**Most recent ASC Reports**

21/2009  *Michael Kaltenbäck, Harald Woracek*
Pontryagin Spaces of Entire Functions V

20/2009  *Christoph Erath, Stefan Funken, Petra Goldenits, Dirk Praetorius*
Simple Error Estimations for the Galerkin Bem for some Hypersingular Integral Equation in 2D

19/2009  *Daniel Matthes, Ansgar Jüngel, Giuseppe Toscani*
Convex Sobolev Inequalities Derived from Entropy Dissipation

18/2009  *Roberta Bosi, Maria J. Cáreres*
The BGK Model with External Confining Potential: Existence, Long-time Behaviour and Time-periodic Maxwellian Equilibria

17/2009  *Irene M. Gamba, Ansgar Jüngel, Alexis Vasseur*
Global Existence of Solutions to One-Dimensional Viscous Quantum Hydrodynamic Equations

16/2009  *Claire Chainai-Hillairat, Marguerite Gisclon, Ansgar Jüngel*
A Finite-Volume Scheme for the Multidimensional Quantum Drift-Diffusion Model for Semiconductors

15/2009  *Carsten Carstensen, Dirk Praetorius*
Convergence of Adaptive Boundary Element Methods

14/2009  *Franco Fagnola, Lukas Neumann*
Quantum Fokker-Planck Models: Limiting Case in the Lindblad Condition

13/2009  *Anton Baranov, Harald Woracek*
Majorization in de Branges Spaces II. Banach Spaces Generated by Majorants

12/2009  *Anton Baranov, Harald Woracek*
Majorization in de Branges Spaces I. Representability of Subspaces

---

Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

ISBN 978-3-902627-02-5

Pontryagin spaces of entire functions VI

MICHAEL KALTERNBÄCK, HARALD WORACEK

Abstract
In the theory of two-dimensional canonical (also called ‘Hamiltonian’) systems, the notion of the Titchmarsh-Weyl coefficient associated to a Hamiltonian function plays a vital role. A cornerstone in the spectral theory of canonical systems is the Inverse Spectral Theorem due to Louis de Branges which states that the Hamiltonian function of a given system is (up to changes of scale) fully determined by its Titchmarsh-Weyl coefficient. Much (but not all) of this theory can be viewed and explained using the theory of entire operators due to Mark G.Krein.

Motivated from the study of canonical systems or Sturm-Liouville equations with a singular potential, and from other developments in the indefinite world, it was a long standing open problem to find an indefinite (Pontryagin space) analogue of the notion of canonical systems, and to prove a corresponding analogue of de Branges’ Inverse Spectral Theorem. We gave a definition of an indefinite analogue of a Hamiltonian function and elaborated the operator theory of such ‘indefinite canonical systems’ in previous work. In the present paper we prove the corresponding version of the Inverse Spectral Theorem.

AMS Classification Numbers: 34A55, 46C20, 46E22, 30H05
Keywords: canonical system, Pontryagin space, Inverse Spectral Theorem

1 Introduction
A function \( H : (s_-, s_+) \to \mathbb{R}^{2 \times 2} \) is called a Hamiltonian, if it is locally integrable, \( H(t) \geq 0 \) for \( t \in (s_-, s_+) \) a.e., and if it does not vanish identically on any set of positive measure. A canonical system is a differential equation of the form
\[
y'(t) = z J H(t) y(t), \quad t \in (s_-, s_+),
\]
where \( H \) is a Hamiltonian, \( J \) denotes the symplectic matrix
\[
J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]
and \( z \) is a complex parameter. The spectral theory of canonical systems can be understood with the help of an operator model, which consists of a Hilbert space boundary triplet \( (L^2(H), T_{\max}(H), \Gamma(H)) \), see e.g. [K], [HSW].

We say that a Hamiltonian \( H \) is regular at the endpoint \( s_- \) if, for some \( \epsilon > 0 \),
\[
\int_{s_-}^{s_-+\epsilon} \text{tr} \, H(t) \, dt < \infty.
\]
If \( \int_{s_-}^{s_-+\epsilon} \text{tr} \, H(t) \, dt = \infty \), it is called singular at \( s_- \). The terminology of regular/singular at the endpoint \( s_+ \) is defined analogously. Sometimes one also speaks of Weyl’s limit circle and limit point case, instead of regular and singular, respectively.
Assume that $H$ regular at $s_-$. Then each solution of the equation (1.1) possesses a locally absolutely continuous extension to $[s_-, s_+]$. Hence, in this case, it is meaningful to prescribe initial values at $s_-$. We denote by $W(t, z) = (w_{ij}(t, z))_{i,j=1,2}$ the transposed of the fundamental matrix solution of (1.1), i.e. $W(t, z)$ is the unique $2 \times 2$-matrix function which satisfies

$$\frac{\partial}{\partial t}W(t, z)J = zW(t, z)H(x), \quad t \in [s_-, s_+], \quad W(s_-, z) = I.$$ Dependng whether $H$ is regular or singular at $s_+$, we meet two significantly different situations.

$H$ is regular at $s_+$: The function $W(t, z)$ admits a continuous extension to $s_+$. The matrix function $W(s_+, z)$, sometimes also called the monodromy matrix of $H$, belongs to the class $\mathcal{M}_0$, i.e. the entries of $W(s_+, z)$ are entire functions which are real for real $z$, $\det W(s_+, z) = 1$, and

$$\frac{W(s_+, z)JW(s_+, z)^* - J}{z - \bar{z}} \geq 0, \quad \text{Im } z > 0. \quad (1.3)$$

The chain $\omega_H := (W(t, z))_{t \in [s_-, s_+]}$ is a finite maximal chain going down from $W(s_+, z)$.

$H$ is singular at $s_+$: We have $\lim_{t \to s_+} \text{tr}(W(t, 0)') = +\infty$. Write $W(t, z) = (w_{ij}(t, z))_{i,j=1,2}$. Then for each $\tau \in \mathbb{R} \cup \{\infty\}$, the limit

$$q_H(z) := \lim_{t \to s_+} \frac{w_{11}(t, z)\tau + w_{12}(t, z)}{w_{21}(t, z)\tau + w_{22}(t, z)} \quad (1.4)$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ and does not depend on $\tau$. The function $q_H$ is called the Titchmarsh-Weyl coefficient associated to the Hamiltonian $H$. It belongs to the Nevanlinna class $\mathcal{N}_0$, i.e. is analytic on $\mathbb{C} \setminus \mathbb{R}$, satisfies $q_H(\bar{z}) = \overline{q_H(z)}$, $z \in \mathbb{C} \setminus \mathbb{R}$, and

$$\text{Im } q_H(z) \geq 0, \quad \text{Im } z > 0. \quad (1.5)$$

The following two results are cornerstones in the study of canonical systems, see [dB],[GK2].

1.1. **Inverse Spectral Theorem; positive definite, regular:** The assignment $H \mapsto W(s_+, z)$ establishes a bijective correspondence between the set of all Hamiltonians (up to changes of scale) which are regular at $s_-$ and $s_+$, and the set $\mathcal{M}_0$.

1.2. **Inverse Spectral Theorem; positive definite, singular:** The assignment $H \mapsto q_H$ establishes a bijective correspondence between the set of all Hamiltonians (up to changes of scale) which are regular at $s_-$ and singular at $s_+$, and the set $\mathcal{N}_0$.

The notion of the Nevanlinna class $\mathcal{N}_0$ allows a generalization to an indefinite setting, namely the class $\mathcal{N}_{<\infty}$. Thereby, the positivity condition (1.5) is replaced by the condition that the kernel

$$N_q(w, z) := \frac{q(z) - \overline{q(w)}}{z - \overline{w}}$$

2
has a finite number of negative squares. The actual number of its negative squares will be denoted by \( \text{ind}_- q \). This indefinite analogue of \( N_0 \) has a long history and was systematically studied, e.g. in [KL] with help of the theory of selfadjoint operators in Pontryagin spaces.

Also the class \( M_0 \) admits a generalization to the indefinite setting, namely the class \( M_{<\infty} \) where the positivity condition (1.3) is replaced by the requirement that the kernel

\[
H_W(w, z) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}
\]

has a finite number of negative squares. Again we denote the actual number of its negative squares by \( \text{ind}_- W \). For more on this class see [KW/V, §2] and the references given there.

Finally, the notion of a Hamiltonian has been generalized to an indefinite setting in our previous work [KW/IV]. The accurate definition of this generalization is a bit lengthy, cf. [KW/IV, Definition 8.1]. Intuitively we can think of it as a Hamiltonian with finitely many singularities which do not behave too badly, plus a contribution to the canonical differential equation which happens inside each singularity, plus interface conditions which relate before and after each singularity. The degree of negativity of a general Hamiltonian \( \mathfrak{h} \), denoted by \( \text{ind}_- \mathfrak{h} \), is a number which is composed out of measures for the growth of \( \mathfrak{h} \) towards its singularities and for the size of the contribution happening inside the singularities. It measures the deviation of \( \mathfrak{h} \) from the classical, positive definite, situation. Similar to the distinction between positive definite Hamiltonians being regular at \( s_- \) and \( s_+ \) or regular at \( s_- \) and singular at \( s_+ \), one can distinguish cases that a general Hamiltonian \( \mathfrak{h} \) is regular or singular.

If \( \mathfrak{h} \) is regular, a \( 2 \times 2 \)-matrix function \( \omega(B(\mathfrak{h})) \in M_{<\infty} \) was associated to \( \mathfrak{h} \) in [KW/V, §4.e]. This is the analogue of the matrizzant of a positive definite regular Hamiltonian. If \( \mathfrak{h} \) is singular, a function \( q_{\mathfrak{h}} \in N_{<\infty} \) was associated to \( \mathfrak{h} \) in [KW/V, Definition 5.2]. This is the analogue of the Weyl coefficient of a positive definite singular Hamiltonian.

Our task in the present paper is to prove the following two results.

1.3. **Inverse Spectral Theorem; indefinite, regular**: The assignment \( \mathfrak{h} \mapsto \omega(B(\mathfrak{h})) \) establishes a bijective correspondence between the set of all regular general Hamiltonians (modulo reparameterization) and the set \( M_{<\infty} \). Thereby

\[ \text{ind}_- \mathfrak{h} = \text{ind}_- \omega(B(\mathfrak{h})). \]

1.4. **Inverse Spectral Theorem; indefinite, singular**: The assignment \( \mathfrak{h} \mapsto q_{\mathfrak{h}} \) establishes a bijective correspondence between the set of all singular general Hamiltonians (modulo reparameterization) and the set \( N_{<\infty} \) of all generalized Nevanlinna functions. Thereby

\[ \text{ind}_- \mathfrak{h} = \text{ind}_- q_{\mathfrak{h}}. \]

This is the point of culmination of our series of papers on ‘Pontryagin spaces of entire functions’. With the proof of these theorems we have completed a full indefinite analogue of canonical system, their operator theory, and their Weyl-theory. Also, we see that the notion of general Hamiltonians introduced in [KW/IV] goes exactly as far as Pontryagin space theory might possibly lead.

Our notation in the present paper will follow the terminology introduced in [KW/0]–[KW/V], in particular [KW/V, §3]. We will, without further notice, use the terms introduced there. Moreover, references to [KW/0]–[KW/V] will

3
be given as the following examples indicate: Lemma 0.2.1 refers to Lemma 2.1 of [KW/0], Theorem IV.8.6 to Theorem 8.6 of [KW/IV], or (V.4.1) to equation (4.1) of [KW/V].

In light of our previous work, the Inverse Spectral Theorems 1.3 and 1.4 appear as two-step-results:

1° We have shown in [KW/II], for a more concise formulation see also V.3.6 and V.3.9, that the set of finite maximal chains of matrices (modulo reparameterization) corresponds bijectively to $\mathcal{M}_{<\infty}$ and that the set of maximal chains of matrices (modulo reparameterization) corresponds bijectively to $\mathcal{N}_{<\infty}$.

2° By Theorem V.5.1 to each general Hamiltonian $h$ a chain $\omega_h$ is associated. This chain is finite maximal or maximal, depending whether $h$ is regular or singular.

In order to prove 1.3 and 1.4 it is thus enough to establish the following two theorems.

1.5 Theorem. Let $\omega$ be a maximal chain or a finite maximal chain, and assume that $t \circ \omega$ and $(t \circ \omega|_J)^{-1}$ are locally absolutely continuous for any maximal interval contained in the domain of $\omega$. Then there exists a general Hamiltonian $h(\omega)$ such that $\omega = \omega_h(\omega)$.

For each (finite) maximal chain $\omega$ there exists a reparameterization which satisfies the above continuity condition.

1.6 Theorem. Let $h$ and $h'$ be general Hamiltonians. Then $\omega_h$ is a reparameterization of $\omega_{h'}$ if and only if $h$ is a reparameterization of $h'$.

### Table of contents

1. Introduction p.1
2. The converse construction; core of the argument p.4
3. Completion of the converse construction p.30
4. Bijectivity modulo reparameterization p.33

In Section 2 we carry out the construction of $h(\omega)$ for chains of a specific form which, roughly speaking, correspond to elementary indefinite Hamiltonians of kind (A). This is the hardest part of the converse construction. In Section 3 we settle other particular cases and employ the usual splitting-and-pasting technique to complete the proof of Theorem 1.5. Finally, in Section 4, we turn to the proof of Theorem 1.6. We will employ a fairly elementary method, based on a set of recurrence relations for the power series coefficients of the Potapov-Ginzburg transform of $\omega(t)$. In this way we also obtain some knowledge on how the data of $h$ concentrated inside a singularity influences $\omega(t)$, a topic we already have touched upon in [LW].

### 2. The converse construction; core of the argument

As already said, we will in this section deal with chains of a particular form.
2.1. Overall assumption: Throughout this section let \(\omega\) be a finite maximal chain defined on 
\[ I = [s_-, \sigma] \cup (\sigma, s_+] \]
and assume that
\[ \phi(\sigma) = 0, \quad \sup \left( I_{\text{reg}} \cap (s_-, \sigma) \right) = \sigma, \] \tag{2.1}
\[ (\sigma, s_+) \text{ not indivisible} \Rightarrow \inf \left( I_{\text{reg}} \cap (\sigma, s_+) \right) = \sigma, \] \tag{2.2}
\[ \omega \text{ does not start with an indivisible interval of type 0 at } s_-. \] \tag{2.3}

2.2 Remark. Let us comment on the role played by the conditions (2.1)–(2.3).
The first condition in (2.1) is just a normalization, it corresponds to (I), cf. [KW/IV, p.734].
The second condition in (2.1) is of more intrinsic nature. It means that it cannot happen that an indivisible interval adjoins \(\sigma\) at the left. This case could also be treated explicitly, but this would require development of more, and quite different, tools. Hence, we prefer to reduce it to the case (2.1) with help of the operation \(\text{rev}\). The condition (2.2) is of technical nature. It could be avoided on the cost of increasing effort, but there is no need to do so. The condition (2.3) is important. Roughly speaking, its presence allows to identify the chain of spaces \(\mathcal{R}(\omega(t))\) with the chain of dB-subspaces of a de Branges spaces, cf. Corollary I.9.7 together with Proposition I.8.3. Thus it makes it possible to apply the knowledge on the structure of the dB-subspaces of a given de Branges Pontryagin space, especially on the degenerated members of this chain, cf. [KW/III].

