f\colon \mathbb{R} \to \mathbb{C}$ is called *positive definite* if $f(-t) = \overline{f(t)}$, $t \in \mathbb{R}$, and if the kernel

$$K_f(s,t) := f(t-s), \qquad t,s \in \mathbb{R},$$

is positive definite, i.e. the matrices $\big(K_f(t_i,t_j)\big)_{i,j=1}^n$ are positive semi-definite for all choices of $n \in \mathbb{N}$ and $t_1,\dots,t_n \in \mathbb{R}$.

(ii) An entire functions is said to belong to the *Hermite–Biehler class* $\mathcal{HB}_0$ if $|E(\overline{z})| < |E(z)|$, $z \in \mathbb{C}^+$, where $\mathbb{C}^+$ denotes the upper half-plane. Equivalently, we could require that the kernel

$$K_E(w,z) := \frac{i}{2}\,\frac{E(z)\overline{E(w)} - \overline{E(\overline{z})}E(\overline{w})}{z-\overline{w}}\,, \qquad z,w \in \mathbb{C},$$

is positive definite (for $z = \overline{w}$ this formula has to interpreted appropriately as a derivative).

Each function $E \in \mathcal{HB}_0$ generates a reproducing kernel Hilbert space $\mathcal{H}(E)$ via the kernel K_E whose elements are entire functions. The reproducing kernel property is

$$F(w) = \big(F, K_E(w,\cdot)\big)_{\mathcal{H}(E)}, \qquad F \in \mathcal{H}(E),\ w \in \mathbb{C}.$$

[2] This contrasts the work [GS2] on Schrödinger operators, which does give a new proof of the classical Borg–Marchenko Theorem.

[3] We do not know if the Schrödinger case could also be deduced from 1.2.

[4] Only the power in (iii) has to be adapted.

The space $\mathcal{H}(E)$ is called the *de Branges space* generated by E.

(iii) An entire 2×2-matrix-valued function $W(z)$ is said to belong to the class $\mathcal{M}_0$ if $W(\overline{z}) = \overline{W(z)}$, $W(0) = I$ and if the kernel

$$K_W(w,z) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}\,, \qquad z, w \in \mathbb{C},$$

is positive definite (for $z = \overline{w}$ this formula again has to interpreted appropriately as a derivative). Each function $W \in \mathcal{M}_0$ generates a reproducing kernel Hilbert space $\mathfrak{K}(W)$ via the kernel K_W whose elements are 2-vector-valued entire functions.

//

2.2. We are going to explain the relations among all these objects according to the following diagram:

$$\begin{tikzcd}
\big(H(t)\big)_{t\in[s_-,s_+)} \arrow[r, leftrightarrow, "(1)"] \arrow[d, leftrightarrow, "(3)"'] & \big(W(t,z)\big)_{t\in[s_-,s_+)} \arrow[r, leftrightarrow, "(2)"] & \big(E_{W(t,\cdot)}(z)\big)_{t\in[s_-,s_+)} \arrow[d, leftrightarrow, "(4)"] \\
q_H(z) \arrow[r, leftrightarrow, "(5)"] & f_H(x) \arrow[r, leftrightarrow, "(6)"] & \big(\mathcal{H}(E_{W(t,\cdot)})\big)_{t\in[s_-,s_+)}
\end{tikzcd} \tag{2.1}$$

Relation (1)*:* A Hamiltonian $H(t)$ gives rise to a family of matrix functions $W(t,z)$ via (1.2). These matrices have the property that $W(s,z)^{-1}W(t,z) \in \mathcal{M}_0$ whenever $s,t \in [s_-,s_+)$, $s \leq t$; in particular, $W(t,z) \in \mathcal{M}_0$, $t \in [s_-,s_+)$. Conversely, the fundamental solution of a Hamiltonian system determines its Hamiltonian uniquely. These classical facts can be found, e.g. in [GK], [HSW], [O], [dB] [5].

Note that for two Hamiltonians H_1, H_2 we have $H_1 \sim H_2$ with reparameterization φ, i.e. $H_2(t) = H_1(\varphi(t))\varphi'(t)$ if and only if $W_2(t) = W_1(\varphi(t))$ where W_j is the fundamental solution corresponding to H_j.

