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Adaptive hp-FEM for the contact problem with Tresca friction in linear elasticity: The primal-dual formulation and a posteriori error estimation

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Abstract

We present an *a priori* analysis of the hp-version of the finite element method for the primal-dual formulation of frictional contact in linear elasticity. We employ a novel hp-mortar projection operator, which is uniformly stable in the mesh width and grows slowly in the polynomial degree. We derive an hp-FEM residual error indicator, develop an hp-adaptive strategy that is based on testing for analyticity, and show in numerical examples that the adaptive algorithm can lead to exponential rates of convergence.

Key words: Finite elements, hp-adaptivity, linear elasticity, Tresca friction, residual error indicator
2000 MSC: 65N30, 65N50, 74B05, 74M10, 74M15, 74S05

1. Introduction

This paper is a continuation of the analysis of hp-FEM for frictional contact problems started in [10]. While the primal formulation used in [10] as the basis of an hp-approximation is appealing in theory, efficient numerical solution strategies are not straightforward. The alternative is to dualise on the continuous level. This leads us to a mixed finite element method for the primal-dual formulation of the frictional contact problem, where a non-conforming approximation for the Lagrange multiplier on the boundary is used. This approach was applied in [6] in the context of hp-boundary element methods and is the basis of the present article.

The first part of this article is devoted to *a priori* error estimates for the hp-version of the finite element method (hp-FEM) in 2D and 3D. The key ingredient is a novel hp-mortar projection operator with a bound independent of the mesh widths h and growing slowly in the polynomial degree p. While the construction of the mortar projection operator is motivated by the present primal-dual formulation of a friction problem, it is also of interest for the analysis of the hp-FEM for mortar methods (see, e.g., [32] for related results). The mortar projection operator allows us to obtain an inf-sup condition for

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appropriately chosen approximation spaces, and thus leads to *a priori* error estimates. In particular, if the discretisation resolves the transition points well enough, the convergence rates resulting from the numerical scheme are optimal up to the reduction following from the non-uniform stability of the mortar projection operators.

In the second part of the paper, we generalise the residual error indicator for the frictional contact problem introduced in [15] to our high-order context using methods from [28]. Here, we restrict our attention to the 2D situation as is done in [28]; we mention that we believe an extension to 3D is possible. We prove reliability, and show that efficiency holds true provided the non-conformity of the Lagrange multiplier, which can be estimated in our scheme, is negligible.

Finally, we provide numerical results for two two-dimensional model problems from [15]. We demonstrate numerically that *hp*-adaptivity can lead to exponential convergence for the non-linear friction problem under consideration. Our *hp*-adaptive strategy is based on testing for analyticity as suggested in [13].

2. Problem formulation

Let $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, be a polygonal or polyhedral Lipschitz domain. We decompose its boundary Γ with normal vector $\vec{\nu}$ into three relatively open, disjoint parts Γ_D, Γ_N, and Γ_C. For simplicity, we assume that $|\Gamma_\mathrm{D}| > 0$. On Γ_D, we prescribe homogeneous Dirichlet conditions, on Γ_N Neumann conditions with given traction $\vec{t}$, and on Γ_C contact conditions with Tresca friction and a friction coefficient g, which we assume to be constant. The volume forces are denoted by $\vec{F}$. Furthermore, we assume that contact holds on the entirety of Γ_C. In the following, we will assume that Γ_C is a single edge (if $d = 2$) or face (if $d = 3$) of the polyhedral domain Ω.

We denote by $\mathrm{H}^s(\Omega)$ the usual Sobolev spaces on Ω, and similarly on the boundary parts, with norms defined through the Slobodeckij seminorms (see, e.g., [31]). In particular, for $s > 0$, we denote by $\mathrm{H}^s_{00}(\Gamma_\mathrm{C})$ the functions in $\mathrm{H}^s(\Gamma_\mathrm{C})$ whose extension to $\partial\Omega$ by zero is in $\mathrm{H}^s(\partial\Omega)$. The dual spaces of $H^s_{00}(\Gamma_\mathrm{C})$ and $\mathrm{H}^s(\Gamma_\mathrm{C})$ are denoted $\mathrm{H}^{-s}(\Gamma_\mathrm{C})$ and $\mathrm{H}^{-s}_{00}(\Gamma_\mathrm{C})$ respectively: $(\mathrm{H}^s_{00}(\Gamma_\mathrm{C}))' = H^{-s}(\Gamma_\mathrm{C})$ and $(\mathrm{H}^s(\Gamma_\mathrm{C}))' = \mathrm{H}^{-s}_{00}(\Gamma_\mathrm{C})$. The Besov spaces $\mathrm{B}^s_{2,q}(\Omega)$, $s \in (k, k+1)$, $k \in \mathbb{N}_0$, $q \in [1, \infty]$, are defined as the interpolation spaces $(\mathrm{H}^k(\Omega), \mathrm{H}^{k+1}(\Omega))_{s-k,q}$ (note that the J- and the K-method of interpolation generate the same spaces with equivalent norms, see e.g. [34, Lemma 24.3]). For $q = 2$, the Besov space $\mathrm{B}^s_{2,2}(\Omega)$ and the Sobolev space $\mathrm{H}^s(\Omega)$ coincide with equivalent norms, which yields that fractional order Sobolev spaces can be defined by interpolation.

We apply standard notation of linear elasticity: $\vec{\varepsilon}_{ij}(\vec{v}) := \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ denotes the small strain tensor and $\vec{\sigma}(\vec{v}) := \mathbb{C}\vec{\varepsilon}(\vec{v})$ the stress tensor, where $\mathbb{C}$ is the Hooke tensor, which is assumed to be uniformly positive definite. For a vector field $\vec{\mu}$ on Γ_C, $\mu_n := \vec{\mu}\cdot\vec{\nu}$ is its normal component and $\vec{\mu}_t := \vec{\mu} - (\vec{\mu}\cdot\vec{\nu})\vec{\nu}$ its tangential component. With the trace operator $\gamma_{0,\Gamma_\mathrm{D}} \colon (\mathrm{H}^1(\Omega))^d \to (\mathrm{H}^{1/2}(\Gamma_\mathrm{D}))^d$, we set

$$V := \left\{\vec{v} \in \left(\mathrm{H}^1(\Omega)\right)^d : \gamma_{0,\Gamma_\mathrm{D}}(\vec{v}) = 0\right\}, \tag{2.1}$$

$$W := \left\{\vec{\mu} \in \left(\mathrm{H}^{-1/2}_{00}(\Gamma_\mathrm{C})\right)^d : \vec{\mu}\cdot\vec{\nu} = 0\right\}, \tag{2.2}$$

$$\Lambda := \left\{\vec{\mu} \in W \colon \|\vec{\mu}\|_{\mathrm{L}^\infty(\Gamma_\mathrm{C})} \le 1\right\}, \tag{2.3}$$

define the bilinear forms $a\colon V\times V\to\mathbb{R}$ and $b\colon V\times W\to\mathbb{R}$ by

$$a(\vec{v},\vec{w}) := \int_\Omega \boldsymbol{\sigma}(\vec{v}) : \boldsymbol{\varepsilon}(\vec{w})\mathrm{d}\vec{x}, \tag{2.4a}$$

$$b(\vec{v},\vec{\mu}) := \int_{\Gamma_\mathrm{C}} g\vec{v}_t\cdot\vec{\mu}\mathrm{d}s_{\vec{x}}, \tag{2.4b}$$

the linear form $L\colon V\to\mathbb{R}$ and the convex, nondifferentiable functional $j\colon V\to\mathbb{R}$ by

$$L(\vec{v}) \quad := \quad \int_\Omega \vec{F}\cdot\vec{v}\mathrm{d}\vec{x} + \int_{\Gamma_\mathrm{N}} \vec{t}\cdot\vec{v}\mathrm{d}s_{\vec{x}}, \tag{2.5}$$

$$j(\vec{v}) \quad := \quad \int_{\Gamma_\mathrm{C}} g|\vec{v}_t|\mathrm{d}s_{\vec{x}}. \tag{2.6}$$

The primal version of the continuous version of the linearly elastic contact problem with Tresca friction then reads:

Find $\vec{u}\in V$ such that for all $\vec{v}\in V$

$$a(\vec{u},\vec{v}-\vec{u}) + j(\vec{v}) - j(\vec{u}) \geq L(\vec{v}-\vec{u}); \tag{2.7}$$

this variational inequality is equivalent to the minimisation problem for the energy functional $J(\vec{v}) := \frac{1}{2}a(\vec{v},\vec{v}) - L(\vec{v}) + j(\vec{v})$ (see [12]).

Following [19, pp. 197], we can derive an equivalent primal-dual or saddle point formulation:

Find $(\vec{u},\vec{\lambda})\in V\times\Lambda$ such that for all $(\vec{v},\vec{\mu})\in V\times\Lambda$

$$a(\vec{u},\vec{v}) \qquad + b(\vec{v},\vec{\lambda}) = L(\vec{v}), \tag{2.8a}$$

$$b(\vec{u},\vec{\mu}-\vec{\lambda}) \qquad \leq 0. \tag{2.8b}$$

The unique solvability of the two formulations follows by standard arguments since the Hooke tensor $\mathbb{C}$ is uniformly positive definite and Γ_D has positive measure, see [19, 15, 22].

Choosing a discrete subspace $V_N\subseteq V$ and a discretisation $j_N\colon V_N\to\mathbb{R}$ of j, we obtain the discrete primal formulation:

Find $\vec{u}_N\in V_N$ such that for all $\vec{v}_N\in V_N$

$$a(\vec{u}_N,\vec{v}_N-\vec{u}_N) + j_N(\vec{v}_N) - j_N(\vec{u}_N) \geq L(\vec{v}_N-\vec{u}_N). \tag{2.9}$$

With a discrete subset $\Lambda_N\subseteq W$, the discrete saddle point problem reads:

Find $(\vec{u}_N,\vec{\lambda}_N)\in V_N\times\Lambda_N$ such that for all $(\vec{v}_N,\vec{\mu}_N)\in V_N\times\Lambda_N$

$$a(\vec{u}_N,\vec{v}_N) \qquad + b(\vec{v}_N,\vec{\lambda}_N) = L(\vec{v}_N), \tag{2.10a}$$

$$b(\vec{u}_N,\vec{\mu}_N-\vec{\lambda}_N) \qquad \leq 0. \tag{2.10b}$$

3. A priori error estimates: The primal-dual formulation

3.1. hp-FEM discretisation

Let $\mathcal{T}_N$ be partitions of Ω consisting of affine quadrilateral or hexahedral elements, together with polynomial degrees $p_{N,K}\geq 2$ for all $K\in\mathcal{T}_N$, and assume that the boundary parts Γ_{C}, Γ_{D} and Γ_{N} are resolved by the mesh. As it is standard, we denote by $h_{N,K}$ the diameter of element $K \in \mathcal{T}_N$. We denote the set of edges ($d = 2$) or faces ($d = 3$) on the contact boundary by $\mathcal{E}_{\mathrm{C},N}$; for example, for $d = 2$,

$$\mathcal{E}_{\mathrm{C},N} := \{E \colon E \subset \Gamma_{\mathrm{C}} \text{ is an edge of } \mathcal{T}_N\}\,. \tag{3.1}$$

The sets of edges (or faces) on Γ_{N} and Γ_{D} are denoted by $\mathcal{E}_{\mathrm{N},N}$ and $\mathcal{E}_{\mathrm{D},N}$, and we let $\mathcal{E}_{\mathrm{I},N}$ be the set of interior edges (or faces). We see that for every $E \in \mathcal{E}_{\mathrm{C},N}$, there exists a unique $K_E \in \mathcal{T}_N$ such that E is an edge or face of K_E. We denote by $\mathcal{Q}^r$ the tensor product polynomials of degree $r \in \mathbb{N}_0$. For each element $K \in \mathcal{T}_N$, let $p_{N,K} \in \mathbb{N}$ be a polynomial degree. We now set

$$V_N := \{\vec{v}_N \in V \colon \vec{v}_N|_K \in \mathcal{Q}^{p_{N,K}} \text{ for all } K \in \mathcal{T}_N\}\,, \tag{3.2a}$$

$$W_N := \left\{\vec{\mu}_N \in W \colon \vec{\mu}_N|_E \in \mathcal{Q}^{p_{N,K_E}-2} \text{ for all } E \in \mathcal{E}_{\mathrm{C},N}\right\}, \tag{3.2b}$$

$$\Lambda_N := \left\{\vec{\mu}_N \in W_N \colon |\vec{\mu}_N(\vec{x})| \leq 1 \text{ for all } \vec{x} \in \mathrm{G}_{E,p_{N,K_E}-2} \text{ and all } E \in \mathcal{E}_{\mathrm{C},N}\right\}, \tag{3.2c}$$

where $\mathrm{G}_{E,q}$ denotes the Gaussian quadrature points with $(q+1)^{d-1}$ points on E.