This section is divided into subsections according to the following schedule:

- **a.** We define Hamiltonians \(H_-\) and \(H_+\), and show that \(H_+\) and \(H_-\) satisfy (HS) and that \(\phi(H_-) = \phi(H_+)\).
- **b.** We define numbers \(d_0(\omega) \in \mathbb{N} \cup \{0\}\) and \(b_1(\omega), \ldots, b_{\delta+1}(\omega) \in \mathbb{R}\).
- **c.** We define a map \(\varphi(\omega) : \mathcal{R}(\omega(s_+)) \to \mathcal{M}(I)/\omega_H\).
- **d.** We define numbers \(d_j(\omega), j \geq 0\).
- **e.** The so far constructed data constitutes an elementary indefinite Hamiltonian \(\mathfrak{h}\) of kind (A). We show that \(\omega_\mathfrak{h} = \omega\).

**a. Construction of the Hamiltonian function.**
The definition of the Hamiltonian function of the to-be-constructed elementary indefinite Hamiltonian \(\mathfrak{h}(\omega)\) is based on Proposition V.3.23.

**2.3 Definition.** Let \(H_-\) and \(H_+\) denote the Hamiltonians defined on \((s_-, \sigma)\) and \((\sigma, s_+)\), respectively, which satisfy
\[ \omega_{H_-} = \omega|_{(s_-, \sigma)}, \quad \omega_{\text{rev} H_+} = (\text{rev} \omega)|_{(-s_+, -\sigma)}. \]
Moreover, we will set
\[ H(t) := \begin{cases} H_-(t), & t \in [s_-, \sigma] \\ H_+(t), & t \in (\sigma, s_+] \end{cases}, \]
\[ L^2(H) := L^2(H_-) \oplus L^2(H_+), \]
\[ T_{\text{max}}(H) := T_{\text{max}}(H^-) \oplus T_{\text{max}}(H^+) \, , \]

\[ \Gamma(H) := \left\{ (f_- + f_+; g_- + g_+); (a, b) : ((f_-; g_-); a) \in \Gamma(H^-), ((f_+; g_+); b) \in \Gamma(H^+) \right\} \, . \]

Note that, by the definition of \( H_{\pm} \), the function \( \omega \) is a solution of the differential equation

\[ \frac{\partial}{\partial t} \omega(t)(z)J = z\omega(t)(z)H(t), \quad t \in [s_-, \sigma) \cup (\sigma, s_+] \, . \]

In order to know that \( H_{\pm} \) qualify for constituting the Hamiltonian function of an elementary indefinite Hamiltonian of kind (A), we need to check several properties, cf. (IV.4.1) and Definition IV.4.1.

The condition (A) in Definition IV.4.1 holds because by the second property in (2.1) no interval of the form \((s, \sigma), s \in [s_-, \sigma)\), is indivisible. The conditions in the first two lines of (IV.4.1) are easy to see, and will be deduced right below.

The requirement in the third line of (IV.4.1), however, is much more delicate and will be seen only later, cf. Proposition 2.28.

2.4 Proposition. The Hamiltonian \( H_{\pm} \) is regular at \( s_{\pm} \) and singular at \( \sigma \). It satisfies (I) and (HS\( \pm \)).

Proof. It is enough to consider the Hamiltonian \( H^- \), since passing from \( \omega \) to \( \text{rev} \omega \) will exchange the roles of \( H^- \) and \( H^+ \). The fact that \( H^- \) is regular at \( s_- \) and singular at \( \sigma \) follows from

\[ \lim_{t \nearrow s_-} \omega(t) = I, \quad \lim_{t \nearrow \sigma} t(\omega(t)) = +\infty, \]

since \( t(H^-) = t \circ \omega \) when \( t(H^-) \) is chosen such that \( t(H^-)(s_-) = 0 \), cf. Proposition V.3.23, (i).

Let \( q \) denote the Weyl-coefficient of the positive definite canonical system with Hamiltonian \( H^- \). Then \( q \) is an intermediate Weyl-coefficient in the sense of [KW/III, §5], see also Proposition V.3.10. This implies that \( q \) is meromorphic in \( \mathbb{C} \). Moreover, it is seen from Theorem III.7.4 and Corollary III.7.9, that the sequences \((a_k^+)\) and \((a_k^-)\) of poles of \( q \) located in \((0, \infty)\) or \((-\infty, 0)\), respectively, satisfy

\[ \lim_{k \to \infty} \frac{k}{a_k^+} = \lim_{k \to \infty} \frac{k}{a_k^-} \in \mathbb{R} \, . \]  

(2.4)

Since \( q \) is meromorphic, the selfadjoint extensions of \( T_{\text{min}}(H) \) have discrete spectrum and, hence, compact resolvents. By (2.4), they belong to each Neumann-van Schatten class \( \mathcal{S}_{1+\epsilon} \) with \( \epsilon > 0 \), cf. [GK1]. In particular, they are Hilbert-Schmidt operators.

By [KW, Theorem 2.4], see also Theorem IV.2.27, there exists a unique angle \( \phi_- \in [0, \pi) \) such that \( \xi_{\phi_-} \in L^2(H^-) \). However, since \( \circ_{-\phi_-} \omega|_{[s_-, \sigma]} \) satisfies the canonical differential equation with Hamiltonian \( \circ_{-\phi_-} H_- \), we have

\[ \frac{d}{ds} [\circ_{-\phi_-} \omega(t)]_{12}(0) = \int_{s_-}^t \left( \begin{array}{c} 0 \\ 1 \end{array} \right)^* \circ_{-\phi_-} H_-(s) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) ds = \int_{s_-}^t \xi_{\phi_-}^* H_-(s) \xi_{\phi_-} \, ds \, . \]
Letting $t$ tend to $\sigma$, we obtain
\[
\lim_{t \to \sigma} |\phi_\omega(t)|_{L^2(H^-)} = \|\xi_\omega\|_{L^2(H^-)} < \infty,
\]
and conclude that $\phi^- = \phi(\sigma) = 0$. \hfill \Box

2.5 Remark. The construction of $H_\pm$ and the fact that these Hamiltonians satisfy $(HS_\pm)$ did not use (2.1)–(2.3). The condition (I) is just the first condition in (2.1).

b. The elements $D_k$.

Let us consider the structure of the chain of dB-subspaces of the space $\mathfrak{P}(E_{\omega(s_+)}$, cf. Remark 2.2. By Lemma III.3.15, this chain is equal to
\[
\{ \mathfrak{P}(E_t) : t \in [s_-, \sigma) \cap I_{\text{reg}} \} \cup \{ \mathfrak{P}_0, \ldots, \mathfrak{P}_a \} \cup \{ \mathfrak{P}(E_t) : t \in (\sigma, s_+) \cap I_{\text{reg}} \},
\]
where $\mathfrak{P}_0, \ldots, \mathfrak{P}_a$ are all degenerated dB-subspaces of $\mathfrak{P}(E_{\omega(s_+)}$. We assume that these spaces are enumerated in such a way that $\mathfrak{P}_0 \subset \ldots \subset \mathfrak{P}_a$. The singularity $\sigma$ is not of polynomial type, and hence does certainly not lie inside an indivisible interval with negative length. We conclude from Corollary III.3.17, and the notice after it, that there exists at least one degenerated dB-subspace of $\mathfrak{P}(E_{\omega(s_+)}$, i.e. $a \geq 0$. Moreover, by Remark III.3.13, the second part of (2.1), and (2.3) we have
\[
\mathfrak{P}_0 = \text{clos}(\mathfrak{P}(E_{s_+}) \bigcup_{t \in [s_-, \sigma) \cap I_{\text{reg}}} \mathfrak{P}(E_t),
\]
\[
\dim \left( \mathfrak{P}_{j+1} / \mathfrak{P}_j \right) = 1, \quad j = 0, \ldots, a - 1,
\]
\[
\dim \left( \bigcap_{t \in (\sigma, s_+) \cap I_{\text{reg}}} \mathfrak{P}(E_t) / \mathfrak{P}_a \right) = \begin{cases} 1, & (\sigma, s_+) \text{ indivisible} \\ 0, & \text{otherwise} \end{cases} \tag{2.5}
\]

2.6 Definition. Define numbers
\[
\Delta^- (\omega) := \dim \mathfrak{P}_0^\omega, \quad \Delta^+ (\omega) := \dim \mathfrak{P}_a^\omega
\]
\[
\bar{\omega}(\omega) := a - |\Delta^+ (\omega) - \Delta^- (\omega)|
\]
Moreover, put $\Delta(\omega) := \max\{\Delta^- (\omega), \Delta^+ (\omega)\}$. \hfill \Box

2.7 Remark. Note that the symbols $\Delta_\pm, \Delta$, were already used to measure the growth of the Hamiltonians $H_\pm$ towards $\sigma$, cf. Definition IV.3.1. It will turn out later that actually $\Delta_\pm (\omega) = \Delta(H_\pm)$, cf. Proposition 2.28. However, before knowing this, the explicit notation of the argument $\omega$ in $\Delta_\pm$ and $\Delta$ is necessary to distinguish these notions and will be dropped only when no confusion is possible.

We continue with a closer investigation of the structure of the spaces $\mathfrak{P}_j$, as done in [KW/III]. We know from Theorem III.2.3 that there exists an element $F \in \mathfrak{P}(E_{\omega(s_+)})$ such that $\mathfrak{P}_0^\omega = \text{span}\{F(z), \ldots, z^{\Delta^- (\omega) - 1}F(z)\}$. It is important for our present purposes, that $F$ can be chosen such that $F(0) = -1$. This fact is seen by an analysis of the proof of Theorem III.2.3. In view of our later needs we have to explain the situation in detail.

The proof of Theorem III.2.3 is based on the existence of a dB-Pontryagin space $\mathfrak{P}$ with the following properties:
(i) \( \mathfrak{P} \) contains \( \mathfrak{P}_a \) isometrically as a dB-subspace,

(ii) \( \dim \mathfrak{P}/\mathfrak{P}_a < \infty \),

(iii) all dB-subspaces \( \mathcal{L} \) of \( \mathfrak{P} \) with \( \mathfrak{P}_a \subseteq \mathcal{L} \subseteq \mathfrak{P} \) are degenerated and \( \dim \mathcal{L}^\circ \leq \dim \mathfrak{P}_a^\circ \).

The proof of this fact can be found on [KW/III, p.256]. We will now take a closer look at the chain of dB-subspaces of \( \mathfrak{P} \). It is of the form

\[
\{ \mathfrak{P}(E_t) : t \in [s_-, \sigma) \cap I_{\text{reg}} \} \cup \{ \mathfrak{P}_0, \ldots, \mathfrak{P}_a \} \cup \{ \mathfrak{P} \},
\]

where \( \mathfrak{P}_0, \ldots, \mathfrak{P}_a \) are all degenerated dB-subspaces of \( \mathfrak{P} \). We assume that these spaces are enumerated in such a way that \( \mathfrak{P}_0 \subseteq \ldots \subseteq \mathfrak{P}_a \). Note that, clearly, \( \tilde{a} \geq a \) and \( \mathfrak{P}_j = \mathfrak{P}_j, j = 0, \ldots, a \). We can picture the situation as follows:

The spaces \( \mathfrak{P}_j \) can be described by means of the operator \( \tilde{S} \) of multiplication by \( z \) in the space \( \mathfrak{P} \). Write \( \mathfrak{P} = \mathfrak{P}(\tilde{E}) \) with some function \( \tilde{E} \in \mathcal{H}B_{< \infty}, \tilde{E}(0) = -i \), and let \( \psi \in [0, \pi) \) be the unique number such that \( \tilde{S}_\psi \in \mathfrak{P}(\tilde{E}) \), cf. Corollary I.6.3. Denote by \( \tilde{n} \) the number

\[
\tilde{n} := \sup \{ k \in \mathbb{N}_0 : z^k \tilde{S}_\psi(z) \in \mathfrak{P} \},
\]

which is finite by Lemma I.7.1. By Lemma II.5.19, we have

\[
\overline{\text{dom}} \tilde{S}_k = \text{span} \{ \tilde{S}_\psi(z), \ldots, z^{k-1} \tilde{S}_\psi(z) \}^\perp, \quad k = 1, \ldots, \tilde{n} + 1, \tag{2.6}
\]

and

\[
\overline{\text{dom}} \tilde{S}_k^\circ \neq \{ 0 \}, \quad k = 1, \ldots, \tilde{n}.
\]

Note that the spaces \( \overline{\text{dom}} \tilde{S}_k \) are dB-subspaces of \( \mathfrak{P} \), cf. [KW/I]. However, (2.5) implies that there exists no nondegenerated dB-subspace of \( \mathfrak{P} \) with finite codimension, hence also \( \overline{\text{dom}} \tilde{S}_{\tilde{n}+1} \neq \{ 0 \} \). By Lemma II.5.19 this implies \( \overline{\text{dom}} \tilde{S}_{\tilde{n}+2} = \overline{\text{dom}} \tilde{S}_{\tilde{n}+1} \). Since, by (2.6),

\[
\dim \left( \overline{\text{dom}} \tilde{S}_{k-1}/ \overline{\text{dom}} \tilde{S}_k \right) = 1, \quad k = 1, \ldots, \tilde{n} + 1,
\]

we conclude that

\[
\tilde{n} = \tilde{a} \quad \text{and} \quad \mathfrak{P}_j = \overline{\text{dom}} \tilde{S}_{\tilde{n}+1-j} = \text{span} \{ \tilde{S}_\psi(z), \ldots, z^{\tilde{a}-j} \tilde{S}_\psi(z) \}^\perp, \quad j = 0, \ldots, \tilde{a}. \tag{2.7}
\]

Next we describe the isotropic part of \( \mathfrak{P}_j \). Set \( \tilde{d}_j := \dim \mathfrak{P}_j^\circ \). We obtain from (2.7) that

\[
\mathfrak{P}_j^\circ = \text{span} \{ \tilde{S}_\psi(z), \ldots, z^{\tilde{a}-j} \tilde{S}_\psi(z) \}^\circ, \quad j = 0, \ldots, \tilde{a}, \tag{2.8}
\]
hence $d_j \leq \tilde{a} + 1 - j$. In particular, we have $\tilde{d}_{\tilde{a}} = 1$.