Relation (2)*:* If $W(z) = (w_{ij}(z))_{i,j=1,2}$ belongs to the class $\mathcal{M}_0$, then the function

$$E_W(z) := w_{22}(z) + iw_{21}(z)$$

is a Hermite–Biehler function. It satisfies $E_W(0) = 1$ and the de Branges space generated by E_W is invariant under forming difference quotients, i.e.

$$F \in \mathcal{H}(E_W) \quad \Longrightarrow \quad \frac{F(z) - F(w)}{z - w} \in \mathcal{H}(E_W),\ w \in \mathbb{C}\,.$$

Conversely, if $E \in \mathcal{HB}_0$ possesses these two additional properties, then there exists a function $W \in \mathcal{M}_0$ with $E = E_W$. The function W can be chosen such that the projection π_2 onto the second component is an isometric isomorphism of the space $\mathfrak{K}(W)$ onto $\mathcal{H}(E)$, and with this additional requirement it becomes unique. This can be found in [dB, Theorem 27] [6].

[5] for the indefinite case see [KW/V, Theorem 5.1], [KW/VI, Theorem 1.5, Theorem 1.6]

[6] [KW/I, Proposition 8.3, Lemma 8.6], [KW/I, Proposition 10.3, Corollary 10.4]

Relation (3)*:* The Hamiltonian H corresponds bijectively (up to reparameterization) to its Titchmarsh–Weyl coefficient q_H via de Branges' inverse spectral theorem, cf. [dB], [W1, Theorem 1] [7].

For our present task it is important to know that the following three conditions are equivalent, see [W2], [dB] [8]:

– The Hamiltonian H satisfies

$$\exists\,\varepsilon > 0: \quad H(t) = h(t)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ t \in [s_-, s_- + \varepsilon] \text{ a.e.}, \tag{2.2}$$

with some scalar integrable function h.

– For one, and hence for all, values $t \in (s_-, s_+)$ the reproducing kernel space $\mathfrak{K}(W(t,\cdot))$ contains the constant function $\binom{0}{1}$.

– The Titchmarsh–Weyl coefficient q_H of H satisfies

$$\lim_{y\to+\infty} q_H(iy) = 0 \quad \text{and} \quad \lim_{y\to+\infty} y|\operatorname{Im} q_H(iy)| < \infty.$$

Relation (4)*:* A function $E \in \mathcal{HB}_0$ gives rise to the de Branges space $\mathcal{H}(E)$. Two different functions E_1 and E_2 of Hermite–Biehler class may induce the same Hilbert space, meaning that

$$F \in \mathcal{H}(E_1) \Leftrightarrow F \in \mathcal{H}(E_2) \quad \text{and} \quad [F,G]_{\mathcal{H}(E_1)} = [F,G]_{\mathcal{H}(E_2)},\ F,G \in \mathcal{H}(E_1).$$

In fact, this is the case if and only if there exists a matrix $U \in \mathbb{R}^{2\times 2}$ with $\det U = 1$, such that

$$(A_2, B_2) = (A_1, B_1)U \tag{2.3}$$

where $A_j(z) := \frac{1}{2}(E_j(z) + \overline{E_j(\overline{z})})$ and $B_j(z) := \frac{i}{2}(E_j(z) - \overline{E_j(\overline{z})})$. This fact is shown in [dB] [9].

It is clear that, if besides $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ we require that $E_1(0) = E_2(0) = 1$, then the matrix U in (2.3) must be of the form

$$U = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

We need the following, more specific and less elementary statement.