Remark 3.1. In [10], the primal formulation is considered and its discretisation is solved by a primal-dual method. That method is different from the present approach, which is based on discretising the primal-dual formulation. In fact, while uniqueness of the discrete Lagrange multiplier $\vec{\lambda}_N$ follows from Theorem 3.9 here, it is given in the setting of [10] only for special geometric arrangements and choices of quadrature formulae.

The convergence analysis of the present section rests on a mortar projection operator, which in turn, relies on the existence of a polynomial approximation operator with optimal (in h and p) approximation properties. Since the mortar projection operator is constructed on meshes consisting of quadrilateral/hexahedra, the presence of hanging nodes is mandatory in an adaptive setting. Since polynomial approximation operators of Clément type do not seem to exist in the literature for this setting, we formulate their existence as an assumption:

Assumption 3.2. There exists a family of Scott-Zhang type quasi-interpolation operators $(I_N)_N$, $I_N \colon V \to V_N$, preserving piecewise polynomials on Γ_{C}

$$\sum_{K \in \mathcal{T}_N} \frac{p_{N,K}^2}{h_{N,K}^2} \|u - I_N u\|_{\mathrm{L}^2(K)}^2 + |u - I_N u|_{\mathrm{H}^1(K)}^2 \lesssim \|u\|_{H^1(\Omega)}^2 \qquad \forall u \in H^1(\Omega). \tag{3.3}$$

Remark 3.3. The existence of an hp-Scott-Zhang type operator I_N is stipulated as an assumption since the such operators do not seem to exist in the literature for the 3D situation and meshes with hanging nodes. Nevertheless, Assumption 3.2 is reasonable in view of the following facts: In two space dimensions and on regular meshes such operators have been constructed in [27] under the assumption that $p_{N,K} \sim p_{N,K'}$ for all elements K, $K' \in \mathcal{T}_N$ with $\overline{K} \cap \overline{K'} \neq \emptyset$. For the three-dimensional case with hanging nodes, some results for lower order elements are provided in [18].

3.2. A priori estimates in 2D and 3D

For the choice of V_N, W_N, and Λ_N given in (3.2) one can show convergence of the discrete approximations $(\vec{u}_N, \vec{\lambda}_N) \in V_N \times \Lambda_N$ to the exact solutions $(\vec{u}, \lambda) \in V \times \Lambda$ if $\sup_{K \in \mathcal{T}_N} h_{N,K}/p_{N,K} \to 0$ as $N \to \infty$. This convergence result can be obtained using Glowinski's theorem (see [19, Section 1.1.52, Theorem 5.3] and [14, Chapter I, Theorem 5.2]) in a way similar to the procedure in [25] for the Signorini problem. The aim of the present section is, however, the *a priori* result of Theorem 3.4 below that can lead to convergence rates. Setting

$$\beta_N := \max_{E \in \mathcal{E}_{\mathrm{C},N}} \left[p_{N,E}^{3/4} (1 + \log^2 p_{N,E}) \right]^{-1}, \tag{3.4}$$

the main result of this section is:

Theorem 3.4. *Let Assumption 3.2 be valid, and let V_N, W_N, Λ_N be given by (3.2). Let $(\vec{u}, \vec{\lambda}) \in V \times \Lambda$ be the solution of* (2.8) *and let $(\vec{u}_N, \vec{\lambda}_N)$ be the solution of* (2.10). *Then there exists a constant $C > 0$ independent of N such that for all $(\vec{v}_N, \vec{\mu}_N) \in V_N \times \Lambda_N$ there holds*

$$\begin{aligned} \|\vec{u} - \vec{u}_N\|^2_{\mathrm{H}^1(\Omega)} \leq C \Big[& b(\vec{u}, \vec{\lambda}_N - \vec{\mu}) + b(\vec{u}, \vec{\lambda} - \vec{\mu}_N) \\ & + \beta_N^{-2} \left(\|\vec{u} - \vec{v}_N\|^2_{\mathrm{H}^1(\Omega)} + \|\vec{\lambda} - \vec{\mu}_N\|^2_{\mathrm{H}_{00}^{-1/2}(\Gamma_\mathrm{C})} \right) \Big] \quad \text{and} \end{aligned} \tag{3.5}$$

$$\|\vec{\lambda} - \vec{\lambda}_N\|_{\mathrm{H}_{00}^{-1/2}(\Gamma_\mathrm{C})} \leq C \beta_N^{-1} \left(\|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(\Omega)} + \|\vec{\lambda} - \vec{\mu}_N\|_{\mathrm{H}_{00}^{-1/2}(\Gamma_\mathrm{C})} \right). \tag{3.6}$$

Proof. The proof of the theorem relies on several auxiliary results and can be found at the end of Section 3.4. As in the case of linear saddle point problems, the key step towards showing error estimates are bounds for the inf-sup constant of the bilinear form b in (2.4b). We will obtain these bounds in Theorem 3.9 with the aid of a so-called Fortin projector $P_N \colon V \to V_N$ (see, e.g., [4, Section 12.5]). Specifically, we will construct this operator only for the case $d = 3$ and merely state the corresponding result for the case $d = 2$. The construction is based on the work done in [8] for the case $d = 2$, but contains improved estimates even for that case. □

3.3. An hp-mortar projection operator

Consider a rectangle $\Omega = (a, b) \times (c, d) \subset \mathbb{R}^2$, and a sequence of partitions $\mathcal{T}_N$ of Ω into rectangles, together with a vector of polynomial degrees $(p_{N,K})_{K \in \mathcal{T}_N}$. We admit hanging nodes of arbitrary order in $\mathcal{T}_N$ but assume shape-regularity of the elements. Set $\tilde{V} := \mathrm{H}^{1/2}(\Omega)$. We denote by $\tilde{V}_N$ the space of all functions $v \in \tilde{V}$ such that $v|_K \in \mathcal{Q}^{p_{N,K}}$ for all $K \in \mathcal{T}_N$. On the interelement edges, we use a minimum rule, and resolve hanging nodes by matching the polynomials on the two respective edges. Finally, $W_N := \left\{ \mu_N \in W \colon \mu_N|_K \in \mathcal{Q}^{p_{N,K}-2} \text{ for all } K \in \mathcal{T}_N \right\}$.

3.3.1. The operator $\tilde{\Pi}_{\hat{K},q}$

Let $\hat{K} := I^2 = [-1, +1]^2$ be the reference element, $\hat{E} := [-1, +1]$ the reference edge, $\Pi_{\hat{K},q}$ the projection onto $\mathcal{Q}^q$ with respect to the scalar product of $\mathrm{L}^2(\hat{K})$ and $\Pi_{\hat{E},q}$

correspondingly for $\hat{E}$, and $(L_j)_{j\geq 0}$ the Legendre polynomials as given in [31]. We define a skew projection operator $\tilde{\Pi}_{\hat{K},q}\colon \mathrm{H}^{1/2}(\hat{K}) \to \mathcal{Q}^q$ for $v \in \mathrm{H}^{1/2}(\hat{K})$ by

$$\begin{aligned}\tilde{\Pi}_{\hat{K},q}v(x_1,x_2) = \Pi_{\hat{K},q-2}v(x_1,x_2)\\ &+ \kappa_{-,q}(v)(x_1)L_{q-1}(x_2) + \kappa_{+,q}(v)(x_1)L_q(x_2)\\ &+ \kappa_{q,-}(v)(x_2)L_{q-1}(x_1) + \kappa_{q,+}(v)(x_2)L_q(x_1)\\ &+ \sum_{i=q-1}^{q}\sum_{j=q-1}^{q}\varphi_{q,i,j}(v)L_i(x_1)L_j(x_2),\end{aligned} \tag{3.7}$$

where

$$\kappa_{-,q}(v) := -\frac{1}{2}\Big[(-1)^{q-1}\Pi_{\hat{K},q-2}v(\cdot,-1) + \Pi_{\hat{K},q-2}v(\cdot,+1)\Big], \tag{3.8a}$$
$$\kappa_{+,q}(v) := -\frac{1}{2}\Big[(-1)^{q}\Pi_{\hat{K},q-2}v(\cdot,-1) + \Pi_{\hat{K},q-2}v(\cdot,+1)\Big], \tag{3.8b}$$
$$\kappa_{q,-}(v) := -\frac{1}{2}\Big[(-1)^{q-1}\Pi_{\hat{K},q-2}v(-1,\cdot) + \Pi_{\hat{K},q-2}v(+1,\cdot)\Big], \tag{3.8c}$$
$$\kappa_{q,+}(v) := -\frac{1}{2}\Big[(-1)^{q}\Pi_{\hat{K},q-2}v(-1,\cdot) + \Pi_{\hat{K},q-2}v(+1,\cdot)\Big], \tag{3.8d}$$

and

$$\varphi_{q,q-1,q-1}(v) := -\frac{1}{2}\left[(-1)^{q-1}\kappa_{q,-}(v)(-1) + \kappa_{q,-}(v)(+1)\right], \tag{3.9a}$$
$$\varphi_{q,q-1,q}(v) := -\frac{1}{2}\left[(-1)^{q}\kappa_{q,-}(v)(-1) + \kappa_{q,-}(v)(+1)\right], \tag{3.9b}$$
$$\varphi_{q,q,q-1}(v) := -\frac{1}{2}\left[(-1)^{q-1}\kappa_{q,+}(v)(-1) + \kappa_{q,+}(v)(+1)\right], \tag{3.9c}$$
$$\varphi_{q,q,q}(v) := -\frac{1}{2}\left[(-1)^{q}\kappa_{q,+}(v)(-1) + \kappa_{q,+}(v)(+1)\right], \tag{3.9d}$$

A direct calculation shows that $\tilde{\Pi}_{\hat{K},q}$ satisfies

$$\Pi_{\hat{K},q-2}\tilde{\Pi}_{\hat{K},q}v = \Pi_{\hat{K},q-2}v \tag{3.10}$$

and

$$\tilde{\Pi}_{\hat{K},q}v|_{\partial\hat{K}} = 0 \quad \text{for all } v \in \mathrm{H}^{1/2}(\hat{K}). \tag{3.11}$$

Furthermore, by definition, it is clear that

$$\tilde{\Pi}_{\hat{K},q}\Pi_{\hat{K},q-2}v = \tilde{\Pi}_{\hat{K},q-2}v \quad \text{for all } v \in \mathrm{H}^{1/2}(\hat{K}). \tag{3.12}$$

3.3.2. Auxiliary results

Lemma 3.5 ($\mathrm{B}^{1/2}_{2,1}$-H^s *p*-version inverse inequality). *For every $d \geq 1$ and $s \in [0, 1/2]$, there exists a constant $C > 0$ such that for all $q \in \mathbb{N}$ and all polynomials $v_q \in \mathcal{Q}^q$,*

$$\|v_q\|_{\mathrm{B}^{1/2}_{2,1}(I^d)} \lesssim q^{1-2s}\|v_q\|_{\mathrm{H}^s(I^d)} \quad \textit{for } s < 1/2$$
$$\|v_q\|_{\mathrm{B}^{1/2}_{2,1}(I^d)} \lesssim \left(1 + \sqrt{\ln q}\right)\|v_q\|_{\mathrm{H}^{1/2}(I^d)}.$$

Proof. We use the characterisation of fractional Sobolev and Besov spaces in terms of the K-functional given by $K(t,w) := \inf_{v\in H^1(I^d)} \|w-v\|_{L^2(I^d)} + t\|v\|_{H^1(I^d)}$. By [7, p.193, equation (7.4)], we have, for every $\varepsilon \in (0,1)$ that

$$\|v_q\|_{\mathrm{B}^{1/2}_{2,1}(I^d)} \sim \int_0^1 t^{-1/2} K(t,v_q)\frac{\mathrm{d}t}{t} = \int_0^\varepsilon t^{-1/2} K(t,v_q)\frac{\mathrm{d}t}{t} + \int_\varepsilon^1 t^{-1/2} K(t,v_q)\frac{\mathrm{d}t}{t}. \quad (3.13)$$

For the first term in (3.13), we estimate

$$\int_0^\varepsilon t^{-1/2} K(t,v_q)\frac{\mathrm{d}t}{t} \le \int_0^\varepsilon t^{1/2}\|v_q\|_{\mathrm{H}^1(I^d)}\frac{\mathrm{d}t}{t} = 2\sqrt{\varepsilon}\|v_q\|_{\mathrm{H}^1(I^d)}.$$

By tensorising the inverse inequality of [3, Proposition 4.1],

$$\sqrt{\varepsilon}\|v_q\|_{\mathrm{H}^1(I^d)} \le Cq^{2(1-s)}\sqrt{\varepsilon}\|v_q\|_{\mathrm{H}^s(I^d)}.$$

For the second term in (3.13), we apply, for $s > 0$, the Cauchy-Schwarz inequality to obtain

$$\int_\varepsilon^1 t^{-1/2} K(t,v_q)\frac{\mathrm{d}t}{t} \le \sqrt{\int_\varepsilon^1 t^{2s-3}\mathrm{d}t}\,\|v_q\|_{\mathrm{H}^s(I^d)}.$$