Since $\tilde{S}$ is symmetric, we have $[z^k \tilde{S}_\psi, z^l \tilde{S}_\psi] = \gamma_{k+l}$, $k, l = 0, \ldots, \tilde{a}$, with some numbers $\gamma_0, \ldots, \gamma_{2\tilde{a}}$. The fact that the corresponding Gram-matrix has Hankel-type yields
\[
\gamma_0 = \cdots = \gamma_{\tilde{a}+\tilde{d}_0-1} = 0, \quad \gamma_{\tilde{a}+\tilde{d}_0} \neq 0, \quad (2.9)
\]
and
\[
\hat{P}_j = \text{span} \{ \tilde{S}_\psi(z), \ldots, z^{\tilde{d}_j-1} \tilde{S}_\psi(z) \}, \quad j = 0, \ldots, \tilde{a}. \quad (2.10)
\]
Moreover, the dimension $\tilde{d}_j$ can be computed as
\[
\tilde{d}_j = \begin{cases} 
\tilde{d}_0 + j & , j = 0, \ldots, \lfloor \frac{\tilde{a} - \tilde{d}_0}{2} \rfloor \\
\tilde{a} + 1 - j & , j = \lceil \frac{\tilde{a} - \tilde{d}_0}{2} \rceil + 1, \ldots, \tilde{a}
\end{cases}, \quad (2.11)
\]
and we have
\[
\tilde{d}_j = \begin{cases} 
\tilde{d}_0 + j & , \tilde{a} - \tilde{d}_0 \text{ even} \\
\tilde{d}_0 + \tilde{d}_j & , \tilde{a} - \tilde{d}_0 \text{ odd}
\end{cases}. \quad (2.12)
\]
Note here that $\lfloor \frac{\tilde{a} - \tilde{d}_0}{2} \rfloor \geq -1$. The maximal degree of degeneracy is assumed for $j = \lfloor \frac{\tilde{a} - \tilde{d}_0}{2} \rfloor + 1$, and
\[
\max_{j=0, \ldots, \tilde{a}} \tilde{d}_j = \tilde{a} - \lfloor \frac{\tilde{a} - \tilde{d}_0}{2} \rfloor = \lfloor \frac{\tilde{a} + \tilde{d}_0 + 1}{2} \rfloor.
\]
We can picture the situation as indicated in the following diagrams:

\[
\text{dim } \hat{P}_j : \quad \begin{array}{c}
\tilde{d}_j \\
\text{max } d_i \\
\tilde{d}_0 + 1 \\
\tilde{d}_0
\end{array}
\]

\[
\begin{array}{c}
\tilde{a} - \tilde{d}_0 \text{ even} \\
\tilde{a} - \tilde{d}_0 \text{ odd}
\end{array}
\]

Finally, it also follows that
\[
z^{\tilde{d}_0+j+1} \tilde{S}_\psi(z) \in \overline{\text{dom } \tilde{S}^{\tilde{a}-j-1}} \setminus \overline{\text{dom } \tilde{S}^{\tilde{a}-j-1+1}}, \quad j = 0, \ldots, \tilde{a} - \tilde{d}_0, \quad (2.13)
\]
and we conclude that
\[
\hat{P}_j = \hat{P}_0 + \text{span} \{ z^{\tilde{d}_0} \tilde{S}_\psi(z), \ldots, z^{\tilde{d}_0+j-1} \tilde{S}_\psi(z) \}, \quad j = 1, \ldots, \tilde{a} - \tilde{d}_0 + 1.
\]
The numbers $\tilde{a}$ and $\hat{d}_j$ are in various ways related to the magnitudes $\Delta_{\pm}$, $a$, and $\hat{a}$ associated with $\omega$. First of all, it is trivial that $d_0 = \Delta_-$ and $\hat{d}_a = \Delta_+$. In conjunction with property (iii) of $\mathfrak{P}$, the formulas (2.11) and (2.12) imply

\[
a \geq \left[ \frac{\tilde{a} - \Delta_{-}}{2} \right] \quad \text{and} \quad a = \left[ \frac{\tilde{a} - \Delta_{-}}{2} \right] \iff \tilde{a} - \Delta_{-} \text{ even, } a = \Delta_{+} - \Delta_{-},
\]

\[
\tilde{a} - a = \begin{cases} \Delta_{+} & , \tilde{a} - \Delta_{-} \text{ even, } a = \Delta_{+} - \Delta_{-} \\ \Delta_{+} - 1 & , \text{otherwise} \end{cases}
\]

(2.13)

Since always $|\hat{d}_j - \hat{d}_{j+1}| \leq 1$, we see that $\hat{o} = a - |\hat{d}_a - \hat{d}_0| \geq 0$. From (2.11) and (2.12), we conclude that

\[
\hat{o} = 0 \iff \left( \tilde{a} = \Delta_{-} - 1 \lor a = \left[ \frac{\tilde{a} - \Delta_{-}}{2} \right] \lor a = \left[ \frac{\tilde{a} - \Delta_{-}}{2} \right] + 1, \tilde{a} - \Delta_{-} \text{ odd} \right)
\]

(2.14) 2.8 Lemma. Denote $n := \max \{k \in \mathbb{N}_0 : z^k \tilde{S}_\psi(z) \in \mathfrak{P}_a \}$. Then

\[
n = \Delta + \hat{o} - 1 = \min \{\tilde{a}, \Delta_{+} + a - 1\} = \begin{cases} \tilde{a} & , \Delta_{+} > \Delta_{+} \\ \Delta_{+} + a - 1 & , \Delta_{-} \leq \Delta_{+} \end{cases}
\]

Moreover, we have

\[
\mathfrak{P}_a[\tilde{a}] = \text{span} \{ \tilde{S}_\psi(z), \ldots, z^\tilde{a} \tilde{S}_\psi(z) \}
\]

(2.15)

Proof. Let $k \leq \tilde{a}$. Then we obtain from (2.7) and (2.9) that

\[
z^k \tilde{S}_\psi(z) \in \mathfrak{P}_a \iff z^k \tilde{S}_\psi(z) \perp \tilde{S}_\psi(z), \ldots, z^{\tilde{a} - a} \tilde{S}_\psi(z) \iff k < \Delta_{-} + a
\]

(2.16)

Thus $z^k \tilde{S}_\psi(z) \in \mathfrak{P}_a$ for all $k \leq \min \{\tilde{a}, \Delta_{-} + a - 1\}$. On the other hand,

\[
z^k \tilde{S}_\psi(z) \begin{cases} \notin \mathfrak{P} & , \tilde{a} < k \\ \in \mathfrak{P} \setminus \mathfrak{P}_a & , \Delta_{-} + a \leq k \leq \tilde{a} \end{cases}
\]

and it follows that $n = \min \{\tilde{a}, \Delta_{-} + a - 1\}$.

Consider the case that $\Delta_{-} \leq \Delta_{+}$. Then $\Delta = \Delta_{+}$ and $\hat{o} = a - \Delta_{-} + \Delta_{+}$. By (2.13), we have $\tilde{a} \geq a + \Delta_{+} - 1 \geq a + \Delta_{-} - 1$, and it follows that

\[
n = \Delta_{-} + a - 1 = \Delta + \hat{o} - 1.
\]

If $\Delta_{-} > \Delta_{+}$, we have $\Delta = \Delta_{-}$ and $\hat{o} = a - \Delta_{-} + \Delta_{+}$. Moreover, since $\hat{d}_a < \max \hat{d}_i$, we must have $\left[ \frac{\hat{a} - \Delta_{-}}{2} \right] + 1 < a$. In particular, in (2.13), we must be in the second case, and it follows that $\tilde{a} = a + \Delta_{+} - 1 < \Delta_{-} + a - 1$. Thus

\[
n = \tilde{a} = a + \Delta_{+} - 1 = \Delta + \hat{o} - 1.
\]

This proves (2.14). In order to see the first assertion in (2.15), we compute

\[
\mathfrak{P}_a[\tilde{a}] = \text{span} \{ \tilde{S}_\psi(z), \ldots, z^{\tilde{a} - a} \tilde{S}_\psi(z) \}^\perp \cap \text{span} \{ \tilde{S}_\psi(z), \ldots, z^{\tilde{a}} \tilde{S}_\psi(z) \}.
\]

The second equivalence in (2.16) yields the desired formula.
We come to the proof of the second assertion in (2.15). First note that, by (2.8), we have $(\Psi_\sigma[-]\Psi_\sigma) = \tilde{\Psi}_{\sigma-n}^\circ$. Hence we have to show that $\tilde{d}_{a-n} = \Delta$.

We distinguish four cases:

1°: $\Delta_+ > \Delta_-. \quad$ In this case $n = \tilde{a}$, and hence $\tilde{d}_{a-n} = \tilde{d}_a = \Delta_+ = \Delta$.

2°: $\Delta_- \leq \Delta_+$ and $\tilde{a} - \Delta_-$ even, $a = \Delta_+ - \Delta_-$. We have

$$\tilde{a} - n = (a + \Delta_+) - (\Delta_+ + a - 1) = \Delta_+ - \Delta_+ = a + 1,$$

and $a = \lfloor \frac{\tilde{a} - \Delta_-}{2} \rfloor$. Hence

$$\tilde{d}_{a+1} = \tilde{d}_a = \Delta_- + a = \Delta_+ = \Delta.$$

3°: $\Delta_- \leq \Delta_+$ and $\tilde{a} - \Delta_-$ odd, $a = \lfloor \frac{\tilde{a} - \Delta_-}{2} \rfloor + 1$. We have

$$\Delta_+ = \tilde{d}_a = \tilde{d}_{a-1} + 1 = \Delta_- + a,$$

and

$$\tilde{a} - n = (a + \Delta_+ - 1) - (\Delta_- + a - 1) = \Delta_+ + \Delta_- = a.$$

4°: $\Delta_- \leq \Delta_+$ but neither 2° nor 3°: Since we are not in case 2°,

$$\tilde{a} - n = (a + \Delta_+ - 1) - (\Delta_- + a - 1) = \Delta_+ - \Delta_-,$$

and since we are not in case 3°,

$$\Delta_+ \leq \Delta_- + \lfloor \frac{\tilde{a} - \Delta_-}{2} \rfloor.$$

Hence $\tilde{d}_{\Delta_+ - \Delta_-} = \Delta_- + (\Delta_+ - \Delta_-) = \Delta$. \[\square\]

After having elaborated the geometry of the spaces $\Psi_0, \ldots, \Psi_n$, we return to our original aim, namely to show that the element $F$ with $\Psi_0 = \text{span}\{F(z), \ldots, z^{\Delta-1}F(z)\}$ can be chosen such that $F(0) = -1$. In view of (2.16) this will follow from the next statement.

2.9 Lemma. We have $\hat{S}_\psi(0) = 1$.

Proof. Recall that we have chosen $\tilde{E}$ such that $\tilde{\Psi} = \Psi(\tilde{E})$ and $\tilde{E}(0) = -i$. By Corollary I.10.4 there exists a matrix $\tilde{W} \in \mathcal{M}_{\infty}$ with $\mathfrak{h}_\sigma(\tilde{W}) = \mathfrak{h}(\tilde{W})$, such that $E_{\tilde{W}} = \tilde{E}$. Let $\tilde{\omega}$ be the finite maximal chain going downwards from $\tilde{W}$ as constructed in Theorem II.7.1. As we know from the proof of this theorem, the chain $\tilde{\omega}$ is, if parameterized appropriately, given as

$$\tilde{\omega}(t) = \begin{cases} \omega(t), & t \in [s_-, \sigma) \\ \tilde{W}W_{l(t), \psi}, & t \in (\sigma, \sigma + 1) \end{cases} \quad (2.17)$$

where $l(t) := \frac{1}{\sigma - t} + 1$.

By our first assumption in (2.1), the value $\tilde{\omega}(t)_{12}^\sigma(0)$ remains bounded if $t$ approaches $\sigma$ from below, cf. the proof of Proposition 2.4. Hence, by Proposition III.5.8, $\tilde{\omega}(t)_{12}^\sigma(0)$ also remains bounded when $t$ approaches $\sigma$ from above. This implies that $\psi = \frac{\pi}{2}$, i.e. $\hat{S}_\psi = \frac{1}{2}(\tilde{E} - \tilde{E}^\#)$. In particular, we have $\hat{S}_\psi(0) = 1$. \[\square\]
2.10 Corollary. There exists a unique element \( D_0 \in \Psi(E_{s+}) \) such that

\[
\Psi_0 = \text{span} \left\{ D_0(z), \ldots, z^{\Delta_n - 1} D_0(z) \right\}
\]

\( D_0(0) = -1 \) \hspace{1cm} (2.18)

Proof. We have already noted that the element \( D_0 := -\tilde{S}_\psi \) satisfies (2.18). The uniqueness assertion holds true since any two elements satisfying the equality in the first line of (2.18) must be scalar multiples of each other. \( \square \)

Throughout the following we will denote

\[
D_k(z) := -z^k \tilde{S}_\psi(z), \quad k = 0, \ldots, n.
\]

and

\[
c_j := \begin{cases} 
[D_{\Delta+j-1}, D_n] = \gamma_{\Delta+j-1+n}, & j = 1, \ldots, 6 \\
0, & \text{otherwise}
\end{cases}
\]

Due to the symmetry of \( \tilde{S} \) and (2.15), we have

\[
[D_k, D_l] = c_{k+l+2-2\Delta-\delta}, \quad k, l = 0, \ldots, n.
\]

2.11 Definition. Let \( b_1, \ldots, b_{\tilde{\delta}} \) be the unique numbers such that

\[
(c_1, \ldots, c_{\tilde{\delta}}) \begin{pmatrix} b_1 & \cdots & b_{\tilde{\delta}} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_1 \end{pmatrix} = (-1, 0, \ldots, 0),
\]

and put \( b_{\tilde{\delta}+1} := 0 \). Moreover, denote \( B(z) := \sum_{l=1}^{\tilde{\delta}} b_l D_{\Delta + \delta - l}(z) \).

2.12 Remark. \( \quad \)

(i) By Proposition I.6.1 and the discussion after it the entire function \( \tilde{S}_\psi(z) \) is real. This shows that the numbers \( c_j \), and hence also the numbers \( b_j \), are real.

