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New Velocity Formulation of Euler Equations with Third-order Derivatives: Application to Viscous Korteweg-type and Quantum Navier-Stokes Models

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New velocity formulation of Euler equations with third-order derivatives: application to viscous Korteweg-type and quantum Navier-Stokes models

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Abstract

A formulation of certain barotropic Euler equations with third-order derivatives, modeling viscous Korteweg-type or quantum Navier-Stokes flows, as a viscous Euler system is proposed, using a new velocity variable. This formulation allows for the derivation of an energy identity and a global existence result for the one-dimensional quantum Navier-Stokes equations. *To cite this article: A. Jüngel, Comptes Rendus Math. (2009).*

Résumé

Nouvelle formulation des équations d'Euler avec des dérivées d'ordre trois : application aux modèles visqueux de type Korteweg et de Navier-Stokes quantique. Une formulation des équations d'Euler barotropiques avec des dérivées d'ordre trois, qui modèlisent des flux visqueux de type Korteweg et de Navier-Stokes quantique, comme un système d'Euler visqueux est proposée. Cette formulation permet de dériver une égalité d'énergie et de démontrer l'existence globale des équations de Navier-Stokes quantique en une dimension. *Pour citer cet article : A. Jüngel, Comptes Rendus Math. (2009).*

Key words: Korteweg-type models, quantum hydrodynamic equations, viscous Euler system, energy estimates.

Version française abrégée

On considère des équations d'Euler avec des dérivées d'ordre trois, qui incluent des modéles particuliers visqueux de type Korteweg [3] et de Navier-Stokes quantique [4]. Pour ces modèles, il est très difficile de

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dériver des estimations (d'energie) a priori et de démontrer l'existence des solutions à cause des termes fortement non linéaires. Le résultat principale de cette note est la découverte que, sous des conditions appropriées, les termes d'ordre trois peuvent être *éliminés* en utilisant une nouvelle variable de type vitesse. Ce résultat permet de dériver une égalité d'énergie et de démontrer l'existence globale des équations de Navier-Stokes quantique en une dimension.

Les équations d'Euler barotropiques pour la densité des particules n et la vitesse u sont données par (1)-(3), où $p(n)$ est la pression, f la force, S la tenseur des contraintes visqueuses, et K la tenseur des contraintes de type Korteweg, données par (4). Dans cette définition, μ et λ sont les coefficients visqueux de Lamé, $D(u) = (\nabla u + \nabla u^\top)/2$ est le gradient de vitesse, et $\nabla^2\xi$ est la matrice hessienne de ξ.

Korteweg [12] avait proposé l'équation constitutive (5) pour la tenseur des contraintes K, où a_i sont des fonctions dépendant de la densité et $\mathbb{I}$ est la matrice unité dans $\mathbb{R}^{d\times d}$. L'expression pour K dans (4) est obtenue pour $a_1 = \mu\xi''$, $a_4 = \mu\xi'$ et $a_2 = a_3 = 0$.

Un deuxième exemple est donné par les équations de Navier-Stokes quantique, dérivées par Brull et Méhats [4]. Dans ce modèle, $\mu(n) = \varepsilon^2 n$, $\lambda(n) = 0$ et $\xi(n) = \varepsilon^2 \log n$. Le modèle non locale, donné dans (22), est dérivé de l'équation de Wigner-BGK. Pour obtenir les équations locales, on fait une expansion en ε^2. En effet, en utilisant l'expansion (23) et $S = 2\mathrm{div}(nD(u)) + O(\varepsilon^2)$ [4, Remark 1] ainsi que les hypothèses $\tau = \varepsilon^2$ et $\mathrm{curl}(u) = O(\varepsilon)$, on arrive après un calcul aux équations (1) et (6).

Le premier résultat de cette article est une formulation de (1)-(2) en utilisant la vitesse "effective" $w = u + \nabla\xi(n)$. Sous des conditions structurelles sur μ, λ et ξ, le systéme en variable (n, w) ne contient plus des dérivées d'ordre trois.

Theorem 0.1 *Soit* (n, u) *une solution classique de* (1)-(3) *et suppose* (8). *Alors,* (n, w) *est une solution de* (9)-(11) *et vice versa. En outre, avec l'énergie définie dans* (7)*, on a l'identité* (12).

Cette formulation nous permet de démontrer l'existence des solutions globales pour les équations de Navier-Stokes quantique en une dimension.