This yields

$$\int_\varepsilon^1 t^{-1/2} K(t,v_q)\frac{\mathrm{d}t}{t} \le \begin{cases} \sqrt{\log\varepsilon^{-1}}\|v_q\|_{\mathrm{H}^{1/2}(I^d)}, & \text{if } s = 1/2, \\ \frac{1}{1-2s}\sqrt{\varepsilon^{2s-1}-1}\|v_q\|_{\mathrm{H}^s(I^d)} & \text{if } 0 < s < 1/2, \\ \frac{1}{2}(\varepsilon^{-1/2}-1)\|v_q\|_{\mathrm{L}^2(I^d)} & \text{if } s = 0. \end{cases}$$

Here, the case $s = 0$ follows from the trivial estimate $K(t,v_q) \le \|v_q\|_{\mathrm{L}^2(I^d)}$. Selecting $\varepsilon := q^{-2}$ and inserting the resulting bound in (3.13) allows us to conclude the proof. □

Lemma 3.6. *The operator $\tilde{\Pi}_{\hat{K},q}$ of (3.7) satisfies the stability estimate*

$$\|\tilde{\Pi}_{\hat{K},q} v_q\|_{\mathrm{H}^{1/2}(\hat{K})} \le C(1+\ln q)q^{1/2}\|v_q\|_{\mathrm{H}^{1/2}(\hat{K})} \quad \textit{for all } v_q \in \mathcal{Q}^{q-2}. \quad (3.14)$$

Proof. As, similarly as in [23, p. 52, Theorem 10.2],

$$\|w\|_{\mathrm{H}^{1/2}(I^2)} \le C\left(\int_I \left(\|w(\cdot,t)\|^2_{\mathrm{H}^{1/2}(I)} + \|w(t,\cdot)\|^2_{\mathrm{H}^{1/2}(I)}\right)\mathrm{d}t\right)^{1/2}, \quad (3.15)$$

we see that

$$\begin{aligned}\|\kappa_{-,q}(v_q)(x_1)L_{q-1}(x_2)\|^2_{\mathrm{H}^{1/2}(\hat{K})} \lesssim &\|\kappa_{-,q}(v_q)\|_{\mathrm{L}^2(\hat{E})}\|L_{q-1}\|_{\mathrm{H}^{1/2}(\hat{E})} \\ &+ \|\kappa_{-,q}(v_q)\|_{\mathrm{H}^{1/2}(\hat{E})}\|L_{q-1}\|_{\mathrm{L}^2(\hat{E})}. \end{aligned} \quad (3.16)$$

As $v_q \in \mathcal{Q}^{q-2}$, by [34, (32.8)],

$$\|\kappa_{-,q}(v_q)\|_{\mathrm{L}^2(\hat{E})} \le \frac{1}{2}\left(\|v_q(\cdot,-1)\|_{\mathrm{L}^2(\hat{E})} + \|v_q(\cdot,-1)\|_{\mathrm{L}^2(\hat{E})}\right) \lesssim \|v_q\|_{\mathrm{B}^{1/2}_{2,1}(\hat{K})}, \quad (3.17)$$

and therefore by Lemma 3.5 with $s = 1/2$,

$$\|\kappa_{-,q}(v_q)\|_{\mathrm{L}^2(\hat{E})} \lesssim (1 + \sqrt{\ln q})\|v_q\|_{\mathrm{H}^{1/2}(\hat{K})}. \tag{3.18}$$

For the other term in (3.16), apply [3, Proposition III.4.1] to obtain

$$\|\kappa_{-,q}(v_q)\|_{\mathrm{H}^{1/2}(\hat{E})} \lesssim q\|\kappa_{-,q}(v_q)\|_{\mathrm{L}^2(\hat{E})}. \tag{3.19}$$

As $\mathrm{B}^{1/2}_{2,1}(\hat{E})$ is continuously embedded in $\mathrm{C}^0(\hat{E})$, we obtain from Lemma 3.5 with $s = 0$

$$|\varphi_{q,q-1,q-1}(v)| \lesssim \|\kappa_{q,-}(v)\|_{\mathrm{B}^{1/2}_{2,1}(\hat{E})} \lesssim q\|\kappa_{-,q}(v_q)\|_{\mathrm{L}^2(\hat{E})}. \tag{3.20}$$

Analogous results hold true for the other terms. Since $\|L_j\|^2_{\mathrm{L}^2(\hat{E})} = 2/(2j+1)$ and $\|L_j\|_{\mathrm{H}^{1/2}(\hat{E})} \sim \sqrt{\ln(j+1)}$ (see the proof of [1, Lemma 10]), we obtain the result. □

Set $\tilde{\Pi}_{K,q}v := \tilde{\Pi}_{\hat{K},q}(v \circ F_K) \circ F_K^{-1}$, where $F_K : \hat{K} \to K$ is the (affine) element map. With $J_N : \tilde{V} \to \tilde{V}_N$ an arbitrary operator, we define the operator P_N on $\tilde{V}$ elementwise by

$$(P_N v)_K := J_N v|_K + \tilde{\Pi}_{K,p_{N,K}}((v - J_N v)|_K). \tag{3.21}$$

It follows by (3.11) that $P_N : \tilde{V} \to \tilde{V}_N$ is well-defined (as $P_N v|_{\partial K} = J_N v|_{\partial K}$ for all $K \in \mathcal{T}_N$) and by (3.10) that

$$\langle v - P_N v, \mu_N\rangle = 0 \quad \text{for all } \mu_N \in W_N. \tag{3.22}$$

Proposition 3.7 (von Petersdorff inequality). *There exists a constant $C > 0$ independent of the mesh such that, if $v|_K \in \mathrm{H}^{1/2}_{00}(K)$ for all $K \in \mathcal{T}_N$, then*

$$|v|^2_{\mathrm{H}^{1/2}_{00}(\Omega)} \le C \sum_{K\in\mathcal{T}_N} |v|^2_{\mathrm{H}^{1/2}_{00}(K)}. \tag{3.23}$$

For the proof, see [2, Theorem 4.1].

Lemma 3.8. *The operators (P_N) defined in (3.21) satisfy the stability estimate*

$$\|P_N v\|_{\mathrm{H}^{1/2}(\Omega)} \le \|J_N v\|_{\mathrm{H}^{1/2}(\Omega)} + C\left\{\sum_{K\in\mathcal{T}_N} \left[p^{3/4}_{N,K}(1+\log^2 p_{N,K})\right]^2 \left(h^{-1}_{N,K}\|v - J_N v\|^2_{\mathrm{L}^2(K)} + |v - J_N v|^2_{\mathrm{H}^{1/2}(K)}\right)\right\}^{1/2}.$$

Proof. By the triangle inequality,

$$\|P_N v\|_{\mathrm{H}^{1/2}(\Omega)} \le \|J_N v\|_{\mathrm{H}^{1/2}(\Omega)} + \|P_N v - J_N v\|_{\mathrm{H}^{1/2}(\Omega)}.$$

Clearly, $(P_N v - J_N v)|_K = \tilde{\Pi}_{K,p_{N,K}}((v - J_N v)|_K)$ vanishes on ∂K for every $K \in \mathcal{T}_N$, so we can apply Proposition 3.7 to obtain

$$\|P_N v - J_N v\|^2_{\mathrm{H}^{1/2}(\Omega)} \le C \sum_{K\in\mathcal{T}_N} \|\tilde{\Pi}_{K,p_{N,K}}((v - J_N v)|_K)\|^2_{\mathrm{H}^{1/2}_{00}(K)}.$$

Scaling and [17, Lemma 6] yields

$$\begin{aligned}&\|\tilde{\Pi}_{K,p_{N,K}}((v-J_Nv)|_K)\|_{\mathrm{H}^{1/2}_{00}(K)} \lesssim h_{N,K}^{1/2}\|\tilde{\Pi}_{\hat{K},p_{N,K}}((v-J_Nv)|_K\circ F_K)\|_{\mathrm{H}^{1/2}_{00}(\hat{K})}\\ &\quad\lesssim h_{N,K}^{1/2}(1+\ln p_{N,K})\|\tilde{\Pi}_{\hat{K},p_{N,K}}((v-J_Nv)|_K\circ F_K)\|_{\mathrm{H}^{1/2}(K)}.\end{aligned}$$

By (3.12) and Lemma 3.6 we see that

$$\begin{aligned}&\|\tilde{\Pi}_{\hat{K},p_{N,K}}((v-J_Nv)|_K\circ F_K)\|_{\mathrm{H}^{1/2}(K)}\\ &\quad\lesssim \|\tilde{\Pi}_{\hat{K},p_{N,K}}\Pi_{\hat{K},p_{N,K}-2}((v-J_Nv)|_K\circ F_K)\|_{\mathrm{H}^{1/2}(\hat{K})}\\ &\quad\lesssim (1+\ln p_{N,K})p_{N,K}^{1/2}\|\Pi_{\hat{K},p_{N,K}-2}((v-J_Nv)|_K\circ F_K)\|_{\mathrm{H}^{1/2}(\hat{K})},\end{aligned}$$

Finally, the $\mathrm{H}^{1/2}$-stability of $\Pi_{\hat{K},q}$, which follows from an interpolation argument using [5, Theorem 2.4], reads

$$\|\Pi_{\hat{K},p_{N,K}-2}((v-J_Nv)|_K\circ F_K)\|_{\mathrm{H}^{1/2}(\hat{K})} \lesssim p_{N,K}^{1/4}\|(v-J_Nv)|_K\circ F_K\|_{\mathrm{H}^{1/2}(\hat{K})}.$$

The result now follows by a scaling argument. □

3.4. Proof of Theorem 3.4

We are now ready to give an *a priori* convergence rate result for the finite element method formulated in (2.10). We first prove an inf-sup condition using the results of the last section. For simplicity, we restrict ourselves to the situation that Γ_C consists of a single rectangular surface.

Theorem 3.9. *Let Assumption 3.2 be valid. Then we have the discrete inf-sup condition*

$$\inf_{\vec{\mu}_N\in W_N}\sup_{\vec{v}_N\in V_N}\frac{b(\vec{v}_N,\vec{\mu}_N)}{\|\vec{v}_N\|_{\mathrm{H}^1(\Omega)}\|\vec{\mu}_N\|_{\mathrm{H}^{-1/2}_{00}(\Gamma_\mathrm{C})}} \geq C\beta_N, \tag{3.24}$$

where the constant $C>0$ is independent of N and β_N is defined in (3.4).

Proof. Given an affine mapping $F\colon [0,1]\times[0,1]\to\Gamma_\mathrm{C}$, the trivial extension by zero $Z\colon \mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{C})\to\mathrm{H}^{1/2}(\Gamma)$, a lifting operator $L\colon \mathrm{H}^{1/2}(\Gamma)\to\mathrm{H}^1(\Omega)$ and the Scott-Zhang operator I_N of Assumption 3.2, we set

$$\tilde{P}_N v := I_N LZ(P_N(\gamma_{0,\Gamma_\mathrm{C}}v\circ F)\circ F^{-1}),$$

where P_N is the operator given in Lemma 3.8 with $J_N := \gamma_{0,\Gamma_\mathrm{C}}I_N L$. As $P_N(v\circ F)|_{\partial\Gamma_\mathrm{C}} = 0$, the operator $\tilde{P}_N\colon V\to V_N$ is well-defined. Due to the fact that g is constant, b is just a scalar multiple of the L^2-scalar product on Γ_C, and thus, by Lemma 3.8.

$$\begin{gathered}\|\tilde{P}_N v\|_{\mathrm{H}^1(\Omega)} \lesssim \beta_N^{-1}\|v\|_{\mathrm{H}^1(\Omega)} \quad\text{and}\\ b(v-\tilde{P}_N v,\mu_N)=0 \quad\text{for all } v\in V \text{ and } \mu_N\in\Lambda_N.\end{gathered}$$

We conclude the proof with an appeal to [4, Lemma 12.5.22]. □

Remark 3.10. Analysing the proofs of Lemmas 3.6 and 3.8, we see that the same result can be proved in dimensions other than $d = 3$ in a similar way. Then, the bounds β_N are given by

$$\beta_N = \max_{E \in \mathcal{E}_{C,N}} \left[p_{N,E}^{(d-2)/2+1/4} \left(1 + \ln^2 p_{N,E}\right)\right]^{-1}. \tag{3.25}$$

The following abstract a *priori* error estimate for mixed variational inequalities follows in the same way as in [16, Theorem 6]:

Proposition 3.11. *Let $\mathcal{V}$, $\mathcal{W}$ be Banach spaces and let $a\colon \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ and $b\colon \mathcal{V}\times\mathcal{W} \to \mathbb{R}$ be continuous bilinear forms. Suppose that a is elliptic and that b satisfies an inf-sup condition. Given continuous linear functionals $F : \mathcal{V} \to \mathbb{R}$ and $G : \mathcal{W} \to \mathbb{R}$ and a closed convex set $\Lambda \subset \mathcal{W}$, let $(u, \lambda) \in \mathcal{V} \times \Lambda$ satisfy:*

$$a(u, v) + b(v, \lambda) = F(v) \qquad \forall v \in \mathcal{V} \tag{3.26a}$$

$$b(u, \mu - \lambda) \leq G(\mu - \lambda) \qquad \forall \mu \in \Lambda. \tag{3.26b}$$

Let $(\mathcal{V}_N)_N \subset \mathcal{V}$, $(\mathcal{W}_N)_N \subset \mathcal{W}$ be sequences of finite-dimensional subspaces and let $(\Lambda_N)_N \subset \mathcal{W}$ be a sequence of closed, convex subsets. Assume that b satisfies a non-uniform discrete inf-sup condition on $(\mathcal{V}_N, \mathcal{W}_N)_N$ with constants $(\beta_N)_N$, and that Λ and $(\Lambda_N)_N$ are uniformly bounded, i.e., there exists a constant $C > 0$ such that for all $\mu \in \Lambda$, $\|\mu\|_W < C$ and for all N and $\mu_N \in \Lambda_N$, $\|\mu_N\|_W < C$.