(ii) Note that, by our choice of \( B(z) \),

\[
[B, D_k] = \begin{cases} 
-1, & k = \Delta \\
0, & \text{otherwise}
\end{cases}
\]

\hspace{1cm} (2.19)

If the interval \( (\sigma, s_+) \) is indivisible, then we can choose \( \Psi := \Psi(E_{\omega(s_+)} \) in the above construction. Doing so, things become more explicit.

2.13 Corollary. Assume that \( (\sigma, s_+) \) is indivisible. Write

\[
\omega(s_+) = (\omega(s_+)(ij))_{i,j=1}^{2},
\]

and let \( \pi_2 : \Re(\omega(s_+)) \to \Psi(E_{\omega(s_+)} \) be the isomorphism of projecting onto the second component. Then

\[
\pi_2^{-1} D_0 = -\begin{pmatrix} \omega(s_+)(12) \\ \omega(s_+)(22) \end{pmatrix}.
\]
Proof. As we saw in Lemma 2.9

\[ D_0 = -S_p = -\frac{i}{2}(E_{\omega(s_+)} - E_{\omega(s_+)}^\#) = -\omega(s_+)^{22}. \]

Corollary II.7.4 implies that

\[ (\omega(s_+)_{12}, \omega(s_+)_{22}) = \omega(s_+) \in \mathcal{R}(\omega(s_+))^1. \]

Clearly, \( \pi (\omega(s_+)_{12}) = \omega(s_+)_{22} \), and thus the assertion follows. \( \square \)

The elements \( D_k \) provide examples of elements of \( \Gamma(\omega(s_+)) \), cf. Lemma 2.15 and Lemma 2.16 below. Thereby, the first one is specific for the case that \( (\sigma, s_+) \) is indivisible. Boundary values will be computed with the help of a simple but useful general fact.

2.14 Lemma. Let \( (\mathcal{P}_1, T_1, \Gamma_1) \) and \( (\mathcal{P}_2, T_2, \Gamma_2) \) be boundary triplets with defect 2, and assume that \( \text{mul} \Gamma_1 = \text{mul} \Gamma_2 = \{0\} \). Let \( ((f; g); (a; b)) \in \Gamma_1 \uplus \Gamma_2 \), then

\[ f, g \perp \mathcal{P}_1 \Rightarrow a = 0, \quad f, g \perp \mathcal{P}_2 \Rightarrow b = 0 \]

Proof. Assume that \( f, g \perp \mathcal{P}_1 \). Let \( v \in \mathbb{C}^2 \) be given. Since \( \Gamma_1 \) is surjective, there exist \( F, G \in \mathcal{P}_1 \) such that

\[ ((F; G); (v; 0)) \in \Gamma_1. \]

It follows that also \( ((F; G); (v; 0)) \in \Gamma_1 \uplus \Gamma_2 \), and hence the abstract Green’s identity gives

\[ 0 = [g, F] - [f, G] = v^*Ja. \]

Since \( v \) was arbitrary, this implies that \( a = 0 \).

The second implication follows in the same way. \( \square \)

In the following we set \( \delta(\omega)_k := \pi_2^{-1}D_k, \; k = 0, \ldots, \Delta + \delta - 1 \). We will also write \( \delta_k \) instead of \( \delta_k(\omega) \) if no confusion can occur.

2.15 Lemma. Assume that \( (\sigma, s_+) \) is indivisible. Then we have

\[ ((0; \delta_0); (0; \begin{pmatrix} 1 \\ 0 \end{pmatrix})) \in \Gamma(\omega(s_+)). \]  

(2.20)

Proof. Let us again choose \( \mathcal{P} := \mathcal{P}(E_{\omega(s_+)}) \) in the construction of the element \( D_0 \). It follows that \( D_0 \) spans the multivalued part of the selfadjoint extension \( A_{D_0} \) of \( \mathcal{S}(E_{\omega(s_+)}) \). By Lemma V.2.16, we get \( (0; \pi_2^{-1}D_0) \in \Gamma(\omega(s_+)) \).

Let \( a, b \in \mathbb{C}^2 \) be such that \( ((0; \delta_0); (a; b)) \in \Gamma(\omega(s_+)) \), and write \( b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \). Choose \( r \in (s-, \sigma) \cap \text{Ireg} \) such that \( (s-r, r) \) is not indivisible. By Lemma 2.8 \( D_0 \perp \mathcal{P}_0 \supseteq \mathcal{P}(E_{\omega(r)}) \), and hence \( \delta_0 \subseteq \mathcal{R}(\omega(r)) \). By Lemma 2.14, we have \( a = 0 \).

Using the abstract Green’s identity with the element

\[ ((H_{\omega(s_+)}(0, \cdot) \begin{pmatrix} 0 \\ 1 \end{pmatrix}; 0); ((\begin{pmatrix} 0 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \end{pmatrix})) \in \Gamma(\mathcal{R}(\omega(s_+))), \]

(2.21)
gives
\[-b_1 = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}^* Jb = [\pi_2^{-1} D_0, H_{\omega(s_+)}(0,\cdot) \begin{pmatrix} 0 \\ 1 \end{pmatrix}] = D_0(0) = -1,\]
cf. (2.18). Finally, we use the element and remember Corollary 2.13, to obtain
\[-b_2 = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}^* Jb = [\pi_2^{-1} D_0, H_{\omega(s_+)}(0,\cdot) \begin{pmatrix} 1 \\ 0 \end{pmatrix}] = 0.\]
\[
\fbox{2.16 Lemma. We have}
\[(\delta_k; \delta_{k+1}); (0; 0)) \in \Gamma(\omega(s_+)), \ k = 0, \ldots, \Delta + \delta - 2.\]
\[
\text{Proof. Since } (D_k; D_{k+1}) \in S(E_{\omega(s_+)}), \text{ Lemma V.2.16 implies } (\delta_k; \delta_{k+1}) \in T(\omega(s_+)). \text{ Let } a, b \in \mathbb{C}^2 \text{ be such that } ((\delta_k; \delta_{k+1}); (a; b)) \in \Gamma(\omega(s_+)). \text{ The same argument as in the proof of Lemma 2.15 yields that } a = 0.
\]
Assume that \((\sigma, s_+)\) is not indivisible. Choose \(r \in (\sigma, s_+) \cap I_{\text{reg}}\) such that \((r, s_+)\) is not indivisible. Lemma 2.8 tells us that \(D_k, D_{k+1} \in \mathfrak{F}_a \subseteq \mathfrak{F}(E_{\omega(r)}),\) and hence orthogonal to \(\omega(r) \cdot \mathfrak{F}(\omega(r)^{-1}\omega(s_+)).\) Lemma 2.14 gives \(b = 0.\)
Finally, assume that \((\sigma, s_+)\) is indivisible. The abstract Green’s identity with the pair (2.21) gives \(-b_1 = D_{k+1}(0) = 0.\) Applying it with the pair (2.20) gives
\[0 = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}^* Jb = b_2.\]
\[
\fbox{c. The map } \psi(\omega).
\]
We will next define a map \(\psi(\omega) : \mathfrak{F}(\omega(s_+)) \to \mathcal{M}(I)_{\equiv H}\) which relates \(\mathfrak{F}(\omega(s_+))\) to \(L^2(\mathcal{H}), T_{\text{max}}(\mathcal{H})\) and \(\Gamma(\mathcal{H}).\) Here \(\mathcal{M}(I)\) denotes the set of all measurable functions of \(I\) into \(\mathbb{C}^2\) which have the property that, if \((\alpha_-; \alpha_+) \subseteq I\) is an indivisible interval of type \(\phi,\) then \(\xi_{\alpha}^\phi f(t)\) is constant on \((\alpha_-, \alpha_+).\)

The definition of \(\psi(\omega)\) is based on the following observation. For \(r \in I_{\text{reg}}\) denote by \(P_r\) the orthogonal projection of \(\mathfrak{F}(\omega(s_+))\) onto
\[\left\{ \begin{array}{ll} \mathfrak{F}(\omega(r)) & , \ r \in [s_-, \sigma) \\ \omega(r) \cdot \mathfrak{F}(\omega(r)^{-1}\omega(s_+)) & , \ r \in (\sigma, s_+] \end{array} \right.\]
Moreover, recall the notation \(\Theta_{r_-, r_+}\) from Corollary V.3.27.

\[
\fbox{2.17 Lemma. Let } F \in \mathfrak{F}(\omega(s_+)) \text{ be given. Then there exists a unique element } f \in \mathcal{M}(I)_{\equiv H} \text{ such that}
\]
\[
(i) \text{ If } r \in [s_-, \sigma) \cap I_{\text{reg}} \text{ is such that } (s_-, r) \text{ is not indivisible, then}
\]
\[
\Theta_{s_-, r}(f|_{[s_-, r)}) = P_r F. \]
(ii) If \( r \in (\sigma, s_{+}] \cap I_{\text{reg}} \) is such that \((r,s_{-})\) is not indivisible, then
\[
\omega(r) \cdot \Theta_{r,s_{+}}(f|_{(r,s_{+})}) = P_{r} F.
\]

(iii) If \((\sigma, s_{+}]\) is indivisible, then
\[
f|_{(\sigma,s_{+})} = -[\pi_{2} F, D_{0}] \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Proof. Let \( r \in [s_{-}, \sigma) \cap I_{\text{reg}} \). By Corollary V.3.27, \( \Theta_{s_{-},r} \) maps \( L^{2}(H|_{(s_{-},r)}) \) bijectively onto \( \mathcal{R}(\omega(r)) \). Hence there exists a unique element \( f_{r} \in L^{2}(H|_{(s_{-},r)}) \) such that
\[
\Theta_{s_{-},r} f_{r} = P_{r} F.
\]

Let \( r, r' \in [s_{-}, \sigma) \cap I_{\text{reg}}, r < r' \). Then, by Lemma V.3.34, we have
\[
\begin{array}{c}
\mathcal{R}(\omega(s_{+})) \\
\mathcal{R}(\omega(r)) \\
\mathcal{R}(\omega(r'))
\end{array}
\begin{array}{c}
P_{r'} \\
\Theta_{s_{-},r'} L^{2}(H|_{(s_{-},r')}) \\
\Theta_{s_{-},r} L^{2}(H|_{(s_{-},r)})
\end{array}
\]

By uniqueness of the elements satisfying (2.23), it follows that \( f_{r'} = \rho_{s_{-},r'} f_{r} \). Since \( \sup([s_{-}, \sigma) \cap I_{\text{reg}}) = \sigma \), an element \( f_{-} \in \mathcal{M}((s_{-}, \sigma)]_{\text{reg}} \) is well-defined by the requirement that always \( (f_{-})|_{(s_{-},r)} = f_{r} \).

Consider the case that \((\sigma, s_{+}]\) is not indivisible. We will argue similarly as in the preceding paragraph. Let \( r \in (\sigma, s_{+}] \cap I_{\text{reg}} \). By Corollary V.3.27, \( \Theta_{r,s_{+}} \) maps \( L^{2}(H|_{(r,s_{+})}) \) bijectively onto \( \mathcal{R}(\omega(r)^{-1}\omega(s_{+})) \). Hence there exists a unique element \( f_{r} \in L^{2}(H|_{(r,s_{+})}) \) such that
\[
\omega(r) \cdot \Theta_{s_{-},r} f_{r} = P_{r} F.
\]

Let \( r, r' \in (\sigma, s_{+}] \cap I_{\text{reg}}, r < r' \). Denote by \( \tilde{P}_{r'} \) the orthogonal projection of \( \mathcal{R}(\omega(r)^{-1}\omega(s_{+})) \) onto \( \omega(r)^{-1}\omega(r') \cdot \mathcal{R}(\omega(r')^{-1}\omega(s_{+})) \). The fact that
\[
\mathcal{R}(\omega(s_{+})) = \mathcal{R}(\omega(r)) \oplus \omega(r) \mathcal{R}(\omega(r)^{-1}\omega(s_{+})) = \\
\mathcal{R}(\omega(r)) \oplus \omega(r) \left[ \mathcal{R}(\omega(r)^{-1}\omega(r')) \oplus \omega(r') \cdot \mathcal{R}(\omega(r')^{-1}\omega(s_{+})) \right] = \\
\mathcal{R}(\omega(r)) \oplus \omega(r) \mathcal{R}(\omega(r')^{-1}\omega(r')) \oplus \omega(r') \cdot \mathcal{R}(\omega(r')^{-1}\omega(s_{+}))
\]
just means that
\[
P_{r'} = \omega(r) \circ \tilde{P}_{r'} \circ \omega(r')^{-1} \circ P_{r}.
\]
Using Lemma V.3.34, we obtain

\[
\begin{aligned}
P_r, & \quad \omega(r) \cdot \mathcal{R}((\omega(s_+))^{-1} \omega(s_+) ) \quad \xrightarrow{\rho_r} \quad \mathcal{R}(\omega(r))^{-1} \omega(s_+) \quad \xrightarrow{\Theta_{r,s_+}} \quad L^2(H_{|[r,s_+],}) \\
P_r' & \quad \omega(r') \cdot \mathcal{R}((\omega(s_+))^{-1} \omega(s_+) ) \quad \xrightarrow{\rho_{r'}} \quad \mathcal{R}(\omega(r'))^{-1} \omega(s_+) \quad \xrightarrow{\Theta_{r',s_+}} \quad L^2(H_{|[r',s_+],})
\end{aligned}
\]

By uniqueness of the elements satisfying (2.24), it follows that \( f_{r'} = \rho_{r',s_+} f_r \).
Since \( \inf((\sigma,s_+) \cap \text{I}_{\text{reg}}) = \sigma \), an element \( f_+ \in \mathcal{M}((\sigma,s_+)) / \omega \) is well-defined by the requirement that always \( (f_+)_{\mathcal{H}_{|[r,s_+]}} = f_r \). Defining \( f \in \mathcal{M}(\mathcal{H}_{|\omega}) \) by \( f|_{(s_-,\sigma)} = f_- \) and \( f|_{(\sigma,s_+)} = f_+ \), clearly gives us an element with the desired properties.

If \( (\sigma,s_+) \) is indivisible, we simply take on the right half of the interval the formula (2.22) as a definition, and again obtain an element with the desired properties.

2.18 Definition. Let \( \psi(\omega) : \mathcal{R}(\omega(s_+)) \to \mathcal{M}(\mathcal{H}_{|\omega}) \) be the map which assigns to each element \( F \in \mathcal{R}(\omega(s_+)) \) the unique element \( f \) given by Lemma 2.17.