Theorem 0.2 *Soient* $d = 1$, $n_I \in W^{1,\infty}(\mathbb{R})$, $u_I \in L^\infty(\mathbb{R})$ *et* $n_I \geq \delta > 0$ *dans* $\mathbb{R}$. *Suppose que* $\mu(n) = n$ *pour* $n \geq 0$, (8) *et* $f = 0$. *Alors, il existe une solution classique bornée* (n, u) *de* (1)-(3) *telle que* $n(x, t) \geq c(\delta, t) > 0$ *pour* $(x, t) \in \mathbb{R} \times [0, \infty)$.

On note que la vitesse "effective" apparaît dans plusieurs modèles. Bresch et Desjardins [2] ont dérivé des nouvelles estimations a priori pour les modèles visqueux de type Korteweg et de "shallow water" avec la tenseur K satisfaisant $\mathrm{div}\, K = n\nabla(\sigma'(n)\Delta\sigma(n))$. Un exemple typique est $\sigma(n) = \kappa n$ où $\kappa > 0$, ce qui donne $\mathrm{div}\, K = \kappa^2 n\nabla\Delta n$. Si $\sigma = \mu$, le système (1)-(2) peut être reformulé comme (13) où $A(u) = (\nabla u - \nabla u^\top)/2$. On peut montré que la tenseur K est inclut dans notre modèle si et seulement si $\sigma(n) = 2\sqrt{n}$ donnant le modèle de Navier-Stokes quantique.

En plus, la vitesse w apparaît dans les équations hydrodynamique quantique visqueuses (15) avec $w = u - \nu\nabla \log n$ et dans les équations de Navier-Stokes modifiées (14) où u et $w = u - \nu\nabla \log n$ sont interpretés comme la vitesse de volume et de masse, respectivement.

Théorème 0.1 est démontré comme suit. Equation (9) est une conséquence de la définition de w. Pour montrer (10), nous calculons (16)-(18). La somme de (17) et (18) donne (19). Puis, l'identité d'énergie résulte de (20) et (21), en utilisant (9)-(10) et $\mu'(n)H''(n) = \xi'(n)p'(n)$.

L'existence des solutions locales suit des résultats de DiPerna [8]. Pour l'existence globale, on observe qu'il existe des invariants de Riemann pour (9)-(10) [6] et alors des bornes L^∞ pour n et u. La borne inférieure pour la densité initiale donne une borne inférieure positive pour n ; voir [5,8]. Ces deux résultats suffisent de conclure le Théorème 0.2.

1. Introduction

Euler equations with third-order derivatives occur, for instance, in the theory of capillarity with diffuse interfaces [9], in viscous shallow-water models [3], and quantum hydrodynamic semiconductor equations [10]. In this note, we consider a specific class of third-order derivatives including special viscous Korteweg-type models and quantum Navier-Stokes equations. For these models, it is usually far from being trivial to derive suitable a priori (energy) estimates and to prove the existence of solutions, due to the strongly nonlinear third-order derivatives. The main result of this note is the discovery that, under suitable assumptions, the third-order expression can be *eliminated* by using a so-called effective velocity variable. The particle density and the new velocity then solve a viscous Euler system, which allows for the derivation of energy estimates and a global-in-time existence result in one space dimension.

We consider the barotropic Euler equations for the particle density n and the fluid velocity u,

$$n_t + \mathrm{div}(nu) = 0, \tag{1}$$

$$(nu)_t + \mathrm{div}(nu \otimes u) + \nabla p(n) = nf + \mathrm{div}(S + K) \quad \text{in } \mathbb{R}^d,\ t > 0, \tag{2}$$

$$n(\cdot, 0) = n_I, \quad u(\cdot, 0) = u_I \quad \text{in } \mathbb{R}^d, \tag{3}$$

where $p(n)$ is the (density dependent) pressure and f describes some forces. The viscous stress tensor S and the Korteweg-type stress tensor K are assumed to have the special form

$$\mathrm{div}\, S = 2\mathrm{div}(\mu(n)D(u)) + \nabla(\lambda(n)\mathrm{div}\, u), \quad K = \mu(n)\nabla^2\xi(n) = \mu(n)\xi''(n)|\nabla n|^2 + \mu(n)\xi'(n)\nabla^2 n, \tag{4}$$

where μ and λ are the Lamé viscosity coefficients, $D(u) = (\nabla u + \nabla u^\top)/2$ is the velocity gradient, and $\nabla^2\xi$ is the Hessian of the scalar function ξ.

