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MAPPING PROPERTIES OF HELMHOLTZ BOUNDARY INTEGRAL OPERATORS AND THEIR APPLICATION TO THE HP-BEM

MAIKE LÖHNDORF* AND JENS MARKUS MELENK†

Abstract. For the Helmholtz equation (with wavenumber k) and analytic curves or surfaces Γ we analyze the mapping properties of the single layer, double layer as well combined potential boundary integral operators. A k-explicit regularity theory for the single layer and double layer potentials is developed, in which these operators are decomposed into three parts: the first part is the single or double layer potential for the Laplace equation, the second part is an operator with finite shift properties, and the third part is an operator that maps into a space of piecewise analytic functions. For all parts, the k-dependence is made explicit. We also develop a k-explicit regularity theory for the inverse of the combined potential operator $A = \pm 1/2 + K - \mathbf{i}\eta V$ and its adjoint, where V and K are the single layer and double layer operators for the Helmholtz kernel and $\eta \in \mathbb{R}$ is a coupling parameter with $|\eta| \sim |k|$. Under the assumption that $\|A^{-1}\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)}$ grows at most polynomially in k, the inverse A^{-1} is decomposed into an operator $A_1 : L^2(\Gamma) \to L^2(\Gamma)$ with bounds independent of k and a smoothing operator A_2 that maps into a space of analytic functions on Γ. The k-dependence of the mapping properties of A_2 is made explicit. We show quasi-optimality (in an $L^2(\Gamma)$-setting) of the hp-version of the Galerkin BEM applied to A or A' under the assumption of scale resolution, i.e., the polynomial degree p is at least $O(\log k)$ and kh/p is bounded by a number that is sufficiently small, but independent of k. Under this assumption, the constant in the quasi-optimality estimate is independent of k. Numerical examples in 2D illustrate the theoretical results.

1. introduction. Acoustic and electromagnetic scattering problems are often treated with boundary integral equation (BIE) methods. In a time-harmonic setting, these BIEs depend on the wavenumber k under consideration. An understanding of how the boundary integral operators (BIOs) and the solutions of the BIEs depend on k is crucial for the design and analysis of efficient numerical schemes based on such BIEs, especially in the high frequency regime. Key components of efficient numerical methods are (a) approximation properties of the ansatz spaces and (b) the stability of the method. As discussed in the recent survey article [5], notable progress has been made in the construction of highly efficient approximation spaces that are capable of capturing the oscillatory nature of the solution. The situation is less developed for the stability analysis of numerical methods based on BIEs, particularly in the high frequency regime. Partly, this is due to an insufficient understanding of the wavenumber dependence of the mapping properties of the relevant BIOs. The present paper addresses this latter issue for the specific case of two types of combined field BIEs for the Helmholtz equation, namely, those usually attributed to Burton & Miller, [11] and those commonly associated with the names of Brakhage & Werner [4], Leis [18], and Panič [26].

Our k-explicit regularity theory takes the form of an additive decomposition of the operators into several terms with different mapping properties. Section 4 provides these decompositions for the classical single and double layer potentials. These operators are decomposed into three parts: the first part is the corresponding operator for the Laplace equation and therefore k-independent; the other two terms have smoothing properties but their operator norms depend on k. Our principal decomposition results for the layer potential are for analytic geometries (see Theorems 4.3, 4.4); however, it is also possible to obtain similar results for Lipschitz boundaries, which is worked out

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in Theorems 4.1, 4.2. Section 6 is at the heart of this paper and provides the additive decompositions for the inverses of the combined field operators in Theorems 6.7, 6.8; here, we restrict our attention to analytic geometries.

At first glance the stability theory for Galerkin discretizations of combined field BIEs on smooth geometries does not seem to pose difficulties since these BIOs are compact perturbations of the identity and hence, by general functional analytic arguments, *asymptotic* quasi-optimality is ensured. However, these general arguments give no indication of how the wavenumber k enters in the estimates and, in particular, affects the onset of quasi-optimal convergence. The k-explicit regularity theory developed in Section 6 allows us be explicit at this point for the hp-version of the BEM in Corollaries 7.19, 7.22. For analytic geometries and under the assumption that the solution operator for the combined field BIE grows at most polynomially in the wavenumber k, a scale resolution condition of the form

$$\frac{kh}{p} \text{ sufficiently small} \qquad \text{and} \qquad p \geq C \log k \tag{1.1}$$

ensures quasi-optimality of the hp-BEM. We stress that, by [9], the assumption of polynomial growth of the norm of the inverse of the combined field BIO is ensured for star-shaped domains so that the present paper provides a complete k-explicit convergence theory for the case of star-shaped domains with analytic boundary. It is worth rephrasing the scale resolution condition (1.1) as follows: If the approximation order p is selected as $p = O(\log k)$, then the onset of quasi-optimality is achieved for $h = O(p/k)$, i.e., for a fixed number of degrees of freedom per wavelength. The numerical results of Section 8 illustrate that indeed a scale resolution condition of the form (1.1) ensures quasi-optimality of the hp-BEM. The side condition $p = O(\log k)$ in (1.1) may be viewed as expressing the possibility of "pollution". However, our numerical experiments show that the weaker condition "kh/p small" alone is often sufficient for quasi-optimality of the hp-BEM. Put differently: in contrast to the finite element method, the BEM appear not to be very susceptible to "pollution".

To the knowledge of the authors, the only other k-explicit stability analysis for discretizations of combined field BIOs is provided in [13], where the special cases of circular or spherical geometries are studied; in that setting the double layer and single layer operators can be diagonalized simultaneously by Fourier techniques, which allows [13] to show that the combined field BIOs are even L^2-elliptic.

The result of the present paper have counterparts in the context of differential equations and finite elements. Decomposition results analogous to those of the present paper have recently been obtained in [24, 25] for several Helmholtz boundary value problems. A k-explicit convergence theory for the hp-version of the finite element method has also been developed in [24, 25] using similar techniques; also there, the key scale resolution condition on the mesh size h and the approximation order p takes the form (1.1).

The paper is organized as follows: the remainder of this first section introduces general notation and various boundary integral operators. Section 2 collects mapping properties of the classical single layer and double layer potential operators on Lipschitz domains. In particular, the limiting cases studied in Lemmata 2.1, 2.2 appear to be new. Section 3 studies the mapping properties of the Newton potential for the Helmholtz equation. Section 4 provides decomposition results for the Helmholtz single layer and double layer potential operators both for Lipschitz domains and domains with analytic boundaries. Section 5 applies the results of Section 4 to the combined field operators. Section 6 is a key section of the paper in that it provides decomposition results for the inverses of the combined field operators. Section 7 shows how the regularity theory of Sections 5 and 6 permits a k-explicit stability and convergence analysis of the hp-BEM. In Section 8 finally, we present numerical results for the hp-BEM in 2D.

1.1. notation and general assumptions.

1.1.1. general notation. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded Lipschitz domain with a connected boundary. We set $\Gamma := \partial\Omega$ and $\Omega^+ := \mathbb{R}^d \setminus \Omega$. Throughout the paper, we assume that the open ball $B_R := B_R(0)$ of radius R around the origin contains $\overline{\Omega}$, i.e., $\overline{\Omega} \subset B_R$. We set $\Omega_R := (\Omega \cup \Omega^+) \cap B_R = B_R \setminus \Gamma$. We will denote by γ_0^{int} and γ_0^{ext} the interior and exterior trace operator on Γ. The interior and exterior co-normal derivative operators are denoted by γ_1^{int}, γ_1^{ext}, i.e., for sufficiently smooth functions u, we set $\gamma_1^{int} u := \gamma_0^{int} \nabla u \cdot \vec{n}$ and $\gamma_1^{ext} u := \gamma_0^{ext} \nabla u \cdot \vec{n}$, where, in both cases $\vec{n}$ is the unit normal vector point out of Ω. As is standard, we introduce the jump operators

$$[u] = \gamma_0^{ext} u - \gamma_0^{int} u, \qquad [\partial_n u] = \gamma_1^{ext} u - \gamma_1^{int} u.$$

For linear operators $\widetilde{\mathcal{A}}$ that map into spaces of piecewise defined functions, we define the operators $[\widetilde{\mathcal{A}}]$ and $[\partial_n \widetilde{\mathcal{A}}]$ in an analogous way, e.g., $[\widetilde{\mathcal{A}}]\varphi = [\widetilde{\mathcal{A}}\varphi]$. Sobolev spaces H^s are defined in the standard way, [1, 30]. We stress, however, that if an open set $\omega \subset \mathbb{R}^d$ consists of $m \in \mathbb{N}$ components of connectedness ω_i, $i = 1, \dots, m$, then the space $H^s(\omega)$ can be identified with the product space $\prod_{i=1}^m H^s(\omega_i)$ equipped with the norm $\left(\sum_{i=1}^m \|u\|^2_{H^s(\omega_i)}\right)^{1/2}$. For a domain $\omega \subset \mathbb{R}^d$, we will also employ the Besov spaces $B^s_{1/2,\infty}(\omega)$, which are defined in the standard way by the real method of interpolation (see, e.g., [3,30,31]). Sets of analytic functions will play a very important role in our theory. We therefore introduce the following definition.

Definition 1.1. *For an open set T and constant C_f, $\gamma_f > 0$ we set*

$$\mathfrak{A}(C_f, \gamma_f, T) := \{f \in L^2(T) \mid \|\nabla^n f\|_{L^2(T)} \le C_f \gamma_f^n \max\{n+1, k\}^n \qquad \forall n \in \mathbb{N}_0\}.$$

Here, $|\nabla^n u(x)|^2 = \displaystyle\sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = n} \frac{n!}{\alpha!} |D^\alpha u(x)|^2.$

For domains $\omega \subset \mathbb{R}^d$, it is convenient to introduce the k-dependent norm $\|u\|_{\mathcal{H},\omega}$ by

$$\|u\|^2_{\mathcal{H},\omega} := \|u\|^2_{L^2(\omega)} + k^2 \|\nabla u\|^2_{L^2(\omega)}.$$

Tubular neighborhoods T of Γ are open sets of such that $T \supset \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, \Gamma) < \varepsilon\}$ for some $\varepsilon > 0$.

Throughout the paper, we will use the following conventions:

Convention 1.2.

(i) We assume $|k| \ge k_0 > 0$ for some fixed $k_0 > 0$.

(ii) If the wavenumber k appears outside the boundary integral operators and potentials such as V_k and $\widetilde{V}_k$ and the expressions of (1.7), then it is just a short-hand for $|k|$. In particular, k stands for $|k|$ in estimates. For example, $k \ge k_0$ means $|k| \ge k_0$.

1.1.2. layer potentials. In recent years, boundary element methods (BEM) and BIOs have been made accessible to a wider audience through several monographs, e.g., [14,20,27,29]. We refer to these books for more information about the operators studied here.
We denote by V, K, K' the usual single layer, double layer, and adjoint double layer operators for the Helmholtz equation. The single layer and double layer potentials are denoted by $\widetilde{V}$ and $\widetilde{K}$. More specifically, we define the Helmholtz kernel G_k by

$$G_k(x,y) := \begin{cases} \frac{\mathbf{i}}{4} H_0^{(1)}(k|x-y|), & d=2, \\ \frac{e^{\mathbf{i}k|x-y|}}{4\pi|x-y|}, & d=3, \end{cases} \quad \text{for } k>0,$$
$$G_k := \overline{G_{-k}} \quad \text{for } k<0,$$

where $H_0^{(1)}$ is the first kind Hankel function of order zero. The limiting case $k=0$ corresponds to the Laplace operator and is defined as $G_0(x,y) = -1/(2\pi)\ln|x-y|$ for the case $d=2$ and $G_0(x,y) = 1/(4\pi|x-y|)$ for the case $d=3$. The potential operators $\widetilde{V}$ and $\widetilde{K}$ are defined by

$$(\widetilde{V}\varphi)(x) := \int_\Gamma G_k(x,y)\varphi(y)\,ds_y, \qquad (\widetilde{K}\varphi)(x) := \int_\Gamma \partial_{n_y} G_k(x,y)\varphi(y)\,ds_y, \quad x \in \mathbb{R}^d\backslash\Gamma.$$

From these potentials, the single layer, double layer, and adjoint double layer operators are defined as follows:

$$V := \gamma_0^{int}\widetilde{V}, \qquad K := \frac{1}{2}\left(\gamma_0^{int}\widetilde{K} + \gamma_0^{ext}\widetilde{K}\right), \qquad K' := \gamma_1^{int}\widetilde{V} - \frac{1}{2}\operatorname{Id}. \tag{1.2}$$

If need be, we will write V_k, K_k, K'_k to clarify the k-dependence. We mention in passing that for $k \neq 0$, the potentials $\widetilde{V}_k$ and $\widetilde{K}_k$ are solutions of the homogeneous Helmholtz equation on $\mathbb{R}^d\backslash\Gamma$; for $k>0$ they satisfy the *outgoing* Sommerfeld radiation condition while for $k<0$, they satisfy the *incoming* radiation condition.
We finally turn to the definition of adjoint operators. We have for all $k \in \mathbb{R}$ for the $L^2(\Gamma)$ scalar product and all φ, $\psi \in H^{1/2}(\Gamma)$:

$$(V_k\varphi,\psi)_{L^2(\Gamma)} = (\varphi, V_{-k}\psi)_{L^2(\Gamma)}, \tag{1.3a}$$
$$(K_k\varphi,\psi)_{L^2(\Gamma)} = (\varphi, K'_{-k}\psi)_{L^2(\Gamma)}, \tag{1.3b}$$

i.e., the adjoints of V_k and K_k are V_{-k} and K'_{-k}, respectively. It is worth pointing out that we have the connections $\widetilde{V}_{-k}\overline{\varphi} = \overline{\widetilde{V}_k\varphi}$ and $\widetilde{K}_{-k}\overline{\varphi} = \overline{\widetilde{K}_k\varphi}$.

1.1.3. combined field operators. For a coupling parameter $\eta \in \mathbb{R}\setminus\{0\}$ we consider four combined field operators. The operator A has one of the following two forms:

$$A = A_k = -\frac{1}{2} + K - \mathbf{i}\eta V \tag{1.4a}$$
$$A = A_k = \frac{1}{2} + K - \mathbf{i}\eta V. \tag{1.4b}$$

The operator A' has one of the following two forms:

$$A' = A'_k = -\frac{1}{2} + K' + \mathbf{i}\eta V, \tag{1.5a}$$
$$A' = A'_k = \frac{1}{2} + K' + \mathbf{i}\eta V. \tag{1.5b}$$

We use the same notation for the operators in (1.4a), (1.4b) and (1.5a), (1.5b) since most of our results will be valid for both cases.
In order to avoid keeping track of the precise dependence of various constants on η, we assume throughout this paper that

$$|\eta| \sim |k| \tag{1.6}$$

On smooth surfaces, it is well-known, [7, 11], that the operators A and A' of the form given in (1.4b), (1.5b), are invertible as operators acting on $L^2(\Gamma)$. In fact, the operator of (1.4b) is invertible on $H^s(\Gamma)$ for $s \geq 0$ and the operator of (1.5b) is invertible on $H^s(\Gamma)$ for $s \geq -1/2$, [8, 9]. We abbreviate (omitting the implicit dependence on η)

$$C(A_k, s, k) := \|A_k^{-1}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)}, \qquad C(A'_k, s, k) := \|(A'_k)^{-1}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)}. \tag{1.7}$$

We will see that in the context of high order Galerkin BEM, a case of particular interest is the one where $C(A, s, k)$ grows only polynomially in k. In view of the following lemma, a polynomial growth of $C(A, s, k)$ can reasonably be expected:
LEMMA 1.3 ([9]). *Let the Lipschitz domain Ω be star-shaped with respect to the origin. Then there exists a constant $C > 0$ independent of k such that for the operators A, A' given in (1.4b), (1.5b), there holds*

$$C(A_k, 0, k) = C(A'_{-k}, 0, -k) \leq C.$$

For ease of future reference, we introduce the following two assumptions.
ASSUMPTION 1.4. *The operator $A : H^{s_A}(\Gamma) \to H^{s_A}(\Gamma)$ is boundedly invertible with* $C(A_k, s_A, k) = \|A_k^{-1}\|_{H^{s_A}(\Gamma)\leftarrow H^{s_A}(\Gamma)}$.
ASSUMPTION 1.5. *The operator $A' : H^{s_A}(\Gamma) \to H^{s_A}(\Gamma)$ is boundedly invertible with* $C(A'_k, s_A, k) = \|(A'_k)^{-1}\|_{H^{s_A}(\Gamma)\leftarrow H^{s_A}(\Gamma)}$.

2. properties of the Laplace single and double layer potentials.

In this section, we collect some mapping properties of the potential operators $\widetilde{V}_0$ and $\widetilde{K}_0$ for the Laplace equation.

2.1. Lipschitz domains.

For Lipschitz domains Ω and $-1 \leq s \leq 1$ one can define the Sobolev spaces $H^s(\Gamma)$ intrinsically. It is then known (see also Lemmata 2.1, 2.2 below) that for $|s| \leq 1/2$ the operators

$$\widetilde{V}_0 : H^{-1/2+s}(\Gamma) \to H^1(B_R) \cap H^{1+s}(\Omega_R) \tag{2.1a}$$
$$\widetilde{K}_0 : H^{1/2+s}(\Gamma) \to H^{1+s}(\Omega_R) \tag{2.1b}$$

are bounded linear operators (relevant literature includes [12,15,16,32]; see also Lemmata 2.1, 2.2 below). The following Lemma 2.1 clarifies into what space of functions defined on the ball B_R (as opposed to Ω_R) the potential operator $\widetilde{V}_0$ maps elements in the limiting cases $s = \pm 1/2$:
LEMMA 2.1 (mapping properties of $\widetilde{V}_0$). *For $-1/2 < s < 1/2$ we have that $\widetilde{V}_0 : H^{-1/2+s}(\Gamma) \to H^{1+s}(B_R)$ is a bounded linear operator. The limiting cases $s = \pm 1/2$ take the forms $\widetilde{V}_0 : H^{-1}(\Gamma) \to B^{1/2}_{2,\infty}(B_R)$ and $\widetilde{V}_0 : L^2(\Gamma) \to B^{3/2}_{2,\infty}(B_R)$.*
Proof. The result for $-1/2 < s < 1/2$ are known in the literature (see, e.g., [20]). The proofs of the limiting cases $s = \pm 1/2$ are relegated to Appendix A. ◻
The potential operator $\widetilde{K}_0$ produces functions that jump across Γ. This implies that, viewed as a function on the ball B_R, one cannot hope for more regularity than $\widetilde{K}_0\varphi \in B^{1/2}_{2,\infty}(B_R)$; this is indeed the case for the limiting case $s = -1/2$:

LEMMA 2.2 (mapping properties of $\widetilde{K}_0$). *For $-1/2 \le s \le 1/2$ we have $\widetilde{K}_0 : H^{1/2+s}(\Gamma) \to H^{1+s}(\Omega_R)$. For the limiting case $s = -1/2$ we have the additional result $\widetilde{K}_0 : L^2(\Gamma) \to B^{1/2}_{2,\infty}(B_R)$.*

Proof. See Appendix A. ◻

2.2. smooth domains. The mapping properties given in (2.1) are restricted to the range $|s| \le 1/2$ for Lipschitz domains. For smooth domains, the range can be extended, for example, to include all $s \ge -1$. To that end, we note

LEMMA 2.3. *Let Γ be of class C^∞. Then there exists $C > 0$ depending only on Ω and R such that for $\varphi \in H^{1/2}(\Gamma)$ there holds*

$$\|\widetilde{V}_0\varphi\|_{L^2(\Omega_R)} \le C\|\varphi\|_{H^{-3/2}(\Gamma)},$$
$$\|\widetilde{K}_0\varphi\|_{L^2(\Omega_R)} \le C\|\varphi\|_{H^{-1/2}(\Gamma)}.$$

Proof. Set $u := \widetilde{V}_0\varphi$. We only aim at estimating $\|u\|_{L^2(\Omega)}$ since $\|u\|_{L^2(\Omega_R\setminus\Omega)}$ is estimated similarly. To that end, let $w \in H^2(\Omega)$ solve

$$-\Delta w = u \quad \text{in } \Omega, \qquad \partial_n w = 0 \quad \text{on } \Gamma.$$

Then $w \in H^2(\Omega)$ together with $\|w\|_{H^2(\Omega)} \lesssim \|u\|_{L^2(\Omega)}$ and therefore

$$\|u\|^2_{L^2(\Omega)} = \left|\int_\Gamma \gamma_1^{int} u w\right| \lesssim \|\gamma_1^{int} u\|_{H^{-3/2}(\Gamma)}\|w\|_{H^2(\Omega)} \lesssim \|\gamma_1^{int} u\|_{H^{-3/2}(\Gamma)}\|u\|_{L^2(\Omega)}.$$

Next, we use the representation

$$\gamma_1^{int} u = \gamma_1^{int}\widetilde{V}_0\varphi = (\frac{1}{2} + K_0')\varphi$$

and [20, Thm. 7.2] to bound $\|\gamma_1^{int} u\|_{H^{-3/2}(\Gamma)} \le C\|\varphi\|_{H^{-3/2}(\Gamma)}$.

We proceed in a similar manner to bound $\|\widetilde{K}_0\varphi\|_{L^2(\Omega)}$. Let $u = (\widetilde{K}_0\varphi)|_\Omega$ and let $w \in H^2(\Omega) \cap H^1_0(\Omega)$ solve

$$-\Delta w = u \quad \text{in } \Omega, \qquad w|_\Gamma = 0.$$

Then $\|w\|_{H^2(\Omega)} \lesssim \|u\|_{L^2(\Omega)}$ and therefore

$$\|u\|^2_{L^2(\Omega)} = \left|\int_\Gamma \gamma_1^{int} w \gamma_0 u\right| \lesssim \|u\|_{H^{-1/2}(\Gamma)}\|\gamma_1^{int} w\|_{H^{1/2}(\Gamma)} \lesssim \|u\|_{H^{-1/2}(\Gamma)}\|u\|_{L^2(\Gamma)}.$$

From the representation $\gamma_0^{int} u = (-\frac{1}{2} + K_0)\varphi$ and the mapping properties of K_0 on smooth domains, [20, Thm. 7.2], we get again $\|u\|_{H^{-1/2}(\Gamma)} \le C\|\varphi\|_{H^{-1/2}(\Gamma)}$. ◻

Lemma 2.3 allows us to extend the operators $\widetilde{V}_0$ and $\widetilde{K}_0$ to operators defined on $H^{-3/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ respectively. We thus have

LEMMA 2.4. *Let Γ be of class C^∞. Then the operators $\widetilde{V}_0$ and $\widetilde{K}_0$ are bounded linear operators*

$$\widetilde{V}_0 : H^{-1/2+s}(\Gamma) \to H^{1+s}(\Omega_R), \qquad \widetilde{K}_0 : H^{1/2+s}(\Gamma) \to H^{1+s}(\Omega_R)$$

for every $s \ge -1$ and every $R > 0$ such that $\overline{\Omega} \subset B_R$.

Proof. The case $s > -1/2$ is shown in [20, Cor. 6.14]. The case $s = -1$ follows from Lemma 2.3. An interpolation argument then provided the intermediate range $-1 \le s \le -1/2$. ◻

2.3. invertibility properties. For future reference, we recall the following results:

LEMMA 2.5. *Let Γ be smooth and $\alpha \in \mathbb{R} \setminus \{0\}$ be fixed. If $d = 2$, assume additionally that* $\operatorname{diam}\Omega < 1$. *Then:*

(i) $-\frac{1}{2} + K_0 : H^s(\Gamma) \to H^s(\Gamma)$ *is boundedly invertible for* $s \geq 0$.

(ii) $\frac{1}{2} + K_0 + \mathbf{i}\alpha V_0 : H^s(\Gamma) \to H^s(\Gamma)$ *is boundedly invertible for* $s \geq 0$.

(iii) $-\frac{1}{2} + K_0' : H^s(\Gamma) \to H^s(\Gamma)$ *is boundedly invertible for* $s \geq -1/2$.

(iv) $\frac{1}{2} + K_0' + \mathbf{i}\alpha V_0 : H^s(\Gamma) \to H^s(\Gamma)$ *is boundedly invertible for* $s \geq -1/2$.

Proof. See Appendix D. ∎

3. Properties of the Helmholtz Newton potential. A key ingredient of our decomposition of the operators $\widetilde{V}$, $\widetilde{K}$, and A, A' are low pass and high pass filters that we introduce now:

LEMMA 3.1 (full space frequency splitting). *Let $q \in (0,1)$. Then one can construct linear operators $H_{\mathbb{R}^d}$ and $L_{\mathbb{R}^d}$ defined on $L^2(\mathbb{R}^d)$ with the following properties:*

(i) $H_{\mathbb{R}^d} + L_{\mathbb{R}^d} = \operatorname{Id}$

(ii) $\|H_{\mathbb{R}^d} f\|_{H^{s'}(\mathbb{R}^d)} \leq C_{s,s'}(qk^{-1})^{s-s'}\|f\|_{H^s(\mathbb{R}^d)}$ *for all* $0 \leq s' \leq s$ *and* $f \in H^s(\mathbb{R}^d)$

(iii) $L_{\mathbb{R}^d} f$ *is entire and*

$$\|\nabla^n L_{\mathbb{R}^d} f\|_{L^2(\mathbb{R}^d)} \leq C(\gamma k)^n \|f\|_{L^2(\mathbb{R}^d)} \qquad \forall n \in \mathbb{N}_0.$$

Here, the constants C, γ depend on the choice of q and s but are independent of $k \geq k_0$.

Proof. See [24, Lemmata 4.2, 4.3] for details. A sketch of the construction is as follows: The operators $H_{\mathbb{R}^d}$ and $L_{\mathbb{R}^d}$ are defined in terms the Fourier transformation $\mathcal{F} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^d)$ by $\mathcal{F}(H_{\mathbb{R}^d}(f)) := \chi_{\mathbb{R}^d \setminus B_{k\eta}(0)}\mathcal{F}(f)$ and $\mathcal{F}(L_{\mathbb{R}^d}(f)) := \chi_{B_{k\eta}(0)}\mathcal{F}(f)$. Here, $\eta > 1$ is a parameter that is selected depending on the chosen $q \in (0,1)$ and χ_E denotes the characteristic function of the set $E \subset \mathbb{R}^d$. ∎

The Newton potential $N_k(f)$ of $f \in L^2(\mathbb{R}^d)$ with compact support is defined by

$$N_k(f) := G_k \star f. \tag{3.1}$$

It is the solution of the inhomogeneous Helmholtz equation with right-hand side f and satisfies the outgoing radiation condition if $k > 0$ and the incoming radiation condition if $k < 0$. For N_k we have the following decomposition result:

LEMMA 3.2 (mapping properties of N_k). *For every $f \in L^2(\mathbb{R}^d)$ there holds*

$$\|N_k(f)\|_{\mathcal{H},B_R} + k^{-1}\|N_k(f)\|_{H^2(B_R)} \leq C_R\|f\|_{L^2(\mathbb{R}^d)}. \tag{3.2}$$

Additionally, the following decomposition result holds: Let $q \in (0,1)$ be arbitrary. Then the high frequency operator $H_{\mathbb{R}^d}$ and the low frequency operator $L_{\mathbb{R}^d}$ can be chosen such that for $s \geq 0$ and $0 \leq s' \leq s+2$ the function $N_k(H_{\mathbb{R}^d} f)$ satisfies

$$\|N_k(H_{\mathbb{R}^d} f)\|_{H^{s'}(B_R)} \leq C_{s,s'}(qk^{-1})^{2+s-s'}\|f\|_{H^s(\mathbb{R}^d)}. \tag{3.3}$$

The constant $C_{s,s'}$ is independent of $q \in (0,1)$ and $k \geq k_0$. The function $N_k(L_{\mathbb{R}^d} f)$ is entire and satisfies

$$\|\nabla^n N_k(L_{\mathbb{R}^d} f)\|_{L^2(B_R)} \leq C(\gamma k)^{n-1}\|f\|_{L^2(\mathbb{R}^d)} \qquad \forall n \in \mathbb{N}_0. \tag{3.4}$$

Here, the constants C, γ are independent of $k \geq k_0$ but depend on q.

Proof. The estimate (3.2) is shown in [25, Lemma 3.5]. Inspection of the procedure in [25, Lemma 3.5] reveals that the function $v_{\mathcal{A}}$ in [25, Lemma 3.5] coincides with $N_k(L_{\mathbb{R}^d} f)$, which shows (3.4). Finally, [25, Lemma 3.5] shows (3.3) for the case $s = 0$. Inspection of the proof shows that it can be extended in a straight forwards way to the case $s > 0$. ∎

An interpolation argument allows us to infer the following result:

COROLLARY 3.3. *Let $s \geq 0$ and $s \notin \mathbb{N}_0$. Fix a cut-off function χ with* $\operatorname{supp}\chi \subset B_{2R}$. *Then for all $f \in B^s_{2,\infty}(B_{2R})$*

$$\|N_k(H_{\mathbb{R}^d}(\chi f))\|_{H^{s'}(B_R)} \leq C_{s,s'}(qk^{-1})^{2+s-s'}\|f\|_{B^s_{2,\infty}(B_{2R})}, \quad 0 \leq s' < 2+s, \tag{3.5}$$

$$\|N_k(H_{\mathbb{R}^d}(\chi f))\|_{B^{2+s}_{2,\infty}(B_R)} \leq C_s\|f\|_{B^s_{2,\infty}(B_{2R})}. \tag{3.6}$$

Proof. The operator $f \mapsto N_k(H_{\mathbb{R}^d}\chi f)$ is linear and, for every $t \geq 0$, we have by Lemma 3.2

$$\|N_k(H_{\mathbb{R}^d}\chi f)\|_{H^{t+2}(B_R)} \leq C_t\|f\|_{H^t(B_{2R})}, \tag{3.7}$$

$$\|N_k(H_{\mathbb{R}^d}\chi f)\|_{L^2(B_R)} \leq C_t(qk^{-1})^{2+t}\|f\|_{H^t(B_{2R})}, \tag{3.8}$$

for a constant $C_t > 0$ that depends solely on t, R, and χ. Since the spaces $B^s_{2,\infty}$ are defined as interpolation spaces between standard Sobolev spaces, the estimates (3.7) imply (3.6). Since $(L^2(B_R), L^2(B_R))_{\theta,\infty} = L^2(B_R)$ for every $\theta \in (0,1)$, the estimate (3.5) for the special case $s' = 0$ follows also from an interpolation argument and (3.8). Finally, the general case in (3.5) follows from the interpolation inequality $\|z\|_{H^{\theta(s+2)}} \leq C\|z\|^{1-\theta}_{L^2}\|z\|^{\theta}_{B^{s+2}_{2,\infty}}$ for $s+2 > 0$ and $\theta \in (0,1)$. ∎

4. decomposition of layer potentials.

The present section focuses on the mapping properties of the layer potentials $\widetilde{V}$ and $\widetilde{K}$ with particular emphasis on making the k-dependence explicit. We do this through an additive decomposition of $\widetilde{V}$ and $\widetilde{K}$ into a leading order part that corresponds to the Laplace operator (i.e., $\widetilde{V}_0$ and $\widetilde{K}_0$) and regularizing parts.

We present two different types of decompositions: the first type is done for Lipschitz domains and formulated in Subsection 4.1. Since the regularizing parts are defined as solutions of transmission problems, the limited regularity of Lipschitz domains imposes restrictions on the Sobolev range for which the decomposition can be done in a meaningful way. We therefore consider in Section 4.2 the case of domains with analytic boundary, where, by a modification of the procedure of Section 4.1, decompositions are obtained that are valid for large ranges of Sobolev spaces.

4.1. decomposition of layer potentials: Lipschitz domains.

4.1.1. decomposition of the single layer potential.