Then, the discretisation of (3.26) obtained by replacing $\mathcal{V}$, $\mathcal{W}$, Λ with $\mathcal{V}_N$, $\mathcal{W}_N$, Λ_N has a unique solution $(u_N, \lambda_N) \in \mathcal{V}_N \times \Lambda_N$ and satisfies the following a priori *estimate: for all $\mu \in \Lambda$, $\mu_N \in \Lambda_N$, and $v_N \in V_N$ there holds*

$$\|u - u_N\|_V^2 \lesssim \Big[b(u, \lambda_N - \mu) - G(\lambda_N - \mu) + b(u, \lambda - \mu_N) - G(\lambda - \mu_N) + \beta_N^{-2} \left(\|u - v_N\|_V^2 + \|\lambda - \mu_N\|_W^2\right)\Big] \quad \textit{and} \tag{3.27}$$

$$\|\lambda - \lambda_N\|_W \lesssim \beta_N^{-1} \left(\|u - u_N\|_V + \|\lambda - \mu_N\|_W\right). \tag{3.28}$$

Proof of Theorem 3.4: Theorem 3.4 now follows by applying Proposition 3.11 together with Theorem 3.9.

3.5. Remarks on the 2D situation

The two-dimensional situation $\Omega \subset \mathbb{R}^2$ differs from the 3D case in that the restriction of any mesh to the contact boundary Γ_C consists of line segments. The construction of the mortar projection operator P_N is then also possible for triangulations of Ω that consist of triangles and quadrilaterals without hanging nodes. In this situation, Assumption 3.2 is satisfied by [27]. For ease of future reference, we formulate this observation as Theorem 3.13 below. The setting is as follows:

Let $\mathcal{T}_N$ be a shape-regular regular mesh consisting of affine quadrilateral and/or affine triangles. We assume that the element size and polynomial degree of neighboring elements is comparable:

Assumption 3.12. There exists a constant $C > 0$ such that

$$h_{N,K} \leq C h_{N,K'} \quad \text{and} \quad p_{N,K} \leq C p_{N,K'} \qquad \forall K, K' \in \mathcal{T}_N \quad \text{with } \overline{K} \cap \overline{K'} \neq \emptyset \tag{3.29}$$

On meshes satisfying Assumption 3.12, we consider the spaces $V_N \subset V$ defined as

$$V_N = \{\vec{v} \in V \mid \vec{v}|_K \in S^{p_{N,K}} \quad \forall K \in \mathcal{T}_N\} \tag{3.30}$$

where $S^r = \mathcal{Q}^r$ if K is a quadrilateral and $S^r = \mathcal{P}^r = \operatorname{span}\{x^i y^j \mid 0 \le i, j, \quad 0 \le i+j \le r\}$ if K is a triangle. The spaces W_N and the sets Λ_N are defined as in (3.2). Within this setting, we have the following reliability result:

Theorem 3.13 (primal-dual apriori estimates in 2D). *Let V_N, W_N, Λ_N be given as above and let Assumption 3.12 be valid. Let $(\vec{u}, \vec{\lambda}) \in V \times W$ and $(\vec{u}_N, \vec{\lambda}_N) \in V_N \times \Lambda_N$ be the solutions of* (2.10)*. Then there exists $C > 0$ such that for all $(\vec{v}_N, \vec{\mu}_N) \in V_N \times \Lambda_N$ the estimates (3.5) and (3.6) are satisfied with β_N given by*

$$\beta_N^{2D} = \max_{E \in \mathcal{E}_{C,N}} \left[p_{N,E}^{1/4} \left(1 + \ln^2 p_{N,E}\right)\right]^{-1}$$

instead of (3.4).

Proof. The proof follows from Remark 3.10 and the observation that Assumption 3.2 is satisfied by [27]. □

4. A posteriori error indication: The residual error indicator in 2*D*

Reasoning as in [15, Theorem 6.6], one can show:

Theorem 4.1. *Let $\vec{u} \in V$ be the solution of the continuous minimisation formulation in Problem* (2.8)*, and let $\vec{w} \in V$ be arbitrary.*

Then, for all $\vec{r} \in \left(\mathrm{L}^2(\Omega)\right)^{d \times d}$,

$$\begin{aligned} \frac{1}{2} a(\vec{u} - \vec{w}, \vec{u} - \vec{w}) &\le \int_\Omega \mathbb{C}\left(\boldsymbol{\varepsilon}(\vec{w}) - \vec{r}\right) : \left(\boldsymbol{\varepsilon}(\vec{w}) - \vec{r}\right) \mathrm{d}\vec{x} \\ &+ \inf_{\vec{\mu} \in \Lambda} \left[\sup_{\vec{v} \in V} \frac{1}{a(\vec{v}, \vec{v})} \left(- \int_\Omega \mathbb{C}\vec{r} : \boldsymbol{\varepsilon}(\vec{v}) \mathrm{d}\vec{x} + L(\vec{v}) - b(\vec{v}, \vec{\mu}) \right)^2 + j(\vec{w}) - b(\vec{w}, \vec{\mu}) \right]. \end{aligned} \tag{4.1}$$

In the following, let $d = 2$ and adopt the setting of Section 3.5. Let $(\vec{u}, \lambda) \in V \times \Lambda$ solve (2.8) and $(\vec{u}_N, \lambda_N) \in V_N \times \Lambda_N$ solve (2.9) with V_N, W_N, and Λ_N specified in Section 2. We denote by $\Pi_\Lambda : \mathrm{L}^2(\Gamma_C) \to \Lambda$ the $\mathrm{L}^2(\Gamma_C)$ projection onto Λ, and we will further abbreviate $\tilde{\vec{\lambda}}_N := \Pi_\Lambda \vec{\lambda}_N$.

Selecting $\vec{w} := \vec{u}_N$, $\vec{r} := \boldsymbol{\varepsilon}(\vec{u}_N)$ and $\vec{\mu} := \tilde{\vec{\lambda}}_N$ in Theorem 4.1 and applying the Korn inequality (see [22, Lemma 6.2]), we have the error estimate

$$\|\vec{u} - \vec{u}_N\|^2_{\mathrm{H}^1(\Omega)} \le C \Big\{ \sup_{\vec{v} \in V} \|\vec{v}\|^{-2}_{\mathrm{H}^1(\Omega)} \left(-a(\vec{u}_N, \vec{v}) + L(\vec{v}) - b(\vec{v}, \tilde{\vec{\lambda}}_N) \right)^2 + j(\vec{u}_N) - b(\vec{u}_N, \tilde{\vec{\lambda}}_N) \Big\}.$$

Inserting the function $\vec{\lambda}_N \in \Lambda_N$ obtained by solving Problem (2.10), we obtain by the definition of the $\mathrm{H}_{00}^{-1/2}$-norm that

$$\begin{aligned} \|\vec{u} - \vec{u}_N\|^2_{\mathrm{H}^1(\Omega)} \le C \Big[&\sup_{\vec{v} \in V} \|\vec{v}\|^{-2}_{\mathrm{H}^1(\Omega)} \left[-a(\vec{u}_N, \vec{v}) + L(\vec{v}) - b(\vec{v}, \vec{\lambda}_N) \right]^2 \\ &+ j(\vec{u}_N) - b(\vec{u}_N, \tilde{\vec{\lambda}}_N) + g^2 \|\vec{\lambda}_N - \tilde{\vec{\lambda}}_N\|^2_{\mathrm{H}_{00}^{-1/2}(\Gamma_C)} \Big]. \end{aligned} \tag{4.2}$$

Applying the definition of the discrete problem, we can insert $\vec{v}_N \in V_N$ and substitute $\vec{v}$ by $-\vec{v}$, which yields

$$\begin{aligned}
&\sup_{\vec{v}\in V}\|\vec{v}\|^{-1}_{\mathrm{H}^1(\Omega)}\left[-a(\vec{u}_N,\vec{v})+L(\vec{v})-b(\vec{v},\vec{\lambda}_N)\right]\\
&\qquad\le \sup_{\vec{v}\in V}\|\vec{v}\|^{-1}_{\mathrm{H}^1(\Omega)}\left[a(\vec{u}_N,\vec{v}-\vec{v}_N)-L(\vec{v}-\vec{v}_N)+b(\vec{v}-\vec{v}_N,\vec{\lambda}_N)\right]
\end{aligned} \tag{4.3}$$

Let $\operatorname{Div}\boldsymbol{\sigma}(\vec{u}_N) := (\boldsymbol{\sigma}_{ji,j}(\vec{u}_N))_{i=1,\dots,d}$ be the vector divergence operator. For $K \in \mathcal{T}_N$, define the interior residuals by

$$\vec{r}_K := -\operatorname{Div}\boldsymbol{\sigma}(\vec{u}_N) - \vec{F} \tag{4.4}$$

and for $E \in \mathcal{E}_N$ the boundary residuals by

$$\vec{R}_E := \begin{cases} \frac{1}{2}\left[\boldsymbol{\sigma}(\vec{u}_N)\cdot\vec{\nu}\right]_E & \text{if } E \in \mathcal{E}_{\mathrm{I},N},\\ (\boldsymbol{\sigma}(\vec{u}_N)\cdot\vec{\nu})_t + g(\vec{\lambda}_N)_t & \text{if } E \in \mathcal{E}_{\mathrm{C},N}\\ \boldsymbol{\sigma}(\vec{u}_N)\cdot\vec{\nu} - \vec{t} & \text{if } E \in \mathcal{E}_{\mathrm{N},N},\\ 0 & \text{if } E \in \mathcal{E}_{\mathrm{D},N}, \end{cases} \tag{4.5}$$

where

$$\left[\boldsymbol{\sigma}(\vec{u}_N)\cdot\vec{\nu}\right]_E := \boldsymbol{\sigma}(\vec{u}_N)|_{K_{E,1}}\cdot\vec{\nu}_{K_{E,1}} + \boldsymbol{\sigma}(\vec{u}_N)|_{K_{E,2}}\cdot\vec{\nu}_{K_{E,2}}$$

is the boundary jump with E the common edge of $K_{E,1}$ and $K_{E,2}$ and $\vec{\nu}_{K_{E,1}}$ pointing from $K_{E,1}$ to $K_{E,2}$, and $\vec{\nu}_{K_{E,2}} = -\vec{\nu}_{K_{E,1}}$. Decomposing the integrals, integrating by parts on each element, applying the Cauchy-Schwarz inequality and regrouping the interior boundary terms as usual, we obtain

$$\begin{aligned}
&a(\vec{u}_N,\vec{v}-\vec{v}_N)-L(\vec{v}-\vec{v}_N)+b(\vec{v}-\vec{v}_N,\vec{\lambda}_N)\\
&\quad=\sum_{K\in\mathcal{T}_N}\left[\int_K \vec{r}_K\cdot(\vec{v}-\vec{v}_N)\mathrm{d}\vec{x} + \sum_{E\subseteq\partial K}\int_E \vec{R}_E\cdot(\vec{v}-\vec{v}_N)\mathrm{d}s_{\vec{x}}\right]\\
&\quad\le\sum_{K\in\mathcal{T}_N}\left[\|\vec{r}_K\|_{\mathrm{L}^2(K)}\|\vec{v}-\vec{v}_N\|_{\mathrm{L}^2(K)} + \sum_{E\subseteq\partial K}\|\vec{R}_E\|_{\mathrm{L}^2(E)}\|\vec{v}-\vec{v}_N\|_{\mathrm{L}^2(E)}\right].
\end{aligned} \tag{4.6}$$

We estimate the remaining terms in (4.2) setting, for $E \in \mathcal{E}_{\mathrm{C},N}$,

$$j_E(\vec{v}) := \int_E g|\vec{v}|\mathrm{d}s_{\vec{x}}, \qquad b_E(\vec{v},\vec{\mu}) := \int_E g\vec{v}_t\cdot\vec{\mu}\mathrm{d}s_{\vec{x}}.$$