Example 1 (Korteweg-type fluid model) Korteweg [12] proposed a constitutive equation for the stress tensor including density gradients, being of the form

$$K = a_1|\nabla n|^2\mathbb{I} + a_2\nabla n \otimes \nabla n + a_3\Delta n\mathbb{I} + a_4\nabla^2 n, \tag{5}$$

where a_i are density-dependent functions and $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{d\times d}$. The expression for K in (4) is obtained after choosing $a_1 = \mu\xi''$, $a_4 = \mu\xi'$, and $a_2 = a_3 = 0$. □

Example 2 (Quantum Navier-Stokes model) Brull and Méhats [4] have derived, starting from a collisional Wigner equation, nonlocal quantum Navier-Stokes equations, whose local version can be written as (1)-(2), where $\mu(n) = \varepsilon^2 n$, $\lambda(n) = 0$, and $\xi(n) = \varepsilon^2 \log n$ (see section 4 for a sketch of the derivation). Then $\mathrm{div}\, K = 2\varepsilon^2 n\nabla(\Delta\sqrt{n}/\sqrt{n})$, and the momentum equation becomes

$$(nu)_t + \mathrm{div}(nu \otimes u) + \nabla p(n) - 2\varepsilon^2 n\nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big) = nf + 2\varepsilon^2\mathrm{div}(nD(u)). \tag{6}$$

For $S = 0$, the system (1) and (6) is called the quantum hydrodynamic model, which is employed in semiconductor simulations [10]. □

The common feature of both examples is that the corresponding free energy contains density gradients. In fact, we prove below that the energy

$$E(t) = \int_{\mathbb{R}^d} \Big(\frac{n}{2}|\nabla(u + \nabla\xi(n))|^2 + H(n)\Big)(x,t)dx, \tag{7}$$

where the internal energy H is a primitive of the enthalpy h, defined by $h'(n) = p'(n)/n$ and $h'(1) = 0$, is nonincreasing in time along smooth solutions to (1)-(3) if $f = 0$. Notice that H is convex if p is nondecreasing since $H''(n) = p'(n)/n$ for $n > 0$.

The first result of this paper is the formulation of (1)-(2) in terms of the effective velocity $w = u + \nabla\xi(n)$ which, under a structural assumption on μ, ξ, and λ, eliminates the third-order derivative appearing in $\mathrm{div}\, K$.

Theorem 1.1 (Viscous Euler formulation) *Let (n,u) be a smooth solution to* (1)-(3) *and let*

$$\mu'(n) = n\xi'(n), \quad \lambda(n) = n\mu'(n) - \mu(n) \quad \text{for all } n \geq 0 \tag{8}$$

and (4) *hold. Then (n,w) with $w = u + \nabla\xi(n)$ is a smooth solution to the* viscous Euler system

$$n_t + \operatorname{div}(nw) = \Delta\mu(n), \tag{9}$$

$$(nw)_t + \operatorname{div}(nw \otimes w) + \nabla p(n) = nf + \Delta(\mu(n)w) \quad \text{in } \mathbb{R}^d,\ t > 0, \tag{10}$$

$$n(\cdot,0) = n_I, \quad w(\cdot,0) = w_I := u_I + \nabla\xi(n_I) \quad \text{in } \mathbb{R}^d. \tag{11}$$

Moreover, if (n,w) is a smooth solution to (9)-(11)*, then (n,u) with $u = w - \nabla\xi(n)$ solves* (1)-(3)*. Finally, the following energy identity holds (recall the definition of E in* (7)*):*

$$\frac{dE}{dt} + \int_{\mathbb{R}^d} \big(\mu(n)|\nabla w|^2 + \xi'(n)p'(n)|\nabla n|^2\big)dx = \int_{\mathbb{R}^d} nf \cdot w dx. \tag{12}$$

This result is also valid using the more general expression

$$\operatorname{div} K = \operatorname{div}(\mu(n)\nabla^2\xi(n)) + n\nabla(\sigma'(n)\Delta\sigma(n)),$$

if $\sigma = \mu$. Indeed, a computation similar to the proof of Theorem 1.1 shows that

$$\frac{d}{dt}\Big(E + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\mu(n)|^2 dx\Big) + \int_{\mathbb{R}^d} \big(\mu'(n)(\Delta\mu(n))^2 + \mu(n)|\nabla w|^2 + \xi'(n)p'(n)|\nabla n|^2\big)dx = \int_{\mathbb{R}^d} nf \cdot w dx.$$

The formulation in the velocity w allows us to conclude a global existence result for the one-dimensional quantum Navier-Stokes model.