THEOREM 4.1 (decomposition of $\widetilde{V}$, Lipschitz domain). *Let $q \in (0,1)$ be given. Then one can write*

$$\widetilde{V} = \widetilde{V}_0 + \widetilde{S}_V + \widetilde{\mathcal{A}}_V,$$

where for every $-1/2 < s < 1/2$ the linear operators $\widetilde{S}_V : H^{-1/2+s}(\Gamma) \to H^{3+s}(B_R)$ and $\widetilde{\mathcal{A}}_V : H^{-1/2+s}(\Gamma) \to H^{3+s}(B_R)$ satisfy the following bounds:

$$\|\widetilde{S}_V\varphi\|_{H^{s'}(B_R)} \leq C_{s,s'}q^2(qk^{-1})^{1+s-s'}\|\varphi\|_{H^{-1/2+s}(\Gamma)}, \qquad 0 \leq s' \leq 3+s,$$

$$\|\nabla^n\widetilde{\mathcal{A}}_V\varphi\|_{L^2(B_R)} \leq C(\gamma k)^{n+1}\|\widetilde{V}_0\varphi\|_{L^2(B_R)} \leq C(\gamma k)^{n+1}\|\varphi\|_{H^{-1}(\Gamma)} \qquad \forall n \in \mathbb{N}_0.$$

Here, the constant $C_{s,s'}$ is independent of q and $k \geq k_0$. The constants C, γ are independent of $k \geq k_0$ but depend on q.
For $s = \pm 1/2$ we have that $\widetilde{S}_V : H^{-1/2+s}(\Gamma) \to B^{3+s}_{2,\infty}(B_R)$ and $\widetilde{\mathcal{A}}_V : H^{-1/2+s}(\Gamma) \to B^{3+s}_{2,\infty}(B_R)$ satisfy the following bounds:

$$\|\widetilde{S}_V\varphi\|_{H^{s'}(B_R)} \leq C_{s,s'}q^2(qk^{-1})^{1+s-s'}\|\varphi\|_{H^{-1/2+s}(\Gamma)}, \qquad 0 \leq s' < 3+s,$$
$$\|\widetilde{S}_V\varphi\|_{B^{s+3}_{2,\infty}(B_R)} \leq C_s q^2(qk^{-1})^{-2}\|\varphi\|_{H^{-1/2+s}(\Gamma)},$$
$$\|\nabla^n\widetilde{\mathcal{A}}_V\varphi\|_{L^2(B_R)} \leq C(\gamma k)^{n+1}\|\widetilde{V}_0\varphi\|_{L^2(B_R)} \leq C(\gamma k)^{n+1}\|\varphi\|_{H^{-1}(\Gamma)} \qquad \forall n \in \mathbb{N}_0.$$

Proof. We will exploit density of $H^{1/2}(\Gamma)$ in $H^{-1/2+s}(\Gamma)$ for $-1/2 \leq s \leq 1/2$. Let therefore $\varphi \in H^{1/2}(\Gamma)$ be given. Set $u := \widetilde{V}\varphi$ and $u_0 := \widetilde{V}_0\varphi$. Let χ be a smooth cut-off function with $\operatorname{supp}\chi \subset B_{2R}$ and $\chi|_{B_R} \equiv 1$. Then the function $\tilde{u} := u - \chi u_0$ satisfies

$$\begin{aligned}
-\Delta\tilde{u} - k^2\tilde{u} = f := -(\Delta\chi)u_0 - 2\nabla\chi\cdot\nabla u_0 + k^2\chi u_0 \qquad &\text{in } \Omega\cup\Omega^+,\\
[\tilde{u}] = 0 \qquad &\text{on } \Gamma \text{ (in } H^{1/2}(\Gamma)),\\
[\partial_n\tilde{u}] = 0 \qquad &\text{on } \Gamma \text{ (in } H^{-1/2}(\Gamma)),\\
\tilde{u} \qquad &\text{satisfies a radiation condition at } \infty,
\end{aligned}$$

and f has compact support. The mapping properties of $\widetilde{V}_0$ on Lipschitz domains of Lemma 2.1 imply for $-1/2 \leq s \leq 1/2$:

$$\|u_0\|_{H^{1+s}(B_R)} \leq C\|\varphi\|_{H^{-1/2+s}(\Gamma)}, \qquad -1/2 < s < 1/2, \tag{4.1}$$
$$\|u_0\|_{B^{1+s}_{2,\infty}(B_R)} \leq C\|\varphi\|_{H^{-1/2+s}(\Gamma)}, \qquad s = \pm 1/2. \tag{4.2}$$

We have therefore an explicit solution formula for $\tilde{u}$, namely,

$$\tilde{u} = N_k(f)$$

Hence, we have the representation

$$u = \chi u_0 + N_k(f) = \chi u_0 + N_k(H_{\mathbb{R}^d}f) + N_k(L_{\mathbb{R}^d}f) =: \chi u_0 + \widetilde{S}_V\varphi + \widetilde{\mathcal{A}}_V\varphi,$$

where the parameter q in the definition of $H_{\mathbb{R}^d}$ is still at our disposal.
We first consider the regularity of $\widetilde{S}_V$. In view of Lemma 3.2 and Corollary 3.3 we have to analyze the regularity properties of f. By interior regularity, we have that u_0 is analytic away from Γ, and we get for $s = \pm 1/2$:

$$\|f\|_{B^{1+s}_{2,\infty}(B_{2R})} \leq Ck^2\|\varphi\|_{H^{-1/2+s}(\Gamma)}.$$

Next, the support properties of f imply that $f = \chi' f$ for some smooth cut-off function χ'. Hence, Corollary 3.3 implies for $s = \pm 1/2$

$$\|N_k(H_{\mathbb{R}^d}f)\|_{L^2(B_R)} \leq C(qk^{-1})^{3+s}k^2\|\varphi\|_{H^{-1/2+s}(\Gamma)}, \tag{4.3}$$
$$\|N_k(H_{\mathbb{R}^d}f)\|_{B^{3+s}_{2,\infty}(B_R)} \leq Ck^2\|\varphi\|_{H^{-1/2+s}(\Gamma)}. \tag{4.4}$$

Interpolation then allows us to conclude for $-1/2 < s < 1/2$

$$\|N_k(H_{\mathbb{R}^d}f)\|_{L^2(B_R)} \leq C(qk^{-1})^{3+s}k^2\|\varphi\|_{H^{-1/2+s}(\Gamma)}, \tag{4.5}$$
$$\|N_k(H_{\mathbb{R}^d}f)\|_{H^{3+s}(B_R)} \leq Ck^2\|\varphi\|_{H^{-1/2+s}(\Gamma)}. \tag{4.6}$$

We have thus shown all the estimates for $\widetilde{\mathcal{S}}_V$ for the cases $s'=0$ and $s'=3+s$. For the remaining intermediate estimates, we simply use another interpolation argument. Specifically, for the case $-1/2 < s < 1/2$ we use the multiplicative interpolation inequality with $\theta = s'/(3+s)$ to get

$$\begin{aligned}\|N_k(H_{\mathbb{R}^d} f)\|_{H^{s'}(B_R)} &\le C\|N_k(H_{\mathbb{R}^d} f)\|^{1-\theta}_{L^2(B_R)} \|N_k(H_{\mathbb{R}^d} f)\|^{\theta}_{H^{3+s}(B_R)} \\ &\le Ck^2(qk^{-1})^{3+s-s'}\|\varphi\|_{H^{-1/2+s}(\Gamma)}.\end{aligned}$$

Let us now turn to the $N_k(L_{\mathbb{R}^d} f)$. From Lemma 3.2 we get

$$\begin{aligned}\|\nabla^n N_k(L_{\mathbb{R}^d} f)\|_{L^2(B_R)} &\le C(\gamma k)^{n-1}\|f\|_{L^2(B_{2R})} \le C(\gamma k)^{n-1}k^2\|u_0\|_{L^2(B_{2R})} \\ &\le C(\gamma k)^{n+1}\|\widetilde{V}_0\varphi\|_{L^2(B_{2R})}.\end{aligned}$$

Density of $H^{1/2}(\Gamma)$ in $H^{-1/2+s}(\Gamma)$ concludes the argument. ∎

4.1.2. decomposition of the double layer potential. The method of proof of Theorem 4.1 is applicable to the double layer potential as well for the end point case $s=-1/2$:

THEOREM 4.2 (decomposition of $\widetilde{K}$, Lipschitz domain). *Let $\Omega \subset B_R$ be a Lipschitz domain and let $q \in (0,1)$ be given. Then*

$$\widetilde{K} = \widetilde{K}_0 + \widetilde{S}_K + \widetilde{\mathcal{A}}_K,$$

where $\widetilde{S}_K : L^2(\Gamma) \to B^{5/2}_{2,\infty}(B_R)$ satisfies

$$\begin{aligned}\|\widetilde{S}_K\varphi\|_{B^{5/2}_{2,\infty}(B_R)} &\le Ck^2\|\varphi\|_{L^2(\Gamma)}, \\ \|\widetilde{S}_K\varphi\|_{L^2(B_R)} &\le Cq^2(qk^{-1})^{1/2}\|\varphi\|_{L^2(\Gamma)}.\end{aligned}$$

Here, the constant C is independent of q and $k \ge k_0$. The linear operator $\widetilde{\mathcal{A}}_K : L^2(\Gamma) \to B^{5/2}_{2,\infty}(B_R)$ maps into a space of analytic functions, viz.,

$$\|\nabla^n\widetilde{\mathcal{A}}_K\varphi\|_{L^2(B_R)} \le C(\gamma k)^{n+1}\|\widetilde{K}_0\varphi\|_{L^2(B_R)} \le C(\gamma k)^{n+1}\|\varphi\|_{L^2(\Gamma)} \qquad \forall n \in \mathbb{N}_0.$$

Here, the constants C, $\gamma > 0$ are independent of $k \ge k_0$ but may depend on q.

Proof. We proceed as in the proof of Theorem 4.1. This implies the form

$$\widetilde{K} = \widetilde{K}_0 + \widetilde{S}_K + \widetilde{\mathcal{A}}_K;$$

here, $\widetilde{S}_K$ and $\widetilde{\mathcal{A}}_K$ are defined by

$$\widetilde{S}_K\varphi + \widetilde{\mathcal{A}}_K\varphi := N_k(H_{\mathbb{R}^d} f) + N_k(L_{\mathbb{R}^d} f),$$

where, for $u_0 = \widetilde{K}_0\varphi$, the function f is given by

$$f = -\Delta\chi u_0 + 2\nabla\chi\cdot\nabla u_0 + k^2\chi u_0$$

The mapping properties of $\widetilde{K}_0$ detailed in Lemma 2.2 imply $\widetilde{K}_0\varphi \in B^{1/2}_{2,\infty}(B_{2R})$. Proceeding as in the proof of Theorem 4.1 we arrive at

$$\begin{aligned}\|N_k(H_{\mathbb{R}^d} f)\|_{L^2(B_R)} &\le C(qk^{-1})^{2+1/2}\|f\|_{B^{1/2}_{2,\infty}(B_{2R})} \\ &\le Cq^2k^{-2}(qk^{-1})^{1/2}k^2\|\widetilde{K}_0\varphi\|_{B^{1/2}_{2,\infty}(B_{2R})} \le Cq^2(qk^{-1})^{1/2}\|\varphi\|_{L^2(\Gamma)}, \\ \|N_k(H_{\mathbb{R}^d} f)\|_{B^{5/2}_{2,\infty}(B_R)} &\le C\|f\|_{B^{1/2}_{2,\infty}(B_{2R})} \le Ck^2\|\varphi\|_{L^2(\Gamma)}.\end{aligned}$$

The estimates for $\widetilde{A}_K\varphi$ are obtained in exactly the same way as in Theorem 4.1. ∎

4.2. decomposition of layer potentials: analytic boundaries. The method of proof in Theorems 4.1 and 4.2 relies on (Sobolev) regularity of $\widetilde{V}_0\varphi$ or $\widetilde{K}_0\varphi$ as a function on the ball B_{2R}. However, these functions are only *piecewise* smooth (higher order derivatives jump across Γ), and the approach of Theorems 4.1, 4.2 could not exploit this piecewise smoothness. In order to exploit it, we need to modify the definition of the operators $\widetilde{S}_V$ and $\widetilde{S}_K$. Our approach to the construction of decompositions will rely on a regularity theory for transmission problem, where the transmission conditions are imposed on Γ. This requires regularity of Γ. We illustrate what kind of result may be expected for the case of analytic Γ.

THEOREM 4.3 (decomposition of $\widetilde{V}$, analytic boundary). *Let Γ be analytic and $q \in (0,1)$. Then*

$$\widetilde{V} = \widetilde{V}_0 + \widetilde{S}_{V,pw} + \widetilde{\mathcal{A}}_{V,pw}$$

where the linear operators $\widetilde{S}_{V,pw}$ and $\widetilde{\mathcal{A}}_{V,pw}$ satisfy the following for every $s \geq -1$:

(i) $\widetilde{S}_{V,pw} : H^{-1/2+s}(\Gamma) \to H^2(B_R) \cap H^{3+s}(\Omega_R)$ *with*

$$\|\widetilde{S}_{V,pw}\varphi\|_{H^{s'}(\Omega_R)} \leq C_{s',s} q^2 (qk^{-1})^{1+s-s'} \|\varphi\|_{H^{-1/2+s}(\Gamma)}, \qquad 0 \leq s' \leq s+3.$$

Here, the constant $C_{s',s} > 0$ is independent of q and $k \geq k_0$.

(ii) $\widetilde{\mathcal{A}}_{V,pw} : H^{-1/2+s}(\Gamma) \to H^2(B_R)$ *maps into a space of piecewise analytic functions and*

$$\|\nabla^n \widetilde{\mathcal{A}}_{V,pw}\varphi\|_{L^2(\Omega_R)} \leq Ck\gamma^n \max\{n+1,k\}^n \|\varphi\|_{H^{-3/2}(\Gamma)} \qquad \forall n \in \mathbb{N}_0.$$

Here, the constants C, $\gamma > 0$ are independent of $k \geq k_0$ but may depend on q.

Proof. We start again as in the proof of Theorem 4.1. We have

$$f = -(\Delta\chi)u_0 - 2\nabla\chi \cdot \nabla u_0 + k^2 \chi u_0,$$

where $u_0 = \widetilde{V}_0\varphi$ and χ is the cut-off function of Theorem 4.1. By the mapping properties of $\widetilde{V}_0$ (cf. Lemma 2.4), we have that f is piecewise in H^{1+s}. More specifically,

$$\|f\|_{H^{1+s}(\Omega_{2R})} \leq Ck^2 \|\varphi\|_{H^{-1/2+s}(\Gamma)}.$$

Let E_Ω and E_{Ω^+} be the Stein extension operators (see [28, Chap. VI.3, Thm. 5]) for the sets Ω and Ω^+. Additionally, let χ_Ω and χ_{Ω^+} be the characteristic functions of Ω and Ω^+. We observe

$$\begin{aligned} f &= H_{\mathbb{R}^d}(E_\Omega(f\chi_\Omega)) + L_{\mathbb{R}^d}(E_\Omega(f\chi_\Omega)) \qquad \text{in } \Omega, \\ f &= H_{\mathbb{R}^d}(E_{\Omega^+}(f\chi_{\Omega^+})) + L_{\mathbb{R}^d}(E_{\Omega^+}(f\chi_{\Omega^+})) \qquad \text{in } \Omega^+. \end{aligned}$$

These formulas suggest to write f in the form $f = f_{H^{1+s}} + f_{\mathcal{A},pw}$, where

$$\begin{aligned} f_{H^{1+s}}|_\Omega &= H_{\mathbb{R}^d}(E_\Omega(f\chi_\Omega))|_\Omega, & f_{H^{1+s}}|_{\Omega^+} &= H_{\mathbb{R}^d}(E_{\Omega^+}(f\chi_{\Omega^+}))|_{\Omega^+}, \\ f_{\mathcal{A},pw}|_\Omega &= L_{\mathbb{R}^d}(E_\Omega(f\chi_\Omega))|_\Omega, & f_{\mathcal{A},pw}|_{\Omega^+} &= L_{\mathbb{R}^d}(E_{\Omega^+}(f\chi_{\Omega^+}))|_{\Omega^+}, \end{aligned}$$

The properties of $H_{\mathbb{R}^d}$ and $L_{\mathbb{R}^d}$ given in Lemma 3.1 then imply

$$\|f_{H^{1+s}}\|_{L^2(\mathbb{R}^d\setminus\Gamma)} \leq C\|f\|_{L^2(\mathbb{R}^d)} \leq Ck^2 \|\widetilde{V}_0\varphi\|_{L^2(B_{2R})}, \tag{4.7}$$

$$\|f_{H^{1+s}}\|_{H^t(\mathbb{R}^d\setminus\Gamma)} \leq C(qk^{-1})^{1+s-t} k^2 \|\varphi\|_{H^{-1/2+s}(\Gamma)}, \qquad t \in \{0, 1+s\}, \tag{4.8}$$

$$\|\nabla^n f_{\mathcal{A},pw}\|_{L^2(\mathbb{R}^d\setminus\Gamma)} \leq Ck^2 (\gamma k)^n \|\widetilde{V}_0\varphi\|_{L^2(B_{2R})} \qquad \forall n \in \mathbb{N}_0. \tag{4.9}$$

It will be advisable to split $f_{H^{1+s}}$ once more, namely, to write

$$f_{H^{1+s}} = H_{\mathbb{R}^d}(f_{H^{1+s}}) + L_{\mathbb{R}^d}(f_{H^{1+s}}) =: f_{fin} + f_{\mathcal{A}}. \tag{4.10}$$

Since $L_{\mathbb{R}^d}(f_{H^{1+s}})$ is an entire function and $f_{H^{1+s}}$ is piecewise smooth, we conclude that $f_{fin} = H_{\mathbb{R}^d}(f_{H^{1+s}})$ is piecewise smooth. Concerning bounds for f_{fin}, we start by noting that Lemma 3.1 implies

$$\|\nabla^n f_{\mathcal{A}}\|_{L^2(\mathbb{R}^d)} \leq C(\gamma k)^n \|f_{H^{1+s}}\|_{L^2(\mathbb{R}^d)} \qquad \forall n \in \mathbb{N}_0.$$

Inserting into this the estimates (4.8) and (4.7) leads to two different bounds:

$$\|\nabla^n f_{\mathcal{A}}\|_{L^2(\mathbb{R}^d)} \leq C(\gamma k)^n q^{1+s} k^{1-s} \|\varphi\|_{H^{-1/2+s}(\Gamma)} \ \forall n \in \mathbb{N}_0, \tag{4.11}$$

$$\|\nabla^n f_{\mathcal{A}}\|_{L^2(\mathbb{R}^d)} \leq C(\gamma k)^n k^2 \|\widetilde{V}_0\varphi\|_{L^2(B_{2R})} \qquad \forall n \in \mathbb{N}_0. \tag{4.12}$$

The estimate (4.11) together with interpolation inequalities implies

$$\|f_{\mathcal{A}}\|_{H^t(\mathbb{R}^d)} \lesssim (qk^{-1})^{1+s-t} k^2 \|\varphi\|_{H^{-1/2+s}(\Gamma)}, \qquad t \in \{0, 1+s\}. \tag{4.13}$$

The bounds (4.8) and (4.13) imply for $f_{fin} = f_{H^{1+s}} - f_{\mathcal{A}}$

$$\begin{aligned} \|f_{fin}\|_{H^t(\mathbb{R}^d\setminus\Gamma)} &\lesssim \|f_{H^{1+s}}\|_{H^t(\mathbb{R}^d\setminus\Gamma)} + \|f_{\mathcal{A}}\|_{H^t(\mathbb{R}^d\setminus\Gamma)} \\ &\lesssim (qk^{-1})^{1+s-t} k^2 \|\varphi\|_{H^{-1/2+s}(\Gamma)}, \qquad t \in \{0, 1+s\}. \end{aligned} \tag{4.14}$$

Next, Lemma 3.2 gives for $N_k(f_{fin}) = N_k(H_{\mathbb{R}^d} f_{H^{1+s}})$

$$\|N_k(f_{fin})\|_{L^2(B_{2R})} \leq C(qk^{-1})^2 \|f_{H^{1+s}}\|_{L^2(\mathbb{R}^d\setminus\Gamma)} \leq Cq^2(qk^{-1})^{1+s} \|\varphi\|_{H^{-1/2+s}(\Gamma)}. \tag{4.15}$$

The regularity theory of Theorem B.6 then implies

$$\begin{aligned} &\|N_k(f_{fin})\|_{H^{(s+1)+2}(\Omega_R)} \lesssim \\ &k^{s+1}\|f_{fin}\|_{L^2(\mathbb{R}^d\setminus\Gamma)} + \|f_{fin}\|_{H^{1+s}(\mathbb{R}^d\setminus\Gamma)} + k^{(s+1)+2}\|N_k(f_{fin})\|_{L^2(B_{2R})} \leq Ck^2\|\varphi\|_{H^{-1/2+s}(\Gamma)}. \end{aligned}$$

This estimate together with (4.15) can be written as

$$\|N_k(f_{fin})\|_{H^t(\Omega_R)} \lesssim k^2 (qk^{-1})^{3+s-t} \|\varphi\|_{H^{-1/2+s}(\Gamma)}, \qquad t \in \{0, 3+s\}.$$

The (piecewise) multiplicative interpolation inequality then gives estimates for the intermediate values $0 \leq s' \leq 3+s$:

$$\begin{aligned} \|N_k(f_{fin})\|_{H^{s'}(\Omega_R)} &\leq C\|N_k(f_{fin})\|_{L^2(\Omega_R)}^{(3+s-s')/(3+s)} \|N_k(f_{fin})\|_{H^{3+s}(\Omega_R)}^{s'/(3+s)} \\ &\leq Ck^2(qk^{-1})^{3+s-s'} \|\varphi\|_{H^{-1/2+s}(\Gamma)}. \end{aligned}$$

Upon setting $\widetilde{S}_{V,pw}\varphi := N_k(f_{fin}))$ we get the desired estimates for $\widetilde{S}_V$. We now turn to the properties of $\mathcal{A}_{V,pw}$, which is defined as $\mathcal{A}_{V,pw}\varphi := N_k(f_{\mathcal{A}}) + N_k(f_{\mathcal{A},pw})$. Lemma 3.2 implies

$$\begin{aligned} &\sum_{j=0}^{1} k^{-j}\|N_k(f_{\mathcal{A},pw})\|_{H^j(B_{2R})} + \sum_{j=0}^{1} k^{-j}\|N_k(f_{\mathcal{A}})\|_{H^j(B_{2R})} \\ &\lesssim k^{-1}\|f_{\mathcal{A}}\|_{L^2(\mathbb{R}^d)} + k^{-1}\|f_{\mathcal{A},pw}\|_{L^2(\mathbb{R}^d)} \leq Ck\|\widetilde{V}_0\varphi\|_{L^2(B_{2R})}. \end{aligned} \tag{4.16}$$

(4.16) and Theorem B.4 produce

$$\|\nabla^{n+2}\widetilde{\mathcal{A}}_{V,pw}\varphi\|_{L^2(\Omega_R)} \le C\max\{n,k\}^{n+2}\gamma^n\left[\|\widetilde{V}_0\varphi\|_{L^2(B_{2R})} + k\|\widetilde{V}_0\varphi\|_{L^2(B_{2R})}\right] \quad \forall n\in\mathbb{N}_0$$

for suitable constants C, $\gamma > 0$ independent of n and k. Together with (4.16) and the observation $\|\widetilde{V}_0\varphi\|_{L^2(\Omega_R)} \le C\|\varphi\|_{H^{-3/2}(\Gamma)}$ (cf. Lemma 2.4) this implies the desired estimates for $\widetilde{A}_{V,pw}\varphi$. □

The proof of Theorem 4.3 relies on two facts, namely, on a piecewise shift theorem for $\widetilde{V}_0$ and regularity theory for Helmholtz transmission problems. The same arguments can therefore be used for the double layer potential $\widetilde{K}$.

THEOREM 4.4 (decomposition of $\widetilde{K}$, analytic boundary). *Let Γ be analytic and $q\in(0,1)$. Then we can decompose $\widetilde{K}$ as*

$$\widetilde{K} = \widetilde{K}_0 + \widetilde{S}_{K,pw} + \widetilde{\mathcal{A}}_{K,pw}$$

such that for every $s\ge -1$:

(i) $\widetilde{S}_{K,pw} : H^{1/2+s}(\Gamma) \to H^2(B_R)\cap H^{3+s}(\Omega_R)$ *with*

$$\|\widetilde{S}_{K,pw}\varphi\|_{H^{s'}(\Omega_R)} \le C_{s',s}q^2(qk^{-1})^{1+s-s'}\|\varphi\|_{H^{1/2+s}(\Gamma)}, \qquad 0\le s'\le s+3$$

Here, the constant $C_{s',s}>0$ is independent of q and $k\ge k_0$.

(ii) $\widetilde{\mathcal{A}}_{K,pw} : H^{1/2+s}(\Gamma)\to H^2(B_R)$ *maps into a space of piecewise analytic functions and*

$$\|\nabla^n\widetilde{\mathcal{A}}_{K,pw}\varphi\|_{L^2(\Omega_R)} \le Ck\gamma^n\max\{n+1,k\}^n\|\varphi\|_{H^{-1/2}(\Gamma)} \qquad \forall n\in\mathbb{N}_0.$$

Here, the constants C, $\gamma>0$ are independent of $k\ge k_0$ but may depend on q.

Proof. The proof is analogous to that of Theorem 4.3. □

4.2.1. further mapping properties of the operators $\widetilde{V}$ and $\widetilde{K}$. The results of Section 4.2 permit us to formulate the following corollary.

COROLLARY 4.5. *Let Γ be analytic. Then*

$$\|\widetilde{V}\varphi\|_{L^2(\Omega_R)} \le Ck\|\varphi\|_{H^{-3/2}(\Gamma)}, \tag{4.17}$$

$$\|\widetilde{V}\varphi\|_{H^1(\Omega_R)} \le C\left[\|\varphi\|_{H^{-1/2}(\Gamma)} + k^2\|\varphi\|_{H^{-3/2}(\Gamma)}\right], \tag{4.18}$$

$$\|\widetilde{K}\varphi\|_{L^2(\Omega_R)} \le Ck\|\varphi\|_{H^{-1/2}(\Gamma)}, \tag{4.19}$$

$$\|\widetilde{K}\varphi\|_{H^1(\Omega_R)} \le C\left[\|\varphi\|_{H^{1/2}(\Gamma)} + k^2\|\varphi\|_{H^{-1/2}(\Gamma)}\right], \tag{4.20}$$

$$k^2\|\widetilde{V}\varphi\|_{H^{-1}(B_R)} \le Ck^2\|\varphi\|_{H^{-3/2}(\Gamma)}, \tag{4.21}$$

$$k^2\|\widetilde{K}\varphi\|_{H^{-1}(B_R)} \le Ck^2\|\varphi\|_{H^{-1/2}(\Gamma)}. \tag{4.22}$$

Furthermore, since for $\varphi\in H^{-1/2}(\Gamma)$ we have $\widetilde{V}\varphi$, $\widetilde{K}\varphi\in L^2(B_R)$, there holds for every open subset $\omega\subset B_R$:

$$\|\widetilde{V}\varphi\|_{H^{-1}(\omega)} \le \|\widetilde{V}\varphi\|_{H^{-1}(B_R)}, \qquad \|\widetilde{K}\varphi\|_{H^{-1}(\omega)} \le \|\widetilde{K}\varphi\|_{H^{-1}(B_R)}. \tag{4.23}$$

Proof. For the L^2- and H^1-bounds, combine Theorems 4.3, 4.4 with Lemma 2.3. For the H^{-1}-estimates, we proceed as follows. For the double layer potential $\widetilde{K}\varphi\in L^2(\Omega_R)$ we use the differential equation to get for $v\in H^1_0(B_R)$

$$k^2\langle\widetilde{K}\varphi, v\rangle = -\int_{\Omega_R}\Delta\widetilde{K}\varphi v = -\int_{\Omega_R}\Delta(\widetilde{S}_K\varphi + \widetilde{\mathcal{A}}_K\varphi)v.$$

An integration by parts and the observations that $\widetilde{S}_K\varphi$ and $\widetilde{\mathcal{A}}_K\varphi \in H^2(B_R)$ (and thus their normal derivative does not jump across Γ) yield together with Theorem 4.4

$$\left|k^2\langle \widetilde{K}\varphi, v\rangle\right| = \left|\int_{\Omega_R} \nabla(\widetilde{S}_K\varphi + \widetilde{\mathcal{A}}_K\varphi)\cdot\nabla v\right|$$
$$\le C\left[q^2(qk^{-1})^s\|\varphi\|_{H^{1/2+s}(\Gamma)} + k^2\|\varphi\|_{H^{-1/2}(\Gamma)}\right]\|\nabla v\|_{L^2(B_R)}.$$

Selecting $s=-1$ leads to the claim estimate. For $\|\widetilde{V}\varphi\|_{H^{-1}(B_R)}$, we proceed analogously. □

For later reference, we collect some interior regularity results for solutions to the homogeneous Helmholtz equation.

LEMMA 4.6. *Let $\omega' \subset\subset \omega \subset \mathbb{R}^d$ be two bounded Lipschitz domains. Let $u \in L^2(\omega)$ solve the homogeneous Helmholtz equation. Then there exists $C>0$ (depending only on* $\operatorname{dist}(\omega',\partial\omega) > 0$*, ω, and k_0) such that*

$$\|u\|_{\mathcal{H},\omega'} \le Ck^2\|u\|_{H^{-1}(\omega)}.$$

If $u \in H^1(\omega)$, then we have

$$\|\partial_n u\|_{H^{-1/2}(\omega)} \le Ck\|u\|_{\mathcal{H},\omega}.$$

Proof. For every smooth cut-off function χ with $\operatorname{supp}\chi \subset \omega$ we have $\|\chi u\|_{H^{-1}(\omega)} \le C\|u\|_{H^{-1}(\omega)}$. Next, classical interior regularity gives us

$$\|\nabla u\|_{L^2(\omega')} \le Ck^2\|u\|_{L^2(\omega'')}$$

for all $\omega' \subset\subset \omega'' \subset\subset \omega$ Next, to get the L^2-estimate we observe that χu satisfies

$$-\Delta(\chi u) + k^2\chi u = 2k^2\chi u - 2\nabla\chi\cdot\nabla u - \Delta\chi u, \qquad \chi u = 0 \quad \text{on } \partial\omega.$$

Lax-Milgram for the operator $-\Delta + k^2\,\mathrm{Id}$ then gives

$$\|\chi u\|_{\mathcal{H},\omega} \le Ck^2\|\chi u\|_{H^{-1}(\omega)} \le Ck^2\|u\|_{H^{-1}(\omega)}.$$

We now turn to the case of $u \in H^1(\omega)$. For $v \in H^1(\omega)$ we have

$$|\langle\partial_n u, v\rangle| = \left|\int_\omega \nabla u\cdot\nabla v + \int_\omega \Delta u v\right| = \left|\int_\omega \nabla u\cdot\nabla v - k^2\int_\omega uv\right| \le \|u\|_{\mathcal{H},\omega}\|v\|_{\mathcal{H},\omega},$$

which implies the stated estimate. □

5. decomposition of combined field operators.

The combined field operators A and A' of (1.4), (1.5) are linear combinations of the operators V and K. Hence, the decompositions of the operators V and K of Section 4 imply decompositions of A and A'. The purpose of the present section is to give these decompositions a form that will be convenient later on. We restrict our attention to the case of analytic boundaries Γ.

5.1. frequency splitting for function spaces on surfaces and domains.

An important tool for the analysis will the "frequency splitting" operators analogous to the operators $H_{\mathbb{R}^d}$ and $L_{\mathbb{R}^d}$ of Lemma 3.1. We have

LEMMA 5.1 (frequency splitting on domains). *Let $q \in (0,1)$ and Ω be a bounded Lipschitz domain. Then one can construct operators L_Ω and H_Ω defined on $L^2(\Omega)$ with the following properties:*

(i) $H_\Omega + L_\Omega = \mathrm{Id}$

(ii) $\|H_\Omega f\|_{H^{s'}(\Omega)} \le C_{s,s'}(qk^{-1})^{s-s'}\|f\|_{H^s(\Omega)}$, *where* $0 \le s' \le s$ *and* $s \ge 0$.

(iii) $L_\Omega f$ *is an entire function on* $\mathbb{R}^d$ *and*

$$\|\nabla^n L_\Omega f\|_{L^2(\mathbb{R}^d)} \le C(\gamma k)^n \|f\|_{L^2(\Omega)} \qquad \forall n \in \mathbb{N}_0.$$

Here, $C_{s,s'}$ *is independent of* k *and* q*; the constants* C, γ *are independent of* k.

Proof. Let $E_\Omega : L^2(\Omega) \to L^2(\mathbb{R}^d)$ be the Stein extension operator. Then define $H_\Omega f = (H_{\mathbb{R}^d} \circ E_\Omega f)|_\Omega$ and $L_\Omega f := (L_{\mathbb{R}^d} \circ E_\Omega f)$. The properties then follow from Lemma 3.1. ∎

Lemma 5.2 (frequency splitting on surfaces). *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain with smooth boundary* Γ. *Let* $s > 0$ *and* $q \in (0,1)$. *Then one can construct operators* $L_\Gamma : H^s(\Gamma) \to H^{1/2+s}(\mathbb{R}^d)$ *and* $H_\Gamma : H^s(\Gamma) \to H^s(\Gamma)$ *with the following properties:*

(i) $H_\Gamma + \gamma_0^{int} L_\Gamma = \mathrm{Id}$

(ii) $\|H_\Gamma f\|_{H^{s'}(\Gamma)} \le C_{s,s'}(qk^{-1})^{s-s'}\|f\|_{H^s(\Gamma)}$, *where* $0 \le s' \le s$.

(iii) $L_\Gamma f$ *is an entire function on* $\mathbb{R}^d$ *and*

$$\|\nabla^n L_\Gamma f\|_{L^2(\mathbb{R}^d)} \le C(\gamma k)^{n-(1/2+s)} \|f\|_{H^s(\Gamma)} \qquad \forall n \in \mathbb{N}_0.$$

Here, the constant $C_{s,s'}$ *is independent of* k *and* q*; the constants* C, γ *are independent of* k.