Then,

$$j(\vec{u}_N) - b(\vec{u}_N,\tilde{\vec{\lambda}}_N) = \sum_{E\in\mathcal{E}_{\mathrm{C},N}}\left[j_E(\vec{u}_N) - b_E(\vec{u}_N,\tilde{\vec{\lambda}}_N)\right].$$

Furthermore, as $\vec{\lambda}_N$, $\tilde{\vec{\lambda}}_N \in \mathrm{H}^{-1/2}_{00}(\Gamma_{\mathrm{C}})$, by [2, Theorem 4.1],

$$\|\vec{\lambda}_N - \tilde{\vec{\lambda}}_N\|^2_{\mathrm{H}^{-1/2}_{00}(\Gamma_{\mathrm{C}})} \le \sum_{E\in\mathcal{E}_{\mathrm{C},N}}\|\vec{\lambda}_N - \tilde{\vec{\lambda}}_N\|^2_{\mathrm{H}^{-1/2}_{00}(E)}.$$

Defining the *local error indicators* by

$$\eta_{N,K}^2 := h_{N,K}^2 p_{N,K}^{-2} \|\vec{r}_K\|_{\mathrm{L}^2(K)}^2 + h_{N,K} p_{N,K}^{-1} \sum_{E \subseteq \partial K} \|\vec{R}_E\|_{\mathrm{L}^2(E)}^2$$
$$+ j_{\partial K \cap \Gamma_\mathrm{C}}(\vec{u}_N) - b_{\partial K \cap \Gamma_\mathrm{C}}(\vec{u}_N, \tilde{\vec{\lambda}}_N) + g^2 \|\vec{\lambda}_N - \tilde{\vec{\lambda}}_N\|_{\mathrm{H}_{00}^{-1/2}(\partial K \cap \Gamma_\mathrm{C})}^2, \tag{4.7}$$

where the last three terms vanish if $\partial K \cap \Gamma_\mathrm{C} = \emptyset$, and the *global error indicator* by

$$\eta_N^2 := \sum_{K \in \mathcal{T}_N} \eta_{N,K}^2, \tag{4.8}$$

Theorem 4.2 (Reliability). *Assume the setting of Section 3.5 and let Assumption 3.12 be valid. Then there exists a constant $C > 0$ such that the residual error indicator given by (4.7), (4.8) satisfies*

$$\|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(\Omega)} \le C \eta_N \quad \text{for all } N. \tag{4.9}$$

Proof. The proof combines (4.2), (4.3), (4.6), and selects for $\vec{v}_N$ in (4.6) the *hp*-Clément interpolant $I_N \vec{v}$ of [27]. □

Remark 4.3 (numerical realization of η_K). Considering the definition of the local error indicator, we see that for edges E on Γ_C, we need to calculate an integral of the absolute value of a polynomial and the $\mathrm{H}_{00}^{-1/2}$-norm of the nonconformity of $\vec{\lambda}_N$. In general, these terms cannot be calculated exactly; however, they can easily be estimated:

For the integration error, we distinguish cases: Assuming that $(\vec{u}_N)_t$ does not change sign, we can actually calculate the integral exactly, as $\int_E |(\vec{u}_N)_t| \mathrm{d}s_{\vec{x}} = |\int_E (\vec{u}_N)_t \mathrm{d}s_{\vec{x}}|$. If $(\vec{u}_N)_t$ does change sign, we estimate the integral by the maximum of $|(\vec{u}_N)_t|$ on the given interval times the interval length, $j_E(\vec{u}_N) \le h_{N,E} \|(\vec{u}_N)_t|_E\|_{\mathrm{L}^\infty(E)}$.

For the nonconformity of $\vec{\lambda}_N$, we proceed similarly: We first estimate the $\mathrm{H}_{00}^{-1/2}$-norm by the L^2-norm, and then the resulting integral by the difference between the maximum of the absolute value of $\vec{\lambda}_N$ minus one times the square root of the interval length.

Note that the maximum of a polynomial can be efficiently estimated using an expansion into a Legendre series in the following way: For a polynomial p of degree $q \ge 2$, let $p = \sum_{j=0}^q c_j L_j$. With x^* chosen such that $p_0 := \sum_{j=0}^2 c_j L_j$ attains the maximum of its absolute value at x^*, we have that

$$\|p\|_{\mathrm{L}^\infty(I)} \le \mathrm{EST} := |p_0(x^*)| + \sum_{j=3}^q |c_j|, \tag{4.10}$$

where we used that $\|L_j\|_{\mathrm{L}^\infty(I)} = 1$ for all $j \ge 0$.

Theorem 4.4 (Efficiency). *Let Assumption 3.12 be valid. For each K let $\bar{\vec{r}}_K \in S^{p_{N,K}}$ be a polynomial approximation of $\vec{r}_K$. For each edge E, let $\bar{\vec{R}}_E$ be a polynomial approximation of $\vec{R}_E$ of degree p_E, where $p_E = \min\{p_{N,K} \,|\, E$ is edge of $K \in \mathcal{T}_N\}$.*

For $K \in \mathcal{T}_N$ denote by K_{patch} the union of elements of $\mathcal{T}_N$ that share an edge with K. Let $\vec{r}_{patch}$ and $\bar{\vec{r}}_{patch}$ be defined on K_{patch} in an elementwise fashion by $\vec{r}_{patch}|_{K'} = r_{K'}$ and $\bar{\vec{r}}_{patch}|_{K'} = \bar{\vec{r}}_{K'}$ for all $K' \subset K_{patch}$.

Let $\beta \in (1/2, 1]$. Then there exists a constant $C > 0$ such that the residual error indicator satisfies

$$\eta_{N,K}^2 \lesssim p_{N,K}^{2\beta}\Big(p_{N,K}\|\vec{u} - \vec{u}_N\|^2_{\mathrm{H}^1(K_{patch})} + h_{N,K}^2 p_{N,K}^{-3+2\beta}\|\bar{\vec{r}}_{K_{patch}} - \vec{r}_{K_{patch}}\|_{\mathrm{L}^2(K_{patch})}$$
$$+ h_{N,K}p_{N,K}^{-1}\sum_{E\subseteq\partial K}\|\bar{\vec{R}}_E - \vec{R}_E\|^2_{\mathrm{L}^2(E)} + g^2 h_{N,K}p_{N,K}^{-1}\|\vec{\lambda}_N - \vec{\lambda}\|^2_{\mathrm{L}^2(\partial K\cap\Gamma_\mathrm{C})}\Big)$$
$$+ g h_{N,\partial K\cap\Gamma_\mathrm{C}}^{1/2}\|\vec{u}_N - \vec{u}\|_{\mathrm{L}^2(\partial K\cap\Gamma_\mathrm{C})} + g\|\vec{u}\|_{\mathrm{L}^2(\partial K\cap\Gamma_\mathrm{C})}\|\vec{\lambda} - \vec{\lambda}_N\|_{\mathrm{L}^2(\partial K\cap\Gamma_\mathrm{C})}$$
$$+ g^2\|\vec{\lambda}_N - \vec{\lambda}\|^2_{\mathrm{L}^2(\partial K\cap\Gamma_\mathrm{C})} \tag{4.11}$$

for all N and all $K \in \mathcal{T}_N$.

Proof. The proof basically follows the efficiency proof in [28] for the terms $\|\vec{r}_K\|_{L^2(K)}$ and $\|\vec{R}_E\|_{L^2(E)}$ in (4.7). See Appendix A for the details.

For the contributions $j_{\partial K\cap\Gamma_C}(\vec{u}_N) - b_{\partial K\cap\Gamma_C}(\vec{u}_N, \tilde{\vec{\lambda}}_N)$ and $\|\vec{\lambda}_N - \tilde{\vec{\lambda}}_N\|_{\mathrm{H}_{00}^{-1/2}(\partial K\cap\Gamma_C)}$ in (4.7) we proceed as follows: For K with $\partial K\cap\Gamma_\mathrm{C} \neq \emptyset$, we have $j_E(\vec{u}) = b_E(\vec{u}, \vec{\lambda})$. Thus by the inverse triangle and Cauchy-Schwarz inequalities and the fact that $\|\tilde{\vec{\lambda}}_N\|_{\mathrm{L}^2(E)} \leq \|\vec{\lambda}_N\|_{\mathrm{L}^2(E)} \leq h_{N,E}^{1/2}$,

$$j_E(\vec{u}_N) - b_E(\vec{u}_N, \tilde{\vec{\lambda}}_N) = j_E(\vec{u}_N) - j_E(\vec{u}) + b_E(\vec{u}, \vec{\lambda}) - b_E(\vec{u}, \tilde{\vec{\lambda}}_N)$$
$$+ b_E(\vec{u}, \tilde{\vec{\lambda}}_N) - b_E(\vec{u}_N, \tilde{\vec{\lambda}}_N)$$
$$\leq g h_{N,E}^{1/2}\|\vec{u}_N - \vec{u}\|_{\mathrm{L}^2(E)} + g\|\vec{u}\|_{\mathrm{L}^2(E)}\|\vec{\lambda} - \tilde{\vec{\lambda}}_N\|_{\mathrm{L}^2(E)} + g\|\vec{u} - \vec{u}_N\|_{\mathrm{L}^2(E)}\|\tilde{\vec{\lambda}}_N\|_{\mathrm{L}^2(E)}.$$

Next, as $\Pi_\Lambda\lambda = \lambda$ and Π_Λ is Lipschitz continuous with respect to the L^2-norm, we obtain

$$\|\vec{\lambda}_N - \tilde{\vec{\lambda}}_N\|_{\mathrm{H}_{00}^{-1/2}(E)} \leq \|\vec{\lambda}_N - \tilde{\vec{\lambda}}_N\|_{\mathrm{L}^2(E)} \leq \|\vec{\lambda}_N - \vec{\lambda}\|_{\mathrm{L}^2(E)} + \|\vec{\lambda} - \tilde{\vec{\lambda}}_N\|_{\mathrm{L}^2(E)}$$
$$\leq 2\|\vec{\lambda}_N - \vec{\lambda}\|_{\mathrm{L}^2(E)},$$

which allows us to conclude the argument. □

Remark 4.5. In the above efficiency result, as we have to take square roots, the terms $g h_{N,E}^{1/2}\|\vec{u}_N - \vec{u}\|_{\mathrm{L}^2(E)}$ and $g\|\vec{u}\|_{\mathrm{L}^2(E)}\|\vec{\lambda} - \vec{\lambda}_N\|_{\mathrm{L}^2(E)}$ are of the wrong order compared to the other terms. Note, however, that these terms result from the terms estimating the error in numerical integration of $\vec{u}_N$ by using $\vec{\lambda}_N$ and the nonconformity of $\vec{\lambda}_N$. Our numerical results confirm that after some steps in the adaptive algorithm, those terms are negligible, showing that for practical problems, the error indicator is efficient.

5. Numerical experiments

We consider the two two-dimensional numerical problems described in [15, Examples 6.12 and 6.13]. We apply the primal-dual formulation (2.9) with the spaces V_N and W_N given in (3.2), i.e., we use affine quadrilaterals and admit hanging nodes. We require the "one hanging node rule" and that all irregular nodes be one-irregular. In the

Algorithm 5.1 decision h or p

% parameter $\delta > 0$
% input: element K to be refined
expand $\vec{u}_N|_K \circ F_K = \sum_{i,j=0}^{p_{N,K}} u_{K,ij} L_i(x) L_j(y)$
compute b_K by a least squares fit of data $u_{K,ij}$ to the law $\ln |u_{K,ij}| \sim C - b_K(i+j)$.
if $b_K < \delta$ **then**
  flag K for h-refinement
else
  if K has no edge on Γ_C **then**
    flag K for p-enrichment
  else
    let E be edge of K on Γ_C
    compute EST as described in Remark 4.3 such that $\|(\vec{\lambda}_N)_t\|_{L^\infty(E)} \leq \text{EST}$
    if EST ≤ 1 **then**
      flag K for p-enrichement
    else
      flag K for h-refinement
    end if
  end if
end if

context of non-uniform polynomial degree distributions, differing polynomial degrees on neighbouring elements are resolved by using the minimum rule on the common edge.

We compare four discretisations: an h-uniform and an h-adaptive method with polynomial degree 2, a p-uniform method, starting with polynomial degree 2, and an hp-adaptive method, starting with polynomial degree 3.

The error indicators η_K of (4.7) are employed and are estimated as described in Remark 4.3. In the h-adaptive methods, we mark all elements for refinement where the local estimated error η_K satisfies $\eta_K \geq \frac{0.5}{\#\mathcal{T}_N} \sum_{K \in \mathcal{T}_N} \eta_K$. In the hp-adaptive scheme, each adaptive step refines those 20% of the elements that have the largest error indicator. In the hp-adaptive setting, the decision of whether to do an h- or a p-refinement, is done based on locally testing for analyticity as done in [13, Strategy II] for triangles and earlier on quadrilaterals in [26, 20]. The details of the procedure are described in Algorithm 5.1. The computations are performed with $\delta = 1$ in Algorithm 5.1. Compared to [13, Strategy II], Algorithm 5.1 checks additionally $\|(\vec{\lambda}_N)_t\|_{L^\infty(E)}$ and enforces h-refinement if $\|(\vec{\lambda}_N)_t\|_{L^\infty(E)} > 1$. The motivation behind this is to "find" the (unknown) transition point from sliding to sticking and the observation in our numerical experiments that $\vec{\lambda}_N$ is non-conforming only in few elements near the transition point. Indeed, in the final refinement step of our numerical experiments the transition point is sufficiently resolved and the Lagrange multiplier actually satisfies $|\vec{\lambda}|_{L^\infty(\Gamma_C)} \leq 1$.