Theorem 1.2 (Quantum Navier-Stokes model) *Let $d = 1$ and $n_I \in W^{1,\infty}(\mathbb{R})$, $u_I \in L^\infty(\mathbb{R})$ such that $n_I \geq \delta > 0$ in $\mathbb{R}$. Assume that $\mu(n) = n$ for $n \geq 0$,* (8)*, and $f = 0$. Then there exists a smooth bounded solution (n,u) to* (1)-(2) *satisfying $n(x,t) \geq c(\delta,t) > 0$ for $(x,t) \in \mathbb{R} \times [0,\infty)$.*

We expect that this theorem can be generalized to adiabatic pressures $p(n) = n^\alpha$ with $\alpha > 1$, more general functions μ, or bounded domains with suitable boundary conditions. On the other hand, the global existence in several space dimensions (with strictly positive density) is less obvious. These issues are currently under investigation.

2. Relations to other models

The effective velocity $w = u + \nabla\xi(n)$ appears also in related models. First, it has been used by Bresch and Desjardins to derive new entropy estimates for viscous Korteweg-type and shallow-water models with the capillary stress tensor K satisfying $\operatorname{div} K = n\nabla\big(\sigma'(n)\Delta\sigma(n)\big)$ [2]. A typical example is $\sigma(n) = \kappa n$, where $\kappa > 0$, which gives $\operatorname{div} K = \kappa^2 n\nabla\Delta n$ [3]. When $\sigma = \mu$, the system (1)-(2) can be reformulated as

$$n_t + \operatorname{div}(nw) = 0, \quad (nw)_t + \operatorname{div}(nu \otimes w) + \nabla p(n) = 2\operatorname{div}(\mu(n)A(u)) + n\nabla\big(\mu'(n)\Delta\mu(n)\big), \tag{13}$$

where $A(u) = (\nabla u - (\nabla u)^\top)/2$. It is not difficult to prove that the above stress tensor is included in our model if and only if $\sigma(n) = 2\sqrt{n}$, $\mu(n) = n$, and $\xi(n) = \log n$, which gives the quantum Navier-Stokes model. Indeed, defining σ by $\sigma'(n)^2 = n\xi'(n)^2$ to dispose of the third-order derivatives, a computation shows that $\operatorname{div}(\mu(n)\nabla^2\xi(n)) = n\nabla(\sigma'(n)\Delta\sigma(n)) + R$, and the remainder R vanishes (for nonconstant functions ξ) if and only if $\mu(n) = n$.

Brenner [1] suggested the modified Navier-Stokes model

$$n_t + \operatorname{div}(nw) = 0, \quad (nu)_t + \operatorname{div}(nu \otimes w) + \nabla p(n) = \operatorname{div} S, \tag{14}$$

which is similar to (13), and interpreted u and w as the volume and mass velocities, respectively, which are related by the constitutive equation $u - w = \nu\nabla\log n$ with the phenomenological constant $\nu > 0$.

The effective velocity also appears in the viscous quantum hydrodynamic model [11, p. 453],

$$n_t + \operatorname{div}(nu) = \nu\Delta n, \quad (nu)_t + \operatorname{div}(nu\otimes u) + \nabla p(n) = nf + \nu\Delta(nu) + 2\varepsilon^2\nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big), \tag{15}$$

by setting $w = u-\nu\nabla\log n$. In [13, p. 328], the expression $\nu\nabla\log n$ is referred to as the quantum diffusive velocity. In the new velocity formulation (n, w), the factor of the third-order quantum term does not vanish but it increases to $2\varepsilon^2 + 2\nu^2$.

3. Proof of Theorems 1.1 and 1.2

First, we prove Theorem 1.1. Let (n, u) be a smooth solution to (1)-(3) and let $w = u+\nabla\xi(n)$. Equation (9) follows directly from $\operatorname{div}(nw) = \operatorname{div}(nu) + \operatorname{div}(n\xi'(n)\nabla n) = \operatorname{div}(nu) + \operatorname{div}(\mu'(n)\nabla n)$. In order to reformulate the momentum equation, we need some auxiliary results. First, we compute, similarly as in [2], $\nabla\xi(n)_t = -\nabla(\xi'(n)\operatorname{div}(nu))$ and, using $\mu'(n) = n\xi'(n)$,

$$\begin{aligned}(n\nabla\xi(n))_t &= -\nabla\xi(n)\operatorname{div}(nu) - n\nabla\big(\xi'(n)\operatorname{div}(nu)\big) = -\nabla\big(n\xi'(n)\operatorname{div}(nu)\big)\\ &= -\nabla\big(\mu'(n)\operatorname{div}(nu)\big) = -\nabla\operatorname{div}(\mu(n)u) + \nabla\big((\mu(n) - n\mu'(n))\operatorname{div} u\big).\end{aligned} \tag{16}$$