Proof. Related frequency splittings have been constructed in [24]. We therefore merely sketch the construction. Let $G : H^s(\Gamma) \to H^{1/2+s}(\mathbb{R}^d)$ be a lifting operator. Define $H_\Gamma := \gamma_0^{int} \circ H_{\mathbb{R}^d} \circ G$ and $L_\Gamma := L_{\mathbb{R}^d} \circ G$. The properties of $H_{\mathbb{R}^d}$ and $L_{\mathbb{R}^d}$ given Lemma 3.1 then imply the statements. For example, the bound for H_Γ follows from the properties of $H_{\mathbb{R}^d}$. Specifically, the multiplicative trace inequality (see, e.g., [23, Thm. A.2]) yields

$$\begin{aligned}\|H_\Gamma \varphi\|_{L^2(\Gamma)} &\le \|H_{\mathbb{R}^d}(G\varphi)\|_{L^2(\Omega)}^{2s/(1+2s)} \|H_{\mathbb{R}^d}(G\varphi)\|_{H^{1/2+s}(\Omega)}^{1/(1+2s)} \\ &\lesssim (qk^{-1})^s \|G\varphi\|_{H^{1/2+s}(\Omega)} \lesssim (qk^{-1})^s \|\varphi\|_{H^s(\Gamma)};\end{aligned}$$

on the other hand, trace inequalities and the stability of $H_{\mathbb{R}^d}$ yield

$$\|H_\Gamma \varphi\|_{H^s(\Gamma)} \lesssim \|H_{\mathbb{R}^d}(G\varphi)\|_{H^{1/2+s}(\Omega)} \lesssim \|G\varphi\|_{H^{1/2+s}(\Omega)} \lesssim \|\varphi\|_{H^s(\Gamma)}.$$

Thus, the limiting cases $s' \in \{0, s\}$ are proved. The intermediate cases $0 < s' < s$ follow by interpolation arguments. ∎

The frequency splitting in Lemma 5.2 relies on a frequency splitting in a domain and the trace operator. This precludes a direct extension of the construction to negative-index Sobolev spaces. Nevertheless, splittings can be defined on such spaces, and the following lemma presents one possible construction.

Lemma 5.3 (frequency splitting on surfaces, negative norms). *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain with an analytic boundary* Γ. *Let* $q \in (0,1)$. *Then one can construct operators* L_Γ^{neg}, H_Γ^{neg} *on* $H^{-1}(\Gamma)$ *with the following properties:*

(i) $L_\Gamma^{neg} + H_\Gamma^{neg} = \mathrm{Id}$

(ii) *for* $-1 \le s' \le s \le 1$*:*

$$\|H_\Gamma^{neg} f\|_{H^{s'}(\Gamma)} \le C(q/k)^{s-s'} \|f\|_{H^s(\Gamma)}$$

(iii) $L_\Gamma^{neg} f$ *is the restriction to* Γ *of a function that is analytic on a tubular neighborhood* T *of* Γ *and satisfies*

$$\|\nabla^n L_\Gamma^{neg} f\|_{L^2(T)} \le C k^{d/2} \gamma^n \max\{k,n\}^n \|f\|_{H^{-1/2}(\Gamma)} \qquad \forall n \in \mathbb{N}_0.$$

Proof. Consider on the compact manifold Γ for the Laplace-Beltrami operator Δ_Γ the eigenvalue problem

$$-\Delta_\Gamma \varphi - \lambda^2 \varphi = 0 \quad \text{ on } \Gamma.$$

There are countably many eigenfunctions φ_m, $m \in \mathbb{N}_0$, with associated eigenvalues $\lambda_m \ge 0$, which we assume to be sorted in ascending order. Without loss of generality, we impose the normalization $\|\varphi_m\|_{L^2(\Gamma)} = 1$. We have Weyl's formula (see [10, p. 155])

$$N(\lambda) := \operatorname{card}\{\lambda_m \,|\, \lambda_m \le \lambda\} \sim C_\Gamma \lambda^{d-1},$$

where the constant C_Γ depends solely on Γ. Additionally, we have from Lemma C.1 the existence of a tubular neighborhood T of Γ and constants C, $\gamma > 0$ such that

$$\|\nabla^n \varphi_m\|_{L^2(T)} \le C\gamma^n \{\lambda_m, n\}^n \qquad \forall n \in \mathbb{N}_0. \tag{5.1}$$

Furthermore, the functions $(\varphi_m)_{m=0}^\infty$ are an orthonormal basis of $L^2(\Gamma)$ and an orthogonal basis of $H^1(\Gamma)$:

$$\|u\|_{L^2(\Gamma)}^2 = \sum_{m=0}^\infty |\langle u, \varphi_m\rangle_{L^2(\Gamma)}|^2 \quad \forall u \in L^2(\Gamma),$$
$$\|u\|_{H^1(\Gamma)}^2 = \sum_{m=0}^\infty (1+\lambda_m^2)|\langle u, \varphi_m\rangle_{L^2(\Gamma)}|^2 \quad \forall u \in H^1(\Gamma).$$

By interpolation, we get for $0 \le s \le 1$ and $u \in L^2(\Gamma)$:

$$\|u\|_{H^s(\Gamma)}^2 \sim \sum_{m=0}^\infty (1+\lambda_m^2)^s |\langle u, \varphi_m\rangle_{L^2(\Gamma)}|^2.$$

By duality, distributions $f \in H^s(\Gamma)$ with $s \in [-1, 0]$ can be identified with sequences $(f_m)_{m=0}^\infty$ such that $\|f\|_{H^s(\Gamma)}^2 \sim \sum_{m=0}^\infty (1+\lambda_m^2)^s |f_m|^2$. We will write (formally) $f = \sum_{m=0}^\infty f_m \varphi_m$ to express this identification.
We now define the operators H_Γ^{neg} and L_Γ^{neg} by

$$H_\Gamma^{neg} f := \sum_{m:\lambda_m > \eta k} f_m \varphi_m, \qquad L_\Gamma^{neg} f := \sum_{m:\lambda_m \le \eta k} f_m \varphi_m$$

Then clearly $H_\Gamma^{neg} + L_\Gamma^{neg} = \mathrm{Id}$. Next, in the tubular neighborhood T of Γ we have

$$\begin{aligned}
\|\nabla^n L_\Gamma^{neg} f\|_{L^2(T)} &\le \sum_{m:\lambda_m \le \eta k} |f_m| \|\nabla^n \varphi_m\|_{L^2(T)} \le C\gamma^n \max\{\eta k, n\}^n \sum_{m:\lambda_m \le \eta k} |f_m| \\
&\le C\gamma^n \max\{\eta k, n\}^n \sqrt{\sum_{m:\lambda_m \le \eta k} (1+\lambda_m^2)^{1/2}} \sqrt{\sum_{m:\lambda_m \le \eta k} (1+\lambda_m^2)^{-1/2} |f_m|^2} \\
&\le C\gamma^n \max\{\eta k, n\}^n (1+(\eta k)^2)^{1/4} \sqrt{N(\eta k)} \|f\|_{H^{-1/2}(\Gamma)} \\
&\le C k^{d/2} \gamma^n \max\{\eta k, n\}^n \|f\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

For $H_\Gamma^{neg} f$, we compute for $-1 \le s' \le s \le 1$:

$$\|H_\Gamma^{neg} f\|^2_{H^{s'}(\Gamma)} \le C \sum_{m:\lambda_m > \eta k} (1+\lambda_m)^{2s'} |f_m|^2 \le C(1+\eta k)^{2(s'-s)} \|f\|^2_{H^s(\Gamma)},$$

which finishes the proof. ∎

REMARK 5.4. The factor $k^{d/2}$ in the estimates for L_Γ^{neg} is not optimal and can be reduced (see Remark C.2). Also, the proof shows that the term $\|f\|_{H^{-1/2}(\Gamma)}$ in the bounds for L_Γ^{neg} can be reduced to $\|f\|_{H^{-1}(\Gamma)}$ at the expense of further powers of k.

5.2. decomposition of A and A'. We recall the definition of $\mathfrak{A}(C, \gamma, T \setminus \Gamma)$ given in Definition 1.1 and the definition of the jump operator $[\cdot]$ in Section 1.1.1.

LEMMA 5.5 (decomposition of A). *Let Γ be analytic and let $s \ge 0$. Fix $q \in (0,1)$. Then the operator A can be written as*

$$A = \pm \frac{1}{2} + K_0 + R_A + k[\widetilde{\mathcal{A}}_A]$$

where $R_A : H^s(\Gamma) \to H^{s+1}(\Gamma)$ and $\widetilde{\mathcal{A}}_A$ satisfy for some constant C, which is independent of $k \ge k_0$ and q, and a constant $\gamma > 0$, which is independent of $k \ge k_0$,

$$\|R_A\|_{H^{s+1}(\Gamma) \leftarrow H^s(\Gamma)} \le Ck, \qquad \|R_A\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)} \le q,$$
$$\widetilde{\mathcal{A}}_A \varphi \in \mathfrak{A}(CC_\varphi, \gamma, \Omega_R)), \qquad C_\varphi = \|\varphi\|_{H^{-1/2}(\Gamma)} + k\|\varphi\|_{H^{-3/2}(\Gamma)}.$$

Proof. Before turning to the proof, we point out that, since only the jump of the potential $\widetilde{A}_A \varphi$ across Γ appears in the decomposition of A, there is some freedom in the choice of $\widetilde{A}_A$. In particular, $\widetilde{\mathcal{A}}_A$ can be selected such that $(\widetilde{\mathcal{A}}_A f)|_{\Omega^+} = 0$ or $(\widetilde{\mathcal{A}}_A f)|_\Omega = 0$. Indeed, we will construct $\widetilde{\mathcal{A}}_A$ such that $\widetilde{\mathcal{A}}_A f = 0$ on Ω^+ if $A = -1/2 + K - \mathbf{i}\eta V$ is considered and $\widetilde{\mathcal{A}}_A f = 0$ on Ω if $A = 1/2 + K - \mathbf{i}\eta V$.

We will only consider the operator A given in (1.4a) (i.e., the case $A = -1/2 + K - \mathbf{i}\eta V$), the other case being handled analogously. Since $A = \gamma_0^{int}(\widetilde{K} - \mathbf{i}\eta \widetilde{V})$, the decompositions of $\widetilde{K}$ and $\widetilde{V}$ of Theorems 4.4, 4.3 produce

$$A = \left\{-\frac{1}{2} + K_0\right\} + \left\{\gamma_0^{int}\left(\widetilde{S}_{K,pw} - \mathbf{i}\eta \widetilde{S}_{V,pw}\right) - \mathbf{i}\eta V_0\right\} + \left\{\gamma_0^{int}\left(\widetilde{\mathcal{A}}_{K,pw} - \mathbf{i}\eta \widetilde{\mathcal{A}}_{V,pw}\right)\right\}.$$

With the aid of the high and low frequency operators H_Γ and L_Γ of Lemma 5.2, we write $V_0 = H_\Gamma V_0 + \gamma_0^{int} L_\Gamma V_0$ and therefore arrive at the decomposition

$$A = -1/2 + K_0 + R_A + \gamma_0^{int} \widetilde{\mathcal{A}}_A,$$
$$R_A = \gamma_0^{int}\left(\widetilde{S}_{K,pw} - \mathbf{i}\eta \widetilde{S}_{V,pw}\right) - \mathbf{i} H_\Gamma \eta V_0,$$
$$\widetilde{\mathcal{A}}_A = -\mathbf{i}\eta k^{-1} L_\Gamma V_0 + k^{-1} \widetilde{\mathcal{A}}_{K,pw} - \mathbf{i}\eta k^{-1} \widetilde{\mathcal{A}}_{V,pw}.$$

It remains to obtain the stated bounds. Theorems 4.3, 4.4 and Lemma 5.2 produce (for notational convenience, we employ the same parameter $q \in (0,1)$ in the splittings of Theorems 4.3, 4.4 and Lemma 5.2)

$$\|\gamma_0^{int} \widetilde{S}_{V,pw}\|_{H^{1+s}(\Gamma) \leftarrow H^s(\Gamma)} \le Cq^2, \qquad \|\gamma_0^{int} \widetilde{S}_{V,pw}\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)} \le Cq^3 k^{-1},$$
$$\|\gamma_0^{int} \widetilde{S}_{K,pw}\|_{H^{1+s}(\Gamma) \leftarrow H^s(\Gamma)} \le Cqk, \qquad \|\gamma_0^{int} \widetilde{S}_{K,pw}\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)} \le Cq^2,$$
$$\|H_\Gamma V_0\|_{H^{1+s}(\Gamma) \leftarrow H^s(\Gamma)} \le C, \qquad \|H_\Gamma V_0\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)} \le Cqk^{-1}.$$

By selecting q sufficiently small, we can obtain the desired bounds for R_A. For $\widetilde{\mathcal{A}}_A$ we see that Theorems 4.3, 4.4, and Lemma 5.2 together with the mapping properties of V_0 yield

$$\varphi \in H^{-1/2}(\Gamma) \quad \Longrightarrow \quad \widetilde{\mathcal{A}}_A\varphi \in \mathfrak{A}(CC_\varphi, \gamma, \Omega_R), \qquad C_\varphi := \|\varphi\|_{H^{-1/2}(\Gamma)} + k\|\varphi\|_{H^{-3/2}(\Gamma)}.$$

This concludes the proof. ∎

Remark 5.6. *The operator $-1/2 + K_0$ is invertible while the operator $1/2 + K_0$ has a one-dimensional kernel. It will be convenient to have decompositions with invertible leading term. By Lemma 2.5, the operator $1/2 + K_0 - \mathbf{i}V_0$ is invertible. Inspection of the proof of Lemma 5.5 shows that we can achieve a decomposition of the following form:*

$$1/2 + K - \mathbf{i}\eta V = \frac{1}{2} + K_0 + \mathbf{i}V_0 + R_A + k[\widetilde{\mathcal{A}}_A]$$

where the operators R_A and $\widetilde{\mathcal{A}}_A$ have the regularity properties stated in Lemma 5.5.

The next two lemmas provide decompositions of A'—the difference between these two results lies in the range of Sobolev spaces on which they are defined: While Lemma 5.7 covers the case $s \geq 0$, Lemma 5.9 extends the range to $s \geq -1/2$ at the expense of further powers of k.

Lemma 5.7 (decomposition of A'). *Let Γ be analytic and let $s \geq 0$. Fix $q \in (0, 1)$. Then the operator A' can be written in the form*

$$A' = \pm\frac{1}{2} + K_0' + R_{A'} + k[\widetilde{\mathcal{A}}_{A',1}] + [\partial_n \widetilde{\mathcal{A}}_{A',2}]$$

where $R_{A'} : H^s(\Gamma) \to H^{s+1}(\Gamma)$ and $\widetilde{\mathcal{A}}_{A'}$ satisfy for some constants C, $\gamma > 0$ that are independent of $k \geq k_0$

$$\begin{aligned}
&\|R_{A'}\|_{H^{s+1}(\Gamma)\leftarrow H^s(\Gamma)} \leq Ck, \qquad \|R_{A'}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} \leq q, \\
&\widetilde{\mathcal{A}}_{A',i}\varphi \in \mathfrak{A}(CC_\varphi, \gamma, \Omega_R)), \qquad C_\varphi = k\|\varphi\|_{H^{-3/2}(\Gamma)}, \qquad i \in \{1, 2\}.
\end{aligned}$$

Proof. We consider the case $A' = \frac{1}{2} + K' + \mathbf{i}\eta V$, the case $A' = -1/2 + K' + \mathbf{i}\eta V$ being handled by analogous arguments. We recall that the operator A' is given by $A'\varphi = \gamma_1^{int}\widetilde{V}\varphi - \mathbf{i}\eta\gamma_0^{int}\widetilde{V}\varphi$. In view of $\gamma_1^{int}\widetilde{V}_0 = 1/2 + K_0'$ we can write with the decomposition of Theorem 4.3

$$A' = \frac{1}{2} + K_0' + \gamma_1^{int}\left(\widetilde{S}_{V,pw} + \widetilde{\mathcal{A}}_{V,pw}\right) + \mathbf{i}\eta\gamma_0^{int}\left(\widetilde{V}_0 + \widetilde{S}_{V,pw} + \widetilde{\mathcal{A}}_{V,pw}\right). \tag{5.2}$$

Here, the parameter q appearing in the definition of the decomposition of Theorem 4.3 is still at our disposal. Using the high and low frequency operators H_Ω of L_Ω (the parameter q appearing in their definition will be selected shortly) we can set

$$\begin{aligned}
R_{A'} &= \gamma_1^{int}\widetilde{S}_{V,pw} + \mathbf{i}\eta\gamma_0^{int}\widetilde{S}_{V,pw} + \mathbf{i}\eta\gamma_0^{int}H_\Omega\widetilde{V}_0, \\
\widetilde{\mathcal{A}}_{A',1} &= -k^{-1}\chi_\Omega\left(\mathbf{i}\eta\widetilde{\mathcal{A}}_{V,pw} + \mathbf{i}\eta L_\Omega\widetilde{V}_0\right), \\
\widetilde{\mathcal{A}}_{A',2} &= -\chi_\Omega\widetilde{\mathcal{A}}_{V,pw},
\end{aligned}$$

where χ_Ω denotes the characteristic function for Ω. Theorem 4.3 yields

$$
\begin{aligned}
&\|\gamma_1^{int}\widetilde{S}_{V,pw}\|_{H^{1+s}(\Gamma)\leftarrow H^s(\Gamma)} \le Cqk, &&\|\gamma_1^{int}\widetilde{S}_{V,pw}\|_{H^{s}(\Gamma)\leftarrow H^s(\Gamma)} \le Cq^2\\
&\|\gamma_0^{int}\widetilde{S}_{V,pw}\|_{H^{1+s}(\Gamma)\leftarrow H^s(\Gamma)} \le Cq^2, &&\|\gamma_0^{int}\widetilde{S}_{V,pw}\|_{H^{s}(\Gamma)\leftarrow H^s(\Gamma)} \le Cq^3k^{-1}\\
&\|\gamma_0^{int}H_\Omega\widetilde{V}_0\|_{H^{1+s}(\Gamma)\leftarrow H^s(\Gamma)} \le C, &&\|\gamma_0^{int}H_\Omega\widetilde{V}_0\|_{H^{s}(\Gamma)\leftarrow H^s(\Gamma)} \le Cqk^{-1}.
\end{aligned}
$$

Selecting q appropriately gives the desired bounds for $R_{A'}$. From Theorem 4.3, Lemma 5.1, and Lemma 2.4 we infer

$$
\widetilde{\mathcal{A}}_{A',2}\varphi \quad\text{and}\quad \widetilde{\mathcal{A}}_{A',1}\varphi \in \mathfrak{A}(CC_\varphi,\gamma,\Omega_R), \qquad C_\varphi := k\|\varphi\|_{H^{-3/2}(\Gamma)}.
$$

□

REMARK 5.8. *The operator $-1/2 + K_0'$ is invertible while the operator $1/2 + K_0'$ has a one-dimensional kernel. By Lemma 2.5, the operator $1/2 + K_0' + \mathbf{i}V_0$ is invertible. Inspection of the proof of Lemma 5.7 shows that we can achieve a decomposition of the following form:*

$$
1/2 + K' + \mathbf{i}\eta V = \frac{1}{2} + K_0' - \mathbf{i}V_0 + R_A + k[\widetilde{\mathcal{A}}_{A',1}] + [\partial_n \widetilde{\mathcal{A}}_{A',2}],
$$

where the operators R_A and $\widetilde{\mathcal{A}}_{A',i}$, $i \in \{1,2\}$ have the regularity properties stated in Lemma 5.7.

LEMMA 5.9 (decomposition of A'). *Let Γ be analytic and let $-1/2 \le s \le 0$. Fix $q \in (0,1)$. Then the operator A' can be written in the form*

$$
A' = \pm\frac{1}{2} + K_0' + R_{A'} + k[\widetilde{\mathcal{A}}_{A',1}] + [\partial_n \widetilde{\mathcal{A}}_{A',2}]
$$

where $R_{A'} : H^s(\Gamma) \to H^{s+1}(\Gamma)$ and $\widetilde{\mathcal{A}}_{A'}$ satisfy for some constants C, $\gamma > 0$ and a tubular neighborhood T of Γ that are all independent of $k \ge k_0$

$$
\begin{aligned}
&\|R_{A'}\|_{H^{s+1}(\Gamma)\leftarrow H^s(\Gamma)} \le Ck, &&\|R_{A'}\|_{H^{s}(\Gamma)\leftarrow H^s(\Gamma)} \le q,\\
&\widetilde{\mathcal{A}}_{A',1}\varphi \in \mathfrak{A}(CC_\varphi,\gamma,T)), &&C_\varphi = k\|\varphi\|_{H^{-3/2}(\Gamma)} + k^{d/2}\|\varphi\|_{H^{-1}(\Gamma)},\\
&\widetilde{\mathcal{A}}_{A',2}\varphi \in \mathfrak{A}(C\widetilde{C}_\varphi,\gamma,T)), &&\widetilde{C}_\varphi = k\|\varphi\|_{H^{-3/2}(\Gamma)}.
\end{aligned}
$$

Proof. The proof is very similar to that of Lemma 5.7. We start from (5.2). Using the frequency splitting operators H_Γ^{neg} and L_Γ^{neg} of Lemma 5.3, we can define

$$
\begin{aligned}
R_{A'} &= H_\Gamma^{neg}\left(\gamma_1^{int}\widetilde{S}_{V,pw} + \mathbf{i}\eta\gamma_0^{int}\widetilde{S}_{V,pw} + \mathbf{i}\eta V_0\right),\\
\widetilde{\mathcal{A}}_{A',1} &= k^{-1}\chi_\Omega\left(-\mathbf{i}\eta\widetilde{\mathcal{A}}_{V,pw} - L_\Gamma^{neg}\left(\gamma_1^{int}\widetilde{S}_{V,pw} + \mathbf{i}\eta\gamma_0^{int}\widetilde{S}_{V,pw} + \mathbf{i}\eta V_0\right)\right),\\
\widetilde{\mathcal{A}}_{A',2} &= -\chi_\Omega\widetilde{\mathcal{A}}_{V,pw}.
\end{aligned}
$$

Using the mapping properties of $\widetilde{S}_{V,pw}$ and V_0 we can infer from Lemma 5.3 that $R_{A'}$ has the desired mapping properties. For the operators $\widetilde{\mathcal{A}}_{A',1}$, $\widetilde{\mathcal{A}}_{A',2}$ we get from Theorem 4.3 and the mapping properties of V_0 that

$$
\|\gamma_1^{int}\widetilde{S}_{V,pw} + \mathbf{i}\eta\gamma_0^{int}\widetilde{S}_{V,pw} + \mathbf{i}\eta V_0\|_{L^2(\Gamma)\leftarrow H^{-1}(\Gamma)} \le Ck.
$$

From Lemma 5.3 we therefore get

$$\widetilde{A}_{A',1}\varphi \in \mathfrak{A}(CC_\varphi, \gamma, T), \qquad C_\varphi = k^{d/2}\|\varphi\|_{H^{-1}(\Gamma)} + k\|\varphi\|_{H^{-3/2}(\Gamma)}$$

and an analogous estimate for $\widetilde{\mathcal{A}}_{A',2}$. ◻

Remark 5.10. *The proof of Lemma 5.9 shows that in the context of smooth domains, further decompositions are possible. In particular, it is possible to exploit the smoothing properties of K_0 and K_0'. Since $K_0 : L^2(\Gamma) \to H^1(\Gamma)$ and $K_0' : L^2(\Gamma) \to H^1(\Gamma)$ we see that the splittings $K_0 = H_\Gamma^{neg} K_0 + L_\Gamma^{neg} K_0$ and $K_0' = H_\Gamma^{neg} K_0' + L_\Gamma^{neg} K_0'$ lead, for example, to*

$$\|H_\Gamma^{neg} K_0\|_{L^2 \leftarrow L^2} \leq Cq/k, \qquad \|H_\Gamma^{neg} K_0'\|_{L^2 \leftarrow L^2} \leq Cq/k.$$

Inserting this in the decompositions of Lemmata 5.5, 5.9 shows that the operators A, A' can be we written as sums of three terms: $\pm 1/2\,\mathrm{Id}$, an operator that is small (as an operator $L^2(\Gamma) \to L^2(\Gamma)$, and an operator that maps into a trace class of analytic functions.

6. decomposition of the inverse of combined field operators.

We turn to the mapping properties of the operators A^{-1} and $(A')^{-1}$, where A and A' are defined in (1.4) and (1.5). Put differently, we seek estimates for the solution φ of the following problems:

$$A\varphi = f \tag{6.1}$$
$$A'\varphi = f. \tag{6.2}$$

Here, f is in an appropriate Sobolev space to specified below. We will focus on the case of analytic boundaries Γ.

6.1. analytic regularity.

In this section, we study (6.1), (6.2) for analytic Γ and analytic right-hand side f. The solution φ is then likewise analytic and the aim of the present section is to study the k-dependence of the solution φ.

6.1.1. the operator A.

Lemma 6.1. *Let Γ be analytic and let T be a tubular neighborhood of Γ. Suppose $g \in \mathfrak{A}(C_g, \gamma_g, T\backslash\Gamma)$ for some C_g, $\gamma_g > 0$. Let $\varphi \in H^{1/2}(\Gamma)$ satisfy*

$$\left(\pm\frac{1}{2} + K - \mathbf{i}\eta V\right)\varphi = \gamma_0^{ext} g - \gamma_0^{int} g$$

Then $\varphi = \gamma_0^{ext} u - \gamma_0^{int} u$, where, with the operator $\widetilde{A}$ defined in (6.3),

$$u \in \mathfrak{A}(CC_u, \gamma, \Omega_R), \qquad C_u = C_g + k^{-1}\|\nabla \widetilde{A}\varphi\|_{L^2(\Omega_R)} + \|\widetilde{A}\varphi\|_{L^2(\Omega_R)}.$$

The constants C and γ depend solely on Γ, γ_g, k_0, and the choice of R.

Proof. Before proving the lemma, we stress the following points: First, the existence of φ is stipulated as an assumption. Second, as will be discussed in more detail below, $k^{-1}\|\nabla \widetilde{A}\varphi\|_{L^2(\Omega_R)} + \|\widetilde{A}\varphi\|_{L^2(\Omega_R)}$ grows only algebraically in k under appropriate assumptions. Thirdly, it is allowed to select g such that it vanishes in Ω or in Ω^+; in fact, this is how Lemma 6.1 will be employed below. Finally, in view of Lemma B.5 it is possible to select u such that it vanishes on Ω or Ω^+.

We define the potential u on $\Omega \cup \Omega^+$ by

$$u = \widetilde{A}\varphi := \widetilde{K}\varphi - \mathbf{i}\eta\widetilde{V}\varphi. \tag{6.3}$$

Then u satisfies the homogeneous Helmholtz equation on $\Omega \cup \Omega^+$ together with

$$\gamma_0^{int} u = [g] \qquad \text{if } (-\tfrac{1}{2} + K - \mathbf{i}\eta V)\varphi = [g], \tag{6.4}$$
$$\gamma_0^{ext} u = [g] \qquad \text{if } (\tfrac{1}{2} + K - \mathbf{i}\eta V)\varphi = [g]. \tag{6.5}$$

We will only consider the first case (corresponding to an interior Dirichlet problem)—the method of proof can be applied to the second case as well. Also, for simplicity of notation we assume that $g = 0$ on Ω^+. This is not a restriction and can realized with the aid of Lemma B.5.
The jump relations satisfied by $\widetilde{K}$ and $\widetilde{V}$ (see [20, Thm. 6.11]) give us on Γ:

$$[u] = \varphi, \qquad \gamma_1^{ext} u - \gamma_1^{int} u = \mathbf{i}\eta\varphi. \tag{6.6}$$

The first jump relation shows that we have to prove $u \in \mathfrak{A}(C_u, \gamma_u, \Omega_R)$. To that end, we note that u solves by (6.4)

$$-\Delta u - k^2 u = 0 \quad \text{on } \Omega, \qquad \gamma_0^{int} u = g.$$

In view of the analyticity of Γ and g, Theorem B.2 implies the existence of a tubular neighborhood T of Γ such that $u \in \mathfrak{A}(C_1, \gamma, T \cap \Omega)$, where $C_1 \leq C\left(C_g + k^{-1}\|u\|_{\mathcal{H},\Omega}\right)$ for a $C > 0$ independent of u and k.
The jump relations (6.6) imply the Robin boundary conditions

$$\gamma_1^{ext} u - \mathbf{i}\eta\gamma_0^{ext} u = \gamma_1^{int} u - \mathbf{i}\eta\gamma_0^{int} u =: \widetilde{g}. \tag{6.7}$$

The analyticity of Γ implies the existence of a tubular neighborhood of Γ (again denoted T) and an analytic function $G^- \in \mathfrak{A}(CC_1 k, \gamma, T \cap \Omega)$ with $\gamma_0^{int} G^- = \widetilde{g}$. Next, Lemma B.5 implies the existence of a function G and a tubular neighborhood of Γ (again denoted T) with $G \in \mathfrak{A}(CC_1 k, \gamma, T \cap \Omega^+)$ and $\gamma_0^{ext} G = \gamma_0^{int} G^- = \widetilde{g}$. Then, Theorem B.3 gives $u \in \mathfrak{A}(CC_2, \gamma, T \cap \Omega^+)$, where $C_2 = C_1 + k^{-1}\|u\|_{\mathcal{H},\Omega^+ \cap B_R}$. Since $u = \widetilde{A}\varphi$, we have so far obtained $u \in \mathfrak{A}(CC_u, \gamma, T \backslash \Gamma)$ with C_u defined in the statement of the lemma. Interior regularity (see [22, Prop. 5.5.1]) finally gives estimates for u not only near Γ but in all of Ω_R, i.e., $u \in \mathfrak{A}(CC_u, \gamma_u, \Omega_R)$ for suitable C, $\gamma_u > 0$. ∎
The existence of φ is stipulated as an assumption in Lemma 6.1. We formulated $\varphi \in H^{1/2}(\Gamma)$ since this readily implies $\widetilde{A}\varphi \in H^1(\Omega_R)$ and the constant C_u can be estimated in terms of $\|\varphi\|_{H^{1/2}(\Gamma)}$. However, it will be more convenient in the following to bound C_u in terms of $\|\varphi\|_{L^2(\Gamma)}$ and $\|A\varphi\|_{H^{1/2}(\Gamma)}$, which we now show how to do:
Lemma 6.2. *Assume the hypotheses of Lemma 6.1. If $\varphi \in H^{1/2}(\Gamma)$ then*

$$\|\widetilde{A}\varphi\|_{L^2(\Omega_R)} + k^{-1}\|\nabla\widetilde{A}\varphi\|_{L^2(\Omega_R)} \leq C\left[k^{-1}\|\varphi\|_{H^{1/2}(\Gamma)} + k\|\varphi\|_{H^{-1/2}(\Gamma)} + k^2\|\varphi\|_{H^{-3/2}(\Gamma)}\right].$$

If $\varphi \in L^2(\Gamma)$ and $A\varphi \in H^{1/2}(\Gamma)$ then

$$\|\widetilde{A}\varphi\|_{L^2(\Omega_R)} + k^{-1}\|\nabla\widetilde{A}\varphi\|_{L^2(\Omega_R)} \leq C\left[\|A\varphi\|_{H^{1/2}(\Gamma)} + k^2\|\varphi\|_{H^{-1/2}(\Gamma)} + k^3\|\varphi\|_{H^{-3/2}(\Gamma)}\right].$$

Proof. If $\varphi \in H^{1/2}(\Gamma)$, then we can insert the result of Corollary 4.5 to get

$$\|\widetilde{A}\varphi\|_{L^2(\Omega_R)} \leq C\left[k\|\varphi\|_{H^{-1/2}(\Gamma)} + k^2\|\varphi\|_{H^{-3/2}(\Gamma)}\right],$$
$$\|\nabla\widetilde{A}\varphi\|_{L^2(\Omega_R)} \leq C\left[\|\varphi\|_{H^{1/2}(\Gamma)} + k^2\|\varphi\|_{H^{-1/2}(\Gamma)} + k^3\|\varphi\|_{H^{-3/2}(\Gamma)}\right],$$

which is the first estimate. For the second one, we consider again the case where $\widetilde{A}\varphi$ (see (6.3)) solves an *interior* Dirichlet problem. If $\varphi \in L^2(\Gamma)$, then it is a priori not clear that $\widetilde{A}\varphi \in H^1(\Omega_R)$. However, this can be inferred as follows: We write $A = \pm 1/2 + K_0 + S$, where, by Theorem 4.4, the operator $S : L^2(\Gamma) \to H^1(\Gamma)$. Since likewise $K_0 : L^2(\Gamma) \to H^1(\Gamma)$, we conclude from $A\varphi \in H^{1/2}(\Gamma)$ that $1/2\varphi \in H^{1/2}(\Gamma)$. In particular, $\widetilde{A}\varphi \in H^1(\Omega_R)$. To get bounds for $u := \widetilde{A}\varphi$, we restrict our attention to the case $A = -1/2 + K - \mathbf{i}\eta V$ as in the proof of Lemma 6.1 and note that (4.21)–(4.23) of Corollary 4.5 produce

$$k^2\|u\|_{H^{-1}(\Omega)} + k^2\|u\|_{H^{-1}(B_{2R})} \leq C\left[k^2\|\varphi\|_{H^{-1/2}(\Gamma)} + k^3\|\varphi\|_{H^{-3/2}(\Gamma)}\right]. \quad (6.8)$$

Next, u is the solution of the following interior Dirichlet problem:

$$-\Delta u = k^2\widetilde{A}\varphi \in L^2(\Omega) \quad \text{in } \Omega, \qquad \gamma_0^{int} u = g := A\varphi.$$

Standard a priori bounds for Laplace Dirichlet problems together with (6.8) and (6.8) imply

$$\|u\|_{\mathcal{H},\Omega} \leq C\left[\|g\|_{H^{1/2}(\Gamma)} + k^2\|\varphi\|_{H^{-1/2}(\Gamma)} + k^3\|\varphi\|_{H^{-3/2}(\Gamma)}\right].$$

Lemma 4.6 allows us to infer

$$\|\gamma_1^{int} u\|_{H^{-1/2}(\Gamma)} \leq Ck\|u\|_{\mathcal{H},\Omega}, \qquad \|u\|_{H^{1/2}(\partial B_R)} \leq Ck^2\|u\|_{H^{-1}(B_{2R})}. \quad (6.9)$$

The jump condition (6.6) satisfied by u reads $\gamma_1^{ext}u - \gamma_1^{int}u = \mathbf{i}\eta\varphi$. Rewriting this as $\gamma_1^{ext}u = \gamma_1^{int}u + \mathbf{i}k\varphi$, we infer that u solves in Ω^+

$$-\Delta u = k^2 u \quad \text{on } \Omega^+, \qquad \gamma_1^{ext}u = \gamma_1^{int}u + \mathbf{i}k\varphi, \qquad u|_{\partial B_R} = u|_{\partial B_R}.$$

A priori bounds for the Laplace operator together with (6.9) give us

$$\begin{aligned}\|u\|_{H^1(\Omega_R\setminus\overline{\Omega})} &\leq C\left[\|k^2u\|_{H^{-1}(\Omega_R\setminus\overline{\Omega})} + \|\gamma_1^{ext}u\|_{H^{-1/2}(\Gamma)} + k\|\varphi\|_{H^{-1/2}(\Gamma)} + \|u\|_{H^{1/2}(\partial B_R)}\right]\\ &\leq Ck\left[\|g\|_{H^{1/2}(\Gamma)} + k^2\|\varphi\|_{H^{-1/2}(\Gamma)} + k^3\|\varphi\|_{H^{-3/2}(\Gamma)}\right],\end{aligned}$$

which concludes the argument. ∎

If the operator A is invertible and Assumption 1.4 is true, then we obtain the following regularity assertion for A^{-1}:

COROLLARY 6.3. *Let Γ be analytic, T be a tubular neighborhood of Γ, and C_g, $\gamma_g > 0$. Let Assumption 1.4 be satisfied for some $s_A \geq 0$. Then there exist constants C, $\gamma > 0$ such that for every $g \in \mathfrak{A}(C_g, \gamma_g, T\setminus\Gamma)$ the solution $\varphi \in H^{s_A}(\Gamma)$ of $A\varphi = [g]$ satisfies*

$$\varphi = [u], \qquad u \in \mathfrak{A}(CC_\varphi, \gamma, \Omega_R), \quad C_\varphi := C_g k(1 + k^\beta C(A, s_A, k)), \qquad \beta := \frac{5}{2} + s_A.$$

Furthermore, u is given explicitly by (6.3), i.e., $u = \widetilde{A}(A^{-1}[g])$.