All calculations were done using `maiprogs` ([24]). For the static condensation of the internal degrees of freedom, `pardiso` was used ([29, 30, 21]). After a diagonal rescaling of the resulting matrix, the variational inequality of the first kind on the boundary was solved by the MPRGP algorithm suggested in [11], applying the modifications made in

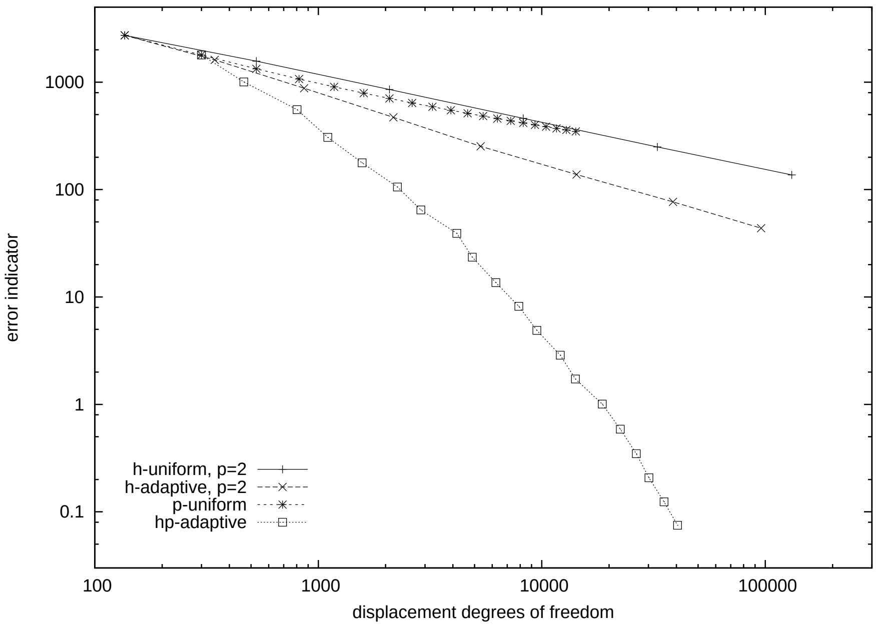

Figure 5.1: Example 5.1, estimated errors vs. problem size

[33] to be able to deal with two-sided constraints.

Example 5.1. Let $\Omega = (0,4) \times (0,4)$, assume homogeneous Dirichlet conditions on $\Gamma_{\mathrm{D}} := \{4\} \times (0,4)$, frictional contact on $\Gamma_{\mathrm{C}} := (0,4) \times \{0\}$ and Neumann conditions on $\Gamma_{\mathrm{N}} := (\{0\} \times (0,4)) \cup ((0,4) \times \{4\})$, where $\vec{t}(0,s) = (150(5-s), -75)\mathrm{daN/mm}^2$ for $s \in (0,4)$ and $\vec{t} = 0$ on $(0,4) \times \{4\}$. The elasticity parameters are chosen to be $E = 1500\mathrm{daN/mm}^2$ and $\nu = 0.4$, the friction coefficient is $g = 450\mathrm{daN/mm}^2$, and plane stress is assumed. The initial mesh is uniform and consists of 16 elements.

Figure 5.1 shows the estimated errors for the h-uniform and adaptive, the p-uniform and the hp-adaptive methods. Assuming an error behaviour of the form $\|u - u_N\|_{\mathrm{H}^1(\Omega)} = CN^{-\alpha}$ for the uniform h- and p-version and the adaptive h-version, we obtain by a least squares fit rates of about $\alpha = 0.44$ for h- and $\alpha = 0.33$ for the p-uniform method and about $\alpha = 0.64$ for the adaptive scheme. Similarly, assuming a behaviour of the form $\|u - u_N\|_{\mathrm{H}^1(\Omega)} = C\exp(-\gamma N^{1/3})$ for the hp-adaptive scheme, a least squares fit yields a convergence rate of about $\gamma = 0.36$, which confirms that the hp-adaptivity as suggested here leads to exponential convergence. The p-refinement near the transition point between sliding and sticking boundary conditions that can be seen in Figure 5.3 can be justified by the fact that the Lagrange multiplier actually is conforming at this refinement level (see Figure 5.2 for the refinement of the entire domain).

Figure 5.4 shows a parameter study for the dependence of the convergence on the refinement control parameter δ of Algorithm 5.1. The fastest convergence of about $\gamma = 0.41$ is obtained for $\delta = 0.6$. For medium values of δ, we recover exponential convergence.

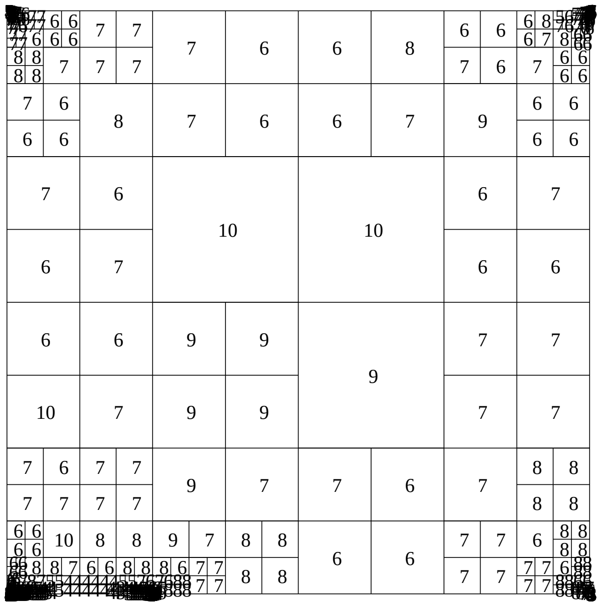

Figure 5.2: Example 5.1, *hp*-adaptivity, final refinement

Example 5.2. In the second example, $\Omega = (0, 10) \times (0, 2)$, we have Dirichlet conditions on $\Gamma_D = (0, 10) \times \{2\}$, frictional contact on $\Gamma_C := (0, 10) \times \{0\}$ and Neumann conditions on $\Gamma_N := (\{0\} \times (0, 2)) \cup (\{10\} \times (0, 2))$, where $\vec{t}(0, s) = (500, 0)\text{daN/mm}^2$ and $\vec{t}(10, s) = (250s - 750, -100)\text{daN/mm}^2$ for $s \in (0, 2)$. The elasticity parameters are $E = 1000\text{daN/mm}^2$ and $\nu = 0.3$, the friction coefficient is $g = 175\text{daN/mm}^2$, and we again assume plane stress. The initial mesh consists of four elements generated by dividing only along the x-axis.

The estimated errors are plotted in Figure 5.5. Using an assumed error behaviour as in Example 5.1, we obtain rates of $\alpha = 0.38$ for the uniform h-version, $\alpha = 0.32$ for the uniform p-version, $\alpha = 0.57$ for the adaptive h-version and $\gamma = 0.33$ for the hp-adaptive scheme. Again, the hp-adaptivity yields exponential convergence. The plots of the refinements in Figures 5.6 and 5.7 show a strong h-refinement near the transition point between sticking and sliding.

6. Acknowledgments

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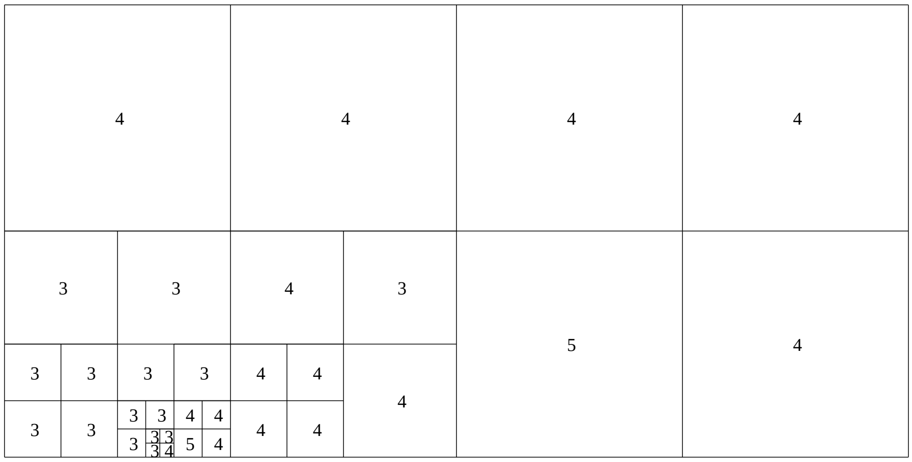

Figure 5.3: Example 5.1, *hp*-adaptivity, final refinement, zoom onto $[0.984375, 1.015625] \times [0.0, 0.015625]$

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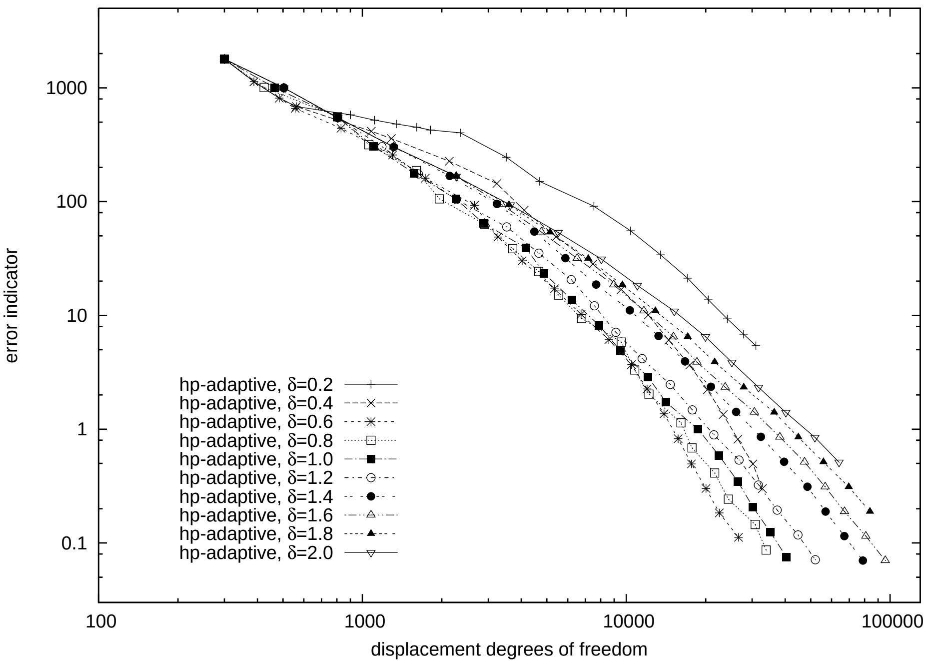

Figure 5.4: Example 5.1, estimated errors vs. problem size, dependence on δ

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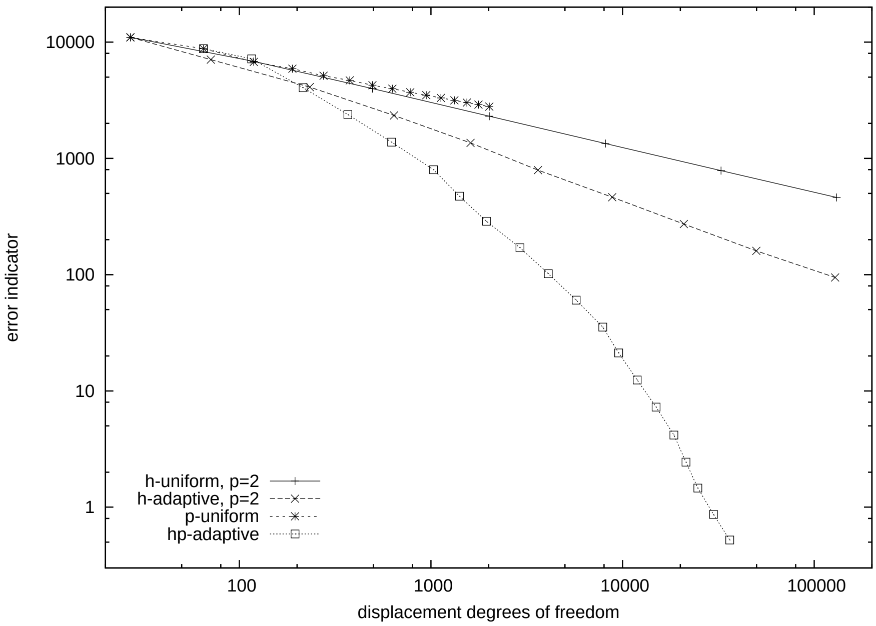