2.3 Lemma. Let H_1 and H_2 be Hamiltonians defined on intervals $[s_-^1, s_+^1)$ and $[s_-^2, s_+^2)$, respectively. Assume that both are in limit circle case and satisfy (2.2)*. Denote by $(W_j(t,z))_{t\in[s_-^j, s_+^j)}$, $j = 1,2$, the respective solutions of* (1.2)*, and set $E_j := E_{W_j(s_+^j,\cdot)}$, $j = 1,2$. Then $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ implies that $H_1 \sim H_2$, and thus $E_1 = E_2$.*

Proof. Set $W_k(z) = (w_{ij}^k(z))_{i,j=1,2} := W_k(s_+^k,\cdot)$, $k = 1,2$. Then $E_k = w_{22}^k + iw_{21}^k$, and $A_k = w_{22}^k$, $B_k = -w_{21}^k$. By the above considerations, the relation $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ thus implies that for some $u \in \mathbb{R}$

$$(w_{21}^2, w_{22}^2) = (w_{21}^1, w_{22}^1)\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}. \tag{2.4}$$

[7] [KW/VI, Theorem 1.4]
[8] [KW/II, Proof of Theorem 7.1], [K, Proposition 5.3]
[9] [KW/I, Corollary 6.2]

Set

$$\tilde{W} := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} W_1 \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix},$$

then $\tilde{W} \in \mathcal{M}_0$ and the map

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$$

is an isometric isomorphism from $\mathfrak{K}(W_1)$ onto $\mathfrak{K}(\tilde{W})$; see, e.g. [Aro] [10]. Since we assume that H_1 satisfies (2.2), it follows that

$$\begin{pmatrix} u \\ 1 \end{pmatrix} \in \mathfrak{K}(\tilde{W}).$$

Since the space $\mathfrak{K}(\tilde{W})$ can contain at most one constant, in particular, the constant $\binom{1}{0}$ does not belong to the space $\mathfrak{K}(\tilde{W})$. Since also H_2 satisfies (2.2), we also have $\binom{1}{0} \notin \mathfrak{K}(W_2)$.

By (2.4) we have $(0,1)W_2 = (0,1)\tilde{W}$, and [dB, Theorem 27][11] implies that for some $v \in \mathbb{R}$

$$W_2 = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \tilde{W} \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u+v \\ 0 & 1 \end{pmatrix} W_1 \begin{pmatrix} 1 & -(u+v) \\ 0 & 1 \end{pmatrix}.$$

Hence the map

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \mapsto \begin{pmatrix} 1 & u+v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$$

is an isometric isomorphism from $\mathfrak{K}(W_1)$ onto $\mathfrak{K}(W_2)$, and we conclude that $\binom{u+v}{1} \in \mathfrak{K}(W_2)$. Again using that H_2 satisfies (2.2), it follows that $u+v=0$, i.e. $W_2 = W_1$ and hence also $E_1 = E_2$. By de Branges' Inverse Spectral Theorem this even implies that $H_1 \sim H_2$. ❏

Relation (5)*:* The class $\mathcal{P}_0$ of positive definite functions corresponds bijectively to the subclass of $\mathcal{N}_0$ which contains all functions $q \in \mathcal{N}_0$ with

$$\lim_{y\to+\infty} q(iy) = 0, \qquad \lim_{y\to+\infty} y|\operatorname{Im} q(iy)| < \infty. \tag{2.5}$$

This bijection is established by the one-sided Fourier transform ($f \in \mathcal{P}_0$)

$$q(z) = i\int_0^\infty f(t)e^{izt}\,dt, \quad \operatorname{Im} z > 0. \tag{2.6}$$

This relationship is best understood via the measures appearing in the Herglotz integral representation of q and in the integral representation of f as a Fourier transform by Bochner's Theorem. In fact, a function q belongs to $\mathcal{N}_0$ and satisfies (2.5) if and only if it can be written as

$$q(z) = \int_{\mathbb{R}} \frac{d\mu_q(t)}{t-z}, \quad \operatorname{Im} z > 0,$$

[10] [ADSR, Theorem 1.5.3]
[11] [KW/I, Corollary 9.8]

with a finite positive Borel measure μ_q on $\mathbb{R}$, cf. [KK1, Theorem S1.4.1][12]. On the other hand, by Bochner's Theorem, a positive definite function f can be represented as

$$f(x) = \int_{\mathbb{R}} e^{-itx}\, d\mu_f(t), \quad x \in \mathbb{R},$$

with some finite positive Borel measure μ_f on $\mathbb{R}$; see, e.g. [AG]. An application of Fubini's Theorem gives that the functions q and f are related by (2.6) if and only if $\mu_q = \mu_f$ [13].