Proof. From the trace inequality (and, in the limiting case $s_A = 0$, a multiplicative trace inequality) we get

$$\|[g]\|_{H^{s_A}(\Gamma)} \leq CC_g k^{s_A+1/2}, \qquad \|[g]\|_{L^2(\Gamma)} \leq CC_g k^{1/2}, \qquad \|[g]\|_{H^{1/2}(\Gamma)} \leq CC_g k.$$

Therefore, by assumption we obtain for $\varphi = A^{-1}[g]$

$$\|\varphi\|_{L^2(\Gamma)} \leq C\|\varphi\|_{H^{s_A}(\Gamma)} \leq CC(A, s_A, k)\|[g]\|_{H^{s_A}(\Gamma)} \leq CC(A, s_A, k)k^{s_A+1/2}C_g.$$

Lemma 6.2 then implies for the function $u = \widetilde{A}\varphi$

$$\|u\|_{L^2(\Omega_R)} + k^{-1}\|\nabla u\|_{L^2(\Omega_R)} \leq Ck^{7/2+s_A}C_g C(A, s_A, k).$$

An appeal to Lemma 6.1 concludes the argument. ∎

6.1.2. the operator A'. For the operator A', one can proceed very similarly as for the operator A.

LEMMA 6.4. *Let T be a tubular neighborhood of Γ and let $g_1 \in \mathfrak{A}(C_{g_1}, \gamma_1, T \setminus \Gamma)$ and $g_2 \in \mathfrak{A}(C_{g_2}, \gamma_2, T \setminus \Gamma)$. Let $\varphi \in H^{-1/2}(\Gamma)$ satisfy*

$$\left(\pm\frac{1}{2} + K' + \mathbf{i}\eta V\right)\varphi = k(\gamma_0^{ext} g_1 - \gamma_0^{int} g_1) + (\gamma_1^{ext} g_2 - \gamma_1^{int} g_2)$$

Then $\varphi = \gamma_1^{ext} u - \gamma_1^{int} u$ for a function

$$u \in \mathfrak{A}(CC_u, \Omega_R) \qquad C_u = C_{g_1} + C_{g_2} + k^{-1}\|\varphi\|_{H^{-1/2}(\Gamma)} + k\|\varphi\|_{H^{-3/2}(\Gamma)}.$$

The constants C, $\gamma > 0$ depend only on Γ, γ_{g_1}, γ_{g_2}, and k_0.

Proof. We introduce the potential $u := \widetilde{V}\varphi$, which satisfies the homogeneous Helmholtz equation in $\Omega \cup \Omega^+$. Additionally, it satisfies the jump conditions $\gamma_0^{int} u = \gamma_0^{ext} u$ and

$$\gamma_1^{int} u + \mathbf{i}\eta u = (\frac{1}{2} + K' + \mathbf{i}\eta V)\varphi \quad \text{and} \quad \gamma_1^{ext} u + \mathbf{i}\eta u = (-\frac{1}{2} + K' + \mathbf{i}\eta V)\varphi \qquad \text{on } \Gamma.$$

Let us assume that $A' = 1/2 + K' + \mathbf{i}\eta V$, since the case of $A' = -1/2 + K' + \mathbf{i}\eta V$ is handled with analogous arguments. For simplicity of notation, we assume, as we may in view of Lemma B.5, that $g_1 = g_2 = 0$ on Ω^+.

Then u solves the homogeneous Helmholtz equation in Ω with Robin boundary condition $\gamma_1^{int} u + \mathbf{i}\eta u = k\gamma_0^{int} g_1 + \gamma_1^{int} g_2$ on Γ. The analyticity of g_1 and g_2 then implies by Theorem B.3 the existence of a tubular neighborhood T' of Γ and a constant $\gamma > 0$ such that

$$u \in \mathfrak{A}(CC'_u, \gamma, T' \cap \Omega), \qquad C'_u := \left[k^{-1}\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} + C_{g_1} + C_{g_2}\right] \tag{6.10}$$

By means of Lemma B.5, we may view $\gamma_0^{int} u$ as the trace $\gamma_0^{ext}\widetilde{u}$ of a function $\widetilde{u} \in \mathfrak{A}(CC'_u, \widetilde{\gamma}, T'' \cap \Omega^+)$, where the tubular neighborhood T'' and the constant $\widetilde{\gamma}$ depend solely on Γ, γ, and k_0. In Ω^+, the function u satisfies the homogeneous Helmholtz equation and, in view of the jump condition $\gamma_0^{ext} u = \gamma_0^{int} u$, on Γ the Dirichlet boundary condition $\gamma_0^{ext} u = \gamma_0^{ext}\widetilde{u}$. Hence, we conclude from Theorem B.2 the existence of a tubular neighborhood (again denoted T) and constants C, $\gamma_u > 0$ that depend solely on Γ and $\widetilde{\gamma}$ such that

$$u \in \mathfrak{A}(CC''_u, \gamma_u, T \cap \Omega^+), \qquad C''_u = C'_u + k^{-1}\|u\|_{\mathcal{H},\Omega^+ \cap B_R}.$$

Corollary 4.5 implies

$$k^{-1}\|u\|_{\mathcal{H},B_{2R}} \leq C\left[k^{-1}\|\varphi\|_{H^{-1/2}(\Gamma)} + k\|\varphi\|_{H^{-3/2}(\Gamma)}\right]$$

so that we conclude $u \in \mathfrak{A}(CC_u, \gamma_u, T \setminus \Gamma)$ with C_u defined in the statement of the lemma. Finally, interior regularity (see [22, Prop. 5.5.1]) gives estimates not only near Γ but in all of Ω_R, i.e., $u \in \mathfrak{A}(CC_\varphi, \gamma_u, \Omega_R)$ for suitable γ_u, $C > 0$. Observing that $\gamma_1^{int} u - \gamma_1^{ext} u = \varphi$ concludes the proof. ∎

COROLLARY 6.5. *Let Γ be analytic, T be a tubular neighborhood of Γ, and C_{g_1}, C_{g_2}, $\gamma_g > 0$. Let Assumption 1.5 be satisfied for some $s_A \geq -1/2$. Then there exist constants C, $\gamma > 0$ independent of $k \geq k_0$ such that for all $g_1 \in \mathfrak{A}(C_{g_1}, \gamma_g, T \setminus \Gamma)$, $g_2 \in \mathfrak{A}(C_{g_2}, \gamma_g, T \setminus \Gamma)$ the solution $\varphi \in H^{s_A}(\Gamma)$ of $A\varphi = k[g_1] + [\partial_n g_2]$ satisfies*

$$\varphi = [\partial_n u], \qquad u \in \mathfrak{A}(CC_\varphi, \gamma, \Omega_R), \quad C_\varphi := (C_{g_1} + C_{g_2})\left(1 + k^\beta C(A', s_A, k)\right),$$
$$\beta = \frac{5}{2} + s_A^+, \qquad s_A^+ := \max\{s_A, 0\}.$$

Furthermore, u is given explicitly as $u = \widetilde{V}((A')^{-1}[g])$.
Proof. We use Lemma 6.4. Using $s_A \geq -1/2$ and Assumption 1.5 gives for the solution φ of $A'\varphi = k[g_1] + [\partial_n g_2]$

$$\begin{aligned}\|\varphi\|_{H^{-1/2}(\Gamma)} &\leq C\|\varphi\|_{H^{s_A}(\Gamma)} \leq CC(A,s_A,k)\|k[g_1]+[\partial_n g_2]\|_{H^{s_A}(\Gamma)} \\ &\leq CC(A,s_A,k)\|k[g_1]+[\partial_n g_2]\|_{H^{s_A^+}(\Gamma)} \leq C(A,s_A,k)k^{s_A^+ + 3/2}(C_{g_1}+C_{g_2}).\end{aligned}$$

Hence, we get

$$C_{g_1}+C_{g_2}+k^{-1}\|\varphi\|_{H^{-1/2}(\Gamma)}+k\|\varphi\|_{H^{-3/2}(\Gamma)} \leq C(C_{g_1}+C_{g_2})\left(1+k^{5/2+s_A^+}C(A',s_A,k)\right).$$

An appeal to Lemma 6.4 concludes the proof. ∎

6.2. finite regularity. This section is the core of the paper and provides decomposition results for the operators A^{-1} and $(A')^{-1}$ as operators acting on Sobolev spaces $H^s(\Gamma)$. These results are formulated as Theorems 6.7, 6.8. Before working out the details, we formulate a lemma that isolates an important structural element of the proof of Theorems 6.7, 6.8.

LEMMA 6.6 ("iteration lemma"). *Let T be a tubular neighborhood of Γ. Let s, $s_B \in \mathbb{R}$, and γ_1, γ_2, $\gamma > 0$ be given. Let $C_{smooth}(k)$, $C_{solve}(k) \geq 0$ be two, possibly k-dependent numbers.*
Assume that $B : H^s(\Gamma) \to H^s(\Gamma)$ satisfies the following conditions:

(i) B can be decomposed as

$$B = B_0 + B_{\mathcal{A}} + R$$

where $B_0 : H^s(\Gamma) \to H^s(\Gamma)$ is boundedly invertible, R is a bounded linear operator with

$$\|RB_0^{-1}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} \leq q < 1$$

and $B_{\mathcal{A}}$ is a bounded linear operator of the form

$$B_{\mathcal{A}} = k[\widetilde{B}_{\mathcal{A},1}] + [\partial_n \widetilde{B}_{\mathcal{A},2}]$$

with

$$\widetilde{B}_{\mathcal{A},i}\varphi \in \mathfrak{A}(C_{smooth}(k)\|\varphi\|_{H^s(\Gamma)}, \gamma, T\setminus\Gamma) \qquad \forall \varphi \in H^s(\Gamma), \qquad i \in \{1,2\}.$$

(ii) B^{-1} is a bounded linear operator in $H^s(\Gamma)$ and $H^{s_B}(\Gamma)$ (with possibly k-dependent norms).

(iii) If $\varphi \in H^s(\Gamma)$ satisfies $B\varphi = k[g_1] + [\partial_n g_2]$ for some $g_1 \in \mathfrak{A}(C_{g_1}, \gamma_1, T\setminus\Gamma)$, $g_2 \in \mathfrak{A}(C_{g_2}, \gamma_2, T\setminus\Gamma)$, then $\varphi = [u]$ (or, $\varphi = [\partial_n u]$) for some $u \in \mathfrak{A}(C_{solve}(k)(C_{G_1} + C_{G_2}), \gamma, \Omega_R)$.

Under these assumptions there exist constants $\widetilde{C}$, $\widetilde{\gamma} > 0$ depending only on γ_1, γ_2, and Γ, and k_0 such that B^{-1} can be written as

$$B^{-1} = B_Z + B_B$$

where B_B has the form $B_B f = [\widetilde{B}_B f]$ (or $B_B f = [\partial_n \widetilde{B}_B f]$) and

$$\begin{aligned}&\|B_Z\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} \leq (1-q)^{-1}\|B_0^{-1}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)}, \\ &\widetilde{B}_B f \in \mathfrak{A}(C_B, \widetilde{\gamma}, \Omega_R), \qquad C_B := \widetilde{C}C_{solve}(k)\,C_{smooth}(k)\|f\|_{H^s(\Gamma)}.\end{aligned}$$

Proof. For $f \in H^s(\Gamma)$ consider the following iteration:

$$\varphi_0^{finite} := 0, \qquad \varphi_0^{\mathcal{A}} := 0, \tag{6.11}$$

$$B_0 \delta_n^{finite} := f - B(\varphi_n^{finite} + \varphi_n^{\mathcal{A}}), \tag{6.12}$$

$$B\delta_n^{\mathcal{A}} := -B_{\mathcal{A}} \delta_n^{finite}, \tag{6.13}$$

$$\varphi_{n+1}^{finite} := \varphi_n^{finite} + \delta_n^{finite}, \qquad \varphi_{n+1}^{\mathcal{A}} := \varphi_n^{\mathcal{A}} + \delta_n^{\mathcal{A}}. \tag{6.14}$$

The sequences $(\varphi_n^{finite})_{n=0}^{\infty}$ and $(\varphi_n^{\mathcal{A}})_{n=0}^{\infty}$ converge as we now show. Define the residual $r_n := f - B(\varphi_n^{finite} + \varphi_n^{\mathcal{A}})$. Then

$$\begin{aligned} r_{n+1} &= f - B(\varphi_{n+1}^{finite} + \varphi_{n+1}^{\mathcal{A}}) = f - B(\varphi_n^{finite} + \delta_n^{finite} + \varphi_n^{\mathcal{A}} + \delta_n^{\mathcal{A}}) \\ &= r_n - B\delta_n^{finite} - B\delta_n^{\mathcal{A}} = r_n - (B_0 + B_{\mathcal{A}} + R)\delta_n^{finite} - B\delta_n^{\mathcal{A}} \\ &= -R\delta_n^{finite} - B_{\mathcal{A}}\delta_n^{finite} - B\delta_n^{\mathcal{A}} = -R\delta_n^{finite} = -RB_0^{-1} r_n. \end{aligned}$$

The assumption $\|RB_0^{-1}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} \leq q < 1$ therefore implies $\|r_n\|_{H^s(\Gamma)} \leq q^n \|r_0\|_{H^s(\Gamma)}$ and thus $\|\delta_n^{finite}\|_{H^s(\Gamma)} \leq q^n \|B_0^{-1}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} \|r_0\|_{H^s(\Gamma)}$. We conclude that the sum $\sum_{n=0}^{\infty} \delta_n^{finite}$ converges in $H^s(\Gamma)$. Since B is a bounded linear operator, also the sum $\sum_{n=0}^{\infty} \delta_n^{\mathcal{A}}$ converges in $H^s(\Gamma)$. We thus define the operators B_Z and B_B by

$$B_Z : f \mapsto \lim_{n\to\infty} \varphi_n^{finite} = \sum_{n=0}^{\infty} \delta_n^{finite}, \qquad B_B : f \mapsto \lim_{n\to\infty} \varphi_n^{\mathcal{A}} = \sum_{n=0}^{\infty} \delta_n^{\mathcal{A}}.$$

It is easy to see that $\|B_Z\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} \leq (1-q)^{-1} \|B_0^{-1}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)}$. Next, in view of $\lim_{n\to\infty} \delta_n^{finite} = 0$, we obtain from (6.12) that $\lim_{n\to\infty} \varphi_n^{finite} + \varphi_n^{\mathcal{A}}$ is the solution of $B\varphi = f$. To obtain the representation $B_B f = [\widetilde{B}_B f]$ (or $B_B f = [\partial_n \widetilde{B}_B]$), we sum the terms in (6.13) to get the relation

$$B(B_B f) = -B_{\mathcal{A}} B_Z f.$$

Thus, by assumptions on the operators B and $B_{\mathcal{A}}$, we see that $B_B f$ has the form $B_B f = [\widetilde{B}_B f]$ (or $[\partial_n \widetilde{B}_B f]$) for an operator $\widetilde{B}_B$ that satisfies

$$\widetilde{B}_B f \in \mathfrak{A}(CC_{solve}(k) C_{smooth}(k) \|f\|_{H^s(\Gamma)}, \widetilde{\gamma}, \Omega_R).$$

for appropriate $\widetilde{\gamma}$. ∎

6.2.1. the operator A. We show that the operator A^{-1} of (1.4) can be decomposed into a zero-th order operator with k-independent bounds and an analytic part:

Theorem 6.7 (decomposition of A^{-1}). *Let Γ be analytic. If $d = 2$, then then assume additionally* $\operatorname{diam} \Omega < 1$. *Let Assumption 1.4 be valid for some $s_A \geq 0$ and some $s \geq 0$. Then there exist constants C, $\gamma > 0$ independent of $k \geq k_0$ with the following properties: The operator A^{-1} can be written as*

$$A^{-1} = A_Z + \gamma_0^{ext} \widetilde{A}_{A^{-1}} - \gamma_0^{int} \widetilde{A}_{A^{-1}}$$

where the linear operators A_Z und $\widetilde{\mathcal{A}}_{A^{-1}}$ satisfy

$$\begin{aligned} &\|A_Z\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} \leq C, \\ &\widetilde{\mathcal{A}}_{A^{-1}} f \in \mathfrak{A}(CC_f, \gamma, \Omega_R), \qquad C_f := k^3 \left(1 + k^{5/2+s_A} C(A, s_A, k)\right) \|f\|_{H^s(\Gamma)}. \end{aligned}$$

Proof. Before turning to the proof, we point out that, since only the jump of $\widetilde{A}_A f$ across Γ appears in the decomposition of A^{-1}, there is some freedom in the choice of $\widetilde{A}_A$. In particular, $\widetilde{\mathcal{A}}_A$ can be selected such that $(\widetilde{\mathcal{A}}_A f)|_{\Omega^+} = 0$ or $(\widetilde{\mathcal{A}}_A f)|_{\Omega} = 0$. In fact, the proof shows that we construct $\widetilde{\mathcal{A}}_A$ such that $\widetilde{\mathcal{A}}_A f = 0$ on Ω^+ if $A = -1/2 + K - \mathbf{i}\eta V$ is considered and $\widetilde{\mathcal{A}}_A f = 0$ on Ω if $A = 1/2 + K - \mathbf{i}\eta V$.
Our starting point is Lemma 2.5, which asserts that $-1/2 + K_0$ and $1/2 + K_0 + \mathbf{i}V_0$ are invertible operators on $H^s(\Gamma)$. Lemma 5.5 and Remark 5.6 permit us to write

$$
\begin{aligned}
A &= -\frac{1}{2} + K_0 + R_A + k[\widetilde{\mathcal{A}}_A], && \text{if } A = -1/2 + K - \mathbf{i}\eta V,\\
A &= \frac{1}{2} + K_0 + \mathbf{i}V_0 + R_A + k[\widetilde{\mathcal{A}}_A], && \text{if } A = 1/2 + K - \mathbf{i}\eta V.
\end{aligned}
$$

with operators R_A and $\widetilde{\mathcal{A}}_A$ having the properties stated in Lemma 5.5. In the notation of Lemma 6.6, we set

$$
R = R_A, \quad \widetilde{B}_{\mathcal{A},1} = \widetilde{\mathcal{A}}_A, \quad \widetilde{B}_{\mathcal{A},2} = 0, \quad B_0 = \begin{cases} \frac{1}{2} + K_0 + \mathbf{i}V_0, & \text{if } A = 1/2 + K - \mathbf{i}\eta V,\\ -\frac{1}{2} + K_0 & \text{if } A = -1/2 + K - \mathbf{i}\eta V \end{cases}
$$

In view of Lemma 5.5, the norm $\|R_A\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)}$ can be made arbitrarily small. We may therefore assume that $\|RB_0^{-1}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} < 1$. Furthermore, Lemma 5.5 together with the trivial embedding $H^s(\Gamma) \subset H^{-1/2}(\Gamma) \subset H^{-3/2}(\Gamma)$ implies that $C_{smooth}(k) \le Ck$. Finally, Corollary 6.3 provides us, again in the terminology of Lemma 6.6, with

$$
C_{solve}(k) \sim k^2(1 + k^{5/2+s_A} C(A, s_A, k)). \tag{6.15}
$$

Thus, $C_{solve}(k)C_{smooth}(k) \sim k^3(1 + k^{5/2+s_A} C(A, s_A, k))$, and Lemma 6.6 implies the result. □

6.2.2. the operator A'. The operator A' is handled with similar techniques.
Theorem 6.8 (decomposition of $(A')^{-1}$). *Let Γ be analytic. If $d = 2$, then assume additionally* $\operatorname{diam}\Omega < 1$. *Let Assumption 1.5 be valid for some $s_A \ge -1/2$ and some $s \ge 0$. Then there exist constants C, $\gamma > 0$ independent of $k \ge k_0$ with the following properties: The operator $(A')^{-1}$ can be written as*

$$
(A')^{-1} = A'_Z + \gamma_1^{ext}\widetilde{A}_{A',inv} - \gamma_1^{int}\widetilde{A}_{A',inv}
$$

where the linear operators A'_A and $\widetilde{A}_{A',inv}$ satisfy with $s_A^+ := \max\{s_A, 0\}$

$$
\begin{aligned}
&\|A'_Z\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} \le C,\\
&\widetilde{A}_{A',inv} f \in \mathfrak{A}(CC_f, \gamma, \Omega_R), \qquad C_f := \left(1 + k^{5/2+s_A^+} C(A', s_A, k)\right)\|f\|_{H^s(\Gamma)}.
\end{aligned}
$$

Proof. With Lemma 5.7 and Remark 5.8 we write

$$
A' = \begin{cases} -\frac{1}{2} + K'_0 + R_{A'} + k[\widetilde{\mathcal{A}}_{A',1}] + [\partial_n \widetilde{\mathcal{A}}_{A',2}] & \text{if } A' = -1/2 + K' + \mathbf{i}\eta V,\\ \frac{1}{2} + K'_0 + \mathbf{i}V_0 + R_{A'} + k[\widetilde{\mathcal{A}}_{A',1}] + [\partial_n \widetilde{\mathcal{A}}_{A',2}] & \text{if } A' = 1/2 + K' + \mathbf{i}\eta V. \end{cases}
$$

This has the form required in Lemma 6.6, if we set

$$B_0 = \begin{cases} -\frac{1}{2} + K_0' & \text{if } A' = -1/2 + K + \mathbf{i}\eta V, \\ \frac{1}{2} + K_0' + \mathbf{i}V_0 & \text{if } A' = 1/2 + K + \mathbf{i}\eta V \end{cases} \qquad R = R_{A'}, \qquad \widetilde{B}_{\mathcal{A},i} = \widetilde{\mathcal{A}}_{A',i}, \quad i \in \{1,2\}.$$

By Lemma 2.5, the operator B_0 is invertible on $H^s(\Gamma)$. Hence, selecting q in Lemma 5.7 appropriately, we may assume $\|RB_0^{-1}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} < 1$. Lemma 5.7 provides the necessary information about the mapping properties of $\widetilde{B}_{\mathcal{A},i}$, $i \in \{1,2\}$. Since $s \geq 0$, we conclude that (in the notation of Lemma 6.6) $C_{smooth}(k) \sim k$. From Corollary 6.5 we obtain

$$C_{solve}(k) \sim k^{-1}\left(1 + k^{5/2+s_A^+} C(A', s_A, k)\right). \tag{6.16}$$

Lemma 6.6 then implies the result. ☐

Theorem 6.8 restricts its attention to the case $s \geq 0$. However, the case $s = -1/2$ is particular interest given that it is the energy space for the operator K'. We therefore modify the arguments slightly to cover this case as well:

Theorem 6.9 (decomposition of $(A')^{-1}$, negative norms). *Let Γ be analytic. If $d = 2$, then assume additionally* $\operatorname{diam}\Omega < 1$. *Let Assumption 1.5 be valid for some $s_A \geq -1/2$ and some $-1/2 \leq s \leq 0$. Then the operator $(A')^{-1}$ can be written as*

$$(A')^{-1} = A_Z' + \gamma_1^{ext}\widetilde{A}_{A'} - \gamma_1^{int}\widetilde{A}_{A'}$$

where the linear operators A_A' and $\widetilde{A}_{A'}$ satisfy with $s_A^+ := \max\{s_A, 0\}$

$$\|A_A'\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} \leq C,$$
$$\widetilde{A}_{A'}f \in \mathfrak{A}(CC_f, \gamma, T), \qquad C_f := k^{d/2}\left(1 + k^{5/2+s_A^+}C(A', s_A, k)\right)\|f\|_{H^s(\Gamma)}.$$

Here, C, $\gamma > 0$, and the tubular neighborhood T of Γ are independent of $k \geq k_0$.

Proof. We proceed as in the proof of Theorem 6.8 but replace the decomposition of Lemma 5.7 with that of Lemma 5.9. That lemma leads to $C_{smooth}(k) \leq k^{d/2} + k \sim k^{d/2}$. Since $C_{solve}(k)$ is given by (6.16) we get the desired result. ☐

7. L^2-stability and convergence. Since the operators A and A' will appear now in conjunction with their adjoints, it will use useful to write explicitly the k-dependence, i.e., we write A_k and A_k'. We will use the following additional operators:

$$A_0 = -1/2 + K_0 \qquad \text{if } A = -1/2 + K - \mathbf{i}\eta V, \tag{7.1}$$
$$A_0 = +1/2 + K_0 - \mathbf{i}V_0 \qquad \text{if } A = 1/2 + K - \mathbf{i}\eta V, \tag{7.2}$$
$$A_0' = -1/2 + K_0' \qquad \text{if } A' = -1/2 + K - \mathbf{i}\eta V, \tag{7.3}$$
$$A_0' = +1/2 + K_0' + \mathbf{i}V_0 \qquad \text{if } A' = 1/2 + K - \mathbf{i}\eta V. \tag{7.4}$$

We view these operators as operators acting on $L^2(\Gamma)$ and note that the operators A_0' given in (7.4), (7.3) are the $L^2(\Gamma)$-adjoints of the operators A_0 of (7.2), (7.1) respectively. Associated with these operators are the sesquilinear forms a_k and a_k' (which are linear in the first and anti-linear in the second argument) given by

$$a_k(u,v) := (A_k u, v)_0 = \pm\frac{1}{2}(u,v)_0 + (K_k u, v)_0 - \mathbf{i}\eta(V_k u, v)_0,$$
$$a_k'(u,v) := (A_k' u, v)_0 = \pm\frac{1}{2}(u,v)_0 + (K_k' u, v)_0 + \mathbf{i}\eta(V_k u, v)_0.$$

The operator equations (6.1), (6.2) are discretized as follows: given $X_N \subset L^2(\Gamma)$:

$$\text{find } u_N \in X_N \text{ s.t.} \qquad a_k(u_N, v) = (f, v)_0 \qquad \forall v \in X_N, \tag{7.5}$$
$$\text{find } u'_N \in X_N \text{ s.t.} \qquad a'_k(u'_N, v) = (f, v)_0 \qquad \forall v \in X_N. \tag{7.6}$$

Here and in the following, we use the short-hand $(\cdot,\cdot)_0$ to denote the $L^2(\Gamma)$-inner product. Since A_k and A'_k are compact perturbations of the identity operator, unique solvability of (7.5), (7.6) and quasi-optimality is given if X_N is sufficiently large. The purpose of the present section is to make the k-dependence of the required approximation properties of X_N explicit. We acknowledge here that our technique, which derives the stability of the method from approximation results for suitable adjoint problems, has previously been used in the literature, for example in [21, 24, 25] and [2].