2.19 Remark.

(i) Let \( r \in (s_-,\sigma) \), then \( \psi(F)|_{(s_-,r)} \in L^2(H_{|(s_-,r)}) \). Similarly, if \( r \in (\sigma,s_+) \), then \( \psi(F)|_{(r,s_+)} \in L^2(H_{|(r,s_+)}) \).

(ii) Since \( \Theta_{r,s_+} \) and \( P_r \) are both compatible with the respective involutions, it follows that \( \psi(F^\#) = \overline{\psi(F)} \).

In general it will not be possible to compute \( \psi F \) explicitly. However, for the case of reproducing kernels this can be done.

2.20 Lemma. We have

\[
\psi(H_{\omega(s_+)}(w,\cdot) v) =_{\mathcal{H}} \omega(\cdot) v, \quad w \in \mathbb{C}, v \in \mathbb{C}^2.
\]

Proof. Let \( r \in [s_-,\sigma) \cap \text{I}_{\text{reg}} \) be such that \((s_-,r)\) is not indivisible. By (V.3.6), we have

\[
\Theta_{s_-},r \omega(\cdot) v = H[\omega(r)](w,\cdot) v.
\]

However, since \( P_r \) is the orthogonal projection of \( \mathcal{R}(\omega(s_+)) \) onto \( \mathcal{R}(\omega(r)) \), we have \( P_r H[\omega(s_+)](w,\cdot) = H[\omega(r)](w,\cdot) \). It follows that \( \psi(H[\omega(s_+)](w,\cdot) v)|_{(s_-,\sigma)} = H[\omega(\cdot) v]|_{(s_-,\sigma)} \).

Assume that \((\sigma,s_+)\) is not indivisible, and let \( r \in (\sigma,s_+) \cap \text{I}_{\text{reg}} \) be such that \((r,s_+)\) is not indivisible. The relation (V.3.6) applied with \( u := \omega(r)^* v \) gives

\[
\omega(r) \cdot \Theta_{r,s_+} \omega(\cdot) v = \omega(r) \cdot \Theta_{r,s_+} \omega(\cdot) v = \omega(r) \cdot \Theta_{r,s_+} (\omega(\cdot)^{-1}) u = \omega(r) H[\omega(\cdot)^{-1}] u = \omega(r) H[\omega(\cdot)](w,\cdot) u = \omega(r)^* v.
\]
Since $P_r$ is the orthogonal projection of $\mathcal{R}(\omega(s_+))$ onto $\omega(r) \cdot \mathcal{R}((r)^{-1}\omega(s_+))$, we have

$$P_r H_{\omega(s_+)}(w,.)v = \omega(r) H_{\omega_{(r)^{-1}\omega(s_+)}(w,.)} \omega(r)^* v,$$

cf. (V.2.19). It follows that $\psi(H_{\omega(s_+)}(w,.)v)|_{(\sigma,s_+)} = H \omega(.)v|_{(\sigma,s_+)}$.

Assume next that $(\sigma,s_+)$ is indivisible. Write

$$\omega(t) := \begin{pmatrix} w_{11}(t,.) & w_{12}(t,.) \\ w_{21}(t,.) & w_{22}(t,.) \end{pmatrix}$$

then

$$\omega(t)(w)^* v = \begin{pmatrix} (w_{11}(t,\bar{w}), w_{21}(t,\bar{w}))v \\ (w_{12}(t,\bar{w}), w_{22}(t,\bar{w}))v \end{pmatrix}.$$  

By the definition of $\psi$ and Corollary 2.13, we have

$$\psi(H_{\omega(s_+)}(w,.)v) = [H_{\omega(s_+)}(w,.)v, \begin{pmatrix} w_{12}(s_+,.) \\ w_{22}(s_+,.) \end{pmatrix}] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v^* \begin{pmatrix} w_{12}(s_+,w) \\ w_{22}(s_+,w) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (w_{12}(s_+,\bar{w}), w_{22}(s_+,\bar{w}))v \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

Since we know that the number $\psi$ in (2.17) is equal to $\frac{\tilde{\omega}}{2}$, we have

$$\begin{pmatrix} w_{21}(t,.) \\ w_{22}(t,.) \end{pmatrix} = \begin{pmatrix} w_{21}(s_+,.) \\ w_{22}(s_+,.) \end{pmatrix}, \ t \in (\sigma,s_+).$$

Since the Hamiltonian $H|_{(\sigma,s_+)}$ is of the form

$$H(t) = h(t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ t \in (\sigma,s_+),$$

the asserted equality, i.e. equality modulo $=_{H}$, follows.

Our next task is to show in detail how $\psi$ relates $\mathcal{R}(\omega(s_+))$ with $L^2(H)$. In the following denote by $\rho_- : M(I) \to M([s_-,\sigma])$ and $\rho_+ : M(I) \to M((\sigma,s_+))$ the respective restriction maps, and set

$$\mathcal{R}^-(\omega) := \pi_2^{-1}(\mathcal{P}_0), \quad \mathcal{R}^+(\omega) := \pi_2^{-1}(\mathcal{P}_0)^\perp,$$

$$X^\delta(\omega) := \text{span}\{\delta(\omega)_{\Delta,\ldots,\delta(\omega)_{\Delta\Delta-1}}\},$$

Then, by Lemma 2.8, $X_\delta(\omega) \perp X^\delta(\omega)$, $X^\delta(\omega)$ is nondegenerated, and

$$\begin{align*}
(\mathcal{R}^-(\omega) + \mathcal{R}^+(\omega))^\perp &= \text{span}\{\delta(\omega)_{0,\ldots,\delta(\omega)_{\Delta\Delta-1}}\}, \\
(\mathcal{R}^-(\omega) + \mathcal{R}^+(\omega))^\circ &= \text{span}\{\delta(\omega)_{0,\ldots,\delta(\omega)_{\Delta-1}}\}. 
\end{align*}$$

The above relations then imply

$$\begin{align*}
(\mathcal{R}^-(\omega) + \mathcal{R}^+(\omega) + X^\delta(\omega))^\perp &= (\mathcal{R}^-(\omega) + \mathcal{R}^+(\omega) + X^\delta(\omega))^\circ = \\
&= \text{span}\{\delta(\omega)_{0,\ldots,\delta(\omega)_{\Delta-1}}\}. 
\end{align*}$$

2.21 Proposition. We have

$$\begin{align*}
\mathcal{R}^-(\omega) &= \text{span}\{\delta(\omega)_{\Delta,\ldots,\delta(\omega)_{\Delta\Delta-1}}\}, \\
\mathcal{R}^+(\omega) &= \text{span}\{\delta(\omega)_{0,\ldots,\delta(\omega)_{\Delta\Delta-1}}\}, \\
\mathcal{R}^-(\omega) &= \text{span}\{\delta(\omega)_{0,\ldots,\delta(\omega)_{\Delta-1}}\}. 
\end{align*}$$
(i) \(\ker(\rho_\circ \psi) = (\mathcal{R}^-)^\perp, \quad \ker(\rho_+ \circ \psi) = (\mathcal{R}^+)\perp,\)
\[\ker \psi = \text{span}\{\delta_0, \ldots, \delta_{\Delta+\delta-1}\}.\]

(ii) The restriction \(\psi|_{\mathcal{R}^\pm}\) maps \(\mathcal{R}^\pm\) isometrically onto \(L^2(H_\pm)\). We have \(\varphi, \psi \in \mathcal{R}^\pm\) \(\varphi = \psi\).
\[\begin{align*}
[F, G] &= \int_{\mathcal{I}} (\psi G)^* H(\psi F), \quad F \in X_\delta, G \in (X_\delta + X^\delta)^\perp. \tag{2.27}
\end{align*}\]

(iii) \((\rho_- \circ \psi)^{-1}(L^2(H_-)) = \text{span}\{\delta_0, \ldots, \delta_{\Delta_-1}\}\)
\[(\rho_+ \circ \psi)^{-1}(L^2(H_+)) = \text{span}\{\delta_0, \ldots, \delta_{\Delta_+1}\}\]
\[\psi^{-1}(L^2(H)) = \text{span}\{\delta_0, \ldots, \delta_{\Delta-1}\}\]

Proof (of Proposition 2.21, (i)). By its definition
\[\mathfrak{P}_0 = \mathrm{cls}\{\mathfrak{P}(E_t) : t \in [s_-, \sigma] \cap \text{I}_{\text{reg}}\},\]
and hence
\[\mathcal{R}^- = \mathrm{cls}\{\mathfrak{R}(\omega(t)) : t \in [s_-, \sigma] \cap \text{I}_{\text{reg}}\}. \tag{2.28}\]
Let \(F \in \mathfrak{R}(\omega(s_+))\), then \(F \perp \mathfrak{R}\) if and only if \(F \perp \mathfrak{R}(\omega(t))\) for all \(t \in [s_-, \sigma] \cap \text{I}_{\text{reg}}\). This, however, is by the definition of \(\psi\) equivalent to \((\psi F)(t) = 0\) for all \(t \in [s_-, \sigma]\). Thus, the first part of (i) holds true.

If \((\sigma, s_+)\) is not indivisible, we have
\[\mathfrak{P}_+^\perp = \mathrm{cls}\{\mathfrak{P}(E_t)^\perp : t \in (\sigma, s_+) \cap \text{I}_{\text{reg}}\},\]
and hence
\[\mathcal{R}^+ = \mathrm{cls}\{\mathfrak{R}(\omega(t))^\perp : t \in (\sigma, s_+) \cap \text{I}_{\text{reg}}\}. \tag{2.29}\]
Since \(\mathfrak{R}(\omega(t))^\perp = \omega(t) \cdot \mathfrak{R}(\omega(t)^{-1} \omega(s_+))\), the same argument as in the first paragraph yields that \(F \perp \mathfrak{R}^+\) if and only if \((\psi F)(t) = 0\) for all \(t \in (\sigma, s_+)\).

Consider the case that \((\sigma, s_+)\) is indivisible. By (2.22), we have \((\rho_\circ \psi)F = 0\) if and only if \(F \perp \pi_2^{-1} D_0\). However, \(\{D_0\}^\perp = \mathfrak{P}_a\). This completes the proof of the second part of (i).

Finally, from the already shown facts and Lemma 2.8 we obtain
\[\ker \psi = \ker(\rho_- \circ \psi) \cap \ker(\rho_+ \circ \psi) = \pi_2^{-1}(\mathfrak{P}_0^\perp \cap \mathfrak{P}_a) = \text{span}\{\delta_0, \ldots, \delta_{\Delta+\delta-1}\}.\]

Let \(\mathcal{Q}_1\) and \(\mathcal{Q}_2\) be almost Pontryagin spaces, and let \(L_1 \subseteq \mathcal{Q}_1\), \(L_2 \subseteq \mathcal{Q}_2\), be dense linear subspaces. If \(\lambda : L_1 \rightarrow L_2\) is a surjective and isometric map, then there exists an isometric isomorphism \(\lambda : \mathcal{Q}_1/\mathcal{Q}_1^\perp \rightarrow \mathcal{Q}_2/\mathcal{Q}_2^\perp\) with
\[\begin{align*}
L_1 &\xrightarrow{\lambda} L_2 \\
\text{downwards arrows} &\xrightarrow{\lambda} \mathcal{Q}_1/\mathcal{Q}_1^\perp &\xrightarrow{\lambda} \mathcal{Q}_2/\mathcal{Q}_2^\perp
\end{align*}\]
where the downwards arrows are the canonical maps, i.e. inclusion followed by projection.
Proof. Let \( x \in L_1 \cap Q_1^2 \) and \( y \in L_2 \). Choose \( y_1 \in L_1 \) with \( \lambda(y_1) = y \), then

\[
[\lambda(x), y] = [x, y_1] = 0.
\]

Since \( L_2 \) is dense in \( Q_2 \), it follows that \( \lambda(x) \in Q_2^2 \).

Therefore a map \( \lambda : L_1/\mathbb{Q}_1^2 \to L_2/\mathbb{Q}_2^2 \) with

\[
\begin{array}{ccc}
L_1 & \overset{\lambda}{\longrightarrow} & L_2 \\
\downarrow & & \downarrow \\
L_1/\mathbb{Q}_1^2 & \overset{\lambda}{\longrightarrow} & L_2/\mathbb{Q}_2^2
\end{array}
\]

is well-defined. Clearly, \( \hat{\lambda} \) is isometric. However, \( L_j/\mathbb{Q}_j^2 \) is a dense linear subspace of the Pontryagin space \( Q_j/\mathbb{Q}_j^2 \), and hence \( \hat{\lambda} \) can be extended by continuity to an isometric isomorphism between \( Q_1/\mathbb{Q}_1^2 \) and \( Q_2/\mathbb{Q}_2^2 \).

Proof (of Proposition 2.21, (ii)). Let \( r \in [s_-, \sigma) \cap I_{\text{reg}} \) be such that \( (s_-, r) \) is not indivisible. By the definition of \( \psi \), we have \( \psi|_{\hat{\mathcal{R}}(\omega(r))} = \Theta^{-1}_{s_-, r} \), and therefore \( \psi|_{\hat{\mathcal{R}}(\omega(r))} \) is an isometric isomorphism of \( \hat{\mathcal{R}}(\omega(r)) \) onto \( L^2(H_{(s_-, r)}) \). It follows that \( \psi \) induces a surjective isometry

\[
\lambda := \psi|_{L_r \otimes (\hat{\mathcal{R}}(\omega(r)))} : \bigcup_r \hat{\mathcal{R}}(\omega(r)) \to \bigcup_r L^2(H_{(s_-, r)}),
\]

where unions are taken over the range of all values of \( r \in [s_-, \sigma) \cap I_{\text{reg}} \) such that \( (s_-, r) \) is not indivisible. Lemma 2.22 applied with the map \( \lambda \) and the almost Pontryagin spaces \( \hat{\mathcal{R}} \) and \( L^2(H_-) \), gives an isometric isomorphism \( \hat{\lambda} : \hat{\mathcal{R}} \to L^2(H_-) \).