Relation (6)*:* Let H be a Hamiltonian that satisfies (2.2). Going counterclockwise in the scheme (2.1) along (3) and (5), let $q_H \in \mathcal{N}_0$ be its Titchmarsh–Weyl coefficient and $f_H \in \mathcal{P}_0$ be the Fourier transform of q_H. Going clockwise along (1), (2) and (4), let $W(t,\cdot)$ be the fundamental solution of the Hamiltonian system with Hamiltonian H, and let $E_{W(t,\cdot)}$ and $\mathcal{H}(E_{W(t,\cdot)})$ be the Hermite–Biehler function generated by $W(t,\cdot)$ and the corresponding de Branges space, respectively. Moreover, set

$$s_a := \sup\left\{t \in [s_-, s_+) \colon \int_{s_-}^{t} \sqrt{\det H(x)}\, dx < a\right\}, \quad a > 0.$$

Then, for each $a \in (0, \int_{s_-}^{s_+} \sqrt{\det H(t)}\, dt)$, we have

$$\mathcal{H}(E_{W(s_a,\cdot)}) = \text{c.l.s.}\{e^{-itz} \colon |t| \le a\},$$
$$[e^{-itz}, e^{-iuz}]_{\mathcal{H}(E_{W(s_a,\cdot)})} = f(u-t), \quad |t|, |u| \le a,$$

where c.l.s. stands for 'closed linear span'. See [K, Lemma 5.8] [14] [15] .

We have collected everything that is needed for the proof of the local uniqueness theorem 1.2. //

Proof (of the local uniqueness theorem 1.2).
Step 1: the case that H_1 and H_2 satisfy (2.2).
By Relation (5) there exist positive definite functions f_1 and f_2 with

$$q_{H_j}(z) = i \int_0^\infty f_j(t) e^{itz}\, dt, \qquad \operatorname{Im} z > 0,\ j = 1, 2.$$

Set $f := f_1 - f_2$; then ($\operatorname{Im} z > 0$)

$$q_{H_1}(z) - q_{H_2}(z) = i \int_0^\infty f(t) e^{itz}\, dt = \underbrace{i \int_0^{2a} f(t) e^{itz}\, dt}_{=:F_1(z)} + \underbrace{i \int_{2a}^\infty f(t) e^{itz}\, dt}_{=:F_2(z)}.$$

First we estimate F_2. Since f_j is positive definite, we have $f_j(0) \ge 0$ and $|f_j(t)| \le f_j(0)$, $t \in \mathbb{R}$. This allows us to compute ($\operatorname{Im} z > 0$)

$$|F_2(z)| = \left| i e^{2aiz} \int_{2a}^\infty f(t) e^{i(t-2a)z}\, dt \right| \le e^{-2a \operatorname{Im} z} \cdot \tfrac{f_1(0)+f_2(0)}{\operatorname{Im} z}.$$

[12] In the indefinite case the measure μ_q has to be replaced by some distribution; see [JLT], [KWW].
[13] [KL, Satz 5.3]
[14] We do not know a reference which deals with the positive definite case only.
[15] Density of exponentials depends on de Branges' Ordering Theorem.

Secondly, if $f|_{[0,2a]} \neq 0$, we set $m := \min(\operatorname{supp} f \cap [0,2a])$. Then, by the Theorem of Paley–Wiener on Fourier transforms of functions with compact support combined with the knowledge on regular asymptotic behaviour of functions of bounded type, we have

$$\limsup_{r\to\infty} \frac{\log |F_1(re^{i\theta})|}{r} = -m\sin\theta, \quad \theta \in (0,\pi);$$

see, e.g. [Ko, §3.D, Scholium p. 35], [Boa, §7.2].

Next, note that by Lemma 2.3, the condition $H_1|_{[s_-^1, s_a^1)} \sim H_2|_{[s_-^2, s_a^2)}$ is equivalent to the relation $\mathcal{H}(E_{W_1(s_a^1,\cdot)}) = \mathcal{H}(E_{W_2(s_a^2,\cdot)})$, which by Relation (6) is equivalent to $f_1|_{[0,2a]} = f_2|_{[0,2a]}$, i.e. to $f|_{[0,2a]} = 0$.