7.1. regularity properties of auxiliary operators. In view of (1.3) we have

$$a_k(u, v) = \overline{a'_{-k}(v, u)} \qquad \forall u, v \in L^2(\Gamma),$$

which expresses the fact that the $L^2(\Gamma)$-adjoint of A_k is given by A'_{-k}:

LEMMA 7.1. *For every $k \in \mathbb{R} \setminus \{0\}$ the operators A_k and A'_{-k} are $L^2(\Gamma)$-adjoints of each other.*

We recall from Lemmata 5.5, 5.7 and Remarks 5.6, 5.8 that the operators $A_k - A_0$ and $A'_{-k} - A'_0$ can be decomposed into two parts, namely, a part that is arbitrarily small (as an operator $H^s(\Gamma) \to H^s(\Gamma)$ and therefore, in particular, $L^2(\Gamma) \to L^2(\Gamma)$) and an operator that maps into a class of analytic functions. In view of this observation and the fact that the operators A_k^{-1} and $(A'_{-k})^{-1}$ can, by Theorems 6.7, 6.8, be decomposed into a zero-th order operator (that is uniformly bounded in k) and an operator that maps into a class of analytic functions, we can can formulate the following result:

LEMMA 7.2. *Let Γ be analytic. Let q, $q' \in (0, 1)$ be given. Let $C(A_k, 0, k)$ and $C(A'_{-k}, 0, -k)$ be defined by Assumptions 1.4, 1.5. If $d = 2$, then assume additionally* $\operatorname{diam}\Omega < 1$. *Then*

$$A_k^{-1}(A_k - A_0) = T_A + [\widetilde{\mathcal{A}}_{k,A,inv}],$$
$$(A'_{-k})^{-1}(A'_{-k} - A'_0) = T_{A'} + k[\widetilde{\mathcal{A}}_{-k,A',inv,1}] + [\partial_n \widetilde{\mathcal{A}}_{-k,A',inv,2}],$$

where for some C, $\gamma > 0$ independent of $k \geq k_0$ and all $\varphi \in L^2(\Gamma)$:

$$\|T_A\|_{L^2 \leftarrow L^2} \leq q, \qquad \|T_{A'}\|_{L^2 \leftarrow L^2} \leq q',$$
$$\widetilde{\mathcal{A}}_{k,A,inv}\varphi \in \mathfrak{A}(CC_\varphi, \gamma, \Omega_R), \qquad C_\varphi = (1 + k^{5/2} C(A_k, 0, k))(1 + k^3)\|\varphi\|_{L^2(\Gamma)},$$
$$\widetilde{\mathcal{A}}_{-k,A',inv,i}\varphi \in \mathfrak{A}(CC'_\varphi, \gamma, \Omega_R), \qquad C'_\varphi = (1 + k^{5/2} C(A'_{-k}, 0, -k))\|\varphi\|_{L^2(\Gamma)}, \qquad i \in \{1, 2\}.$$

Proof. We first prove the decomposition result for $(A'_{-k})^{-1}(A'_{-k} - A'_0)$. From Lemma 5.7 (or Remark 5.8) and Theorem 6.8 we get

$$(A'_{-k})^{-1} = A'_Z + [\partial_n \widetilde{\mathcal{A}}_{A',inv}],$$
$$A'_{-k} - A'_0 = R_{A'} + k[\widetilde{\mathcal{A}}_{A',1}] + [\partial_n \widetilde{\mathcal{A}}_{A',2}].$$

Hence, we obtain

$$(A'_{-k})^{-1} = A'_Z R_{A'} + [\partial_n \widetilde{\mathcal{A}}_{A',inv}] R_{A'} + (A'_{-k})^{-1}\left(k[\widetilde{\mathcal{A}}_{A',1}] + [\partial_n \widetilde{\mathcal{A}}_{A',2}]\right).$$

We set $T_A := A'_Z R_{A'}$. From Theorem 6.8 we know that $\|A'_Z\|_{L^2 \leftarrow L^2}$ is bounded uniformly in k. Lemma 5.7 (or Remark 5.8) tells us that $\|R_{A'}\|_{L^2 \leftarrow L^2}$ can be made arbitrarily small. Hence, T_A has the desired property. For $\varphi \in L^2(\Gamma)$ we get from Theorem 6.8 and Lemma 5.7

$$\widetilde{\mathcal{A}}_{A',inv} R_{A'} \varphi \in \mathfrak{A}(CC_{\varphi,1}, \gamma, \Omega_R), \qquad C_{\varphi,1} = \Big(1 + k^{5/2} C(A'_{-k}, 0, -k)\Big) \|\varphi\|_{L^2(\Gamma)},$$
$$k\widetilde{\mathcal{A}}_{A',1}\varphi, \quad \widetilde{\mathcal{A}}_{A',2}\varphi \in \mathfrak{A}(CC_{\varphi,2}, \gamma, \Omega_R), \qquad C_{\varphi,2} = \Big(1 + k^{5/2} C(A'_{-k}, 0, -k)\Big) \|\varphi\|_{L^2(\Gamma)}.$$

Corollary 6.5 then allows us to define the operators $\widetilde{\mathcal{A}}_{-k,A',inv,i}$, $i \in \{1, 2\}$ with the stated properties.
The decomposition of $A_k^{-1}(A_k - A_0)$ is performed in an analogous way by making use of Theorem 6.7, Lemma 5.5, and Corollary 6.3: We can write

$$A_k^{-1} = A_Z + [\widetilde{\mathcal{A}}_{A^{-1}}],$$
$$A_k - A_0 = R_A + k[\widetilde{\mathcal{A}}_A].$$

Therefore, $A_k^{-1}(A_k - A_0) = A_Z R_A + [\widetilde{\mathcal{A}}_{A^{-1}}]R_A + A_k^{-1}k[\widetilde{\mathcal{A}}_A]$. Again, we set $T_A := A_Z R_A$ and see that its norm can be made arbitrarily small. The properties of $\widetilde{\mathcal{A}}_{A^{-1}}$ given in Theorem 6.7 and those of $\widetilde{\mathcal{A}}_A$ given in Lemma 5.5 together with Corollary 6.3 then imply the result. ◻

7.2. abstract convergence analysis. For the approximation space $X_N \subset L^2(\Gamma)$ we denote by $\Pi_N^{L^2} : L^2(\Gamma) \to X_N$ the $L^2(\Gamma)$-projection onto X_N. It will be useful to quantify the approximation of analytic functions from the space X_N:
Definition 7.3. *Let T be a fixed tubular neighborhood of Γ. For every $\gamma > 0$, define $\eta_1(N, k)$, $\eta_2(N, k, \gamma)$, $\eta(N, k, \gamma)$ by*

$$\eta_1(N, k, \gamma) := \sup\{\|k[u] - \Pi_N^{L^2} k[u]\|_{L^2(\Gamma)} \,|\, u \in \mathfrak{A}(1, \gamma, T \setminus \Gamma)\},$$
$$\eta_2(N, k, \gamma) := \sup\{\|[\partial_n u] - \Pi_N^{L^2} [\partial_n u]\|_{L^2(\Gamma)} \,|\, u \in \mathfrak{A}(1, \gamma, T \setminus \Gamma)\},$$
$$\eta(N, k, \gamma) := \eta_1(N, k, \gamma) + \eta_2(N, k, \gamma).$$

We point out that, by linearity, we have for functions $u \in \mathfrak{A}(C_u, \gamma, T \setminus \Gamma)$ the bound $\|k[u] - \Pi_N^{L^2} k[u]\|_{L^2(\Gamma)} \leq C_u \eta_1(N, k, \gamma)$ and an analogous estimate for $\|[\partial_n u] - \Pi_N^{L^2} [\partial_n u]\|_{L^2(\Gamma)}$.
We will also need stability properties of the spaces X_N for the operators A_0 and A'_0; for future reference we formulate these as assumptions:
Assumption 7.4. *The space X_N satisfies a uniform discrete inf-sup condition for the operator $-1/2 + K_0$, i.e., there exists $\gamma_0 > 0$ independent of N such that*

$$0 < \gamma_0 \leq \inf_{0 \neq u \in X_N} \sup_{0 \neq v \in X_N} \frac{|((-1/2 + K_0)u, v)_0|}{\|u\|_0 \, \|v\|_0}. \tag{7.7}$$

The inf-sup condition (7.7) is equivalent to

$$0 < \gamma_0 \leq \inf_{0 \neq u \in X_N} \sup_{0 \neq v \in X_N} \frac{|((-1/2 + K'_0)u, v)_0|}{\|u\|_0 \, \|v\|_0}, \tag{7.8}$$

with the same constant $\gamma_0 > 0$.

ASSUMPTION 7.5. *The space X_N satisfies a uniform discrete inf-sup condition for the operator $1/2 + K_0 - \mathbf{i}V_0$, i.e., there exists $\gamma_0 > 0$ independent of N such that*

$$0 < \gamma_0 \leq \inf_{0\neq u\in X_N} \sup_{0\neq v\in X_N} \frac{|((1/2 + K_0 - \mathbf{i}V_0)u, v)_0|}{\|u\|_0 \,\|v\|_0}. \qquad (7.9)$$

The inf-sup condition (7.9) is equivalent to

$$0 < \gamma_0 \leq \inf_{0\neq u\in X_N} \sup_{0\neq v\in X_N} \frac{|((1/2 + K_0' + \mathbf{i}V_0)u, v)_0|}{\|u\|_0 \,\|v\|_0}, \qquad (7.10)$$

with the same constant $\gamma_0 > 0$.

REMARK 7.6. *For smooth surfaces Γ, the operators $K_0 : L^2(\Gamma) \to L^2(\Gamma)$ and $V_0 : L^2(\Gamma) \to L^2(\Gamma)$ are compact. Hence, Assumptions 7.4, 7.5 are satisfied, for example, for standard hp-BEM spaces, when the discretization is sufficiently fine.*

We close this section with two approximation results.

LEMMA 7.7. *Let Γ be analytic. Let $q \in (0, 1)$ be given and let $\eta(N, -k, \gamma)$ be defined by Definition 7.3. Let Assumption 1.5 be true with $s_A = 0$. Then*

$$\|(\mathrm{Id} - \Pi_N^{L^2})(A'_{-k} - A'_0)\|_{L^2\leftarrow L^2} \leq q + Ck\eta(N, -k, \gamma),$$
$$\|(\mathrm{Id} - \Pi_N^{L^2})(A'_{-k})^{-1}(A'_{-k} - A'_0)\|_{L^2\leftarrow L^2} \leq q + C\left\{1 + k^{5/2}C(A'_{-k}, 0, -k)\right\}\eta(N, -k, \gamma),$$

for a $\gamma > 0$ that is independent of $k \geq k_0$ (but possibly depends on q).

Proof. From Lemma 5.7, we have $A'_{-k} - A'_0 = R_{A'} + k[\widetilde{A}_{A',1}] + [\partial_n \widetilde{A}_{A',2}]$ where $\|R_{A'}\|_{L^2\leftarrow L^2}$ can be made arbitrarily small. Thus, $\|(\mathrm{Id} - \Pi_N^{L^2})R_{A'}\|_{L^2\leftarrow L^2} \leq \|R_{A'}\|_{L^2\leftarrow L^2}$ can be made arbitrarily small. The $L^2(\Gamma)$-approximation of the remaining terms $[\widetilde{\mathcal{A}}_{A',1}]$, $[\partial_n \widetilde{\mathcal{A}}_{A',2}]$ directly lead to the stated estimate.
From Lemma 7.2 we get the decomposition $(A'_{-k})^{-1}(A'_{-k} - A'_0) = T_{A'} + k[\widetilde{\mathcal{A}}_{-k,A',inv,1}] + [\partial_n \widetilde{\mathcal{A}}_{-k,A',inv,2}]$, where $\|T_{A'}\|_{L^2\leftarrow L^2}$ can be made arbitrarily small. It is easy to see that the $L^2(\Gamma)$-approximation of the remaining terms leads to the stated estimate. ∎

LEMMA 7.8. *Let Γ be analytic. Let $q \in (0, 1)$ be given and let $\eta_1(N, k, \gamma)$ be defined by Definition 7.3. Let Assumption 1.4 be true with $s_A = 0$. Then*

$$\|(\mathrm{Id} - \Pi_N^{L^2})(A_k - A_0)\|_{L^2\leftarrow L^2} \leq q + Ck\eta_1(N, k, \gamma),$$
$$\|(\mathrm{Id} - \Pi_N^{L^2})A_k^{-1}(A_k - A_0)\|_{L^2\leftarrow L^2} \leq q + Ck^2\left\{1 + k^{5/2}C(A_k, 0, k)\right\}\eta_1(N, k, \gamma),$$

for a $\gamma > 0$ that is independent of $k \geq k_0$ (but possibly depends on q).

Proof. The proof follows the lines of that of Lemma 7.7. The estimate for $\|(\mathrm{Id} - \Pi_N^{L^2})(A_k - A_0)\|_{L^2\leftarrow L^2}$ follows from Lemma 5.5. Lemma 7.2 finally leads to the second bound. ∎

7.2.1. The case of the operator A. At the heart of our analysis is the following quasi-optimality result:

THEOREM 7.9. *Let Γ be analytic. If $d = 2$, then assume additionally* $\operatorname{diam}\Omega < 1$. *Let Assumption 1.4 be valid with $s_A = 0$. Let η_1, η_2 be defined in Definition 7.3. If $A = -1/2 + K - \mathbf{i}\eta V$, then let Assumption 7.4 be valid; if $A = 1/2 + K - \mathbf{i}\eta V$, then let Assumption 7.5 be satisfied.*
Then there exist constants ε, $\gamma > 0$ independent of k such that under the assumption

$$k\eta_1(N, k, \gamma) \leq \varepsilon, \qquad \left(1 + k^{5/2}C(A'_{-k}, 0, -k)\right)\eta(N, -k, \gamma) \leq \varepsilon \qquad (7.11)$$

the following is true: If $u \in L^2(\Gamma)$ and $u_N \in X_N$ are two functions that satisfy the Galerkin orthogonality

$$a_k(u - u_N, v) = 0 \qquad \forall v \in X_N \tag{7.12}$$

then with γ_0 as stated in Assumptions 7.4, 7.5

$$\|u - u_N\|_{L^2(\Gamma)} \leq 2\left(1 + \frac{\|A_0\|_{L^2 \leftarrow L^2}}{\gamma_0}\right) \inf_{w_N \in X_N} \|u - w_N\|_{L^2(\Gamma)}. \tag{7.13}$$

Proof. We introduce the abbreviation $e := u - u_N$. Let $w_N \in X_N$ be arbitrary. Then by the triangle inequality

$$\|e\|_0 \leq \|u - w_N\|_0 + \|u_N - w_N\|_0. \tag{7.14}$$

Hence, we have to estimate $\|u_N - w_N\|_0$. By the discrete inf-sup condition we can find a $v_N \in X_N$ with $\|v_N\|_0 = 1$ and $\gamma_0 \|u_N - w_N\|_0 \leq (A_0(u_N - w_N), v_N)_0$. With the Galerkin orthogonality $(A_k(u - u_N), v_N)_0 = 0$, we then obtain

$$\begin{aligned}
\gamma_0 \|u_N - w_N\|_0 &\leq ((A_0 - A_k)(u_N - w_N), v_N)_0 + (A_k(u_N - w_N), v_N)_0 \\
&= ((A_0 - A_k)(u_N - w_N), v_N)_0 + (A_k(u - w_N), v_N)_0 \\
&= ((A_k - A_0)e, v_N)_0 + (A_0(u - w_N), v_N)_0 \\
&\leq \|A_0\|_{L^2 \leftarrow L^2} \|u - w_N\|_0 + ((A_k - A_0)e, v_N)_0.
\end{aligned} \tag{7.15}$$

In order to treat the term $((A_k - A_0)e, v_N)_0$ we define $\psi \in L^2(\Gamma)$ by

$$((A_k - A_0)z, v_N)_0 = (z, A'_{-k}\psi)_0 \qquad \forall z \in L^2(\Gamma). \tag{7.16}$$

Lemma 7.2 tells us

$$\psi = (A'_{-k})^{-1}(A'_{-k} - A'_0)v_N \tag{7.17}$$

By selecting $z = e$ in (7.16), using Galerkin orthogonality satisfied by the error e and orthogonality properties of $\Pi_N^{L^2}$ we obtain

$$\begin{aligned}
&((A_k - A_0)e, v_N)_0 \\
&= (e, A'_{-k}\psi)_0 = (A_k e, \psi)_0 = (A_k e, \psi - \Pi_N^{L^2}\psi)_0 \\
&= (A_0 e, \psi - \Pi_N^{L^2}\psi)_0 + ((A_k - A_0)e, \psi - \Pi_N^{L^2}\psi)_0 \\
&= (A_0 e, \psi - \Pi_N^{L^2}\psi)_0 + ((A_k - A_0)e - \Pi_N^{L^2}(A_k - A_0)e, \psi - \Pi_N^{L^2}\psi)_0.
\end{aligned}$$

Hence, from (7.17) and $\|v_N\|_0 = 1$

$$\begin{aligned}
|((A_k - A_0)e, v_N)_0| \leq &\left\{\|A_0\|_{L^2 \leftarrow L^2} + \|(\mathrm{Id} - \Pi_N^{L^2})(A_k - A_0)\|_{L^2 \leftarrow L^2}\right\} \\
&\times \|(\mathrm{Id} - \Pi_N^{L^2})(A'_{-k})^{-1}(A'_{-k} - A'_0)\|_{L^2 \leftarrow L^2} \|e\|_0.
\end{aligned}$$

From Lemmata 7.7, 7.8 we get for arbitrary $q \in (0, 1)$

$$\begin{aligned}
|((A_k - A_0)e, v_N)_0| \leq &\{\|A_0\|_{L^2 \leftarrow L^2} + q + Ck\eta_1(N, k, \gamma)\} \\
&\times \left\{q + C\left(1 + k^{5/2}C(A'_{-k}, 0, -k)\right)\eta(N, -k, \gamma)\right\} \|e\|_0.
\end{aligned} \tag{7.18}$$

Select now $q \in (0,1)$ such that $(\|A_0\|_{L^2 \leftarrow L^2} + q)q < 1/2$. Then the constants C and γ in (7.18) are fixed and independent of $k \geq k_0$. We can furthermore select $\varepsilon > 0$ independent of k such that the assumption (7.11) then guarantees that the product of the two curly braces in (7.18) is bounded by 1/2. Combining (7.14), (7.15), and (7.18) therefore yields

$$\|e\|_0 \leq \left(1 + \frac{\|A_0\|_{L^2 \leftarrow L^2}}{\gamma_0}\right) \|u - w_N\|_0 + \frac{1}{2}\|e\|_0,$$

which leads to the desired estimate. □

Theorem 7.9 provides quasi-optimality under the assumption that $u_N \in X_N$ exists. However, the discrete inf-sup condition follows easily from Theorem 7.9. In particular, we obtain that the discrete inf-sup constant is, up to a constant that is independent of k, and N, the inf-sup constant for the continuous problem. This is a consequence of the following, general result:

THEOREM 7.10. *Let X be a Hilbert space with norm $\|\cdot\|_X$. Let $X_N \subset X$ be a finite-dimensional subspace. Let $a : X \times X \to \mathbb{C}$ be a continuous sesquilinear form that satisfies the inf-sup condition*

$$0 < \gamma_a \leq \inf_{0 \neq u \in X} \sup_{0 \neq v \in X} \frac{|a(u,v)|}{\|u\|_X \|v\|_X}.$$

Let $C_{qopt} > 0$ be such that any pair $(u, u_N) \in X \times X_N$ that satisfies the Galerkin orthogonality

$$a(u - u_N, v) = 0 \qquad \forall v \in X_N$$

enjoys the best approximation property

$$\|u - u_N\|_X \leq C_{qopt} \inf_{v \in X_N} \|u - v\|_X.$$

Then the discrete inf-sup condition holds, i.e.,

$$\inf_{0 \neq u \in X_N} \sup_{0 \neq v \in X_N} \frac{|a(u,v)|}{\|u\|_X \|v\|_X} =: \gamma_N \geq \gamma_a \frac{1}{1 + C_{qopt}} > 0.$$

Proof. We first show that the restriction of the sesquilinear form a to $X_N \times X_N$ induces an injective operator $X_N \to X_N'$. To see this, let $u_N \in X_N$ satisfy $a(u_N, v) = 0$ for all $v \in X_N$. Our assumption is then applicable to the pair $(u, u_N) = (0, u_N)$, and we get $\|u_N\|_X = \|u - u_N\|_X \leq C_{qopt} \inf_{v \in X_N} \|u - v\|_X \leq C_{qopt}\|u\|_X = 0$. By dimension arguments, therefore, the Galerkin projection operator $P_N : X \to X_N$ given by

$$a(u - P_N u, v) = 0 \qquad \forall v \in X_N$$

is well-defined. Additionally, the quasi-optimality assumption produces the stability result $\|P_N u\|_X \leq \|u\|_X + \|u - P_N u\|_X \leq (1 + C_{qopt})\|u\|_X$.

It is known that

$$\inf_{0 \neq u \in X_N} \sup_{0 \neq v \in X_N} \frac{|a(u,v)|}{\|u\|_X \|v\|_X} = \inf_{0 \neq v \in X_N} \sup_{0 \neq u \in X_N} \frac{|a(u,v)|}{\|u\|_X \|v\|_X}.$$

We will therefore just compute the second inf-sup constant. To that end, let $v \in X_N \setminus \{0\}$. Then by Galerkin orthogonality and $v \in X_N$

$$\sup_{0 \not\in u \in X_N} \frac{|a(u,v)|}{\|u\|_X \|v\|_X} = \sup_{0 \not\in u \in X} \frac{|a(P_N u,v)|}{\|P_N u\|_X \|v\|_X} = \sup_{0 \neq u \in X} \frac{|a(u,v)|}{\|P_N u\|_X \|v\|_X}$$
$$\geq \frac{1}{1+C_{qopt}} \sup_{0 \neq u \in X} \frac{|a(u,v)|}{\|u\|_X \|v\|_X} \geq \frac{1}{1+C_{qopt}} \gamma_a.$$

Taking the infimum over all $v \in X_N$ concludes the argument. □

Combining Theorems 7.10 and 7.9 yields:

COROLLARY 7.11. *Assume the hypotheses of Theorem 7.9. If the approximation space X_N satisfies (7.11), then (7.5) is uniquely solvable and the quasi-optimality result (7.13) is true.*

7.2.2. The case of the operator A'. The results of Section 7.2.1 for the discretization of the operator A_k have clearly analogs for the discretization of the operator A'_k. Since the procedure is very similar to that of Section 7.2.1, we merely state the results and leave their proofs to the reader.

THEOREM 7.12. *Let Γ be analytic. If $d = 2$, then assume additionally* $\operatorname{diam}\Omega < 1$. *Let Assumption 1.5 be valid with $s_A = 0$. Let η_1, η be defined in Definition 7.3. If $A' = -1/2 + K' + \mathbf{i}\eta V$, then let Assumption 7.4 be valid; if $A' = 1/2 + K' + \mathbf{i}\eta V$, then let Assumption 7.5 be satisfied.*
Then there exist constants ε, $\gamma > 0$ independent of $k \geq k_0$ such that under the assumption

$$k\eta(N,k,\gamma) \leq \varepsilon, \qquad k^2 \left(1 + k^{5/2} C(A_{-k}, 0, -k)\right) \eta_1(N,-k,\gamma) \leq \varepsilon \tag{7.19}$$

the following is true: If $u \in L^2(\Gamma)$ and $u_N \in X_N$ are two functions that satisfy the Galerkin orthogonality

$$a'_k(u - u_N, v) = 0 \qquad \forall v \in X_N \tag{7.20}$$

then with γ_0 as stated in Assumption 7.4 or 7.5

$$\|u - u_N\|_{L^2(\Gamma)} \leq 2 \left(1 + \frac{\|A'_0\|_{L^2 \leftarrow L^2}}{\gamma_0}\right) \inf_{w_N \in X_N} \|u - w_N\|_{L^2(\Gamma)}. \tag{7.21}$$

Proof. See Appendix E. □

COROLLARY 7.13. *Assume the hypotheses of Theorem 7.12. If the approximation space X_N satisfies (7.19), then (7.6) is uniquely solvable and the quasi-optimality result (7.21) is true.*

7.3. classical *hp*-BEM. The analysis of the preceding section shows that the stability and convergence analysis of discretizations of the operators A and A' can be reduced to questions of approximability. As an example of the abstract theory, we consider the classical *hp*-BEM. We restrict our attention here to a situation in which the h-dependence can be obtained by scaling arguments.

We let $\widehat{K}^{d-1} = \{x \in \mathbb{R}^{d-1} \,|\, 0 < x_i < 1,\ \sum_{i=1}^{d-1} x_i < 1\}$ and $\widehat{K}^d = \{x \in \mathbb{R}^d \,|\, 0 < x_i < 1,\ \sum_{i=1}^{d} x_i < 1\}$ be the references simplices in $\mathbb{R}^{d-1}$ and $\mathbb{R}^d$. By $\mathcal{T}$ we denote a triangulation of Γ into elements $K \in \mathcal{T}$, where the elements K are assumed to be the images of $\widehat{K}^{d-1}$ under smooth *element maps* $F_K : \widehat{K}^{d-1} \to K$. The element maps

F_K are furthermore required to be C^1-diffeomorphisms between $\overline{\widehat{K}^{d-1}}$ and $\overline{K}$. For $p \in \mathbb{N}_0$, we then define the hp-BEM space $S^p(\mathcal{T})$ by

$$S^p(\mathcal{T}) = \{u \in L^2(\Gamma) \,|\, u|_K \circ F_K \in \mathcal{P}_p \quad \forall K \in \mathcal{T}\}, \tag{7.22}$$

where $\mathcal{P}_p$ is the vector space of all polynomials of degree p.

To motivate the class of triangulations of Assumption 7.16 below, we consider the following two examples:

EXAMPLE 7.14. *Let $d = 2$ and $\Gamma = \partial\Omega \subset \mathbb{R}^d$ be an analytic curve. Let the analytic function $R : [0, 1) \to \Gamma$ be a parametrization of Γ. Denote by $\widehat{\mathcal{T}}$ a uniform mesh on $[0, 1)$ with mesh size h. Define the mesh $\mathcal{T}$ by "transporting" the elements of $\widehat{\mathcal{T}}$ to Γ via R. Then the element maps F_K have the form $F_K = R \circ A_K$, where A_K is an affine map with $\|\nabla A_K\| \le Ch$ and $\|(\nabla A_K)^{-1}\| \le Ch^{-1}$. These element maps have the form stipulated in Definition 7.16 below.*

EXAMPLE 7.15. *Let $d = 3$ and $\Gamma = \partial\Omega$ be analytic. Let $\mathcal{T}^d$ be a patchwise constructed mesh on the domain Ω as given in [25, Example 5.1]. There, the element maps $F_K : \widehat{K}^d \to K$ have the form $F_K = R_K \circ A_K$ for an affine map A_K with $\|\nabla A_K\| \le Ch$ and $\|(\nabla A_K)^{-1}\| \le Ch^{-1}$ and the functions R_K satisfy*

$$\|(\nabla R_K)^{-1}\|_{L^\infty(\widetilde{K}^d)} \le C_{\text{metric}}, \qquad \|\nabla^n R_K\|_{L^\infty(\widetilde{K}^d)} \le C_{\text{metric}} \gamma^n n! \qquad \forall n \in \mathbb{N}_0;$$

here, $\widetilde{K}^d = A_K(\widehat{K}^d)$ is the image of the reference simplex $\widehat{K}^d$ under the affine map A_K. The mesh $\mathcal{T}^d$ on the domain Ω induces in a canonical way a mesh mesh on $\Gamma = \partial\Omega$. This trace mesh has the properties specified in the Definition 7.16 below.

The two examples motivate the following assumptions on the triangulation of Γ:

DEFINITION 7.16 (quasi-uniform triangulation). *A triangulation $\mathcal{T}_h$ of the analytic manifold Γ is said to be a quasi-uniform mesh with mesh size h if the following is true: Each element map F_K can be written as $F_K = R_K \circ A_K$, where A_K is an affine map and the maps R_K and A_K satisfy for constants C_{affine}, C_{metric}, $\gamma_{\mathcal{T}} > 0$ independent of h:*

$$\begin{aligned} &\|\nabla A_K\|_{L^\infty(\widehat{K})} \le C_{\text{affine}} h, \qquad &&\|(\nabla A_K)^{-1}\|_{L^\infty(\widehat{K})} \le C_{\text{affine}} h^{-1} \\ &\|(\nabla R_K)^{-1}\|_{L^\infty(\widetilde{K})} \le C_{\text{metric}}, \qquad &&\|\nabla^n R_K\|_{L^\infty(\widetilde{K})} \le C_{\text{metric}} \gamma_{\mathcal{T}}^n n! \qquad \forall n \in \mathbb{N}_0. \end{aligned}$$

Here, $\widetilde{K} = A_K(\widehat{K})$.

LEMMA 7.17. *Let Γ be analytic. Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Γ with mesh size h in the sense of Definition 7.16. Fix a tubular neighborhood T of Γ. Let $X_N = S^p(\mathcal{T}_h)$. Let $\widetilde{C} > 0$ be fixed and assume that h, p, and k satisfy*

$$\frac{kh}{p} \le \widetilde{C}.$$

Then, for every $\gamma > 0$ there exist C, $\sigma > 0$ (independent of h, p, and $k \ge k_0$) such that

$$\eta(N, k, \gamma) \le \eta_1(N, k, \gamma) + \eta_2(N, k, \gamma) \le Ck^{3/2} \left\{ \left(\frac{h}{\sigma + h}\right)^{p+1} + \left(\frac{kh}{\sigma p}\right)^{p+1} \right\}.$$

Proof. We only sketch the arguments for the bound on η_1, which quantifies how well the jump $k[u]$ of a piecewise analytic function can be approximated from $X_N =$

$S^p(\mathcal{T}_h)$. Using Lemma B.5, we may assume that $u|_{\Omega^+} = 0$. Denote by $\vec{n}(x)$ the outer normal vector of Ω at the point $x \in \Gamma$.

1. step: Let T_h be a tubular neighborhood of Γ of width $O(h)$ and $u \in \mathfrak{A}(C_u, \gamma_u, T\setminus\Gamma)$ for a fixed tubular neighborhood of Γ. We assume that h is small (as compared to the width of T). With the aid of [19, Lemma 2.1] and the interpolation inequality $\|v\|_{B^{1/2}_{2,1}(\Omega)} \lesssim \|v\|^{1/2}_{L^2(\Omega)}\|v\|^{1/2}_{H^1(\Omega)}$, we conclude

$$\|\nabla^n u\|_{L^2(T_h)} \leq C\sqrt{kh}C_u(\gamma_u')^n \max\{k, n+1\}^n \qquad \forall n \in \mathbb{N}_0, \tag{7.23}$$

where the constants C, γ_u' are independent of $k \geq k_0$ and h.

2. step: The reference simplex $\widehat{K}^d$ can be written in the form $\widehat{K}^d = \{(\widehat{x}, z) \,|\, 0 < z < 1, \widehat{x} \in z\widehat{K}^{d-1}\}$. The element maps $F_K : \widehat{K}^{d-1} \to \Gamma$ have the form $F_K = R_K \circ A_K$. Define

$$\begin{aligned} A_K^d &: \widehat{K}^d \ni (\widehat{x}, z) \mapsto (A_K(\widehat{x}), hz)\,, \\ R_K^d &: \widetilde{K}^d \ni (\widetilde{x}, \widetilde{z}) \mapsto R_K(\widetilde{x}) - \widetilde{z}\vec{n}(R_K(\widetilde{x})); \end{aligned}$$

here $\widetilde{K}^d$ is the image of $\widehat{K}^d$ under A_K^d, and $\widetilde{x} \in \widetilde{K}$, $\widetilde{z} \in \mathbb{R}$. The assumption on A_K implies readily that $A_K^d : \widehat{K}^d \to \widetilde{K}^d$ satisfies

$$\|\nabla A_K^d\|_{L^\infty(\widehat{K}^d)} \leq Ch, \qquad \|(\nabla A_K^d)^{-1}\|_{L^\infty(\widehat{K}^d)} \leq Ch^{-1}$$

for a constant C that is independent of h. The analyticity of Γ implies furthermore that the function R_K^d satisfies for some constants c_0, C_g, γ_g that depend solely on Γ and the constants C_{metric}, $\gamma_{\mathcal{T}}$

$$\|(\nabla R_K^d)^{-1}\|_{L^\infty(\widetilde{K}^d)} \geq c_0, \qquad \|\nabla^n R_K^d\|_{L^\infty(\widetilde{K}^d)} \leq C_g \gamma_g n! \qquad \forall n \in \mathbb{N}_0.$$

3. step: The images $K^d = (R_K^d \circ A_K^d)(\widehat{K}^d)$ lie in a tubular neighborhood T_h of Γ that has width $O(h)$. Furthermore, geometric considerations imply a finite overlap property, namely, the existence of a constant $M > 0$ such that any $x \in \Omega$ is in no more than M of these sets:

$$\sup_{x\in\Omega} |\{K \in \mathcal{T}_h \,|\, x \in K^d\}| \leq M. \tag{7.24}$$

4. step: Define for each $K \in \mathcal{T}_h$ the constant

$$C_K^2 := \sum_{n\in\mathbb{N}_0} \frac{1}{(2\gamma_u' \max\{k, n\}^n)^2} \|\nabla^n u\|^2_{L^2(K^d)} \tag{7.25}$$

and note that (7.23) and (7.24) imply

$$\sum_{K\in\mathcal{T}_h} C_K^2 \leq M \sum_{n\in\mathbb{N}_0} \left(\frac{1}{2\gamma_u' \max\{k, n+1\}^n}\right)^2 \|\nabla^n u\|^2_{L^2(T_h)} \leq \frac{4}{3} CMC_u^2 kh. \tag{7.26}$$

5. step: We have $u|_{K^d} \in \mathfrak{A}(C_K, 2\gamma_u', K^d)$, and [25, Lemma C.1] implies that the function $u \circ R_K^d$ satisfies $u \circ R_K^d \in \mathfrak{A}(CC_K, \tilde{\gamma}_u, \widetilde{K}^d)$, where the constants C and $\tilde{\gamma}_u$ depend solely on γ_u', γ_g, and C_g. Since the map A_K^d is affine and $F_K^d = R_K^d \circ A_K^d$, we get for constants C, $\overline{\gamma}$ independent of k and h

$$\|\nabla^n (u \circ F_K^d)\|_{L^2(\widehat{K}^d)} \leq CC_K h^{-d/2} (\overline{\gamma}h)^n \max\{k, n\}^n \qquad \forall n \in \mathbb{N}_0.$$

Next, [25, Lemma C.2] gives for constants C, $\sigma > 0$ independent of h, p, and $k \geq k_0$

$$\inf_{\pi \in \mathcal{P}_p} \|u \circ F_K^d - \pi\|_{L^\infty(\widehat{K}^d)} \leq C C_K h^{-d/2} \left(\left(\frac{h}{\sigma + h} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right),$$

where $\mathcal{P}_p$ is the space of d-variate polynomials of degree p. Hence, taking the trace on the $d-1$-dimensional face $\widehat{K}^{d-1}$ produces

$$\inf_{\pi \in \mathcal{P}_p} \|u \circ F_K - \pi\|_{L^\infty(\widehat{K})} \leq C C_K h^{-d/2} \left(\left(\frac{h}{\sigma + h} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right),$$

where $\mathcal{P}_p$ denotes the space of $d-1$-variate polynomials of degree p. Scaling back to the element K and summing over all elements $K \in \mathcal{T}_h$ yields

$$\begin{aligned} \inf_{\pi \in S^p(\mathcal{T}_h)} \|u - \pi\|^2_{L^2(\Gamma)} &\leq \sum_{K \in \mathcal{T}_h} C C_K^2 h^{-d} h^{d-1} \left(\left(\frac{h}{\sigma + h} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right)^2 \\ &\leq C C_u^2 k \left(\left(\frac{h}{\sigma + h} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right)^2. \end{aligned}$$

Recalling that that we are actually interested in the approximation of the function ku instead of u, we see that we have obtained the desired bound for η_1. ☐

THEOREM 7.18 (quasi-optimality for A). *Let Γ be analytic. If $d = 2$, then assume additionally* $\operatorname{diam}\Omega < 1$. *Let $\mathcal{T}_h$ a quasi-uniform mesh on Γ of mesh size h in the sense of Definition 7.16. Let $X_N = S^p(\mathcal{T}_h)$. Then there exist constants C, ε, $\sigma > 0$ independent of h, k, and p such that the following is true: If the* scale resolution *condition*

$$\left\{ k^{5/2} + k^4 C(A'_{-k}, 0, -k) \right\} \left\{ \left(\frac{h}{\sigma + h} \right)^{p+1} + \left(\frac{kh}{\sigma p} \right)^{p+1} \right\} \leq \varepsilon \tag{7.27}$$

is satisfied, then (7.5) has a unique solution u_N which satisfies

$$\|u - u_N\|_{L^2(\Gamma)} \leq C \inf_{v \in S^p(\mathcal{T}_h)} \|u - v\|_{L^2(\Gamma)}, \tag{7.28}$$

where $C > 0$ is independent of $k \geq k_0$.