Let \( F \in \mathcal{R} \) be given. If \( r \in [s_-, \sigma) \cap I_{\text{reg}} \) is such that \( (s_-, r) \) is not indivisible, let \( g \in L^2(H_{(s_-, r)}) \), and choose \( G \in \hat{\mathcal{R}}(\omega(r)) \) such that \( g = \lambda(G) \). Then \( g = \hat{\lambda}(\mathbb{F}[\mathcal{R}^{-1}_{(s_-, r)}) \) and

\[
(\psi F)|(s_-, r) = (\psi P_r F)|(s_-, r) = \hat{\lambda}(\mathbb{F}[\mathcal{R}^{-1}_{(s_-, r)}])|(s_-, r).
\]

We compute

\[
\int_{s_-}^r g^* H \hat{\lambda}(\mathbb{F}[\mathcal{R}^{-1}_{(s_-, r)}]) = [\hat{\lambda}(\mathbb{F}[\mathcal{R}^{-1}_{(s_-, r)}]), \hat{\lambda}(\mathbb{F}[\mathcal{R}^{-1}_{(s_-, r)}])] = [F, G] = [F, P_r G] =
\]

\[
= [P_r F, G] = [\hat{\lambda}(\mathbb{F}[\mathcal{R}^{-1}_{(s_-, r)}]), \hat{\lambda}(\mathbb{F}[\mathcal{R}^{-1}_{(s_-, r)}])] = \int_{s_-}^r g^* H(\psi F),
\]

and get that \( \psi F = \hat{\lambda}(\mathbb{F}|_{\mathcal{R}^{-1}_{(s_-, r)}}) \). Since \( \hat{\lambda} \) is isometric and onto \( L^2(H_-) \), so is \( \psi|_{\hat{\mathcal{R}}} \).

Assume that \( (\sigma, s_+) \) is not indivisible. Let \( r \in (\sigma, s_+) \cap I_{\text{reg}} \) be such that \( (r, s_+) \) is not indivisible, then

\[
\psi|_{\omega(r) \hat{\mathcal{R}}(\omega(r)^{-1} \omega(s_+))} = (\omega(r) \cdot \Theta_{s_-, r})^{-1}.
\]
and therefore the map \( \psi|_{\omega(r) \cdot \mathcal{R}(\omega^{-1}(\sigma, s_0))} \) is an isometric isomorphism of \( \omega(r) \cdot \mathcal{R}(\omega^{-1}(\sigma, s_0)) \) onto \( L^2(H_{r,s}) \). Hence

\[
\psi|_{\mathcal{R}(\omega(r) \cdot \mathcal{R}(\omega^{-1}(\sigma, s_0))) : \bigcup_r \omega(r) \cdot \mathcal{R}(\omega^{-1}(\sigma, s_0)) \to \bigcup_r L^2(H_{r,s})}
\]

is a surjective isometry between dense subspaces of the almost Pontryagin spaces \( \mathcal{R}^+ \) and \( L^2(H_+) \). The desired conclusion follows in exactly the same way as in the above paragraph.

If \((\sigma, s_0)\) is indivisible, matters are trivial. Actually, in this case, \( L^2(H_+) = \{0\} \) and \( \mathcal{R}^+ = \text{span}\{\delta_0\} \). However, by definition \( \psi\delta_0 = 0 \). This completes the proof of the first assertion in \((ii)\).

In order to show \((2.27)\), let \( F \perp X_\delta \) and \( G \perp (X_\delta + X_\delta^\perp) \). Due to \((2.25)\) and \((2.26)\) we can decompose \( F \) as \( F = F_+ + F_1 + F_\perp \) with \( F_\perp \in \mathcal{R}^\perp, F_1 \in X_\delta \), and \( G = G_- + G_\perp \) with \( G_\perp \in \mathcal{R}^\perp \). It follows that

\[
[F, G] = [F_-, G_-] + [F_+, G_+] = \int_{s_-}^\sigma (\psi G_-)^* H(\psi F_-) + \int_{s_-}^\sigma (\psi G_+)^* H(\psi F_+).
\]

The assertions in \((iii)\) are obtained as a corollary of \((ii)\).

**Proof (of Proposition 2.21, \((iii)\)).** Since \( \psi \) maps \( \mathcal{R}^- \) onto \( L^2(H_-) \), we have

\[
(\rho_- \circ \psi)^{-1}(L^2(H_-)) = \mathcal{R}^- + \ker(\rho_- \circ \psi) = \mathcal{R}^- + (\mathcal{R}^-)^\perp.
\]

However, since \( \mathcal{R}(\omega(s_0)) \) is a Pontryagin space, the sum \( \mathcal{R}^- + (\mathcal{R}^-)^\perp \) is closed, and we may compute further

\[
\mathcal{R}^- + (\mathcal{R}^-)^\perp = \left((\mathcal{R}^-)^\perp \cap \mathcal{R}^-\right)^\perp = \text{span}\{\delta_0, \ldots, \delta_{\Delta_+ - 1}\}^\perp.
\]

Similarly,

\[
(\rho_+ \circ \psi)^{-1}(L^2(H_+)) = \mathcal{R}^+ + \ker(\rho_+ \circ \psi) = \mathcal{R}^- + (\mathcal{R}^+)^\perp = \left((\mathcal{R}^+)^\perp \cap \mathcal{R}^+\right)^\perp = \text{span}\{\delta_0, \ldots, \delta_{\Delta_- - 1}\}^\perp.
\]

The last assertion in \((iii)\) is now obvious. We have

The next step in the investigation of \( \psi \) is to show how \( \psi \) relates \( T(\omega(s_0)) \) with differentiation. This is a crucial fact.

**2.23 Proposition.** Let \((\mathcal{R}, (a, b)) \in \Gamma(\omega(s_0))\) be given. Then there exists a locally absolutely continuous representant \( f \) of \( \psi F \), such that \( (g, f) \) is any representant of \( \psi G \)

\[
f' = JHg, \quad f(s_-) = a, \quad f(s_+) = b.
\]

**Proof.** Let \( r \in (s_-, \sigma) \cap I_{\text{reg}} \). By Proposition V.2.11 and Corollary V.3.27, we have

\[
((F_r \otimes F_r \otimes \pi_1) \Gamma(\omega(s_0)) \subseteq ((\text{id} \otimes \text{id}) \otimes \pi_1) \Gamma\omega(r) = ((\text{id} \otimes \text{id}) \otimes \pi_1)((\Theta_{s_0, r} \otimes \Theta_{s_0, r} \otimes \text{id}) \otimes \text{id}) \Gamma(H|_{(s_-, r)}) =
\]

\[
= \Gamma(H) = \Gamma(H_{r,s})
\]
In view of the definition of $\psi$, we must have $f_r = (\psi F)(s_{-r})$ and $g_r = (\psi G)(s_{-r})$. It follows that there exists a unique locally absolutely continuous representant $\tilde{f}_r$ of $(\psi F)(s_{-r})$ satisfying $\tilde{f}_r' = JH(\psi G)(s_{-r})$, cf. [HSW]. Moreover, it has the boundary value $\tilde{f}_r(s_{-r}) = a_r$.

If $r, r' \in (s_{-r}, \sigma) \cap I_{\text{reg}}$, $r' < r$, clearly, $(\tilde{f}_r|_{(s_{-r'}, r')})' = JH(\psi G)|_{(s_{-r'}, r')}$.

Applying Corollary V.3.27, we have

$$(P_2 \otimes P_2) \otimes \pi_r \Gamma(\omega(s_+)) \subseteq (\mathfrak{B}_\omega(r) \otimes \omega^{-1}(\omega(s_+))) = \Gamma(\omega(r) \otimes \omega^{-1}(\omega(s_+)))$$

i.e.,

$$(\mathfrak{B}_\omega(r) \otimes \omega^{-1}(\omega(s_+))) \subseteq (\mathfrak{B}_\omega(r) \otimes \omega^{-1}(\omega(s_+)))$$

Again we see that $f_r = (\psi F)(s_{-r})$ and $g_r = (\psi G)(s_{-r})$ for some element $(f_r; g_r; (a_r, b_r)) \in \Gamma(H|_{(r, s_+)}).$ Hence, the same argument as above will provide us with a locally absolutely continuous representant $f_r$ of $\psi F|_{(\sigma, s_+)}$, such that $(f_r)' = JH(\psi G)|_{(\sigma, s_+)}$ and $f_r(s_+) = b$. Defining $f$ by putting together these two functions, the assertion follows.

Consider the case that $(\sigma, s_+)$ is indivisible. Then, by the abstract Green’s identity and Lemma 2.15, we have

$$-[F; \pi_2^{-1}D_0] = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \int_j b = b_2. \tag{2.30}$$

By the definition (2.22) of $\psi$, we have

$$
\psi F|_{(\sigma, s_+)} = -[\pi_2 F; D_0] \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \psi G|_{(\sigma, s_+)} = -[\pi_2 G, D_0] \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

Since the Hamiltonian $H$ is, on the interval $(\sigma, s_+)$, of the form

$$H(t) = h(t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \in (\sigma, s_+),$$

it follows that the function

$$f_+(t) = \begin{pmatrix} b_1 + [\pi_2 G, D_0] \int_{s_+}^t h(t) \\ \int_{s_+}^t [-\pi_2 F, D_0] \end{pmatrix}$$

is a locally absolutely continuous representant of $\psi F|_{(\sigma, s_+)}$ and satisfies $(f_+)' = JH\psi G|_{(\sigma, s_+)}).$ Moreover, by (2.30), we have $f_+(s_+) = \left(-[\pi_2 F, D_0]\right) = b$. Proceeding similarly as in the above paragraph completes the proof. \[\square\]
2.24 Corollary. Let \((F;G) \in T(\omega(s_+))\), and assume that \(f\) is a locally absolutely continuous representant of \(\psi F\) with \(f' = JH(\psi g)\). Then there exists a constant \(\gamma \in \mathbb{C}\) such that
\[
((F;G); (f(s_-): f(s_+) + \gamma (\begin{array}{c}1 \\ 0 \end{array})) \in \Gamma(\omega(s_+)).
\]
If \((\sigma, s_+)\) is not indivisible, then \(\gamma = 0\).

Proof. Choose a locally absolutely continuous representant \(f_1\) of \(\psi F\) according to Proposition 2.23, so that \((F;G); (f_1(s_-): f_1(s_+)) \in \Gamma(\omega(s_+))\).

Let \(r \in (s_-\cdot \sigma) \cap I_{\text{reg}}\) be such that \((s_-, r)\) is not indivisible, then \(f, f_1, \psi g \in L^2(H(\lfloor s_-\cdot r\lfloor))\), and
\[
f' = f'_1 = JHg, \ t \in (s_-, r) .
\]
Since \(f_1 = H f\), it follows that \(f\mid_{(s_-, \sigma)} = f_1\mid_{(s_-, \sigma)}\), cf. [HSW, Lemma 3.5]. In particular, boundary values at \(s_-\cdot \sigma\) coincide.

The same argument will work for boundary values at \(s_+\) if \((\sigma, s_+)\) is not indivisible. If \((\sigma, s_+)\) is indivisible, we only obtain \(\pi_2 f_1(s_+) = \pi_2 f(s_+)\), cf. (2.22). Still, due to Lemma 2.15 this also yields the present assertion. \(\square\)

2.25 Corollary. Let \(((F_1; G_1); (a_1; b_1)); ((F_2; G_2); (a_2; b_2)) \in \Gamma(\omega(s_+))\).

(i) If \((\psi F_1)\mid_{(s_-, r)} = (\psi F_2)\mid_{(s_-, r)}\) and \((\psi G_1)\mid_{(s_-, r)} = (\psi G_2)\mid_{(s_-, r)}\) for some \(r \in (s_-\cdot \sigma) \cap I_{\text{reg}}\) such that \((s_-, r)\) is not indivisible, then \(a_1 = a_2\).

(ii) If \((\psi F_1)\mid_{(r, s_+)} = (\psi F_2)\mid_{(r, s_+)}\) and \((\psi G_1)\mid_{(r, s_+)} = (\psi G_2)\mid_{(r, s_+)}\) for some \(r \in (\sigma, s_+) \cap I_{\text{reg}}\) such that \((r, s_+)\) is not indivisible, then \(b_1 = b_2\).

(iii) Assume that \((\sigma, s_+)\) is indivisible. If \((\psi F_1)\mid_{(\sigma, s_+)} = (\psi F_2)\mid_{(\sigma, s_+)}\), then \(\pi_2 b_1 = \pi_2 b_2\).

For each prescribed value \(\beta \in \mathbb{C}\), there exists \(((F; G); (a; b)) \in \Gamma(\omega(s_+))\) such that \((\psi F_1)\mid_{(\sigma, s_+)} = (\psi F)\mid_{(\sigma, s_+)}\), \((\psi G_1)\mid_{(\sigma, s_+)} = (\psi G)\mid_{(\sigma, s_+)}\), and \(\pi_2 b_1 = \beta\).

Proof. Assume we are in the situation of (i). Then
\[
((\psi F_1)\mid_{(s_-, r)}; (\psi G_1)\mid_{(s_-, r)}) = ((\psi F_2)\mid_{(s_-, r)}; (\psi G_2)\mid_{(s_-, r)}) \in T_{\text{max}}(H(\lfloor s_-\cdot r\rfloor)).
\]
Choose locally absolutely continuous representants \(f_1\) and \(f_2\) according to Proposition 2.23 applied with \(((F_1; G_1); (a_1; b_1))\) or \(((F_2; G_2); (a_2; b_2))\), respectively. Since \((s_-, r)\) is not indivisible, we must have \(f_1\mid_{(s_-, r)} = f_2\mid_{(s_-, r)}\). It follows that \(a_1 = a_2\).

The assertion (ii) is proved completely similar. Let us investigate the situation that \((\sigma, s_+)\) is indivisible. Then \((\psi F_1)\mid_{(\sigma, s_+)} = (\psi F_2)\mid_{(\sigma, s_+)}\) just means \([F_1, \delta_0] = [F_2, \delta_0]\). Using the abstract Green’s identity with the element (2.20), yields \(\pi_2 b_1 = \pi_2 b_2\). The second part of the assertion in (iii) follows, since we may add arbitrary scalar multiples of the element (2.20) to a given element \(((F_1; G_1); (a_1; b_1))\) of \(\Gamma(\omega(s_+))\) without altering \(\psi F_1\) and \(\psi G_1\). \(\square\)

d. The elements \(P_j\).

Let \(D_0\) be the element constructed in Subsection b., let \(B := B_{-D_0}\) be the bounded selfadjoint operator as introduced in Lemma V.2.15, and recall the notation \(B(z)\) from Definition 2.11. Note that \(BD_k = D_{k-1}\), \(k = 1, \ldots, n\).
2.26 Definition. Let elements $P_j \in \mathfrak{P}(E_s)$, $j \in \mathbb{N} \cup \{0\}$, be defined inductively by

$$P_0(z) := KE_{s+}(0, z),$$

$$P_j(z) := BP_{j-1}(z) - \begin{cases} B(z), & j = \Delta \\ 0, & \text{otherwise} \end{cases},$$

Moreover, set $d_j(\omega) := P_j(0)$.