Now we are in position to establish the equivalences asserted in 1.2. Assume that (i) of 1.2 holds. Then the function F_1 vanishes identically, and hence for each fixed $\alpha \in (0,\pi)$

$$q_{H_1}(z) - q_{H_2}(z) = O\left(\frac{e^{-2a\operatorname{Im} z}}{\operatorname{Im} z}\right), \quad |z| \to \infty, \ z \in \Gamma_\alpha. \tag{2.7}$$

In particular, (iii) holds. Trivially, (iii) implies (ii). Assume next that (ii) holds and $f|_{[0,2a]} \neq 0$. Then $m < 2a$, and hence the summand F_1 dominates the asymptotic behaviour of $q_{H_1} - q_{H_2}$. We obtain that for each $\theta \in (0,\pi)$

$$\limsup_{r\to\infty} \frac{\log |q_{H_1}(re^{i\theta}) - q_{H_2}(re^{i\theta})|}{r} = -m\sin\theta\,.$$

However, by (ii), there exists at least one value of θ such that this limit superior does not exceed $-2a\sin\theta$. It follows that $-2a \geq -m$, and we have reached a contradiction.

Step 2: reduction of the general case to the previous one.
Let H_j, $j = 1,2$, be arbitrary Hamiltonians defined on respective intervals $[s_-^j, s_+^j)$. Define Hamiltonians $\tilde{H}_j$ on the intervals $[s_-^j - 1, s_+^j)$ by

$$\tilde{H}_j(t) := \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & t \in [s_-^j - 1, s_-^j), \\ H_j(t), & t \in [s_-^j, s_+^j). \end{cases}$$

The fundamental matrix solutions of the canonical systems with Hamiltonians $\tilde{H}_j$ are given by

$$\tilde{W}_j(t,z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -(t - s_-^j + 1)z & 1 \end{pmatrix}, & t \in [s_-^j - 1, s_-^j), \\ \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} W_j(t,z), & t \in [s_-^j, s_+^j), \end{cases}$$

and hence

$$q_{\tilde{H}_j}(z) = \frac{q_{H_j}(z)}{-zq_{H_j}(z) + 1} = \frac{-1}{z - \frac{1}{q_{H_j}(z)}}, \qquad j = 1,2.$$

It follows that

$$q_{\tilde{H}_1}(z) - q_{\tilde{H}_2}(z) = \frac{q_{H_1}(z) - q_{H_2}(z)}{q_{H_1}(z)q_{H_2}(z)\big(z - \frac{1}{q_{H_1}(z)}\big)\big(z - \frac{1}{q_{H_2}(z)}\big)} .$$

Denote by $(\tilde{i}), (\tilde{ii}), (\tilde{iii})$ the respective conditions of 1.2 for the pair $(\tilde{H}_1, \tilde{H}_2)$. Since $\det \tilde{H}_j(t) = 0$, $t \in [s_-^j - 1, s_-^j)$, the value of s_a^j remains the same whether computed for H_j or $\tilde{H}_j$. Thus we have $(i) \Leftrightarrow (\tilde{i})$. Next, remember that each Nevanlinna function q satisfies above and below polynomial estimates, in fact there exist constants $\gamma_\pm > 0$ such that[16]

$$\frac{\gamma_-}{|z|} \leq |q(z)| \leq \gamma_+ |z|, \quad z \in \Gamma_\alpha, \operatorname{Im} z \geq 1 .$$

With q_{H_j} also the function $z - \frac{1}{q_{H_1}(z)}$ is a Nevanlinna function. Hence $(ii) \Leftrightarrow (\tilde{ii})$, and (2.7) for $\tilde{H}_j$ instead of H_j implies (iii). The implication $(iii) \Rightarrow (ii)$ is of course trivial. Together with what we showed in Step 1, we have by now established the following implications

$$\begin{tikzcd}
(\tilde{i}) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (2.7) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\tilde{iii}) \arrow[r, Rightarrow] & (\tilde{ii}) \arrow[lll, Rightarrow, bend right] \arrow[d, Leftrightarrow] \\
(i) & (iii) \arrow[rr, Rightarrow] & & (ii)
\end{tikzcd}$$