Proof. Combine Theorem 7.9 with Lemma 7.17. ☐

We now turn to a corollary for the case that the $C(A'_{-k}, 0, -k)$ grows only polynomially in k. This assumption is quite reasonable in view of [9] who showed that for star-shaped Ω and the case $A'_k = 1/2 + K_k + \mathbf{i}\eta V_k$ we have that $C(A'_k, 0, k)$ is bounded uniformly in k (i.e., $\beta = 0$ in the following corollary).

COROLLARY 7.19. *Assume the hypotheses of Theorem 7.18. Assume additionally the existence of C, $\beta \geq 0$ independent of k such that*

$$C(A'_{-k}, 0, -k) \leq C k^\beta. \tag{7.29}$$

Then there exist constants C_1, C_2 independent of h, k, and p such that for

$$\frac{hk}{p} \leq C_1 \qquad \text{and} \qquad p \geq C_2 \log k \tag{7.30}$$

the quasi-optimality assertion (7.28) of Theorem 7.18 is true.

REMARK 7.20. *Corollary 7.19 can be phrased in a different way: the onset of quasi-optimality of the BEM is guaranteed for the choice*

$$p = \lceil C_2 \log k \rceil \qquad \textit{and } h = C_1 \frac{p}{k}.$$

The corresponding problem size $N := \dim S^p(\mathcal{T}_h)$ *is given by*

$$N = \dim S^p(\mathcal{T}_h) \sim h^{-(d-1)} p^{d-1} \sim k^{d-1};$$

i.e., the onset of quasi-optimality of the BEM is achieved with a fixed number of degrees of freedom per wavelength.

Results corresponding to the above ones for the operator A hold for the operator A'. We merely record the statements.

THEOREM 7.21 (quasi-optimality for A'). *Let* Γ *be analytic. If* $d = 2$*, then assume additionally* $\operatorname{diam}\Omega < 1$*. Let* $\mathcal{T}_h$ *a quasi-uniform mesh of mesh size* h *in the sense of Definition 7.16. Let* $X_N = S^p(\mathcal{T}_h)$*. Then there exist constants* C, ε, $\sigma > 0$ *independent of* h, k, *and* p *such that the following is true: If the* scale resolution *condition*

$$\left\{k^{7/2} + k^6 C(A_k, 0, k)\right\} \left\{ \left(\frac{h}{\sigma + h}\right)^{p+1} + \left(\frac{kh}{\sigma p}\right)^{p+1} \right\} \leq \varepsilon \tag{7.31}$$

is satisfied, then (7.6) has a unique solution u_N *which satisfies*

$$\|u - u_N\|_{L^2(\Gamma)} \leq C \inf_{v \in S^p(\mathcal{T}_h)} \|u - v\|_{L^2(\Gamma)}, \tag{7.32}$$

where $C > 0$ *is independent of* k.

COROLLARY 7.22. *Assume the hypotheses of Theorem 7.21. Assume additionally the existence of* C, $\beta \geq 0$ *independent of* k *such that*

$$C(A_k, 0, k) \leq C k^\beta. \tag{7.33}$$

Then there exist constants C_1, C_2 *independent of* h, k, *and* p *such that for*

$$\frac{hk}{p} \leq C_1 \qquad \textit{and} \qquad p \geq C_2 \log k \tag{7.34}$$

the quasi-optimality assertion (7.32) of Theorem 7.21 is true.

REMARK 7.23. *Corollary 7.22 can be phrased in a different way: the onset of quasi-optimality of the BEM is guaranteed for the choice*

$$p = \lceil C_2 \log k \rceil \qquad \textit{and } h = C_1 \frac{p}{k}.$$

The corresponding problem size $N := \dim S^p(\mathcal{T}_h)$ *is given by*

$$N = \dim S^p(\mathcal{T}_h) \sim h^{-(d-1)} p^{d-1} \sim k^{d-1};$$

i.e., the onset of quasi-optimality of the BEM is achieved with a fixed number of degrees of freedom per wavelength. We conclude this section with the classical result that for smooth boundaries Γ, we expect the quasi-optimality constant to be asymptotically 1:

LEMMA 7.24. *Let $T : L^2(\Gamma) \to L^2(\Gamma)$ be compact and assume that $\operatorname{Id} + T$ is invertible. Assume that $(X_N)_{N\in\mathbb{N}} \subset L^2(\Gamma)$ satisfies*

$$\lim_{N\to\infty} \inf_{v\in X_N} \|u - v\|_{L^2(\Gamma)} = 0 \qquad \forall u \in L^2(\Gamma).$$

Then there exists $N_0 > 0$ such that for every $N \geq N_0$ the problem

$$\text{given } u \in L^2(\Gamma) \text{ find } u_N \in X_N \text{ s.t.} \quad (\,(\operatorname{Id} + T)u_N, v_N)_0 = (\,(\operatorname{Id} + T)u, v_N)_0 \qquad \forall v_N \in X_N$$

has a unique solution. Furthermore, for every $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that for $N \geq N_\varepsilon$ we have

$$\|u - u_N\|_{L^2(\Gamma)} \leq (1 + \varepsilon) \inf_{v\in X_N} \|u - v\|_{L^2(\Gamma)}$$

Proof. In view of Theorem 7.10, it suffices to concentrate on the quasi-optimality statement. It will be convenient to recall that $\Pi_N^{L^2} : L^2(\Gamma) \to X_n$ denotes the L^2-projection. Furthermore, we introduce the operator $S : (\operatorname{Id} + T')^{-1}T'$. Since T is compact, the adjoint T' is likewise compact. Since $\operatorname{Id} - \Pi_N^{L^2}$ converges to zero pointwise and S is compact, we conclude

$$\lim_{N\to\infty} \|(\operatorname{Id} - \Pi_N^{L^2})S\|_{L^2\leftarrow L^2} = 0. \tag{7.35}$$

By the same arguments, we have

$$\lim_{N\to\infty} \|(\operatorname{Id} - \Pi_N^{L^2})T\|_{L^2\leftarrow L^2} = 0. \tag{7.36}$$

Let $v_N \in X_N$ be arbitrary and abbreviate $e = u - u_N$ and $\eta = u - v_N$. Then by Galerkin orthogonality

$$\|e\|_0^2 = ((\operatorname{Id} + T)e, e)_0 - (Te, e)_0 = ((\operatorname{Id} + T)e, \eta)_0 - (Te, e)_0 = (e, \eta)_0 + (Te, \eta - e)_0.$$

The invertibility of $\operatorname{Id} + T$ and the compactness of T imply invertibility of $\operatorname{Id} + T'$ and thus that the problem

$$\text{find } \psi \in L^2(\Gamma) \text{ s.t.} \qquad (Tv, \eta - e)_0 = ((\operatorname{Id} + T)v, \psi)_0 \qquad \forall v \in L^2(\Gamma)$$

has a unique solution $\psi = (\operatorname{Id} + T')^{-1}T'(\eta - e) = S(\eta - e)$. Therefore, using again Galerkin orthogonality, we arrive at

$$\begin{aligned}
\|e\|_0^2 &= (e, \eta)_0 + ((\operatorname{Id} + T)e, \psi)_0 = (e, \eta)_0 + ((\operatorname{Id} + T)e, \psi - \Pi_N^{L^2}\psi)_0 \\
&= (e, \eta)_0 + (e, (\operatorname{Id} - \Pi_N^{L^2})\psi)_0 + (Te, (\operatorname{Id} - \Pi_N^{L^2})\psi)_0 \\
&= (e, \eta)_0 + (e, (\operatorname{Id} - \Pi_N^{L^2})\psi)_0 + ((\operatorname{Id} - \Pi_N^{L^2})Te, (\operatorname{Id} - \Pi_N^{L^2})\psi)_0.
\end{aligned}$$

From $\psi = S(\eta - e)$ and the Cauchy-Schwarz inequality we get

$$\begin{aligned}
\|e\|_0^2 \leq \|e\|_0\|\eta\|_0 &+ \|e\|_0\|(\operatorname{Id} - \Pi_N^{L^2})S\| \{\|e\|_0 + \|\eta\|_0\} \\
&+ \|(\operatorname{Id} - \Pi_N^{L^2})T\| \, \|e\|_0\|(\operatorname{Id} - \Pi_N^{L^2})S\| \{\|e\|_0 + \|\eta\|_0\}.
\end{aligned}$$

Let now $\varepsilon > 0$ be given. Then, (7.35), (7.36) imply the existence of $N_\varepsilon > 0$ such that for $N \geq N_\varepsilon$ we have $\|(\operatorname{Id} - \Pi_N^{L^2})S\| \leq \varepsilon$ and $\|(\operatorname{Id} - \Pi_N^{L^2})T\| \leq \varepsilon$. Hence,

$$\|e\|_0 \leq \|\eta\|_0 + (\varepsilon + \varepsilon^2)\{\|e\|_0 + \|\eta\|_0\}.$$

Rearranging terms produces

$$\|e\|_0 \leq \frac{1 + \varepsilon + \varepsilon^2}{1 - \varepsilon - \varepsilon^2}\|\eta\|_0,$$

which shows the desired bound after suitably adjusting ε. ∎

8. Numerical Results. All our numerical examples are based on the operator $A' = 1/2 + K' + \mathbf{i}\eta V$, where the coupling parameter is $\eta = k$ or $\eta = 1$. The ansatz spaces X_N are taken to be standard *hp*-BEM spaces of piecewise polynomials of degree p. Specifically, let $\mathcal{T} = \{K_i \,|\, i = 1, \dots, N\}$ be a partition of Γ into N elements and let $F_K : [-1, 1] \to \Gamma$ be the element maps. Then $S^p(\mathcal{T}) = \{u \in L^2(\Gamma) \,|\, u|_K \circ F_K \in \mathcal{P}_p \quad \forall K \in \mathcal{T}\}$. Here, $\mathcal{P}_p$ denotes the univariate polynomials of degree p. The element maps F_K are constructed as described in Example 7.14, i.e., the uniform mesh $\widehat{\mathcal{T}}$ in parameter space is transported to the curve Γ by its parametrization. The basis of $S^p(\mathcal{T})$ selected for the computations is taken to be the push-forward of the L^2-normalized Legendre polynomials on the reference element $[-1, 1]$. The BEM operators K' and V are set up with an *hp*-quadrature with $p_{max} + 2$ quadrature points in each direction per quadrature cell. Details of the fast quadrature technique employed are described in [17]. Systematically, the number of elements N is taken proportional to k.

Denoting by $P_{\mathcal{T},p} : L^2(\Gamma) \to S^p(\mathcal{T})$ the Galerkin projector, which is characterized by

$$a'_k(u - P_{\mathcal{T},p}u, v) = 0 \qquad \forall v \in S^p(\mathcal{T}),$$

we approximate the Galerkin error $\|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2}$ by the formula

$$\|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2} \approx \sup_{v \in S^{p_{max}}(\mathcal{T})} \frac{\|v - P_{\mathcal{T},p}v\|_{L^2}}{\|v\|_{L^2}}. \tag{8.1}$$

Unless stated otherwise, we select $p_{max} = 20$ for the computation of (8.1).

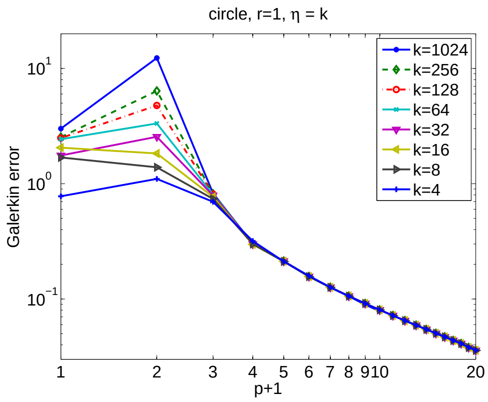
Since for smooth domains we may expect that the quasi-optimality constant to be asymptotically 1 (see Lemma 7.24) we do not present in our numerical examples $\|\operatorname{Id} - P_{\mathcal{T},p}\|_{L^2 \leftarrow L^2}$ of (8.1) but instead the *Galerkin Error Measure*

$$E := \sqrt{\|\operatorname{Id} - P_{\mathcal{T},p}\|^2_{L^2 \leftarrow L^2} - 1}. \tag{8.2}$$

We also report the extremal singular values $\sigma_{min}(\mathbf{M}^{-1}\mathbf{A}')$ and $\sigma_{max}(\mathbf{M}^{-1}\mathbf{A}')$ for $p = 10$, where $\mathbf{M}$ denotes the mass matrix for the space $S^p(\mathcal{T})$ and $\mathbf{A}'$ represents the stiffness matrix for the discretization of A'. These numbers give a very good indication of $1/\|(A')^{-1}\|_{L^2 \leftarrow L^2}$ and $\|A'\|_{L^2 \leftarrow L^2}$. The singular values are computed with the LAPACK-routine ZGESVD.

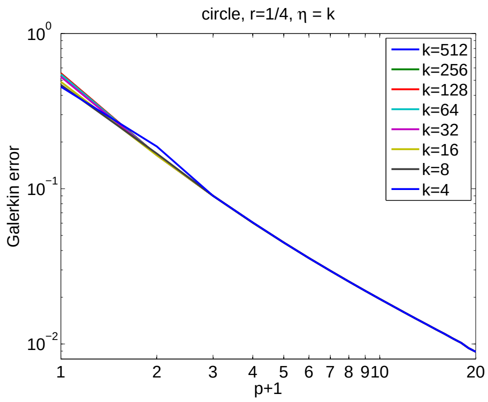
The examples below are selected to illustrate the theoretical results of the paper and to test its limits. The geometries of Examples 8.1, 8.2, 8.3, 8.4 are circles and ellipses and hence fully covered by our theory (recall that $C(A_k, 0, k) = C(A'_{-k}, 0, -k) = O(1)$ by [9]). The geometries in Examples 8.5, 8.6, 8.7, 8.8 are no longer star-shaped. In Examples 8.7, 8.8 we even leave the realm of smooth geometries; these geometries are "trapping domains" as was shown in [6, Thm. 5.1] and the wavenumbers selected in our computations are precisely the critical wavenumbers identified in [6, Thm. 5.1]. Clearly, the choice of the coupling parameter η in (1.4) affects the norm $C(A_k, 0, k)$ and thus, in turn, the conditions on the approximation properties of the discrete spaces X_N for quasi-optimality. We therefore also perform calculations for the choice $\eta = 1$ in Examples 8.6 and 8.8.

EXAMPLE 8.1. $\Omega = B_1(0)$ is a circle with radius $r = 1$. The mesh has $N = k$ elements of equal size. The element maps F_K are obtained with the aid of the parameterization $\{(r \cos\varphi, r \sin\varphi) \,|\, \varphi \in [0, 2\pi)\}$ of the circle. The coupling parameter η is selected as $\eta = k$. Fig. 8.1 shows the Galerkin Error Measure of (8.2) as a function of p; we also give an indication of $\|A'\|_{L^2 \leftarrow L^2}$ and $\|(A')^{-1}\|_{L^2 \leftarrow L^2}$.



k	$\sigma_{max}(\mathbf{M}^{-1}\mathbf{A}')$	$\sigma_{min}(\mathbf{M}^{-1}\mathbf{A}')$
4	1.26835	0.5
8	1.54632	0.5
16	1.89880	0.5
32	2.40042	0.5
64	2.98223	0.5
128	3.76487	0.5
256	4.73099	0.5
1024	7.48469	0.5

FIG. 8.1. *(see Example 8.1) Circle with radius $r = 1$, $\eta = k$. Left: Galerkin Error Measure E (see (8.2)). Right: Estimate of $\|A'\|_{L^2\leftarrow L^2}$ and $1/\|(A')^{-1}\|_{L^2\leftarrow L^2}$.*



k	$\sigma_{max}(\mathbf{M}^{-1}\mathbf{A}')$	$\sigma_{min}(\mathbf{M}^{-1}\mathbf{A}')$
4	1.06802	0.5
8	1.0832	0.5
16	1.26835	0.5
32	1.54632	0.5
64	1.8988	0.5
128	2.40042	0.5
256	2.98223	0.5
512	3.76487	0.5

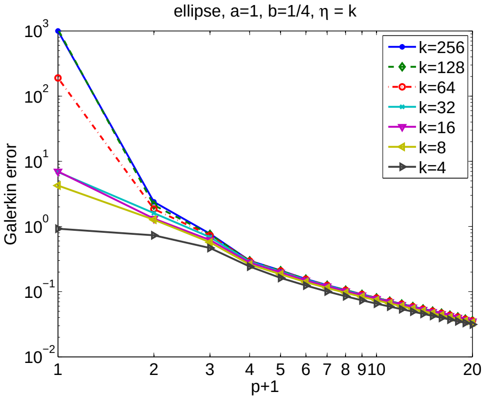
FIG. 8.2. *(see Example 8.2) Circle with radius $r = 1/4$, $\eta = k$. Left: Galerkin Error Measure E (see (8.2)). Right: Estimate of $\|A'\|_{L^2\leftarrow L^2}$ and $1/\|(A')^{-1}\|_{L^2\leftarrow L^2}$.*

EXAMPLE 8.2. The setup is the same as in Example 8.1 except that the radius of the circle is taken to be $r = 1/4$ instead of $r = 1$. The numerical results can be found in Fig. 8.2.

EXAMPLE 8.3. Ω is an ellipse with semi-axes $a = 1$ and $b = 1/4$. The boundary Γ is parametrized in the standard way by $\{(a\cos\varphi, b\sin\varphi)\,|\,\varphi \in [0, 2\pi)\}$. The element maps are obtained by uniformly subdividing the parameter interval $[0, 2\pi)$, and the mesh has $N = k$ elements. The coupling parameter η is $\eta = k$. The numerical results are presented in Fig. 8.3.

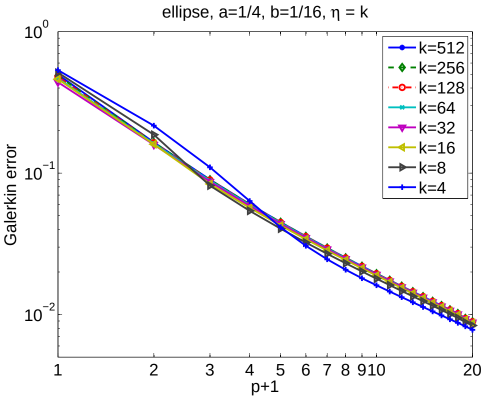
EXAMPLE 8.4. The setup is the same as in Example 8.3 except that the ellipse has been scaled: its semi-axes are $a = 1/4$ and $b = 1/16$. The numerical results are collected in Fig. 8.4.

EXAMPLE 8.5. $\Omega = B_{1/2}(0) \setminus \overline{B_{1/4}(0)}$ is the annular region between two circles of radii 1/2 and 1/4. The normal vector appearing in the definition of K always points outwards. The boundary $\partial\Omega$ is parametrized in the standard way with polar coordinates. The wave number is related to the number of elements N by $N = 2k$, and each of the two components of connectedness of $\partial\Omega$ has $N/2$ elements. The coupling parameter η is $\eta = k$. The results can be found in Fig. 8.5.



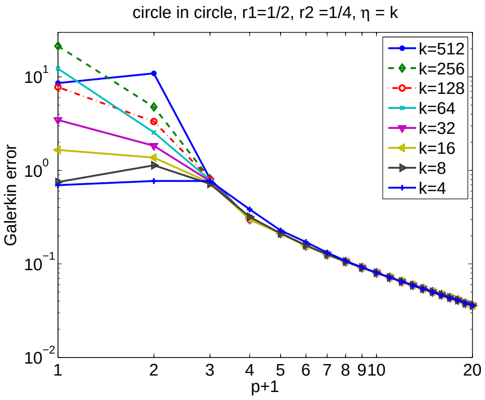
k	$\sigma_{max}(\mathbf{M}^{-1}\mathbf{A}')$	$\sigma_{min}(\mathbf{M}^{-1}\mathbf{A}')$
4	1.41593	0.489
8	1.71889	0.5
16	2.01108	0.5
32	2.64065	0.5
64	3.43955	0.5
128	4.57966	0.5
256	6.0845	0.5

FIG. 8.3. *(see Example 8.3) Ellipse with semiaxes $a = 1$ and $b = 1/4$. Left: Galerkin Error Measure E (see (8.2)). Right: Estimate of $\|A'\|_{L^2 \leftarrow L^2}$ and $1/\|(A')^{-1}\|_{L^2 \leftarrow L^2}$*



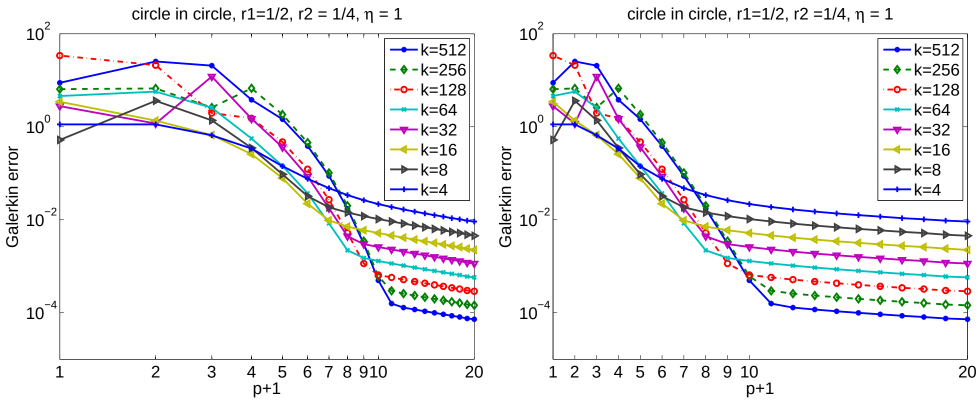
k	$\sigma_{max}(\mathbf{M}^{-1}\mathbf{A}')$	$\sigma_{min}(\mathbf{M}^{-1}\mathbf{A}')$
4	0.986228	0.353
8	1.19612	0.427
16	1.41593	0.489
32	1.71889	0.5
64	2.01108	0.5
128	2.64065	0.5
256	3.43955	0.5
512	4.57966	0.5

FIG. 8.4. *(see Example 8.4) Ellipse with semiaxes $a = 1/4$ and $b = 1/16$. Left: Galerkin Error Measure E (see (8.2)). Right: Estimate of $\|A'\|_{L^2 \leftarrow L^2}$ and $1/\|(A')^{-1}\|_{L^2 \leftarrow L^2}$*



k	$\sigma_{max}(\mathbf{M}^{-1}\mathbf{A}')$	$\sigma_{min}(\mathbf{M}^{-1}\mathbf{A}')$
4	2.36155	0.500129
8	2.35101	0.497189
16	2.54262	0.238509
32	2.81275	0.500153
64	3.2893	0.51368_{-1}
128	3.69209	0.914729_{-1}
256	4.37155	0.884842_{-1}
512	5.1591	0.275835_{-2}

FIG. 8.5. *(see Example 8.5) $\Omega = B_{1/2}(0) \setminus \overline{B_{1/4}(0)}$. Left: Galerkin Error Measure E (see (8.2)). Right: estimate of $\|A'\|_{L^2 \leftarrow L^2}$ and $\|(A')^{-1}\|_{L^2 \leftarrow L^2}$.*



k	$\sigma_{max}(\mathbf{M}^{-1}\mathbf{A}')$	$\sigma_{min}(\mathbf{M}^{-1}\mathbf{A}')$
4	1.84018	0.304956
8	1.64098	0.147632
16	1.67512	0.102911
32	1.65914	0.603251_{-1}
64	1.70917	0.190993_{-1}
128	1.87601	0.911284_{-2}
256	1.99572	0.834516_{-2}

FIG. 8.6. *(see Example 8.6)* $\Omega = B_{1/2}(0)\setminus\overline{B_{1/4}(0)}$. *Coupling parameter* $\eta = 1$. *Galerkin Error Measure* E *(see (8.2)) plotted in loglog-scale (left) and semilogy-scale (right). Bottom: Estimate of* $\|A'\|_{L^2\leftarrow L^2}$ *and* $\|(A')^{-1}\|_{L^2\leftarrow L^2}$.

EXAMPLE 8.6. The setup is as in Example 8.5. However, the coupling parameter η is given by $\eta = 1$ instead of $\eta = k$. The result are presented in Fig. 8.6.
EXAMPLE 8.7. Ω is the C-shaped domain depicted in Fig. 8.7 given by

$$\Omega = ((-r/3, r/3)\times(-r/2, r/2))\setminus((0, r/3)\times(-r/6, r/6)), \qquad r = 1/2.$$

For different values of the parmeter $m \in 3\mathbb{N}$, we select the number of elements N and the wavenumber k according to

$$N = 20m, \qquad k = \frac{3\pi}{r}.$$

The meshes are are uniform on Γ. The coupling parameter $\eta = k$. The results can be found in Fig. 8.8.
EXAMPLE 8.8. The setup is the same as in Example 8.7 with the exception that the coupling parameter η is chosen as $\eta = 1$ instead of $\eta = k$ and that $p_{max} = 15$ instead of $p_{max} = 20$. The numerical results can be found in Fig. 8.9.
EXAMPLE 8.9. The geometry is

$$\Omega = B_{r_1}(0)\setminus\overline{(B_{r_2}(0)\cup\{(r\cos\varphi, r\sin\varphi)\,|\,r>0, |\varphi|<\omega/2\})},$$

where $r_1 = 0.5$, $r_2 = 0.4$, and $\omega = \frac{10}{180}\pi$ (the geometry is drawn to scale in Fig. 8.10). We select $\eta = k$. The discretization is quasi-uniform and $k = N$. The numerical results are reported in Fig. 8.11. We point out, however, that the resolution of the mesh is not very fine: the number of elements in the "outlet" region increases from 1 (corresponding to $k = 56$) to 11 (corresponding to $k = 625$). The width of the "outlet" is 0.1.

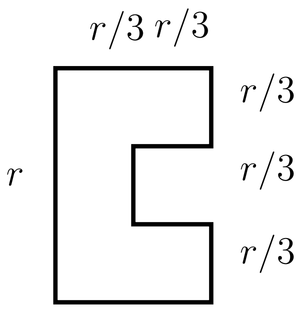
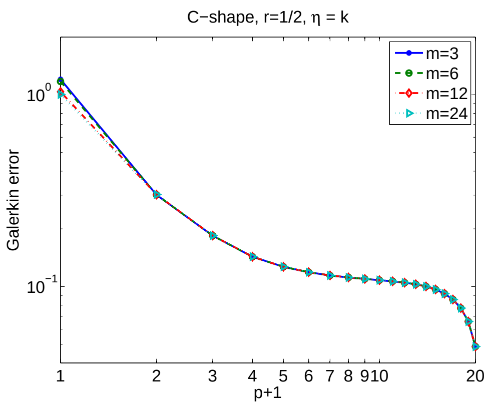
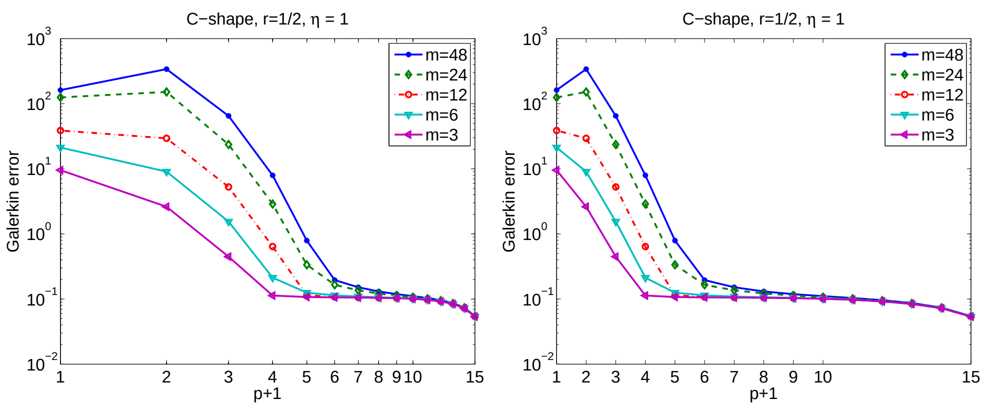


FIG. 8.7. *Geometry of Examples 8.7, 8.8.*



m	$k = 6\pi m$	$\sigma_{max}(\mathbf{M}^{-1}\mathbf{A}')$	$\sigma_{min}(\mathbf{M}^{-1}\mathbf{A}')$
3	56.5487	2.721	2.24795_{-1}
6	113.097	2.99077	1.40383_{-1}
12	226.195	3.59232	7.80885_{-2}
24	452.389	4.86965	4.14679_{-2}

FIG. 8.8. *C-shaped domain (see Example 8.7), $\eta = k$. Left: Galerkin Error Measure E (see (8.2)). Right: estimate of $\|A'\|_{L^2\leftarrow L^2}$ and $\|(A')^{-1}\|_{L^2\leftarrow L^2}$.*



m	$k = 6\pi m$	$\sigma_{max}(\mathbf{M}^{-1}\mathbf{A}')$	$\sigma_{min}(\mathbf{M}^{-1}\mathbf{A}')$
3	56.5487	1.59341	1.24977_{-2}
6	113.097	1.71847	3.42968_{-3}
12	226.195	1.88635	1.02321_{-3}
24	452.389	2.10001	2.27319_{-4}
48	904.779	2.40774	8.30718_{-5}

FIG. 8.9. *C-shaped domain (see Example 8.8), $\eta = 1$. Galerkin Error Measure E (see (8.2)) (left: loglog-scale, right: semilogy-scale). Bottom: Estimate of $\|A'\|_{L^2\leftarrow L^2}$ and $\|(A')^{-1}\|_{L^2\leftarrow L^2}$.*

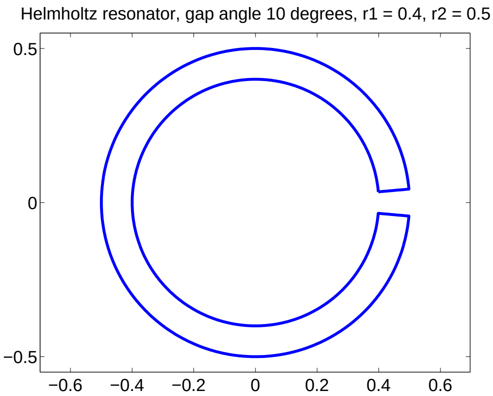
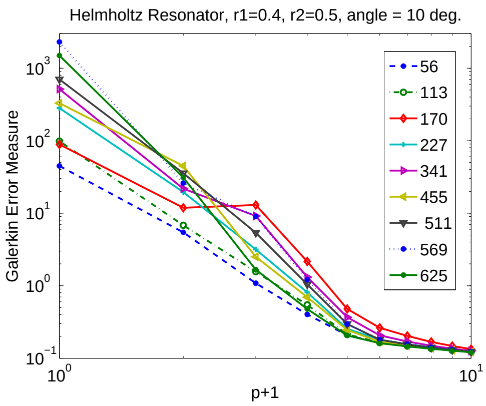


FIG. 8.10. *Geometry of Example 8.9.*



k	$\sigma_{max}(\mathbf{M}^{-1}\mathbf{A}')$	$\sigma_{min}(\mathbf{M}^{-1}\mathbf{A}')$
56	3.84481	0.111343
113	4.38575	0.376075_{-1}
170	4.76636	0.438144_{-2}
227	5.16703	0.198212_{-1}
341	5.69761	0.17661_{-1}
455	6.20524	0.314513_{-1}
511	6.32501	0.125367_{-1}
569	6.3671	0.828561_{-2}
625	6.60512	0.318163_{-1}

FIG. 8.11. *Resonator geometry (see Example 8.9), $\eta = k$. Galerkin Error Measure E (see (8.2)) and estimates of $\|A'\|_{L^2\leftarrow L^2}$ and $\|(A')^{-1}\|_{L^2\leftarrow L^2}$.*

Discussion of the numerical examples.

1. The difference between Examples 8.1 and 8.2 and likewise Examples 8.3 and 8.4 is merely a scaling of the geometry. Alternatively, this could be achieved by changing the wavenumber k by a factor 4. Indeed, comparing corresponding cases in the numerical results shows that the same values are obtained.
2. We recall that in all numerical examples the mesh size h is proportional to $1/k$. In the calculations based on smooth geometries, i.e., Examples 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, we observe that the Galerkin Error Measure E tends to zero as $p \to \infty$. This shows that indeed, asymptotically, the quasi-optimality constant is 1. Closer inspection of the numerical results indicates an $O(1/p)$-behavior, which is consistent with the finite shift properties of V_0 and K_0'. It is noteworthy that in Example 8.6 with $\eta = 1$ the asymptotic behavior of the Galerkin Error Measure appears to be $O(1/(pk))$. Hence, the combined η and k dependence appears to be $O((1+|\eta|)/(kp))$.
3. In the case of circles (Examples 8.1, 8.2), ellipses (8.3, 8.4), and the case of an annular geometry with coupling parameter $\eta = k$ (Example 8.5) we observe

that the condition

$$\frac{kh}{p} \text{ sufficiently small}$$

is already enough to ensure quasi-optimality of the Galerkin *hp*-BEM. The condition $p = O(\log k)$ is not visible. For the special case of a circle, this lack of "pollution" may be expected in view the analysis of [2].