2.27 Lemma. We have

$$[P_j, D_k] = \begin{cases} -1, & j = k \in \{0, \ldots, \Delta - 1\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$[P_k, P_l] = d_{k+l}, \quad k, l \in \mathbb{N}_0.$$ 

Proof. Our definitions together with Lemma V.2.15 give

$$[P_0, D_0] = -1, \quad [P_0, D_k] = [P_k, D_0] = 0, \quad k \geq 1. \quad (2.31)$$

We will show the recurrence relation

$$[P_j, D_k] = [P_{j-1}, D_{k-1}] + \begin{cases} 1, & j = k = \Delta \\ 0, & j \geq 1, k = 1, \ldots, \Delta + \delta - 1 \end{cases}. \quad (2.32)$$

Together with (2.31), this will imply the first assertion of the lemma.

In order to see (2.32) we use the selfadjointness of $B$, and obtain together with (2.19), (2.31) and $BD_k(z) = D_{k-1}(z)$, that for $j \geq 1, k = 1, \ldots, \Delta + \delta - 1$,

$$[P_j, D_k] = [BP_{j-1} - B, D_k] = [P_{j-1}, D_{k-1}] + \begin{cases} 1, & j = k = \Delta \\ 0, & \text{otherwise} \end{cases}.$$ 

(only if $j = \Delta$)

In order to see the second assertion, note that by the already proved part of the present assertion $B \perp P_k$, $k \in \mathbb{N} \cup \{0\}$. Since $B$ is selfadjoint,

$$[P_k, P_{k+1}] = [P_k, BP_k] = [BP_k, P_l] = [P_{k+1}, P_l], \quad k, l \in \mathbb{N} \cup \{0\}.$$ 

The asserted formula follows inductively from the definition $d_k = [P_k, P_0]$. 

We are now in position to establish the condition $\Delta(H) < \infty$. From now on let $b(\omega) := \pi_2^{-1} B(z)$. Again, we will drop the argument $\omega$ if no confusion may occur.

2.28 Proposition. We have $\Delta(H_-) = \Delta_-(\omega)$ and $\Delta(H_+) = \Delta_+(\omega)$. Let functions $w_k$, $k \in \mathbb{N}_0$, be defined as in [KW/IV, §4.1], then

$$w_k = \psi(\pi_2^{-1} P_k), \quad k \in \mathbb{N}_0,$$

and

$$((\pi_2^{-1} P_k + b : \pi_2^{-1} P_{k-1} + d_{k-1} \delta_0); (w_k(s_-); w_k(s_+))) \in \Gamma(\omega(s_+)).$$ 

(only if $k = \Delta$)
Proof. Since \( P_0 = K_{E(s_+)}(0,.) = \pi_2 H_{\omega(s_+)}(0,.)(0_1^0) \), we know from Lemma 2.20 that
\[
\psi(\pi_2^{-1} P_0) = H \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]
Moreover, by definition of \( \Gamma(\omega(s_+)) \),
\[
((\pi_2^{-1} P_0; 0): \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \in \Gamma(\omega(s_+)).
\]
(2.33)

By Lemma V.2.15, the pair
\[
(P_k + B; P_k - 1 + d_{k-1} D_0)
\]
only if \( k = \Delta \)
belongs to some self-adjoint extension of \( S(E_\omega(s_+)) \). Thus Lemma V.2.16 gives
\[
((\pi_2^{-1} P_k + b; \pi_2^{-1} P_{k-1} + d_{k-1} \delta_0); (a_k; b_k)) \in \Gamma(\omega(s_+)),
\]
only if \( k = \Delta \)

Let \( a_k, b_k \in \mathbb{C}^2 \) be such that
\[
((\pi_2^{-1} P_k + b; \pi_2^{-1} P_{k-1} + d_{k-1} \delta_0); (a_k; b_k)) \in \Gamma(\omega(s_+)),
\]
and put \( w_0 := (0) \). We have \( b = \pi_2^{-1} B(z) \), \( \delta_0 \in \ker \psi \). Hence, by Proposition 2.23, there exist locally absolutely continuous representants \( w_k \) of \( \psi(\pi_2^{-1} P_k) \), \( k \in \mathbb{N} \), such that
\[
w_{k+1} = J H w_k, w_k(s-) = a_k, w_k(s+) = b_k, \quad k \in \mathbb{N}_0.
\]
By the definition of \( T_1(\omega(s_+)) \), we have \( \pi_1 a_k = 0 \). Using the Green’s identity with the pair (2.33), it follows that \( \pi_1 b_k = 0 \). By Proposition 2.21, (iii), and Lemma 2.27, we have
\[
\rho_- w_k \in L^2(H_-) \iff k \geq \Delta_-(\omega)
\]
\[
\rho_+ w_k \in L^2(H_+) \iff k \geq \Delta_+(\omega)
\]
We conclude that \( \Delta(H_-) = \Delta_-(\omega) \), \( \Delta(H_+) = \Delta_+(\omega) \), and that \( w_k = w_k = \psi P_k \).
\[\square\]

We set \( p_j := \pi_2^{-1} P_j \).

2.29 Corollary. The elements \( p_0, \ldots, p_{\Delta-1} \) are linearly independent modulo \( (\mathcal{R}^- + \mathcal{R}^+)^+ X^\delta \), and together with this space these elements span all of \( \mathcal{R}(\omega(s_+)) \).
Proof. The image of \( p_0, \ldots, p_{\Delta-1} \) under the linear map \( \psi \) is linearly independent modulo \( L^2(H) \), and
\[
\psi^{-1}(L^2(H)) = \text{span}\{\delta_0, \ldots, \delta_{\Delta-1}\} = \\
= \mathcal{R}^- + \mathcal{R}^+ + \text{span}\{\delta_{\Delta}, \ldots, \delta_{\Delta+\bar{\omega}-1}\}.
\]
Since, by (2.26), \( \text{dim}(\mathcal{R}(\omega(s_+))/(\mathcal{R}^- + \mathcal{R}^+ + X^\delta)) = \Delta \), the last assertion follows.
\[\square\]
e. Realization as model of $\mathfrak{h}$.

We have by the time collected all the necessary ingredients to define an elementary Hamiltonian $\mathfrak{h}$ of kind (A): Let $\mathfrak{h}$ be given by the data

$$H_{-}(\omega), H_{+}(\omega), \partial(\omega), b_{0}(\omega), \ldots, b_{0+1}(\omega), d_{0}(\omega), \ldots, d_{2\Delta(\omega)-1}(\omega).$$

Our aim is to prove that $\omega_{h} = \omega$. Since both, $\omega_{h}|_{[s_{-},\sigma]}$ and $\omega_{h}|_{[s_{-},\sigma]}$, are solutions of the initial value problem (V.3.5) for the Hamiltonian $H_{-}(\omega)$, we already know that $\omega_{h}(t) = \omega(t)$, for $t \in [s_{-},\sigma]$. In order to establish equality also on the interval $(\sigma,s_{+})$, it is enough to show

$$\omega(\mathfrak{B}(\mathfrak{h})) = \omega(s_{+}).$$

(2.34)

In fact, once having establish this equality, we may argue that both, $\omega_{h}|_{[\sigma,s_{+}]}$ and $\omega|_{[\sigma,s_{+}]}$, are solutions of the differential equation in (V.3.5) for the Hamiltonian $H_{+}(\omega)$ with the same initial value at the regular endpoint $s_{+}$. Thus equality on the whole interval $(\sigma,s_{+})$ follows from (2.34).

2.30 Remark. Note that, by the uniqueness part of V.3.9, the relation (2.34) immediately implies that $\omega$ and $\omega_{h}$ are reparameterizations of each other. However, the above argument ensures actual equality.

Below we will show that there exists an isomorphism $(\omega; \text{id})$ between the boundary triplets $\mathfrak{B}(\omega(s_{+}))$ and $\mathfrak{B}(\mathfrak{h})$. Remembering Proposition V.4.7 and Proposition V.4.27, this is sufficient to obtain (2.34).

Our first task is to construct the map $\omega$. We need the following result, which supplements to Proposition IV.4.14.

2.31 Lemma. Let notation be as in Proposition IV.4.14. The linear space $\bigcup_{J} \text{ran} \iota_{J}$, where the union is taken over all sets of the form (IV.4.20), is dense in $\mathcal{C}$.

Proof. Let $y \in \mathfrak{P}(\mathfrak{h})$ be given, and assume that $y \perp \bigcup_{J} \text{ran} \iota_{J}$. Then, by (IV.4.21), we have $\psi y = 0$. However, $\ker \psi = \mathcal{C}^{\perp}$.

Consider the linear space

$$\mathcal{L} := \bigcup_{r_{-},r_{+}} \left( L^{2}(H|_{(s_{-},r_{-})}) \oplus L^{2}(H|_{(r_{+},s_{+})}) \right)$$

where the union is taken over all $r_{-} \in (s_{-},\sigma) \cap I_{\text{reg}}$ such that $(s_{-},r)$ is not indivisible and all $r_{+} \in (\sigma,s_{+}) \cap I_{\text{reg}}$ such that $(r_{+},s_{+})$ is not indivisible, and let $\mathcal{L}$ be endowed with the $L^{2}(H)$-inner product. Note here that, if $(\sigma,s_{+})$ is indivisible, this definition, as well as the following considerations, are understood in such a way that the summands $L^{2}(H|_{(r_{+},s_{+})})$ are not present.

By Proposition IV.4.14 and Lemma V.3.34 we have isometric maps $\iota : \mathcal{L} \rightarrow C(\mathfrak{h})$ and $\theta : \mathcal{L} \rightarrow \mathcal{R}^{-}(\omega) + \mathcal{R}^{+}(\omega)$ which are defined such that for all admissible values of $r_{-},r_{+}$ the following diagram commutes.
The ranges $\theta(L)$ and $\iota(L)$ are dense in the respective almost Pontryagin spaces $R^-(\omega) + R^+(\omega)$ and $C(\mathfrak{h})$. For the map $\theta$ this follows from (2.28), (2.29) and, in case that $(\sigma, s_+)$ is indivisible, because of $R^+(\omega) = \text{span}\{\delta_{0}(\omega)\} \subseteq R^-(\omega)$. For the map $\iota$, this is Lemma 2.31. Thus

$$(R^-(\omega) + R^+(\omega), \theta) \quad \text{and} \quad (C(\mathfrak{h}), \iota)$$

are both almost Pontryagin space completions of $L$ in the sense of [KWW1, Definition 4.2].

2.32 Lemma. The completions $(R^-(\omega) + R^+(\omega), \theta)$ and $(C(\mathfrak{h}), \iota)$ are isomorphic, i.e. there exists an isometric and bicontinuous isomorphism $\varpi : R^-(\omega) + R^+(\omega) \to C(\mathfrak{h})$ such that

\[
\begin{array}{c}
\theta \\
\downarrow \\
R^-(\omega) + R^+(\omega)
\end{array} \xrightarrow{\varpi} \begin{array}{c}
\iota \\
\downarrow \\
C(\mathfrak{h})
\end{array}
\]

Proof. We are going to apply the uniqueness part of [KWW1, Proposition 4.4]. Let $f_1, \ldots, f_\Delta : L \to C$ be the linear functionals

$$f_k(F) := \int_{\{s_-, \sigma\} \cup (\sigma, s_+)} w^* k \cdot H F, \quad F \in L.$$ 

Since no nontrivial linear combination of $w_0, \ldots, w_{\Delta - 1}$ belongs to $L^2(H)$, no nontrivial linear combination of $f_1, \ldots, f_\Delta$ is continuous with respect to the topology induced by the $L^2(H)$-inner product given on $L$. However, we have

\[
\begin{array}{c}
\theta \\
\downarrow \\
R^-(\omega) + R^+(\omega)
\end{array} \xrightarrow{f_k} \begin{array}{c}
\iota \\
\downarrow \\
C(\mathfrak{h})
\end{array} \xrightarrow{\varpi} \begin{array}{c}
\theta \\
\downarrow \\
R^-(\omega) + R^+(\omega)
\end{array}
\]

Thereby, the left side of the diagram commutes by (2.27) which is applicable since each element of ran $\theta$ has compact support in $I$. The right side of the diagram follows from (IV.4.21). Moreover, by (2.25), we have

$$\dim(R^-(\omega) + R^+(\omega))^\circ = \Delta = \dim C(\mathfrak{h})^\circ.$$ 

Hence we may apply [KWW1, Proposition 4.4], and obtain an isomorphism $\varpi$ with the desired properties. \hfill \square

2.33 Definition. Define an extension of $\varpi$ to a map from $R(\omega(s_+))$ to $\mathcal{P}(\mathfrak{h})$ by linearity and

$$p(\omega)_k \mapsto p(\mathfrak{h})_k, \quad k = 0, \ldots, \Delta - 1, \quad \delta(\omega)_k \mapsto \delta(\mathfrak{h})_k, \quad k = \Delta, \ldots, \Delta + \delta - 1.$$ 

We will denote this map again by $\varpi$. //
First of all note that, due to Corollary 2.29, the map \( \varpi \) is well-defined and bijective.

**2.34 Lemma.** The map \( \varpi : \mathfrak{R}(\omega(s_+)) \rightarrow \mathcal{P}(\mathfrak{h}) \) is an isometric isomorphism.

**Proof.** The restriction \( \varpi|_{\mathfrak{R}^-(\omega)} \) is isometric by Lemma 2.32, the restriction \( \varpi|_{\text{span}(P_0, \ldots, P_{\Delta-1}, \delta_\Delta, \ldots, \delta_{\Delta+\omega-1})} \) by the definition of \( b_j(\mathfrak{h}) \), \( d_j(\mathfrak{h}) \) and Lemma 2.27. Moreover, \( \mathfrak{R}^-(\omega) + \mathfrak{R}^+(\omega) \perp X^\delta(\mathfrak{h}) \) and \( \mathcal{C}(\mathfrak{h}) \perp X^\delta(\mathfrak{h}) \).