Second, we observe that

$$\operatorname{div}\big(n\nabla\xi(n)\otimes\nabla\xi(n)\big) = \Delta(\mu(n)\nabla\xi(n)) - \operatorname{div}(\mu(n)\nabla^2\xi(n)), \tag{17}$$

$$\operatorname{div}\big(n\nabla\xi(n)\otimes u + nu\otimes\nabla\xi(n)\big) = \Delta(\mu(n)u) - 2\operatorname{div}(\mu(n)D(u)) + \nabla\operatorname{div}(\mu(n)u). \tag{18}$$

Adding these two identities, using (16), and observing that $\lambda(n) = n\mu'(n) - \mu(n)$, we arrive to

$$\begin{aligned}(nw)_t + \operatorname{div}(nw\otimes w) - \Delta(\mu(n)w) &= (nu)_t + \operatorname{div}(nu\otimes u) - 2\operatorname{div}(\mu(n)D(u)) - \nabla(\lambda(n)\operatorname{div} u)\\ &\quad + \nabla\big((\lambda(n) + \mu(n) - n\mu'(n))\operatorname{div} u\big) - \operatorname{div}(\mu(n)\nabla^2\xi(n))\\ &= -\nabla p(n) + nf.\end{aligned} \tag{19}$$

For the derivation of the energy identity, we differentiate the energy:

$$\frac{d}{dt}\int_{\mathbb{R}^d}\Big(\frac{n}{2}|\nabla w|^2 + H(n)\Big)dx = \int_{\mathbb{R}^d}\Big(n_t\Big(-\frac{1}{2}|w|^2 + H'(n)\Big) + (nw)_t\cdot w\Big)dx. \tag{20}$$

Inserting (9)-(10) and integrating by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d}\Big(\frac{n}{2}|\nabla w|^2 + H(n)\Big)dx = -\int_{\mathbb{R}^d}\big(\mu(n)|\nabla w|^2 + \mu'(n)H''(n)|\nabla n|^2\big)dx. \tag{21}$$

By the definitions of μ and H, $\mu'(n)H''(n) = n\xi'(n)h'(n) = \xi'(n)p'(n)$, which proves the claim.

Finally, it remains to prove Theorem 1.2. We give only a sketch of the proof since the techniques are well known. The local existence of smooth solutions to (9)-(11) follows from the results of DiPerna [8]. Introduce the Riemann invariants $w = ne^u$ and $z = ne^{-u}$. By the theory of positive invariant regions from [6], we find that $\{(w, z) : w \le \text{const.}, z \le \text{const.}\}$ is an invariant region of (9)-(10). This implies L^∞ bounds for (w, z) and also for (n, u). Furthermore, there exists a positive lower bound for n since the initial density is strictly positive; see [5,8]. The L^∞ bounds together with the lower bound on n give the global existence.

4. Derivation of the quantum Navier-Stokes model

By employing a Chapman-Enskog expansion in the Wigner-BGK equation and a maximum quantum entropy closure, Brull and Méhats [4] derived the following nonlocal model:

$$n_t + \mathrm{div}(nu) = 0, \quad (nu)_t + \mathrm{div}(nB \otimes u) + n\nabla B(u - B) + n\nabla(V - A) = \tau S, \tag{22}$$

where $\tau > 0$ is the relaxation time of the BGK collision operator in the Wigner equation, and (n, u) and (A, B) are related by a nonlocal expression. We do not need the precise relation of this expression but only an expansion of A and B in powers of ε^2 [7, Lemma 3.1]:

$$A = -\log n + \frac{\varepsilon^2}{6}\frac{\Delta\sqrt{n}}{\sqrt{n}} - \frac{\varepsilon^2}{24}|\mathrm{curl}(u)|^2 + O(\varepsilon^4), \quad nB = nu + \frac{\varepsilon^2}{12}\mathrm{curl}(n\,\mathrm{curl}\,u) + O(\varepsilon^4). \tag{23}$$

The local model is derived under the assumptions that $\tau = \varepsilon^2$ and that the flow is nearly irrotational, $\mathrm{curl}\,u(\cdot, 0) = O(\varepsilon)$, which implies that $\mathrm{curl}\,u(\cdot, t) = O(\varepsilon)$ for all $t > 0$. Then, inserting the expansions (23) and $S = 2\mathrm{div}(nD(u)) + O(\varepsilon^2)$ [4, Remark 1] into formula (49) of [7], which is equivalent to (22), we arrive to (1) and (6).

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