This gives the equivalence of (i), (ii) and (iii). ☺

Proof (of Corollary 1.3). We apply Theorem 1.2 with H_1, H_2, and a number a which is arbitrarily chosen in (α, β). Since $a < \beta$, the present assumption (1.4) implies that the condition (ii) of 1.2 is satisfied. Hence $H_1|_{[s_-^1, s_a^1)} \sim H_2|_{[s_-^2, s_a^2)}$. However, since $a > \alpha$, we have $s_a^1 = s_+^1$. It follows that $H_2|_{[s_-^2, s_a^2)}$ is in the limit point case, and hence that $s_a^2 = s_+^2$. We see that $H_1 \sim H_2$. ❑

3 Sturm–Liouville equations without potential

From Theorem 1.2 we can easily deduce a local Borg–Marchenko uniqueness result for Sturm–Liouville equations without potential. Let us consider an equation of the form

$$-(p(x)y'(x))' = \lambda w(x)y(x) \tag{3.1}$$

on the interval $[0, b)$ with $0 < b \leq \infty$, where p and w are measurable functions, $p(x) > 0$, $w(x) > 0$ almost everywhere, and $\frac{1}{p}, w \in L^1_{\text{loc}}([0, b))$. Note that here y denotes a scalar function. Equation (3.1) can be written as a Hamiltonian system (1.1) with

$$H(x) = \begin{pmatrix} w(x) & 0 \\ 0 & \frac{1}{p(x)} \end{pmatrix} \tag{3.2}$$

[16] In the indefinite case one has $\gamma_- |z|^{-N} \leq |q(z)| \leq \gamma_+ |z|^N$ with some appropriate $N \in \mathbb{N}$.

and $\lambda = z^2$ because y is a solution of (3.1) if and only if $\binom{y_1}{y_2}$ with

$$y_1(x) = y(x),$$
$$y_2(x) = -\frac{1}{z}p(x)y'(x)$$

is a solution of (1.1) with H from (3.2).

We say that (3.1) is in limit point case if (3.1) for $\lambda \in \mathbb{R} \setminus \mathbb{C}$ has only one linearly independent solution in $L^2_w(0,b)$, where $L^2_w(0,b)$ is the L^2 space with weight w. In this case also H in (3.2) is in limit point case because otherwise every solution of (1.1) is in $L^2(H)$ and hence the first component in $L^2_w(0,b)$. The converse (i.e. H in (3.2) is in limit point case $\Rightarrow$ (3.1) is in limit point case) is not true as the example $p(x) = 1$, $w(x) = (1+x)^{-4}$ shows.

The Titchmarsh–Weyl coefficient of (3.1) is defined as follows. Assume that (3.1) is in limit point case. Let $\theta(x,\lambda)$, $\phi(x,\lambda)$ be solutions of (3.1) that satisfy the initial conditions

$$\theta(0,\lambda) = 1, \qquad p(x)\theta'(x,\lambda)|_{x=0} = 0,$$
$$\phi(0,\lambda) = 0, \qquad p(x)\phi'(x,\lambda)|_{x=0} = 1.$$

Since we have limit point case, there exists a unique number $m(\lambda)$ for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that

$$\theta(x,\lambda) + m(\lambda)\phi(x,\lambda) \tag{3.3}$$

is in $L^2_w(0,b)$. It is easily seen that

$$\theta(x,z^2) = w_{11}(x,z), \qquad \phi(x,z^2) = -\frac{1}{z}w_{21}(x,z).$$

Comparing (3.3) and (1.3) we obtain

$$m(z^2) = zq_H(z). \tag{3.4}$$

In the following corollary we describe in terms of the Titchmarsh–Weyl coefficients when two equations of the form (3.1) coincide on some interval $(0,s)$ up to reparameterization.