Figure 5.5: Example 5.2, estimated errors vs. problem size

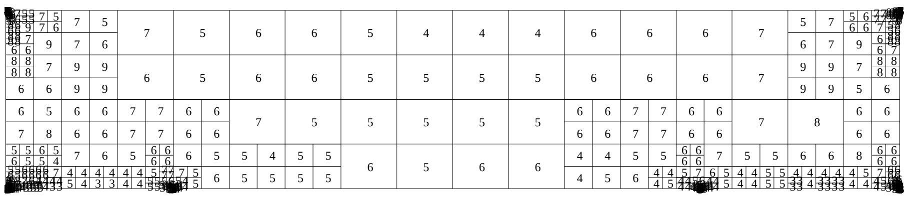

Figure 5.6: Example 5.2, *hp*-adaptivity, final refinement

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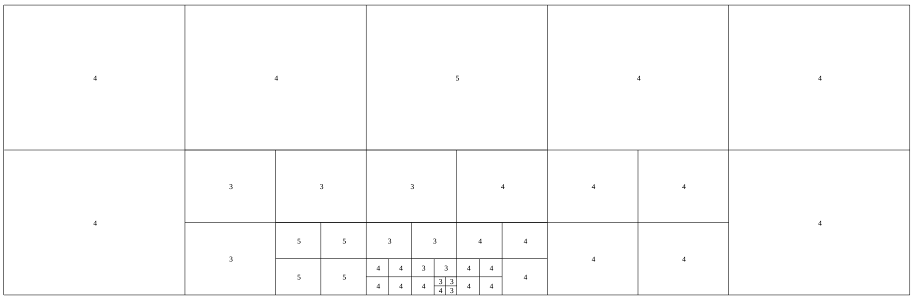

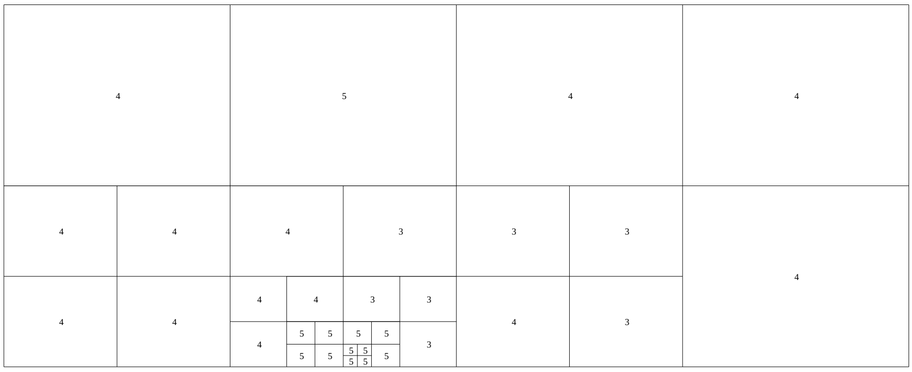

Figure 5.7: Example 5.2, hp-adaptivity, final refinement, zooms onto $[1.7773438, 1.9335938] \times [0.0, 0.0625]$ (top) and $[7.6953125, 8.8515625] \times [0.0, 0.0625]$ (bottom)

A. Proof of Theorem 4.4

In this section, we simplify the notation and write h_K and p_K for the element size and the polynomial degree of an element $K \in \mathcal{T}_N$, i.e., we drop the explicit reference to the index N.

A.1. Preliminaries

Let $F_K \colon S \to K$ be the *element map* for K, that is, F_K is one-to-one and onto and bilinear, and assume that F_K maps I, interpreted as an edge of the reference element, to the edge E of K. Then, using the bubble functions on the reference interval $I := [-1, 1]$,

$$\psi_I(x) := \operatorname{dist}(x, \partial I), \tag{A.1}$$

and reference element $S := [-1, 1]^2$,

$$\psi_S(\vec{x}) := \operatorname{dist}(\vec{x}, \partial S), \tag{A.2}$$

we define the *element bubble function* on K and the *edge bubble function* on E by

$$\psi_K := c_K \psi_S \circ F_K^{-1}, \quad \psi_E := c_E \psi_I \circ F_K^{-1}, \tag{A.3}$$

where the scaling factors c_K, $c_E > 0$ are chosen in such a way that

$$\int_K \psi_K \mathrm{d}\vec{x} = \operatorname{meas} K, \qquad \int_E \psi_E \mathrm{d}s_{\vec{x}} = \operatorname{meas} E. \tag{A.4}$$

A.2. Proof of Theorem 4.4

Consider (2.8a) and (2.10a) and integrate by parts on each element to obtain

$$\begin{aligned} a(\vec{u}-\vec{u}_N,\vec{v}) &= a(\vec{u},\vec{v}) - a(\vec{u}_N,\vec{v}) = L(\vec{v}) - b(\vec{v},\vec{\lambda}) - a(\vec{u}_N,\vec{v}) \\ &= -\sum_{K\in\mathcal{T}_N}\left[\int_K \vec{r}_K\vec{v}\mathrm{d}\vec{x} + \sum_{E\subseteq\partial K}\int_E \vec{R}_E\vec{v}\mathrm{d}s_{\vec{x}}\right] + g\int_{\Gamma_{\mathrm{C}}}(\vec{\lambda}_N-\vec{\lambda})\vec{v}\mathrm{d}s_{\vec{x}}. \end{aligned} \tag{A.5}$$

The remainder of this section is devoted to bounding $\vec{r}_K$ and $\vec{R}_E$ in terms of $\vec{u}-\vec{u}_N$. As is standard in residual based error estimation, we start with $\vec{r}_K$:

Lemma A.1. *Let $\beta \in (1/2, 1]$ and let Assumption 3.12 be valid. Let $\bar{\vec{r}}_K \in S^{p_K}$. Then there exists $C > 0$ such that*

$$\|\vec{r}_K\|_{L^2(K)} \le C\left[p_K^{\beta}\|\bar{\vec{r}}_K - \vec{r}_K\|_{L^2(K)} + h_{N,K}^{-1}p_K^2\|\vec{u}-\vec{u}_N\|_{H^1(K)}\right].$$

Proof. Let $\vec{v} := \psi_K^{\beta}\bar{\vec{r}}_K$, where $\bar{\vec{r}}_K$ is a polynomial approximation of $\vec{r}_K$ of degree p_K. Plugging this into (A.5) yields

$$a(\vec{u}-\vec{u}_N,\psi_K^{\beta}\bar{\vec{r}}_K) = -\int_K \vec{r}_K\psi_K^{\beta}\bar{\vec{r}}_K\mathrm{d}\vec{x}. \tag{A.6}$$

Thus,

$$\int_K \psi_K^{\beta}|\bar{\vec{r}}_K|^2\mathrm{d}\vec{x} = \int_K \psi_K^{\beta}\bar{\vec{r}}_K(\bar{\vec{r}}_K-\vec{r}_K)\mathrm{d}\vec{x} - a(\vec{u}-\vec{u}_N,\psi_K^{\beta}\bar{\vec{r}}_K), \tag{A.7}$$

and the Cauchy-Schwarz inequality and the boundedness of a give, together with the Poincaré inequality, which is applicable due to the fact that $\psi_K^{\beta}\bar{\vec{r}}_K = 0$ on Γ,

$$\begin{aligned} \int_K \psi_K^{\beta}|\bar{\vec{r}}_K|^2\mathrm{d}\vec{x} &\le \|\psi_K^{\beta/2}\bar{\vec{r}}_K\|_{\mathrm{L}^2(K)}\|(\bar{\vec{r}}_K-\vec{r}_K)\psi_K^{\beta/2}\|_{\mathrm{L}^2(K)} \\ &\quad + C\|\vec{u}-\vec{u}_N\|_{\mathrm{H}^1(K)}\|\psi_K^{\beta}\bar{\vec{r}}_K\|_{\mathrm{H}^1(K)} \\ &\le \|\psi_K^{\beta/2}\bar{\vec{r}}_K\|_{\mathrm{L}^2(K)}\|(\bar{\vec{r}}_K-\vec{r}_K)\psi_K^{\beta/2}\|_{\mathrm{L}^2(K)} \\ &\quad + C\|\vec{u}-\vec{u}_N\|_{\mathrm{H}^1(K)}|\psi_K^{\beta}\bar{\vec{r}}_K|_{\mathrm{H}^1(K)}. \end{aligned} \tag{A.8}$$

Applying the inverse inequalities in [28, Theorem 2.5] together with a scaling argument, we see that

$$\begin{aligned} |\psi_K^{\beta}\bar{\vec{r}}_K|^2_{\mathrm{H}^1(K)} &= \|\nabla(\psi_K^{\beta}\bar{\vec{r}}_K)\|^2_{\mathrm{L}^2(K)} \le 2\left[\|(\nabla\psi_K^{\beta})\bar{\vec{r}}_K\|^2_{\mathrm{L}^2(K)} + \|\psi_K^{\beta}\nabla\bar{\vec{r}}_K\|^2_{\mathrm{L}^2(K)}\right] \\ &\le C\left[h_K^{-2}\|\psi_K^{\beta-1}\bar{\vec{r}}_K\|^2_{\mathrm{L}^2(K)} + h_K^{-2}p_K^{2(2-\beta)}\|\psi_K^{\beta/2}\bar{\vec{r}}_K\|^2_{\mathrm{L}^2(K)}\right] \\ &\le Ch_K^{-2}p_K^{2(2-\beta)}\|\psi_K^{\beta/2}\bar{\vec{r}}_K\|^2_{\mathrm{L}^2(K)}. \end{aligned} \tag{A.9}$$

Inserting this in (A.8), we get

$$\begin{aligned}\|\psi_K^{\beta/2}\bar{\vec{r}}_K\|_{\mathrm{L}^2(K)} &\leq C\left[\|(\bar{\vec{r}}_K-\vec{r}_K)\psi_K^{\beta/2}\|_{\mathrm{L}^2(K)}+h_K^{-1}p_K^{2-\beta}\|\vec{u}-\vec{u}_N\|_{\mathrm{H}^1(K)}\right]\\ &\leq C\left[\|\bar{\vec{r}}_K-\vec{r}_K\|_{\mathrm{L}^2(K)}+h_K^{-1}p_K^{2-\beta}\|\vec{u}-\vec{u}_N\|_{\mathrm{H}^1(K)}\right]. \end{aligned}\tag{A.10}$$

Finally, by the triangle inequality and [28, Theorem 2.5], we arrive at

$$\begin{aligned}\|\vec{r}_K\|_{\mathrm{L}^2(K)} &\leq \|\bar{\vec{r}}_K-\vec{r}_K\|_{\mathrm{L}^2(K)}+\|\bar{\vec{r}}_K\|_{\mathrm{L}^2(K)} \leq \|\bar{\vec{r}}_K-\vec{r}_K\|_{\mathrm{L}^2(K)}+Cp_K^{\beta}\|\psi_K^{\beta/2}\bar{\vec{r}}_K\|_{\mathrm{L}^2(K)}\\ &\leq C\left[(1+p_K^{\beta})\|\bar{\vec{r}}_K-\vec{r}_K\|_{\mathrm{L}^2(K)}+h_K^{-1}p_K^{2}\|\vec{u}-\vec{u}_N\|_{\mathrm{H}^1(K)}\right]\\ &\leq C\left[p_K^{\beta}\|\bar{\vec{r}}_K-\vec{r}_K\|_{\mathrm{L}^2(K)}+h_K^{-1}p_K^{2}\|\vec{u}-\vec{u}_N\|_{\mathrm{H}^1(K)}\right]. \end{aligned}\tag{A.11}$$

□

The next step of the standard procedure in residual error estimation is to estimate the edge contributions $\vec{R}_E$.

Lemma A.2. *Assume the hypotheses of Lemma A.1. Let $E\in\mathcal{E}_{\mathrm{I},N}$ and set $p_E:=\min\{p_{K_E},p_{K'_E}\}$, where K_E, K'_E are the elements that share edge E. Set $E_{patch}:=K\cup K'\cup E$ and $p_E:=\min\{p_{K_E},p_{K'_E}\}$. Let $\bar{\vec{R}}_E$ be a polynomial approximation to $\vec{R}_E$ or degree p_E. Then there exists $C>0$ such that*

$$\begin{aligned}\|\vec{R}_E\|_{L^2(E)} \quad\leq\quad & C\big[p_E^{\beta}\|\bar{\vec{R}}_E-\vec{R}_E\|_{L^2(E)}+h_E^{-1/2}p_E^{1+\beta}\|\vec{u}-\vec{u}_N\|_{H^1(E_{patch})}\\ &+h_E^{1/2}p_E^{-1+2\beta}\|\vec{r}_{patch}\|_{L^2(E_{patch})}\big],\end{aligned}$$

where the function r_{patch} is defined as $\vec{r}_{K_E}$ on K_E and as $\vec{r}_{K'_E}$ on K'_E.