4. The C-shaped geometry in the Examples 8.7, 8.8 is not smooth. Hence, the operator K' is no longer smoothing and one cannot expect the Galerin Error Measure E of (8.2) to tend to zero. This is indeed visible in Figs. 8.8, 8.9. The sharp decrease of the the Galerkin Error Measure E for large p is likely to be a numerical artefact since E is obtained by comparing lower values of p with the result for $p_{max} = 20$ in the case of Fig. 8.8 and $p_{max} = 15$ in Fig. 8.9.
5. The work [9] shows that $C(A'_k, 0, k) = \|(A'_k)^{-1}\|_{L^2 \leftarrow L^2}$ is bounded uniformly in k for star-shaped geometries. Indeed, the numerical results for the case of a circle (Examples 8.1, 8.2) and an ellipse (Examples 8.3, 8.4) confirm this. In contrast, the geometries of Examples 8.5 and 8.7 are not star-shaped and we observe in Figs. 8.5, 8.6, 8.8, 8.9 that $C(A'_k, 0, k)$ is not bounded uniformly in k but grows algebraically. The norm $\|A'\|_{L^2 \leftarrow L^2}$ is seen to grow (mildly) in k in all examples. This is in accordance with known results. For example, [13] shows $\|A'\|_{L^2 \leftarrow L^2} = O(k^{1/3})$ for the case of a circle and [6] proves $\|A'\|_{L^2 \leftarrow L^2} = O(k^{1/2})$ for general 2D Lipschitz domains. For the convenience of the reader, we present the tables of Figs. 8.1–8.9 in the form of graphs in Fig. 8.12.
6. For the C-shaped geometry of Examples 8.7, 8.8, a lower bound for $C(A'_k, 0, k)$ is given in [6, Thm. 5.1] as

$$C(A'_k, 0, k) \geq C k^{9/10} \left(1 + \frac{|\eta|}{k}\right)^{-1}.$$

We observe in particular that selecting $\eta = O(1)$ instead of $\eta = O(k)$ leads to an increase of the bound by a factor k. Our numerical examples (see the tables in Figs. 8.7, 8.8 or the graphs in Fig. 8.12) indicate that the lower bounds of [6, Thm. 5.1] are essentially sharp.
7. In the case of circular/elliptic geometries and even in the case of the non-convex geometry of an annulus, we did not observe a "pollution" effect, i.e., quasi-optimality of the Galerkin BEM takes place as soon as kh/p is below a (geometry-dependent) threshold. The more stringent scale resolution condition (1.1) that stipulates $p = O(\log k)$ might, however, be needed in more general situations. This is the purpose of selecting $\eta = 1$ in the Examples 8.6, 8.8. It has the effect of increasing $C(A'_k, 0, k)$, which, according to the analysis of Section 7, puts conditions on the approximation properties of the *hp*-BEM spaces. Indeed, the semilogarithmic plots in Figs. 8.6, 8.9 indicate that a condition $p = O(\log k)$ is necessary to achieve a given quasi-optimality constant for the Galerkin BEM.

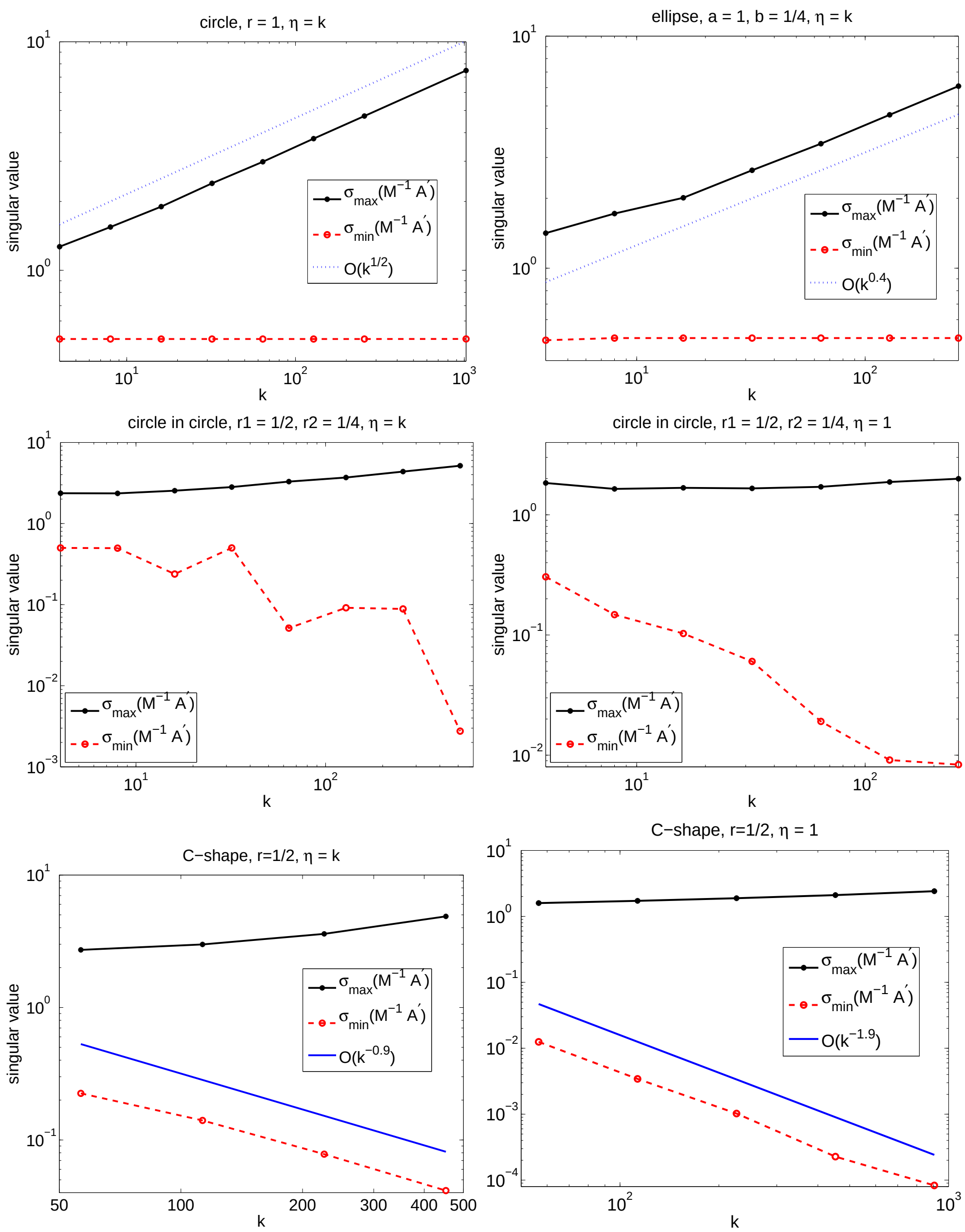


FIG. 8.12. *Extremal singular values of* $\mathbf{M}^{-1}\mathbf{A}'$ *for Examples 8.1 (top left), 8.3 (top right), 8.5 (middle left), 8.6 (middle right), 8.7 (bottom left), 8.8 (bottom right).*

Appendix A. Proofs of Lemmata 2.1, 2.2. *Proof of Lemma 2.1:* The result for $-1/2 < s < 1/2$ being known in the literature (see, e.g., [20]), we restrict our attention to the limiting cases $s = \pm 1/2$. We start with the case $s = 1/2$. Set $u := \widetilde{V}_0\varphi$ for $\varphi \in L^2(\Gamma)$. Then $u \in H^{3/2}(\Omega_R)$ with $\|u\|_{H^{3/2}(\Omega_R)} \leq C\|\varphi\|_{L^2(\Gamma)}$, which can be seen as follows: By [32, Thms. 3.3, 4.11] we have $\|V_0\varphi\|_{H^1(\Gamma)} \leq C\|\varphi\|_{L^2(\Gamma)}$. Since $\gamma_0^{int/ext}\widetilde{V}_0\varphi = V_0\varphi$, the uniqueness assertion of [16, Thm. 5.15] implies that $u = \widetilde{V}_0\varphi \in H^{3/2}(\Omega_R)$. Next, [16, Thm. 5.6, Cor. 5.7] imply

$$\|u\|_{H^{3/2}(\Omega_R)} + \|\sqrt{\delta}\nabla^2 u\|_{L^2(\Omega_R)} + \|u^*\|_{L^2(\Gamma)} + \|(\nabla u)^*\|_{L^2(\Gamma)} \leq C\|\varphi\|_{L^2(\Gamma)}; \qquad \text{(A.1)}$$

here, the notation v^* denotes the non-tangential maximal functions (see [32]) and $\delta(x) = \operatorname{dist}(x, \Gamma)$ denotes the distance from Γ.

Additionally, we have from [16, Prop. 2.18] that $u \in B^{3/2}_{2,\infty}(B_R)$ if and only if $u \in L^2(B_R)$ and $\nabla u \in B^{1/2}_{2,\infty}(B_R)$. It therefore remains to assert $\nabla u \in B^{1/2}_{2,\infty}(B_R)$. To that end, consider $v = \partial_i u$ for a fixed i and let $v_\varepsilon := v \star \rho_\varepsilon$ be its regularization, where ρ_ε is a standard mollifier with length scale ε. We have by standard arguments for each fixed $x \in B_R$ such that $v \in H^1(B_{2\varepsilon}(x))$:

$$\begin{aligned} \|v - v_\varepsilon\|_{L^2(B_\varepsilon(x))} &\leq \varepsilon\|\nabla v\|_{L^2(B_{2\varepsilon}(x))}, \\ \|\nabla v_\varepsilon\|_{L^2(B_\varepsilon(x))} &\leq \|\nabla v\|_{L^2(B_{2\varepsilon}(x))}. \end{aligned}$$

For $\varepsilon > 0$ we denote by $S_\varepsilon := \cup_{x\in\Gamma} B_\varepsilon(x)$ the tubular neighborhood of Γ of width ε. Covering the set $B_R \setminus S_{3\varepsilon} \subset \cup_{x \in B_R\setminus S_{3\varepsilon}} B_\varepsilon(x)$ we infer with the aid of Besicovitch's covering theorem

$$\begin{aligned} \|v - v_\varepsilon\|_{L^2(B_R\setminus S_{3\varepsilon})} &\leq C\varepsilon\|\nabla v\|_{L^2(B_R\setminus S_\varepsilon)} \leq \varepsilon^{1/2}\|\delta^{1/2}\nabla^2 u\|_{L^2(\Omega_R)} \leq \varepsilon^{1/2}\|\varphi\|_{L^2(\Gamma)}, \\ \|\nabla v_\varepsilon\|_{L^2(B_R\setminus S_{3\varepsilon})} &\leq C\|\nabla v\|_{L^2(B_R\setminus S_\varepsilon)} \leq C\varepsilon^{-1/2}\|\delta^{1/2}\nabla v\|_{L^2(\Omega_R)} \leq C\varepsilon^{-1/2}\|\varphi\|_{L^2(\Gamma)}. \end{aligned}$$

For the regularized function v_ε we have with the definition of the non-tangential maximal function and (A.1)

$$\|v_\varepsilon\|_{L^2(S_\varepsilon)} \leq C\|v\|_{L^2(S_{2\varepsilon})} \leq C\varepsilon^{1/2}\|v^*\|_{L^2(\Gamma)} \leq C\varepsilon^{1/2}\|\varphi\|_{L^2(\Gamma)}.$$

Finally, for the derivative we compute

$$\|\nabla v_\varepsilon\|_{L^2(S_{3\varepsilon})} \leq C\varepsilon^{-1}\|v\|_{L^2(S_{4\varepsilon})} \leq C\varepsilon^{-1/2}\|v^*\|_{L^2(\Gamma)} \leq C\varepsilon^{-1/2}\|\varphi\|_{L^2(\Gamma)}.$$

Thus, we obtain the following estimate for the K-functional:

$$K(v, \varepsilon) \leq \|v - v_\varepsilon\|_{L^2(B_R)} + \varepsilon\|v_\varepsilon\|_{H^1(B_R)} \leq C\varepsilon^{1/2}\|\varphi\|_{L^2(\Gamma)}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude $v \in B^{1/2}_{2,\infty}(B_R)$.

For the case $s = -1/2$ we start by noting that $V_0 : H^{-1}(\Gamma) \to L^2(\Gamma)$, which follows from the self-adjointness of V_0, the above cite result by Verchota that $V_0 : L^2(\Gamma) \to H^1(\Gamma)$, and a duality argument. Next, we approximate $\varphi \in H^{-1}(\Gamma)$ by functions $(\varphi_n)_{n\in\mathbb{N}} \subset L^2(\Gamma)$. As above, [16, Thm. 5.15] implies that the functions $\widetilde{V}_0\varphi_n$ are the unique harmonic functions with Dirichlet data $V_0\varphi_n$. Combining an estimate due to Dahlberg (see [16, Thm. 5.3]) and [16, Cor. 5.5] implies that $\widetilde{V}_0\varphi_n \in H^{1/2}(\Omega_R)$ together with

$$\|\widetilde{V}_0\varphi_n\|_{H^{1/2}(\Omega_R)} \leq C\|V_0\varphi_n\|_{L^2(\Gamma)} \leq C\|\varphi_n\|_{H^{-1}(\Gamma)}.$$

By linearity of $\widetilde{V}_0$, the sequence $(\widetilde{V}_0\varphi_n)_n$ is a Cauchy sequence in $H^{1/2}(\Omega_R)$. Furthermore, it converges pointwise to $\widetilde{V}_0\varphi$. We conclude that $\widetilde{V}_0\varphi \in H^{1/2}(\Omega_R)$ and $\|\widetilde{V}_0\varphi\|_{H^{1/2}(\Omega_R)} \leq C\|\varphi\|_{H^{-1}(\Gamma)}$. Appealing once more to [16, Cor. 5.5], we get for $u := \widetilde{V}_0\varphi$ that $\|u^*\|_{L^2(\Gamma)} + \|u\|_{H^{1/2}(\Omega_R)} \leq C\|\varphi\|_{H^{-1}(\Gamma)}$. Using now the same arguments as in the case $s = 1/2$, we conclude $\|u\|_{B^{1/2}_{2,\infty}(B_R)} \leq C\|\varphi\|_{H^{-1}(\Gamma)}$.
The remaining cases $-1/2 < s < 1/2$ can now be inferred from the limiting cases $s = \pm 1/2$ by an interpolation argument. □

Proof of Lemma 2.2: The proof is very similar to that of Lemma 2.1. The case $s = 1/2$ is see as follows: For $\varphi \in H^1(\Gamma) \subset H^{1/2}(\Gamma)$, we have $\widetilde{K}_0\varphi \in H^1(\Omega_R)$. We have $\gamma_0^{int/ext}\widetilde{K}_0\varphi = (\mp 1/2 + K_0)\varphi \in H^{1/2}(\Gamma) \subset L^2(\Gamma)$. By [16, Cor. 5.5], the interior and exterior non-tangential limits $\mathrm{Tr}^{int/ext}\,\widetilde{K}_0\varphi$ on Γ exist and are in $L^2(\Gamma)$. These must coincide with the interior and exterior traces $\gamma_0^{int/ext}\widetilde{K}_0\varphi$ and we conclude $\mathrm{Tr}^{int/ext}\,\widetilde{K}_0\varphi = \gamma_0^{int/ext}\widetilde{K}_0\varphi = (\mp 1/2 + K_0)\varphi$. By [32, Thm. 3.3] we have $(\mp 1/2 + K_0)\varphi \in H^1(\partial\Omega)$, so that [16, Thm. 5.15] implies $\widetilde{K}_0\varphi \in H^{3/2}(\Omega_R)$. Then [16, Cor. 5.7] implies $\widetilde{K}_0\varphi \in H^{3/2}(\Omega_R)$ with $\|\widetilde{K}_0\varphi\|_{H^{3/2}(\Omega_R)} \leq C\|\varphi\|_{H^1(\Gamma)}$.
For the case $s = -1/2$, we proceed as in the proof of Lemma 2.1. First, we show for $\varphi \in L^2(\Gamma)$ that

$$\|\widetilde{K}_0\varphi\|_{H^{1/2}(\Omega_R)} + \|(\widetilde{K}_0\varphi)^*\|_{L^2(\Gamma)} \leq C\|\varphi\|_{L^2(\Gamma)}.$$

The assertion $\widetilde{K}_0\varphi \in B^{1/2}_{2,\infty}(B_R)$ follows from this in the same way as in the proof of Lemma 2.1. Finally, for $-1/2 < s < 1/2$ the assertion $\widetilde{K}_0 : H^{1/2+s}(\Gamma) \to H^{1+s}(\Omega_R)$ follows by an interpolation argument from $\widetilde{K}_0 : H^{1/2+s}(\Gamma) \to H^{1+s}(\Omega_R)$ for the limiting cases $s = \pm 1/2$, which have just been proved. □

Appendix B. regularity assertions for parameter-dependent elliptic PDEs.

B.1. analytic regularity. We start with a lemma that shows that membership in the class $\mathfrak{A}$ of analytic functions is preserved under analytic changes of variables:
LEMMA B.1. *Let G, $G_1 \subset \mathbb{R}^d$ be bounded open sets. Assume that $g : \overline{G_1} \to \mathbb{R}^d$ is analytic, $|\det g'| > 0$ on $\overline{G_1}$ and that $g(G_1) \subset G$. Let $f_1 : \overline{G_1} \to \mathbb{C}$, $f_2 : \overline{G} \to \mathbb{C}$ be analytic and assume that $f_2 \in \mathfrak{A}(C_f, \gamma_f, G)$. Then the function $F : x \mapsto f_1(x)(f_2 \circ g)(x)$ satisfies $F \in \mathfrak{A}(CC_f, \gamma', G)$ for some constants C, γ' that depend solely on γ, f_1, g, and k_0.*
Proof. The case $d = 2$ is taken directly from [22, Lemma 4.3.1]. Inspection of the proof of [22, Lemma 4.3.1] shows that it can be generalized to $d > 2$. □
Next, we recall that if a function u satisfies the differential equation

$$-\nabla \cdot (B\nabla u) + k^2 cu = f \tag{B.1}$$

and if the function F provides a sufficiently smooth change of variables, then the transformed function $\widehat{u} := u \circ F$ solves

$$-\nabla \cdot (\widehat{B}\nabla\widehat{u}) + k^2 \det F'\widehat{c}\widehat{u} = \det F'\widehat{f},$$

where $\widehat{B} = B \circ F$, $\widehat{c} = c \circ F$, and $\widehat{f} = f \circ F$. Finally, for the convenience of referring to the assumptions on the coefficients B, c, we make the following assumptions: The matrix-valued function B is pointwise symmetric positive definite and

$$0 < \lambda_{min} < B(x) \quad \forall x \in \omega, \tag{B.2a}$$

$$\|\nabla^n c\|_{L^\infty(\omega)} \leq C_c \gamma_c^n n!, \qquad \|\nabla^n B\|_{L^\infty(\omega)} \leq C_B \gamma_B^n n! \qquad \forall n \in \mathbb{N}_0. \tag{B.2b}$$

Theorem B.2 (Dirichlet b.c.). *Let $\omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with analytic boundary. Assume (B.2). Let $u \in H^1(\omega)$ solve (B.1) on ω for an $f \in \mathfrak{A}(C_f, \gamma_f, \omega)$. Assume that u satisfies $u|_{\partial\omega} = G|_{\partial\omega}$ for a $G \in \mathfrak{A}(C_G, \gamma_G, \omega \cap T')$, where T' is a tubular neighborhood of $\partial\omega$. Fix a tubular neighborhood T of $\partial\omega$ with $\overline{T} \subset T'$. Then u satisfies*

$$u \in \mathfrak{A}(CC_u, \gamma_u, \omega \cap T), \qquad C_u := k^{-2}C_f + C_G + k^{-1}\|u\|_{\mathcal{H},T'\cap\omega}.$$

where the constants C and γ_u depend solely on γ_G, γ_f, $\partial\omega$, k_0, and the constants of (B.2).

Proof. Consider the function $z := u - G$. Since $G \in \mathfrak{A}(C_G, \gamma_G, \omega \cap T')$, it suffices to establish $z \in \mathfrak{A}(CC_u, \gamma_u, \omega \cap T)$. The function z satisfies

$$-k^{-2}\nabla \cdot (B\nabla z) - cz = \widetilde{f} := k^{-2}f - k^{-2}\nabla \cdot (B\nabla G) - cG \quad \text{on } T' \cap \omega, \qquad z|_{\partial\omega} = 0.$$

The assumptions on f and G and Lemma B.1 imply $\widetilde{f} \in \mathfrak{A}(C(k^{-2}C_f + C_G), \widetilde{\gamma}, T' \cap \omega)$ for some constants C, $\widetilde{\gamma}$. From [22, Props. 5.5.1, 5.5.2] we get $z \in \mathfrak{A}(C(k^{-2}C_f + C_G + k^{-1}\|z\|_{\mathcal{H},T'\cap\omega}), \gamma, T \cap \omega)$. Since $k^{-1}\|z\|_{\mathcal{H},T'\cap\omega} \leq C\left(C_G + k^{-1}\|u\|_{\mathcal{H},T'\cap\omega}\right)$, the desired result now follows. ∎

Theorem B.3 (Robin b.c.). *Let $\omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with analytic boundary. Assume (B.2). Let $u \in H^1(\omega)$ solve (B.1) on ω for an $f \in \mathfrak{A}(C_f, \gamma_f, \omega)$. Assume that u satisfies*

$$\gamma_1^{int} u = \gamma_0^{int} G_1 + \mathbf{i}k(\gamma_0^{int} G_2)\gamma_0^{int} u$$

where, for some tubular neighborhood T' of $\partial\omega$ we have $G_1 \in \mathfrak{A}(C_{G_1}, \gamma_{G_1}, \omega \cap T')$ and G_2 is analytic on T'. Here, the trace operators γ_0^{int} and γ_1^{int} are understood with respect to ω. Fix a tubular neighborhood T of $\partial\omega$ with $\overline{T} \subset T'$. Then u satisfies

$$u \in \mathfrak{A}(CC_u, \gamma_u, \omega \cap T), \qquad C_u := k^{-2}C_f + k^{-1}C_{G_1} + k^{-1}\|u\|_{\mathcal{H},T'\cap\omega},$$

where C and γ_u depend solely on γ_{G_1}, γ_f, $\partial\omega$, G_2, k_0, and the constants of (B.2).

Proof. The proof is sketched for a related 2D problem in [22, Prop. 5.4.5, Rem. 5.4.6]. The key observation is again that Lemma B.1 allows us to locally flatten the boundary while preserving the structure of the differential equation and the boundary conditions. Then the technique employed in [22, Prop. 5.4.5] is applicable. ∎

Theorem B.4 (transmission conditions). *Let ω', $\omega \subset \mathbb{R}^d$ be two bounded domains with $\omega' \subset\subset \omega$. Denote $\gamma := \partial\omega'$ and assume that γ is analytic. Assume (B.2). Let $u \in H^1(\omega)$ solve (B.1) on ω for an $f \in \mathfrak{A}(C_f, \gamma_f, \omega \setminus \gamma)$. Fix $\omega'' \subset\subset \omega$. Then*

$$u \in \mathfrak{A}(CC_u, \gamma_u, \omega'' \setminus \gamma), \qquad C_u := k^{-2}C_f + k^{-1}\|u\|_{\mathcal{H},\omega}$$

for some constants C, $\gamma_u > 0$ that depend solely on γ_f, ω', ω'', ω, k_0, and the constants of (B.2).

Proof. The interesting estimates are those near the boundary γ. Here, the standard procedure of locally flattening γ can be brought to bear in view of Lemma B.1. Then, [22, Prop. 5.5.4] is applicable. ∎

Lemma B.5. *Let $\omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with analytic boundary $\partial\omega$. Set $\omega^+ := \mathbb{R}^d \setminus \overline{\omega}$. Let T be a tubular neighborhood of $\partial\omega$. Let $G \in \mathfrak{A}(C_G, \gamma_G, T \cap \omega)$. Then there exists a tubular neighborhood $\widetilde{T}$ of $\partial\omega$ and constants C, $\gamma_{\widetilde{G}}$ that depend solely on γ_G, $\partial\omega$, k_0 with the following property: There exists a $\widetilde{G} \in \mathfrak{A}(CC_G, \gamma_{\widetilde{G}}, \widetilde{T} \cap \omega^+)$ with $\gamma_0^{ext}\widetilde{G} = \gamma_0^{int} G$. Here, γ_0^{ext} and γ_0^{int} are the trace operators with respect to ω.*

Proof. The idea is to define $\widetilde{G}$ by reflection at $\partial\omega$. One can define boundary fitted coordinates $\psi : \partial\omega \times (-\varepsilon, \varepsilon) \to \mathbb{R}^d$ via $\psi(x, \rho) := x + \rho\vec{n}(x)$, where $\vec{n}(x)$ is the (outer) normal vector of $\partial\omega$ at $x \in \partial\omega$. Since $\partial\omega$ is assumed to be analytic, ψ is likewise analytic. For $\varepsilon > 0$ sufficiently small, the range of ψ is a tubular neighborhood (denoted T) of $\partial\omega$ and restricted to T, the inverse ψ^{-1} of ψ exists and is analytic. We write $\psi^{-1}(x) = (\gamma(x), \rho(x))$. For $x \in T \cap \omega^+$ we then define $G^+(x)$ by $G^+(x) := G(\psi(a(x), -\rho(x)))$. The analyticity of ψ^{-1} and Lemma B.1 then implies the result. ∎

B.2. finite regularity. Theorem B.6. *Let ω' and $\omega \subset \mathbb{R}^d$ be two bounded domains with $\omega' \subset\subset \omega$. Denote by $\gamma := \partial\omega'$ and assume that γ is analytic. Assume (B.2). Let $u \in H^1(\omega)$ solve (B.1) on ω for some $f \in H^s(\omega \setminus \gamma)$ with $s \geq 0$. Fix $\omega'' \subset\subset \omega$.*
If $s \in \mathbb{N}_0$, then

$$\sum_{n=0}^{s} k^{-(n+2)} \|\nabla^{n+2} u\|_{L^2(\omega''\setminus\gamma)} \leq C \left[\sum_{j=0}^{s} k^{-j-2} \|\nabla^j f\|_{L^2(\omega\setminus\gamma)} + \|u\|_{L^2(\omega)} \right], \tag{B.3}$$

where the constant C depends on s but is independent of $k \geq k_0$ and u. If we assume $s \geq 0$, then for some $C > 0$ independent of $k \geq k_0$ and u:

$$\|u\|_{H^{s+2}(\omega''\setminus\gamma)} \leq C \left[k^s \|f\|_{L^2(\omega)} + \|f\|_{H^s(\omega)} + k^{s+2} \|u\|_{L^2(\omega)} \right]. \tag{B.4}$$

Proof. We start by observing that standard elliptic regularity (note that the interface γ is smooth) for

$$-\nabla \cdot (B\nabla\widetilde{u}) = \widetilde{f} \quad \text{on } \omega$$

gives for $s \geq 0$ and any domain $\widetilde{\omega}$ with $\omega'' \subset\subset \widetilde{\omega} \subset\subset \omega$

$$\|\widetilde{u}\|_{H^{s+2}(\omega''\setminus\gamma)} \lesssim \|\widetilde{f}\|_{H^s(\widetilde{\omega}\setminus\gamma)} + \|u\|_{L^2(\widetilde{\omega})}.$$

We apply this result with $\widetilde{f} = f + k^2 cu$, multiply through with k^{-s}, and get

$$k^{-s}\|u\|_{H^{s+2}(\omega''\setminus\gamma)} \lesssim k^{-s}\|f\|_{H^s(\widetilde{\omega}\setminus\gamma)} + k^{-(s-2)}\|u\|_{H^s(\widetilde{\omega}\setminus\gamma)} + k^{-s}\|u\|_{L^2(\widetilde{\omega})}. \tag{B.5}$$

For even integer $s \in 2\mathbb{N}_0$, we can iterate (B.5) to get

$$k^{-s}\|u\|_{H^{s+2}(\omega''\setminus\gamma)} \lesssim \sum_{j=0}^{s/2} k^{-2j}\|f\|_{H^{2j}(\omega\setminus\gamma)} + k^2\|u\|_{L^2(\omega\setminus\gamma)}, \quad s \in 2\mathbb{N}_0. \tag{B.6}$$

For odd $s \in 1 + 2\mathbb{N}_0$ we get analogously

$$k^{-s}\|u\|_{H^{s+2}(\omega''\setminus\gamma)} \lesssim \sum_{j=0}^{(s+1)/2-1} k^{-2j-1}\|f\|_{H^{2j+1}(\omega\setminus\gamma)} + k\|u\|_{H^1(\omega\setminus\gamma)} + k^{-1}\|u\|_{L^2(\omega\setminus\gamma)}.$$

The bound (B.6) with $s = 0$ produces $\|u\|_{H^2(\omega''\setminus\gamma)} \lesssim \|f\|_{L^2(\omega\setminus\gamma)} + k^2\|u\|_{L^2(\omega\setminus\gamma)}$. Combining this with the standard (piecewise) interpolation inequality

$$\|u\|_{H^1(\omega\setminus\gamma)} \lesssim \|u\|^{1/2}_{H^2(\omega\setminus\gamma)} \|u\|^{1/2}_{L^2(\omega\setminus\gamma)} \lesssim k^{-1}\|u\|_{H^2(\omega\setminus\gamma)} + k\|u\|_{L^2(\omega\setminus\gamma)}$$

and appropriately adjusting the domains, we can conclude for $s \in \mathbb{N}_0$

$$k^{-s}\|u\|_{H^{s+2}(\omega''\setminus\gamma)} \lesssim \sum_{j=0}^{s} k^{-j}\|f\|_{H^j(\omega\setminus\gamma)} + k^2\|u\|_{L^2(\omega\setminus\gamma)}, \tag{B.7}$$

from which we derive (B.3). For the proof of (B.4) we introduce the notation $\sigma := \lfloor s \rfloor$ and observe the (piecewise) interpolation inequality

$$\|u\|_{H^s(\omega\setminus\gamma)} \lesssim \|u\|^{1-\theta_1}_{L^2(\omega\setminus\gamma)} \|u\|^{\theta_1}_{H^{\sigma+2}(\omega\setminus\gamma)}, \qquad \theta_1 := \frac{s}{\sigma+2}.$$

For every $\varepsilon_1 > 0$ we get from Young's inequality

$$\|u\|_{H^s(\omega\setminus\gamma)} \lesssim \varepsilon_1^{1/(1-\theta_1)} \|u\|_{L^2(\omega\setminus\gamma)} + \varepsilon_1^{-1/\theta_1}\|u\|_{H^{\sigma+2}(\omega\setminus\gamma)}.$$

Selecting $\varepsilon_1 := k^{s(1-\theta_1)}$ we arrive at

$$\|u\|_{H^s(\omega\setminus\gamma)} \lesssim k^s\|u\|_{L^2(\omega\setminus\gamma)} + k^{s-\sigma-2}\|u\|_{H^{\sigma+2}(\omega\setminus\gamma)}. \tag{B.8}$$

Next, we use again a (piecewise) interpolation inequality to bound for $0 \le j \le \sigma < s$ and Young's inequality

$$k^{s-j}\|f\|_{H^j(\omega\setminus\gamma)} \lesssim \left(k^s\|f\|_{L^2(\omega\setminus\gamma)}\right)^{1-j/s} \|f\|^{j/s}_{H^s(\omega\setminus\gamma)} \lesssim k^s\|f\|_{L^2(\omega\setminus\gamma)} + \|f\|_{H^s(\omega\setminus\gamma)}. \tag{B.9}$$

Combining (B.9), (B.8), (B.7) we arrive at the desired bound (B.4). □

Appendix C. regularity of Laplace-Beltrami eigenfunctions. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with an analytic boundary Γ. Let (φ_m, λ_m^2), $m \in \mathbb{N}_0$, be the eigenpairs of the Laplace-Beltrami operator, i.e.,

$$-\Delta_\Gamma \varphi_m = \lambda_m^2 \varphi_m \qquad \text{on } \Gamma.$$

We assume that the eigenvalues $\lambda_m \ge 0$ are sorted in ascending order and that the eigenfunctions $(\varphi_m)_{m\in\mathbb{N}_0}$ are orthonormalized in $L^2(\Gamma)$.