3.1 Corollary. *Let $b_1, b_2 > 0$ and let p_j, w_j be measurable functions defined on $[0,b_j)$, $j = 1,2$, such that $p_j(x) > 0$, $w_j(x) > 0$ almost everywhere and $\frac{1}{p_j}, w_j \in L^1_{\mathrm{loc}}([0,b_j))$ for $j = 1,2$. Moreover, denote by m_j the Titchmarsh–Weyl coefficient for*

$$-(p_j y')' = \lambda w_j y.$$

Let $a > 0$ and set

$$s_a^j := \int_0^a \sqrt{\frac{w_j(x)}{p_j(x)}}\,dx.$$

Then the following statements are equivalent.

(*i*) *There exists an absolutely continuous, increasing function $\varphi\colon (0,s_a^2) \to (0,s_a^1)$ such that φ^{-1} is also absolutely continuous and*

$$\left.\begin{aligned} w_2(t) &= w_1\big(\varphi(t)\big)\varphi'(t) \\ p_2(t) &= p_1\big(\varphi(t)\big)\frac{1}{\varphi'(t)} \end{aligned}\right\} \quad \text{for almost all } t \in (0,s_a^2). \tag{3.5}$$

(ii) *There exists* $\theta \in (0, 2\pi)$ *such that for every* $\varepsilon > 0$,

$$m_1(re^{i\theta}) - m_2(re^{i\theta}) = O\big(e^{(-2a+\varepsilon)r\sin\frac{\theta}{2}}\big), \quad r \to +\infty\,.$$

(iii) *For every* $\alpha \in (0, \pi)$,

$$m_1(\lambda) - m_2(\lambda) = O\Big(|\lambda|^2 e^{-2a\,\mathrm{Im}\sqrt{\lambda}}\Big),$$
$$|\lambda| \to \infty,\ \lambda \in \{z \in \mathbb{C}\colon \alpha \le \arg z \le 2\pi - \alpha\}\,.$$

Proof. We can apply the local uniqueness theorem 1.2 to the Hamiltonians

$$H_j = \begin{pmatrix} w_j & 0 \\ 0 & \frac{1}{p_j} \end{pmatrix}, \qquad j = 1, 2.$$

Observing (3.4) one can easily see that the conditions $(i) - (iii)$ directly correspond to the conditions $(i) - (iii)$ in Theorem 1.2. Note that in (iii) in Theorem 1.2, $(\mathrm{Im}\, z)^3$ can be replaced by $|z|^3$. ❑

Let us explicitly treat Sturm–Liouville equations in impedance form, that is, the special case of (3.1) where $p = w$; see, e.g. [AHM] and the reference therein. In this case, reparameterizations are not necessary.

3.2 Corollary. *Let* $b_1, b_2 > 0$ *and let* p_j *be measurable functions defined on* $[0, b_j)$, $j = 1, 2$, *such that* $p_j(x) > 0$ *almost everywhere and* $p_j, \frac{1}{p_j} \in L^1_{\mathrm{loc}}([0, b_j))$ *for* $j = 1, 2$. *Moreover, denote by* m_j *the Titchmarsh–Weyl coefficient for*

$$-(p_j y')' = \lambda p_j y.$$

For $a > 0$ *the following statements are equivalent.*

(i) $p_1(x) = p_2(x)$ *almost everywhere on* $(0, a)$.

(ii) *There exists* $\theta \in (0, 2\pi)$ *such that for every* $\varepsilon > 0$,

$$m_1(re^{i\theta}) - m_2(re^{i\theta}) = O\big(e^{(-2a+\varepsilon)r\sin\frac{\theta}{2}}\big), \quad r \to +\infty\,.$$

(iii) *For every* $\alpha \in (0, \pi)$,

$$m_1(\lambda) - m_2(\lambda) = O\Big(|\lambda|^2 e^{-2a\,\mathrm{Im}\sqrt{\lambda}}\Big),$$
$$|\lambda| \to \infty,\ \lambda \in \{z \in \mathbb{C}\colon \alpha \le \arg z \le 2\pi - \alpha\}\,.$$

Proof. If (3.5) holds for $w_j = p_j$, then $\varphi'(t) = 1$ for almost all $t \in (0, a)$ and hence $\varphi(t) = t$, $t \in (0, a)$. ❑

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