Proof. Let $\vec{v}$ be an extension to E_{patch} of $\psi_E^{\beta}\bar{\vec{R}}_E$ constructed by applying [28, Lemma 2.6] together with a scaling argument, and patching the results for the two neighbouring elements K_E and K'_E of E together. This extension satisfies by [28, Lemma 2.6] for any $\varepsilon\in(0,1]$:

$$\|\vec{v}\|^2_{L^2(E_{\mathrm{patch}})} \quad\leq\quad C\varepsilon h_E\|\psi^{\beta/2}\bar{\vec{R}}_E\|^2_{L^2(E)},\tag{A.12}$$

$$\|\nabla\vec{v}\|^2_{L^2(E_{\mathrm{patch}})} \quad\leq\quad C\left(\varepsilon^{-1}+\varepsilon p_E^{2(2-\beta)}\right)h_E^{-1}\|\psi^{\beta/2}\bar{\vec{R}}_E\|^2_{L^2(E)}.\tag{A.13}$$

Plugging this choice of $\vec{v}$ into (A.5) yields

$$\begin{aligned}a(\vec{u}-\vec{u}_N,\psi_E^{\beta}\bar{\vec{R}}_E) = -\sum_{K\subseteq E_{\mathrm{patch}}}&\left[\int_K\vec{r}_K\psi_E^{\beta}\bar{\vec{R}}_E\mathrm{d}\vec{x}+\int_E\vec{R}_E\psi_E^{\beta}\bar{\vec{R}}_E\mathrm{d}s_{\vec{x}}\right]\\ &+g\int_{E\cap\Gamma_{\mathrm{C}}}(\vec{\lambda}_N-\vec{\lambda})\psi_K^{\beta}\bar{\vec{R}}_E\mathrm{d}s_{\vec{x}}.\end{aligned}\tag{A.14}$$

Since by assumption $E\in\mathcal{E}_{\mathrm{I},N}$ we have $\operatorname{meas}E\cap\Gamma_{\mathrm{C}}=0$. In view of the definition of $\vec{R}_E$, we get

$$\int_E\psi_E^{\beta}\bar{\vec{R}}_E^2\mathrm{d}\vec{x}=\int_E\psi_E^{\beta}\bar{\vec{R}}_E(\bar{\vec{R}}_E-\vec{R}_E)\mathrm{d}\vec{x}+\int_E\vec{R}_E\psi_E^{\beta}\bar{\vec{R}}_E\mathrm{d}s_{\vec{x}}$$

$$
\begin{aligned}
&= \int_E \psi_E^\beta \bar{\vec{R}}_E (\bar{\vec{R}}_E - \vec{R}_E) \mathrm{d}\vec{x} - \frac{1}{2} a(\vec{u} - \vec{u}_N, \vec{v}) \\
&\quad - \frac{1}{2} \int_{E_{\text{patch}}} \vec{r}_{E_{\text{patch}}} \vec{v} \mathrm{d}\vec{x}. \qquad \text{(A.15)}
\end{aligned}
$$

The Cauchy-Schwarz inequality and the boundedness of a yield, together with the Poincaré inequality,

$$
\begin{aligned}
\int_E \psi_E^\beta \bar{\vec{R}}_E^2 \mathrm{d}\vec{x} &\le \|\psi_E^{\beta/2} \bar{\vec{R}}_E\|_{\mathrm{L}^2(E)} \|(\bar{\vec{R}}_E - \vec{R}_E)\psi_E^{\beta/2}\|_{\mathrm{L}^2(E)} \\
&\quad + C\Big[\|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(E_{\text{patch}})} \|\vec{v}\|_{\mathrm{H}^1(E_{\text{patch}})} \\
&\qquad + \|\vec{r}_{E_{\text{patch}}}\|_{\mathrm{L}^2(E_{\text{patch}})} \|\vec{v}\|_{\mathrm{L}^2(E_{\text{patch}})}\Big] \\
&\le \|\psi_E^{\beta/2} \bar{\vec{R}}_E\|_{\mathrm{L}^2(E)} \|(\bar{\vec{R}}_E - \vec{R}_E)\psi_E^{\beta/2}\|_{\mathrm{L}^2(E)} \\
&\quad + C\Big[\|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(E_{\text{patch}})} |\vec{v}|_{\mathrm{H}^1(E_{\text{patch}})} \\
&\qquad + \|\vec{r}_{E_{\text{patch}}}\|_{\mathrm{L}^2(E_{\text{patch}})} \|\vec{v}\|_{\mathrm{L}^2(E_{\text{patch}})}\Big]. \qquad \text{(A.16)}
\end{aligned}
$$

Inserting the estimate (A.12), (A.13) produces

$$
\begin{aligned}
\|\psi_E^{\beta/2} \bar{\vec{R}}_E\|_{\mathrm{L}^2(E)} &\le C\Big[\|\bar{\vec{R}}_E - \vec{R}_E\|_{\mathrm{L}^2(E)} + h_E^{-1/2} (\varepsilon p_E^{2(2-\beta)} + \varepsilon^{-1})^{1/2} \|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(E_{\text{patch}})} \\
&\quad + h_E^{1/2} \varepsilon^{1/2} \|\vec{r}_{E_{\text{patch}}}\|_{\mathrm{L}^2(E_{\text{patch}})}\Big]. \qquad \text{(A.17)}
\end{aligned}
$$

Choosing $\varepsilon = p_E^{-2}$ yields

$$
\begin{aligned}
\|\psi^{\beta/2} \bar{\vec{R}}_E\|_{\mathrm{L}^2(E)} &\le C\Big[\|\bar{\vec{R}}_E - \vec{R}_E\|_{\mathrm{L}^2(E)} + h_E^{-1/2} p_E \|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(E_{\text{patch}})} \\
&\quad + h_E^{1/2} p_E^{-1} \|\vec{r}_{E_{\text{patch}}}\|_{\mathrm{L}^2(E_{\text{patch}})}\Big]. \qquad \text{(A.18)}
\end{aligned}
$$

Using the triangle inequality, we obtain with [28, Lemma 2.4] that

$$
\begin{aligned}
\|\vec{R}_E\|_{\mathrm{L}^2(E)} &\le \|\bar{\vec{R}}_E - \vec{R}_E\|_{\mathrm{L}^2(E)} + \|\bar{\vec{R}}_E\|_{\mathrm{L}^2(E)} \\
&\le C\left[\|\bar{\vec{R}}_E - \vec{R}_E\|_{\mathrm{L}^2(E)} + p_E^\beta \|\psi_E^{\beta/2} \bar{\vec{R}}_E\|_{\mathrm{L}^2(E)}\right] \\
&\le C\Big[p_E^\beta \|\bar{\vec{R}}_E - \vec{R}_E\|_{\mathrm{L}^2(E)} + h_E^{-1/2} p_E^{1+\beta} \|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(E_{\text{patch}})} \\
&\quad + h_E^{1/2} p_E^{-1+\beta} \|\vec{r}_{E_{\text{patch}}}\|_{\mathrm{L}^2(E_{\text{patch}})}\Big]. \qquad \text{(A.19)}
\end{aligned}
$$

Estimating $\|\vec{r}_{E_{\text{patch}}}\|_{\mathrm{L}^2(E_{\text{patch}})}$ with the aid of (A.11) gives us

$$
\begin{aligned}
\|\vec{R}_E\|_{\mathrm{L}^2(E)} &\le C\Big[p_E^\beta \|\bar{\vec{R}}_E - \vec{R}_E\|_{\mathrm{L}^2(E)} + h_E^{-1/2} p_E^{1+\beta} \|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(E_{\text{patch}})} \\
&\quad + h_E^{1/2} p_E^{-1+\beta} \big(p_E^\beta \|\bar{\vec{r}}_{E_{\text{patch}}} - \vec{r}_{E_{\text{patch}}}\|_{\mathrm{L}^2(E_{\text{patch}})} \\
&\qquad + h_E^{-1} p_K^2 \|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(E_{\text{patch}})}\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\Big[p_E^{\beta}\|\bar{\vec{R}}_E - \vec{R}_E\|_{\mathrm{L}^2(E)} + h_E^{-1/2}p_E^{1+\beta}\|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(E_{\mathrm{patch}})} \\
&\qquad + h_E^{1/2}p_E^{-1+2\beta}\|\bar{\vec{r}}_{E_{\mathrm{patch}}} - \vec{r}_{E_{\mathrm{patch}}}\|_{\mathrm{L}^2(E_{\mathrm{patch}})}\Big].
\end{aligned} \tag{A.20}
$$

□

Lemma A.2 treats the case of interior edges. The case of edges on the Neumann boundary and on the contact boundary are handled in an analogous way. We merely record the result:

Lemma A.3. *Assume the hypotheses of Lemma A.2. For $E \in \mathcal{E}_{\mathrm{N},N} \cup \mathcal{E}_{\mathrm{C},N}$ denote by $K_E \in \mathcal{T}_N$ the element with $E \subset \partial K_E$. Set $p_E := p_{K_E}$. Then:*

(i) *If $E \in \mathcal{E}_{\mathrm{N},N}$, then*

$$
\begin{aligned}
\|\vec{R}_E\|_{L^2(E)} \quad \leq \quad & C\big[p_E^{\beta}\|\bar{\vec{R}}_E - \vec{R}_E\|_{L^2(E)} + h_E^{-1/2}p_E^{1+\beta}\|\vec{u} - \vec{u}_N\|_{H^1(K_E)} \\
& + h_E^{1/2}p_E^{-1+2\beta}\|\vec{r}_{K_E}\|_{L^2(K_E)}\big].
\end{aligned}
$$

(ii) *If $E \in \mathcal{E}_{\mathrm{C},N}$, then*

$$
\begin{aligned}
\|\vec{R}_E\|_{L^2(E)} \quad \leq \quad & C\big[p_E^{\beta}\|\bar{\vec{R}}_E - \vec{R}_E\|_{L^2(E)} + h_E^{-1/2}p_E^{1+\beta}\|\vec{u} - \vec{u}_N\|_{H^1(K_E)} \\
& + h_E^{1/2}p_E^{-1+2\beta}\|\vec{r}_{K_E}\|_{L^2(K_E)}\big] + g p_E^{\beta}\|\vec{\lambda}_N - \vec{\lambda}\|_{L^2(E)}.
\end{aligned}
$$

Proof of Theorem 4.4: The proof of Theorem 4.4 follows from a combination of Lemmas A.1, A.2, and A.3. For elements $K \in \mathcal{T}_N$ with $\partial K \cap \Gamma_C = \emptyset$, we get for $\operatorname{meas} \partial K \cap \Gamma_{\mathrm{C}} = 0$ due to the local comparability of h and p with an adequate element patch $K_{\mathrm{patch}} \supseteq E_{\mathrm{patch}}$ for all $E \subseteq \partial K$, as $\beta > 1/2$,

$$
\begin{aligned}
\eta_{N,K}^2 &= h_K^2 p_K^{-2}\|\vec{r}_K\|_{\mathrm{L}^2(K)}^2 + h_K p_K^{-1}\sum_{E \subseteq \partial K}\|\vec{R}_E\|_{\mathrm{L}^2(E)}^2 \\
&\lesssim \Big[h_K^2 p_K^{-2}\Big(p_K^{2\beta}\|\bar{\vec{r}}_K - \vec{r}_K\|_{\mathrm{L}^2(K)}^2 + h_K^{-2}p_K^4\|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(K)}^2\Big) \\
&\quad + h_K p_K^{-1}\sum_{E \subseteq \partial K}\Big(p_K^{2\beta}\|\bar{\vec{R}}_E - \vec{R}_E\|_{\mathrm{L}^2(E)}^2 + h_K^{-1}p_K^{2(1+\beta)}\|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(E_{\mathrm{patch}})}^2 \\
&\qquad + h_K p_K^{-2(1-2\beta)}\|\bar{\vec{r}}_{E_{\mathrm{patch}}} - \vec{r}_{E_{\mathrm{patch}}}\|_{\mathrm{L}^2(E_{\mathrm{patch}})}^2\Big)\Big] \\
&\lesssim p_K^{2\beta}\Bigg(p_K\|\vec{u} - \vec{u}_N\|_{\mathrm{H}^1(K_{\mathrm{patch}})}^2 + h_K^2 p_K^{-3+2\beta}\|\bar{\vec{r}}_{K_{\mathrm{patch}}} - \vec{r}_{K_{\mathrm{patch}}}\|_{\mathrm{L}^2(K_{\mathrm{patch}})} \\
&\qquad + h_K p_K^{-1}\sum_{E \subseteq \partial K}\|\bar{\vec{R}}_E - \vec{R}_E\|_{\mathrm{L}^2(E)}^2\Bigg).
\end{aligned} \tag{A.21}
$$

Proceeding similarly for $E \subseteq \partial K \cap \Gamma_{\mathrm{C}}$ and noting that the terms below involving $\vec{\lambda}_N$ vanish whenever $\partial K \cap \Gamma_{\mathrm{C}} = \emptyset$, we obtain the result. □