LEMMA C.1 (analytic regularity of φ_m). *Let Γ be* analytic. *Then there exist constants C, $\gamma > 0$ independent of m such that*

$$\|\nabla^n_\Gamma \varphi_m\|_{L^2(\Gamma)} \le C \max\{\lambda_m, n\}^n \gamma^n \qquad \forall n \in \mathbb{N}_0, \tag{C.1}$$

where ∇_Γ denotes the surface gradient. Furthermore, there exists a tubular neighborhood T of Γ (depending solely on Γ) such that all functions φ_m can be extended to analytic functions (again denoted φ_m) on T that satisfy

$$\|\nabla^n \varphi_m\|_{L^2(T)} \le C \max\{\lambda_m, n\}^n \gamma^n \qquad \forall n \in \mathbb{N}_0. \tag{C.2}$$

Proof. Sketch of the proof: If $\gamma : U \to \Gamma$ for some $U \subset \mathbb{R}^{d-1}$ is one of the analytic charts, then the Laplace-Beltrami operator Δ_Γ applied to a function $u : \Gamma \to \mathbb{R}$ has the following form on U:

$$\frac{1}{\sqrt{g}} \sum_{i,j=1}^{d-1} \partial_i \left(\sqrt{g} g^{ij} \partial_j (u \circ \gamma)\right),$$

where $g = \det G$ is the determinant of the metric tensor G given by $G_{ij} := \partial_i\gamma \cdot \partial_j\gamma$ and the matrix $(g^{ij})_{i,j=1}^d$ is the (pointwise) inverse of G. The matrix G is pointwise symmetric positive definite and thus also its inverse $(g^{ij})_{i,j=1}^d$. By the analyticity of the charts, the matrices $(g^{ij})_{i,j=1}^d$ and the function g are analytic. On U, the pull-back $\widehat{\varphi}_m := \varphi \circ \gamma$ of the eigenfunction φ_m satisfies for the analytic, pointwise symmetric positive definite matrix $A_{ij} = \sqrt{g} g^{ij}$

$$-\lambda_m^{-2} \nabla \cdot (A \nabla \widehat{\varphi}_m) - \sqrt{g} \widehat{\varphi}_m = 0,$$

Fix $K \subset\subset K' \subset\subset U$. Then [22, Prop. 5.5.1] gives

$$\|\nabla^{n+2} \widehat{\varphi}_m\|_{L^2(K)} \leq \max\{n, \lambda_m\}^{n+2} \gamma^n \left(\lambda_m^{-1} \|\nabla \widehat{\varphi}_m\|_{L^2(K')} + \lambda_m^{-2} \|\widehat{\varphi}_m\|_{L^2(K')}\right). \quad \text{(C.3)}$$

We have $\|\varphi_m\|_{H^1(\Gamma)} \sim \lambda_m$, and $\|\varphi_m\|_{L^2(\Gamma)} = 1$. Hence,

$$\|\widehat{\varphi}_m\|_{L^2(K')} \leq C, \qquad \|\widehat{\varphi}_m\|_{H^1(K')} \leq C\lambda_m. \quad \text{(C.4)}$$

Combining (C.3), (C.4) we see that

$$\|\nabla^{n+2} \widehat{\varphi}_m\|_{L^2(K)} \leq C \max\{n, \lambda_m\}^{n+2} \gamma^n \qquad \forall n \in \mathbb{N}_0 \cup \{-1, -2\}.$$

Returning to Γ gives (C.1) in view of Lemma B.1. To see (C.2), we define the extension of φ_m in the trivial way: In a tubular neighborhood T of Γ one can define boundary fitted coordinates $\Gamma \times [-\varepsilon, \varepsilon] \to T$ via $(x, \rho) \mapsto x + \rho \vec{n}(x)$, where $\vec{n}(x)$ is the (outer) normal vector at $x \in \Gamma$. For sufficiently small ε, this is a bijection, and we can define the extension by $\varphi_m(x + \rho \vec{n}(x)) = \varphi_m(x)$. ☐

Remark C.2. Taking the trivial extension to the tubular neighborhood T is clearly not the only choice. For example, if one is only interested in extending φ_m only to $\Omega^+ \cap T$ then one can select the extension to be of the form $\varphi_m(x + \rho\vec{n}(x)) = \varphi_m(x) e^{-\rho/\max\{\lambda_m, k\}}$ with slightly improved bounds in (C.2).

Appendix D. Proof of Lemma 2.5.

Proof of Lemma 2.5:

Proof of (i): For $s = 0$, this is shown in [32, Thms. 3.1, 4.2]. For $s = 1/2$, this follows by the contractivity of the double layer potential (see, e.g., [27, Übungsaufgabe 3.8.8]). By interpolation, the cases $0 < s < 1/2$ are therefore covered. For the cases $s > 1/2$, we use regularity theory for transmission problems. Let $f \in H^s(\Gamma)$ and φ solve $(-1/2 + K_0)\varphi = f$. Then $\varphi \in H^{1/2}(\Gamma)$ and therefore the potential $u = \widetilde{K}_0\varphi \in H^1(\Omega)$ satisfies

$$\|u\|_{H^1(\Omega_{2R})} \lesssim \|\varphi\|_{H^{1/2}(\Gamma)} \lesssim \|f\|_{H^{1/2}(\Gamma)}.$$

Furthermore, u solves on Ω the homogeneous Laplace equation and $\gamma_0^{int} u = (-1/2 + K_0)\varphi = f \in H^s(\Omega)$. By the smoothness of Γ and standard elliptic regularity, we conclude $u \in H^{s+1/2}(\Omega)$ and $\|u\|_{H^{s+1/2}(\Omega)} \leq C\|f\|_{H^s(\Gamma)}$. Next, in view of the jump relation $[\partial_n \widetilde{K}_0\varphi] = 0$, we get that u satisfies

$$-\Delta u = 0 \quad \text{in } \Omega^+, \qquad \gamma_1^{ext} u = \gamma_1^{int} u.$$

From $u \in H^{s+1/2}(\Omega)$ we obtain $\gamma_1^{int} u \in H^{s-1}(\Gamma)$. Elliptic regularity then provides $u|_{\Omega^+\cap B_R} \in H^{s+1/2}(\Omega^+ \cap B_R)$ together with

$$\|u\|_{H^{s+1/2}(\Omega_R\cap\Omega^+)} \lesssim \|\gamma_1^{ext} u\|_{H^{s-1}(\Gamma)} + \|u\|_{H^1(\Omega_{2R}\cap\Omega^+)} \lesssim \|f\|_{H^s(\Gamma)}.$$

The jump relation $[\widetilde{K}_0\varphi] = \varphi$ then implies $\varphi \in H^s(\Gamma)$ with $\|\varphi\|_{H^s(\Gamma)} \lesssim \|f\|_{H^s(\Gamma)}$.
Proof of (ii): Since K_0 and V_0 map $L^2(\Gamma) \to H^1(\Gamma)$, they are compact operators on $L^2(\Gamma)$ and $H^{1/2}(\Gamma)$. Hence, to see the invertibility of the operator $1/2 + K_0 + \mathbf{i}\alpha V_0$ on $L^2(\Gamma)$ and $H^{1/2}(\Gamma)$ it suffices to study the uniqueness of the adjoint. Let therefore $\varphi \in L^2(\Gamma)$ satisfy

$$\left(\frac{1}{2} + K_0' - \mathbf{i}\alpha V_0\right)\varphi = 0.$$

Consider the potential $u = \widetilde{V}_0\varphi$. We have $u \in H^{3/2}(\Omega_{2R})$. Furthermore, u satisfies

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad \gamma_1^{int} u - \mathbf{i}\alpha\gamma_0^{int} u = \left(\frac{1}{2} + K_0' - \mathbf{i}\alpha V_0\right)\varphi = 0.$$

This implies $u|_\Omega = 0$. Next, we aim to show $u|_{\Omega^+} = 0$. To that end, we note that the jump relations for $\widetilde{V}_0$ imply $[u] = 0$. Hence, u solves

$$-\Delta u = 0 \quad \text{in } \Omega^+, \qquad u = 0 \quad \text{on } \Gamma.$$

We now distinguish the cases $d = 3$ and $d = 2$.
For $d = 3$, the decay properties of the single layer potential $u = \widetilde{V}_0\varphi$ imply together with [20, Thm. 8.10] that $u|_{\Omega^+} = 0$. Hence, the jump relations $0 = [\partial_n u] = -\varphi$ yield the desired uniqueness assertion.
For $d = 2$, we let w_{eq} be the "equilibrium density", i.e., $w_{eq} \in H^{-1/2}(\Gamma)$ satisfies $V_0 w_{eq} = \text{const}$ with $\langle w_{eq}, 1\rangle = 1$ (see [20, Thm. 8.15]).
We claim that $(1/2 + K_0')w_{eq} = 0$. To see this, let $v \in H^{1/2}(\Gamma)$ be arbitrary. Then, by [20, Lemma 8.14], we can write $v = V_0\widetilde{v} + a$, where $a \in \mathbb{C}$ and $\langle \widetilde{v}, 1\rangle = 1$. Hence, with $V_0K_0 = K_0'V_0$ (see [29, Cor. 6.19]) and $(1/2 + K_0)1 = 0$:

$$\begin{aligned}\langle (1/2 + K_0')w_{eq}, v\rangle &= \langle (1/2 + K_0')w_{eq}, V_0\widetilde{v} + a\rangle = \langle V_0(1/2 + K_0')w_{eq}, \widetilde{v}\rangle \\ &= \langle (1/2 + K_0)V_0 w_{eq}, \widetilde{v}\rangle = 0,\end{aligned}$$

where, in the last step, we used that $V_0 w_{eq}$ is constant.
Next, let $\beta \in \mathbb{C}$ be such that $\langle \varphi - \beta w_{eq}, 1\rangle = 0$. We define the potential $\widetilde{u} := \widetilde{V}_0(\varphi - \beta w_{eq})$. Then, since by assumption $(1/2 + K_0' - \mathbf{i}\alpha V_0)\varphi = 0$, we get

$$\left(\frac{1}{2} + K_0' - \mathbf{i}\alpha V_0\right)(\varphi - \beta w_{eq}) = -\beta\left(\frac{1}{2} + K_0' - \mathbf{i}\alpha V_0\right) w_{eq} = \beta\mathbf{i}\alpha V_0 w_{eq} = \text{constant}$$

As above, we conclude that $\widetilde{u}|_\Omega$ is constant; in fact $\widetilde{u}|_\Omega = -\beta V_0 w_{eq}$. On Ω^+, the function $\widetilde{u}$ solves Laplace's equation, satisfies the decay condition $\widetilde{u}(x) = O(1)$, $|x| \to$

∞, and attains the constant value $-\beta V_0 w_{eq}$ on Γ. The uniqueness assertion [20, Thm. 8.10] therefore provides that $\widetilde{u}$ is constant on $\mathbb{R}^2$. The jump relation $-(\varphi - \beta w_{eq}) = [\partial_n \widetilde{u}] = 0$ yields $\varphi = \beta w_{eq}$. Hence,

$$0 = (1/2 + K_0' - \mathbf{i}\alpha V_0)\varphi = (1/2 + K_0' - \mathbf{i}\alpha V_0)(\beta w_{eq}) = -\mathbf{i}\alpha V_0(\beta w_{eq}) = -\mathbf{i}\alpha V_0 \varphi.$$

Finally, the scaling assumption on Ω ensures the invertibility of V_0 and therefore $\varphi = 0$.

We have thus shown that $1/2 + K_0 + \mathbf{i}\alpha V_0$ is boundedly invertible as an operator on $L^2(\Gamma)$ and $H^{1/2}(\Gamma)$. By interpolation, it is therefore boundedly invertible on $H^s(\Gamma)$ for $0 \leq s \leq 1/2$.

To see the invertibility on the spaces $H^s(\Gamma)$, $s > 1/2$, we exploit elliptic regularity. Let $f \in H^s(\Gamma)$ for $s > 1/2$. Then, the solution $\varphi \in H^{1/2}(\Gamma)$ of $(1/2 + K_0 + \mathbf{i}\alpha V_0)\varphi = f$ induces a potential $u = \widetilde{K}_0\varphi + \mathbf{i}\alpha\widetilde{V}_0\varphi \in H^1(\Omega_{2R})$ that satisfies on Ω

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad (1 + \mathbf{i}\alpha)\gamma_0^{int} u = f.$$

Thus, $u|_\Omega \in H^{s+1/2}(\Omega)$. The jump conditions satisfied by u are

$$[u] = \varphi, \qquad [\partial_n u] = -\mathbf{i}\alpha\varphi.$$

Hence, the potential u satisfies on Ω^+

$$-\Delta u = 0 \quad \text{in } \Omega^+, \qquad \gamma_1^{ext} u + \mathbf{i}\alpha\gamma_0^{ext} u = \gamma_1^{int} u + \mathbf{i}\alpha\gamma_0^{int} u \in H^{s-1}(\Gamma).$$

We conclude $u|_{\Omega_R \cap \Omega^+} \in H^{s+1/2}(\Omega_R \cap \Omega^+)$. The jump relation $\varphi = [u]$ thus leads to the desired $\varphi \in H^s(\Gamma)$.

Proof of (iii): The contractivity properties of $1/2 + K_0'$ (see [29, Cor. 6.30]) imply the convergence of the Neumann series for $-1/2 + K_0'$ in $H^{-1/2}(\Gamma)$. Thus, $-1/2 + K_0'$ is boundedly invertible on $H^{-1/2}(\Gamma)$. To see that it is boundedly invertible on $H^s(\Gamma)$ for $s > -1/2$ we consider $f \in H^s(\Gamma)$ and let $\varphi \in H^{-1/2}(\Gamma)$ with $(-1/2 + K_0')\varphi = f$. The potential $u = \widetilde{V}_0\varphi \in H^1(\Omega_{2R})$ then satisfies the boundary condition $\gamma_1^{ext} u = (-1/2 + K_0')\varphi = f \in H^s(\Gamma)$. Elliptic regularity thus produces $u|_{\Omega_R \cap \Omega^+} \in H^{s+3/2}(\Omega_R \cap \Omega^+)$. On Ω, the potential u satisfies Laplace's equation together with the boundary condition $\gamma_0^{int} u = \gamma_0^{ext} u \in H^{s+1}(\Gamma)$. Again, elliptic regularity leads to $u \in H^{s+3/2}(\Omega)$. The jump condition $-\varphi = [\partial_n u] \in H^s(\Gamma)$ leads to the desired result.

Proof of (iv): The proof resembles that of (ii). The operators $K_0' : H^{-1}(\Gamma) \to L^2(\Gamma)$ and $V_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ are compact on $H^{-1/2}(\Gamma)$. Hence, unique solvability on $H^{-1/2}(\Gamma)$ for $1/2 + K_0' + \mathbf{i}\alpha V_0$ is ensured if we can ascertain uniqueness for the adjoint equation. Let therefore $\varphi \in H^{1/2}(\Gamma)$ satisfy $(1/2 + K_0 - \mathbf{i}\alpha V_0)\varphi = 0$. Consider the potential $u = \widetilde{K}_0\varphi - \mathbf{i}\alpha\widetilde{V}_0\varphi$. Then, $u \in H^1(\Omega_{2R})$ and $\gamma_0^{ext} u - \mathbf{i}\alpha\gamma_0^{ext} u = (1/2 + K_0)\varphi - \mathbf{i}\alpha V_0\varphi = 0$. We note the jump conditions

$$[u] = \varphi, \qquad [\partial_n u] = \mathbf{i}\alpha\varphi. \tag{D.1}$$

We distinguish again the cases $d = 2$ and $d = 3$.

For $d = 3$, the potential u satisfies the decay conditions at ∞ that allow us to conclude with [20, Thm. 8.10] that $u|_{\Omega^+} = 0$. The jump conditions (D.1) therefore imply that $u|_\Omega$ satisfies $\gamma_1^{int} u - \mathbf{i}\alpha\gamma_0^{int} u = \gamma_1^{ext} u - \mathbf{i}\alpha\gamma_0^{ext} u = 0$. Hence, $u|_\Omega = 0$. Thus, the jump conditions (D.1) imply $\varphi = 0$.

For $d = 2$, we first show that $\langle\varphi, 1\rangle = 0$. To that end, we recall $(1/2 + K_0')w_{eq} = 0$ and note that the $1/2 + K_0 - \mathbf{i}\alpha V_0\varphi = 0$ implies

$$0 = \langle w_{eq}, 1/2 + K_0\rangle - \mathbf{i}\alpha\langle w_{eq}, V_0\varphi\rangle = -\mathbf{i}\alpha\langle V_0 w_{eq}, \varphi\rangle.$$

Since $V_0 w_{eq}$ is a constant function (and, in view of the scaling assumption $\operatorname{diam}\Omega < 1$, we have $V_0 w_{eq} \neq 0$), we get that φ has vanishing mean.
Consider now again the potential $\widetilde{u} = \widetilde{K}_0\varphi - \mathbf{i}\alpha\widetilde{V}_0\varphi$. Then

$$\gamma_0^{ext} u = (1/2 + K_0)\varphi - \mathbf{i}\alpha V_0\varphi = 0. \tag{D.2}$$

Hence, the uniqueness assertion of [20, Thm. 8.10] implies that $u|_{\Omega^+} = 0$. The jump conditions satisfied by u read

$$[\widetilde{u}] = \varphi, \qquad [\partial_n \widetilde{u}] = \mathbf{i}\alpha\varphi. \tag{D.3}$$

Hence, $\gamma_1^{int} u - \mathbf{i}\alpha\gamma_0^{int} u = \gamma_1^{ext} u - \mathbf{i}\alpha\gamma_0^{ext} u = 0$. Therefore, $u|_\Omega = 0$. From (D.3) we finally conclude $\varphi = 0$.
The invertibility of $1/2 + K_0' + \mathbf{i}\alpha V_0$ on $H^s(\Gamma)$, $s > -1/2$ now follows by arguments similar to those used above. Let $\varphi \in H^{-1/2}(\Gamma)$ solve $(1/2 + K_0' + \mathbf{i}\alpha V_0)\varphi = f \in H^s(\Gamma)$. Then the potential $u = \widetilde{V}_0\varphi$ satisfies $\gamma_1^{int} u + \mathbf{i}\alpha\gamma_0^{int} u = f$. Elliptic regularity therefore leads to $u|_\Omega \in H^{s+3/2}(\Omega)$. The jump relations for $\widetilde{V}_0$ then given $\gamma_0^{ext} u = \gamma_0^{int} u \in H^{s+1}(\Gamma)$. Elliptic regularity produces $u|_{\Omega_R \cap \Omega^+} \in H^{s+3/2}(\Omega_R \cap \Omega^+)$. The jump relation $-\varphi = [\partial_n u] \in H^s(\Gamma)$ allows us to conclude the proof. □

Appendix E. Proof of Theorem 7.12. *Proof of Theorem 7.12:* We introduce the abbreviation $e := u - u_N$. Let $w_N \in X_N$ be arbitrary. Then by the triangle inequality

$$\|e\|_0 \le \|u - w_N\|_0 + \|u_N - w_N\|_0. \tag{E.1}$$

Hence, we have to estimate $\|u_N - w_N\|_0$. By the discrete inf-sup condition we can find $v_N \in X_N$ with $\|v_N\|_0 = 1$ and $\gamma_0\|u_N - w_N\|_0 \le (A_0'(u_N - w_N), v_N)_0$. With the Galerkin orthogonality $(A_k'(u - u_N), v_N)_0 = 0$, we then produce

$$\begin{aligned}
\gamma_0\|u_N - w_N\|_0 &\le ((A_0' - A_k')(u_N - w_N), v_N)_0 + (A_k'(u_N - w_N), v_N)_0 \\
&= ((A_0' - A_k')(u_N - w_N), v_N)_0 + (A_k'(u - w_N), v_N)_0 \\
&= ((A_k' - A_0')e, v_N)_0 + (A_0'(u - w_N), v_N)_0 \\
&\le \|A_0'\|_{L^2 \leftarrow L^2}\|u - w_N\|_0 + ((A_k' - A_0')e, v_N)_0.
\end{aligned} \tag{E.2}$$

In order to treat the term $((A_k' - A_0')e, v_N)_0$ we define $\psi \in L^2(\Gamma)$ by

$$((A_k' - A_0')z, v_N)_0 = (z, A_{-k}\psi)_0 \qquad \forall z \in L^2(\Gamma). \tag{E.3}$$

Lemma 7.2 tells us

$$\psi = A_{-k}^{-1}(A_{-k} - A_0)v_N \tag{E.4}$$

By selecting $z = e$ in (E.3), using Galerkin orthogonality satisfied by the error e and orthogonality properties of $\Pi_N^{L^2}$ we obtain

$$\begin{aligned}
&((A_k' - A_0')e, v_N)_0 \\
&= (e, A_{-k}\psi)_0 = (A_k' e, \psi)_0 = (A_k' e, \psi - \Pi_N^{L^2}\psi)_0 \\
&= (A_0' e, \psi - \Pi_N^{L^2}\psi)_0 + ((A_k' - A_0')e, \psi - \Pi_N^{L^2}\psi)_0 \\
&= (A_0' e, \psi - \Pi_N^{L^2}\psi)_0 + ((A_k' - A_0')e - \Pi_N^{L^2}(A_k' - A_0')e, \psi - \Pi_N^{L^2}\psi)_0.
\end{aligned}$$

Hence, from (E.4) and $\|v_N\|_0 = 1$

$$|((A_k' - A_0')e, v_N)_0| \le \left\{\|A_0'\|_{L^2\leftarrow L^2} + \|(\mathrm{Id} - \Pi_N^{L^2})(A_k' - A_0')\|_{L^2\leftarrow L^2}\right\} \\ \times \|(\mathrm{Id} - \Pi_N^{L^2})A_{-k}^{-1}(A_{-k} - A_0)\|_{L^2\leftarrow L^2}\, \|e\|_0.$$

From Lemmata 7.7, 7.8 we get for arbitrary $q \in (0,1)$

$$|((A_k' - A_0')e, v_N)_0| \le \{\|A_0'\|_{L^2\leftarrow L^2} + q + Ck\eta(N,k,\gamma)\} \\ \times \left\{q + Ck^2\left(1 + k^{5/2}C(A,0,-k)\right)\eta_1(N,-k,\gamma)\right\}\, \|e\|_0. \tag{E.5}$$

Select now $q \in (0,1)$ such that $(\|A_0'\|_{L^2\leftarrow L^2} + q)q < 1/2$. Then the constants C and γ in (E.5) are fixed and independent of k. We can furthermore select $\varepsilon > 0$ independent of k such that the assumption (7.19) then guarantees that the product of the two curly braces in (E.5) is bounded by 1/2. Combining (E.1), (E.2), and (E.5) therefore yields

$$\|e\|_0 \le \left(1 + \frac{\|A_0'\|_{L^2\leftarrow L^2}}{\gamma_0}\right)\|u - w_N\|_0 + \frac{1}{2}\|e\|_0,$$

which leads to the desired estimate. □

Appendix F. Notes on mapping properties of $\widetilde{V}_0$ and $\widetilde{K}_0$.

LEMMA F.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with connected boundary. Denote by δ the distance from $\partial\Omega$. Then:*

(i) $\widetilde{V}_0 : L^2(\partial\Omega) \to H^{3/2}(\Omega)$ *and*

$$\|\widetilde{V}_0\varphi\|_{H^{3/2}(\Omega)} + \|(\widetilde{V}_0\varphi)^*\|_{L^2(\partial\Omega)} + \|(\nabla\widetilde{V}_0\varphi)^*\|_{L^2(\partial\Omega)} + \|\sqrt{\delta}\nabla^2\widetilde{V}_0\varphi\|_{L^2(\Omega)} \le C\|\varphi\|_{L^2(\partial\Omega)}.$$

Furthermore, the non-tangential limit of $\widetilde{V}_0\varphi$ on $\partial\Omega$ is $V_0\varphi$.

(ii) $\widetilde{V}_0 : H^{-1}(\partial\Omega) \to H^{1/2}(\Omega)$ *and*

$$\|\widetilde{V}_0\varphi\|_{H^{1/2}(\Omega)} + \|(\widetilde{V}_0\varphi)^*\|_{L^2(\partial\Omega)} + \|\sqrt{\delta}\nabla\widetilde{V}_0\varphi\|_{L^2(\Omega)} \le C\|\varphi\|_{H^{-1}(\partial\Omega)}.$$

Furthermore, the non-tangential limit of $\widetilde{V}_0\varphi$ on $\partial\Omega$ is $V_0\varphi$.

Proof. Before starting with the proof, we introduce the non-tangential trace operator Tr, which is defined as $\mathrm{Tr}\, u(x) := \lim_{z\to x, z\in\Gamma(x)} u(z)$, where $\Gamma(x)$ is the non-tangential cone associated with the point $x \in \partial\Omega$.

We start with the proof of part (i):

1. step: By [32, Thm. 3.3, Thm. 4.11] we have $V_0 : L^2(\partial\Omega) \to H^1(\partial\Omega)$.

2. step: We claim that the function $u := \widetilde{V}_0\varphi$ has non-tangential limits for almost all $x \in \partial\Omega$, i.e., $\mathrm{Tr}\,\widetilde{V}_0\varphi$ exists. To see this, we decompose $\varphi = \varphi^+ - \varphi^-$ with φ^+ and $\varphi^- \ge 0$. By sign properties of the fundamental solution the functions $u^+ := \widetilde{V}_0\varphi^+$ and $u^- := \widetilde{V}_0\varphi^-$ are positive harmonic functions in Ω (for the 2D case, we assume here a proper scaling of the domain). By [15, Thm. 2.3] we conclude that u^+ and u^- have non-tangential limit almost everywhere; hence, also $u = u^+ - u^-$ has this property.

3. step: Since $u \in H^1(\Omega)$ it has a trace on $\partial\Omega$, which is $V_0\varphi \in H^1(\partial\Omega)$. Furthermore, this trace coincides with the non-tangential limit. From [16, Thm. 5.15], we therefore get that $\widetilde{V}_0\varphi = u \in H^{3/2}(\Omega)$. Furthermore, $\|u\|_{H^{3/2}(\Omega)} \le C\|V_0\varphi\|_{H^1(\partial\Omega)} \le C\|\varphi\|_{L^2(\partial\Omega)}$. The estimates for $\|u^*\|_{L^2(\partial\Omega)}$ and $\|(\nabla u)^*\|_{L^2(\partial\Omega)}$ follow from [16, Cor. 5.7].

We now turn to the proof of part (ii):
1. step: The following duality argument shows that $V_0 : H^{-1}(\partial\Omega) \to L^2(\partial\Omega)$: For φ, $\psi \in L^2(\partial\Omega)$ we compute

$$|\langle V_0\varphi, \psi\rangle| = |\langle \varphi, V_0\psi\rangle| \le \|\varphi\|_{H^{-1}(\partial\Omega)}\|V_0\psi\|_{H^1(\partial\Omega)} \le C\|\varphi\|_{H^{-1}(\partial\Omega)}\|\psi\|_{L^2(\partial\Omega)},$$

where the last step follows the assertion $V_0 : L^2(\partial\Omega) \to H^1(\partial\Omega)$ of [32]. By density, V_0 can be (uniquely) extended to an operator $H^{-1}(\partial\Omega) \to L^2(\partial\Omega)$.
2. step: We aim to show that the function $u := \widetilde{V}_0\varphi$ satisfies $\|u^*\|_{L^2(\partial\Omega)} \le C\|V_0\varphi\|_{L^2(\partial\Omega)} \le C\|\varphi\|_{H^{-1}(\partial\Omega)}$. To that end, let $(\varphi_n)_{n\in\mathbb{N}} \subset L^2(\partial\Omega)$ be a sequence converging in $H^{-1}(\partial\Omega)$ to φ. By part (i) (see 3. step), the functions $u_n := \widetilde{V}_0\varphi_n$ converge non-tangentially to $V_0\varphi_n$. By [16, Thm. 5.3, Thm. 5.4, Cor. 5.5] we have

$$\|(u_n)^*\|_{L^2(\partial\Omega)} + \|u_n\|_{H^{1/2}(\Omega)} \le C\|V_0\varphi_n\|_{L^2(\partial\Omega)}.$$

Since $(\varphi_n)_n$ is a Cauchy sequence in $H^{-1}(\partial\Omega)$, we have that $(u_n)_n$ converges pointwise to u in Ω and $(u_n)^*$ converges in $L^2(\partial\Omega)$ to a function $\widetilde{u} \in L^2(\partial\Omega)$. We have

$$\|\widetilde{u}\|_{L^2(\partial\Omega)} = \lim_{n\to\infty}\|u_n^*\|_{L^2(\partial\Omega)} \le C\lim_{n\to\infty}\|V_0\varphi_n\|_{L^2(\partial\Omega)} \le C\lim_{n\to\infty}\|\varphi_n\|_{H^{-1}(\partial\Omega)} = C\|\varphi\|_{H^{-1}(\partial\Omega)}.$$

After possibly passing to a subsequence, we may assume that u_n^* converges to $\widetilde{u}$ pointwise almost everywhere. Let $x \in \partial\Omega$ be a point with $\lim_{n\to\infty} u_n^*(x) = \widetilde{u}(x) \in \mathbb{R}$. Then for every $z \in \Gamma(x)$

$$\begin{aligned} |u(z)| &\le \limsup_{n\to\infty}(|u(z) - u_n(z)| + |u_n(z)|) \le \limsup_{n\to\infty}|u(z) - u_n(z)| + \limsup_{n\to\infty}|u_n(z)| \\ &\le \lim_{n\to\infty}|u(z) - u_n(z)| + \limsup_{n\to\infty}\sup_{z\in\Gamma(x)}|u_n(z)| = 0 + \limsup_{n\to\infty}(u_n)^*(x) = \widetilde{u}(x). \end{aligned}$$

We conclude

$$u^*(x) = \sup_{z\in\Gamma(x)}|u(z)| \le \widetilde{u}(x),$$

and hence that $u^* \in L^2(\partial\Omega)$ with $\|u^*\|_{L^2(\partial\Omega)} \le C\|\varphi\|_{H^{-1}(\partial\Omega)}$.
3. step: By [16, Cor. 5.5], we have $u \in H^{1/2}(\Omega)$ and

$$\|u^*\|_{L^2(\partial\Omega)} + \|u\|_{H^{1/2}(\Omega)} + \|\sqrt{\delta}\nabla u\|_{L^2(\Omega)} \le C\|\varphi\|_{H^{-1}(\partial\Omega)}.$$

To see that the non-tangential limit of $\widetilde{V}_0\varphi$ is $V_0\varphi$, we first note that [16, Cor. 5.5] asserts that $\widetilde{V}_0\varphi$ has a non-tangential trace $\operatorname{Tr} u \in L^2(\partial\Omega)$. Let $\overline{u}$ be the unique solution of the boundary value problem

$$-\Delta\overline{u} = 0 \quad \text{in } \Omega, \qquad \operatorname{Tr}\overline{u} = V_0\varphi \in L^2(\partial\Omega).$$

Since $V_0\varphi_n \to V_0\varphi$ in $L^2(\partial\Omega)$ and $\operatorname{Tr} u_n = V_0\varphi_n$, we get from *a priori* estimates

$$\|\overline{u} - u_n\|_{H^{1/2}(\Omega)} \le C\|\operatorname{Tr}(\overline{u} - u_n)\|_{L^2(\partial\Omega)} \le C\|V_0\varphi - V_0\varphi_n\|_{L^2(\partial\Omega)} \to 0.$$

By Caccioppoli inequalities, we infer that $\overline{u}$ is the pointwise limit of the functions u_n. This pointwise limit is also u, and we conclude $u = \overline{u}$. Thus, $\operatorname{Tr} u = \operatorname{Tr}\overline{u} = V_0\varphi$. □
LEMMA F.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with connected boundary. Denote by δ the distance from $\partial\Omega$. Then:*

(i) $\widetilde{K}_0 : H^1(\partial\Omega) \to H^{3/2}(\Omega)$ *and*

$$\|\widetilde{K}_0\varphi\|_{H^{3/2}(\Omega)}+\|(\widetilde{K}_0\varphi)^*\|_{L^2(\partial\Omega)}+\|(\nabla\widetilde{K}_0\varphi)^*\|_{L^2(\partial\Omega)}+\|\sqrt{\delta}\nabla^2\widetilde{K}_0\varphi\|_{L^2(\Omega)} \le C\|\varphi\|_{H^1(\partial\Omega)}.$$

Furthermore, the non-tangential limit of $\widetilde{K}_0\varphi$ *on* $\partial\Omega$ *is* $(-1/2+K_0)\varphi$.

(ii) $\widetilde{K}_0 : L^2(\partial\Omega) \to H^{1/2}(\Omega)$ *and*

$$\|\widetilde{K}_0\varphi\|_{H^{1/2}(\Omega)} + \|(\widetilde{K}_0\varphi)^*\|_{L^2(\partial\Omega)} + \|\sqrt{\delta}\nabla\widetilde{K}_0\varphi\|_{L^2(\Omega)} \le C\|\varphi\|_{L^2(\partial\Omega)}.$$

Furthermore, the non-tangential limit of $\widetilde{K}_0\varphi$ *on* $\partial\Omega$ *is* $(-1/2+K_0)\varphi$.

Proof. We start with part (i): For $\varphi \in H^1(\partial\Omega) \subset H^{1/2}(\partial\Omega)$, we have $\widetilde{K}_0\varphi \in H^1(\Omega)$. We have $\gamma_0\widetilde{K}_0\varphi = (-1/2+K_0)\varphi \in H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$. By [16, Cor. 5.5], the non-tangential trace $\operatorname{Tr}\widetilde{K}_0\varphi$ exists and is in $L^2(\partial\Omega)$. We conclude $\operatorname{Tr}\widetilde{K}_0\varphi = \gamma_0\widetilde{K}_0\varphi = (-1/2+K_0)\varphi$. By [32, Thm. 3.3] we have $(-1/2+K_0)\varphi \in H^1(\partial\Omega)$, so that [16, Thm. 5.15] implies $\widetilde{K}_0\varphi \in H^{3/2}(\Omega)$. Then [16, Cor. 5.7] implies the desired estimate.

We turn to the proof of part (ii): This is proved using the same arguments as part (ii) of Lemma F.1. □

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