It remains to show that \( [\varpi F, p(h)] = [F, p(\omega)k], \) \( F \in \mathfrak{R}^-(\omega) + \mathfrak{R}^+(\omega), \) \( k = 0, \ldots, \Delta - 1 \). However, the definition of \( \varpi \) and (2.35) implies

\[
[\theta f, p(\omega)k] = f_k(f) = [\varpi(\theta f), p(\omega)k], \quad f \in \mathcal{L}.
\]

By continuity, the desired relation follows for all \( F \in \mathfrak{R}^-(\omega) + \mathfrak{R}^+(\omega) \). \( \square \)

**2.35 Corollary.** We have

\[
\varpi \delta(\omega)k = \delta(h)_k, \quad k = 0, \ldots, \Delta - 1,
\]

\[
\varpi p(\omega)k = p(h)_k, \quad k \geq \Delta.
\]

**Proof.** From the definition of \( \varpi \) it is clear that

\[
\varpi(\mathfrak{R}^-(\omega) + \mathfrak{R}^+(\omega) + X^\delta(\omega)) \subseteq \mathcal{C}(\mathfrak{h}) + X^\delta(\mathfrak{h})\,.
\]

Passing to orthogonal complements, gives

\[
\varpi(X_\delta(\omega)) = X_\delta(\mathfrak{h})\.
\]

However, if \( k, l \in \{0, \ldots, \Delta\} \), we obtain from Lemma 2.27

\[
[\varpi \delta(\omega)k, p(h)] = [\varpi \delta(\omega)k, \varpi p(\omega)] = [\delta(\omega)k, p(\omega)] = [\delta(h)_k, p(h)].
\]

It follows that \( \varpi \delta(\omega)_k = \delta(h)_k \).

For \( k \geq \Delta \) we have \( m_k \in L^2(\mathfrak{h}) \). Hence, by Proposition 2.28 and Proposition 2.21 we have \( p(\omega)_k \in X_\delta(\omega)_\perp \). For each \( g \in \mathcal{C}(\mathfrak{h}) \) we have \( G := \varpi^{-1}g \in \mathfrak{R}^-(\omega) + \mathfrak{R}^+(\omega) \) and, hence, by (2.27)

\[
[\varpi p(\omega)_k, g] = [p(\omega)_k, G] = \int_I (\psi(\omega)G)^* H m_k = \int_I (\psi(h)g)^* H m_k = [p(h), g] \,.
\]

Moreover, by Lemma 2.27,

\[
[\varpi p(\omega)_k, \delta(h)] = [p(\omega)_k, \delta(\omega)] = [p(h)_k, \delta(h)] ,
\]

\[
[\varpi p(\omega)_k, p(h)] = [p(\omega)_k, p(\omega)] = [p(h)_k, p(h)] .
\]

Together, these relations imply \( \varpi p(\omega)_k = p(h)_k \). \( \square \)

**2.36 Proposition.** The pair \( (\varpi; \text{id}) \) is an isomorphism between the boundary triplets \( \mathfrak{B}(\omega(s_+)) \) and \( \mathfrak{B}(\mathfrak{h}) \).

In the proof of this result, we will employ the following lemma.
2.37 Lemma. Denote

\[ \mathcal{F}(H) := L^2(H) + \text{span}\{w_0, \ldots, w_{\Delta-1}\}, \]

\[ T_{\mathcal{F}(H)} := \{(f;g) \in \mathcal{F}(H)^2 : \exists \tilde{f} \text{ abs.cont., } [\tilde{f}]_H = f, \tilde{f}' = JHg\} \]

Then \[ T_{\mathcal{F}(H)} = T_{\text{max}}(H) + \text{span}\{ (w_k; w_{k-1}) : k = 0, \ldots, \Delta \} \]

where we have set \( w_{-1} := 0 \).

Proof. The inclusion ‘\( \supseteq \)’ is obvious. Assume that \( (f;g) \in T_{\mathcal{F}(H)} \), and write

\[ f = f_1 + \sum_{j=0}^{\Delta-1} \eta_j w_j, \quad g = g_1 + \sum_{j=0}^{\Delta-1} \mu_j w_j. \]

Consider the element \( (\hat{f}; \hat{g}) := (f_1 - \mu_{\Delta-1} w_{\Delta}; g_1 + \sum_{j=0}^{\Delta-2} (\mu_j - \eta_{j+1}) w_j) \), then

\[ (\hat{f}; \hat{g}) = (f;g) - \sum_{j=0}^{\Delta-1} \eta_j (w_j; w_{j-1}) - \mu_{\Delta-1} (w_{\Delta}; w_{\Delta-1}) \in T_{\mathcal{F}(H)} \]

and \( \hat{f} \in L^2(H) \). Let \( B_{\pm} \) denote the operators from Section 4.1 of [KW/IV], see also (IV.2.22). It follows that for some constants \( c_1^-, c_1^+, c_2 \in \mathbb{C} \),

\[ \hat{f}|_{(s, s, s)} = B_- g_1|_{(s, s, s)} + \begin{pmatrix} c_1^- \\ 0 \end{pmatrix} + c_2 w_0|_{(s, s, s)} + \sum_{j=0}^{\Delta-2} (\mu_j - \eta_{j+1}) w_{j+1}|_{(s, s, s)}, \]

\[ \hat{f}|_{(s, s, s)} = B_+ g_1|_{(s, s, s)} + \begin{pmatrix} c_1^+ \\ 0 \end{pmatrix} + c_2 w_0|_{(s, s, s)} + \sum_{j=0}^{\Delta-2} (\mu_j - \eta_{j+1}) w_{j+1}|_{(s, s, s)}. \]

As \( w_0, \ldots, w_{\Delta-1} \) are linearly independent modulo \( L^2(H) \), we obtain \( c_2 = 0 \) and \( \mu_j - \eta_{j+1} = 0, j = 0, \ldots, \Delta - 2 \), in particular \( \hat{g} \in L^2(H) \). Thus \( (\hat{f}; \hat{g}) \in L^2(H) \), and the assertion follows. \( \square \)

Proof (of Proposition 2.36).

Step 1: We show

\[ ((\pi \otimes \varpi) \otimes \text{id}) \left( \Gamma(\omega(s_+)) \cap (\mathbb{X}_+^2 \times (\mathbb{C}^2)^2) \right) \subseteq \Gamma(\mathfrak{h}). \]  \hspace{1cm} (2.36)

To this end, let \( ((F;G); (a; b)) \in \Gamma(\omega(s_+)) \cap (\mathbb{X}_+^2 \times (\mathbb{C}^2)^2) \) be given. According to Proposition 2.23 \( (\psi F; \psi G) \in T_{\text{max}}(H) \). By Proposition IV.4.17 there exists an element \( (f; g) \in T(\mathfrak{h}) \cap \mathcal{C}(\mathfrak{h})^2 \), such that

\[ \psi(\mathfrak{h}) f = \psi(\omega) F, \quad \psi(\mathfrak{h}) g = \psi(\omega) G. \]

Let \( \bar{f} := \pi_1 \Psi^{ac}(f; g) \), then \( \bar{f} \) is a locally absolutely continuous representant of \( \psi f \) with \( \bar{f}' = JH(\psi g) \). By Corollary 2.24, we have \( ((F;G); (\bar{f}(s_-), \bar{f}(s_+))) \in \Gamma(\omega(s_+)) \), and Corollary 2.25 implies that

\[ a = \bar{f}(s_-) \quad \text{and} \quad \begin{cases} b = \bar{f}(s_+) & , (\sigma, s_+) \text{ not indivisible} \\ \pi_2 b = \pi_2 \bar{f}(s_+) & , (\sigma, s_+) \text{ indivisible} \end{cases} \]
By adding an appropriate linear combination of $(\delta(h)_k; \delta(h)_{k+1})$, $k = 0, \ldots, \Delta - 2$, and a multiple of $(0; \delta_0(h))$ if $(\sigma, s_+)$ is indivisible (remember Lemma 2.15 and Lemma 2.16), we see that the elements $f$ and $g$ can be chosen such that

$$\pi_1 b = \pi_1 \bar{f}(s_+)$$ also if $(\sigma, s_+)$ is indivisible

$$[f, p(h)_k] = [\varpi F, p(h)_k], \quad k = 0, \ldots, \Delta - 2.$$

**Case $\delta = 0$**: We use the Green’s identity with the pairs $(\omega_\Delta; p(h)_{\Delta-1} + d_{\Delta-1}\delta(h)_0)$ and $(p(\omega)_{\Delta}; p(\omega)_{\Delta-1} + d_{\Delta-1}\delta(\omega)_0)$ to obtain

$$[f, p(h)_{\Delta-1}] = [f, p(h)_{\Delta-1} + d_{\Delta-1}\delta(h)_0] = [g, \omega_\Delta] - \omega_\Delta(s_-)^* J \bar{f}(s_-) + \omega_\Delta(s_+)^* J \bar{f}(s_+) =$$

$$= \int_j (\omega_\Delta)^* J(\psi g) - \omega_\Delta(s_-)^* Ja + \omega_\Delta(s_+)^* Jb =$$

$$= [G, p(\omega)_{\Delta-1}] - \omega_\Delta(s_-)^* Ja + \omega_\Delta(s_+)^* Jb =$$

$$= [F, p(\omega)_{\Delta-1} + d_{\Delta-1}\delta(\omega)_0] = [F, p(\omega)_{\Delta-1}] = [\varpi F, p(h)_{\Delta-1}]$$

Thus $f$ and $\varpi F$ are elements of span$(\delta(h)_0, \ldots, \delta(h)_{\Delta-1})$ whose images under $\psi(h)$ and whose inner products with $p(h)_k$, $k = 0, \ldots, \Delta - 1$, coincide. This implies their equality.

Using the Green’s identity with the pairs $(p_0; 0)$ and $(p_k; p_{k-1} + d_{k-1}\delta_0)$, $k = 1, \ldots, \Delta - 1$, yields $[g, p(h)_k] = [\varpi G, p(h)_k], k = 0, \ldots, \Delta - 1$. Thus also $\varpi G = g$. Altogether, we obtain

$$((\varpi F; \varpi G); (a; b)) \in \Gamma(h).$$

**Case $\delta > 0$**: By adding an appropriate linear combination of $(\delta(h)_k; \delta(h)_{k+1})$, $k = \Delta - 1, \ldots, \Delta + \delta - 2$, we see that the elements $f$ and $g$ can be choosen in $C(h) + X^\delta$ such that

$$[f, p(h)_{\Delta-1}] = [\varpi F, p(h)_{\Delta-1}],$$

$$[f, \delta(h)_k] = [\varpi F, \delta(h)_k], \quad k = \Delta + 1, \ldots, \Delta + \delta - 1.$$ Applying the Green’s identity with the pair $(\delta(\cdot)_{\Delta-1}; \delta(\cdot)_{\Delta})$ gives

$$[f, \delta(h)_{\Delta}] = [g, \delta(h)_{\Delta-1}] = 0 = [G, \delta(\omega)_{\Delta-1}] =$$

$$= [F, \delta(\omega)_{\Delta}] = [\varpi F, \delta(h)_{\Delta}].$$

Again it follows that $\varpi F = f$. In the same way as above, we obtain that also $\varpi G = g$.

**Step 2: Finish of proof.** Let $((F; G); (a; b)) \in \Gamma(\omega(s_+))$ be given. Applying $\Psi$ and using Lemma 2.37, we find constants $\lambda_j \in C$ such that

$$((\bar{F}; \bar{G}); (\hat{a}; \hat{b})) := ((F; G); (a; b)) +$$

$$+ \sum_{k=0}^{\Delta-1} \lambda_k \{(p(\omega)_k; p(\omega)_{k-1} + d_{k-1}\delta(\omega)_0); (\omega_k(s_-); \omega_k(s_+))\} +$$

...
+ \lambda \left( (p(\omega) \Delta + b(\omega); p(\omega) \Delta^{-1} + d \Delta^{-1} \delta (\omega) \theta; (\mathbf{w} \Delta (s-); \mathbf{w} \Delta (s_+))) \right)

belongs to \Gamma(\omega(s_+)) \cap ((X^k_\delta)^2 \times (\mathbb{C}^2)^2). By Step 1 of this proof, it follows that \(((\pi \hat{F}; \pi \hat{G}); (a; b)) \in \Gamma(\mathfrak{h})\). By the definition of \pi and Corollary 2.35 it follows that also \(((\hat{F}; \hat{G}); (a; b)) \in \Gamma(\mathfrak{h}).

An application of Lemma V.2.12 completes the proof.

As we have already remarked, Proposition 2.36 is sufficient to obtain \omega_h = \omega. Hence, we have realized each finite maximal chain \omega subject to the conditions (2.1)–(2.3) as \omega_h with an elementary indefinite Hamiltonian of kind \text{(A)}. \qed

3 Completion of the converse construction

In the previous section we have treated chains which correspond to elementary indefinite Hamiltonians of kind (A). In the present short section we will settle the other cases, namely chains which correspond to positive definite or elementary indefinite Hamiltonians of kind (B) or (C). Moreover, we provide the necessary splitting-and-pasting argument to glue together all these ‘piecewise’ results.

We divide this section into two subsections.

a. Let \omega be a positive definite maximal or finite maximal chain, or let \omega be a finite maximal chain defined on

\[ I = [s_-, \sigma) \cup (\sigma, s_+] , \]

such that

\[(s_-, \sigma), (\sigma, s_+) \text{ indivisible of type } \frac{\pi}{2}. \tag{3.1} \]

We construct a Hamiltonian \mathfrak{h}, which is positive definite or elementary indefinite of kind (B) and (C), such that \omega = \omega_h. This is done by references to the classical theory and explicit computations, respectively.

b. The splitting-and-pasting technique is applied, and the proof of Theorem 1.5 is completed.

a. Positive definite and elementary indefinite kind (B), (C).

Let \omega \in \mathcal{M}_0 \cup \mathcal{M}_0. By Proposition V.3.23, \text{(ii)}, we have \omega = \omega_H for some positive definite Hamiltonian H. Define \mathfrak{h} to be positive definite and given by this Hamiltonian function H. By Remark V.5.7, \text{(iii)}, we have \omega_h = \omega_H.

Let \omega : [s_-, \sigma) \cup (\sigma, s_+] \to \mathcal{M}_< \infty be a finite maximal chain, and assume that \(s_-, \sigma) and \(\sigma, s_+) are indivisible of type \frac{\pi}{2}. Then we have, cf. Proposition V.3.10,

\[ 0 = \lim_{t \to \sigma} \omega(t) \ast 0 = \lim_{t \to \sigma} \omega(t) \ast 0 = \omega(s_+) \ast 0 . \]

It follows from [KWW2, Lemma 5.3] that \omega(s_+) is a polynomial matrix of the form

\[ \omega(s_+) = \begin{pmatrix} 1 \\ -p(z) \\ 0 \\ 1 \end{pmatrix} \text{ where } p \in \mathbb{R}[z], \ p(0) = 0. \tag{3.2} \]

The case that \(p(z) = a_1 z\) with \(a_1 \geq 0\) is excluded since, by the existence of a singularity, certainly \(\text{ind}_- \omega(s_+) > 0\). The needed data to compose a general Hamiltonian \mathfrak{h} with the property that \omega_h = \omega can be immediately read off from the formula given in Proposition V.4.31:
Case $\deg p = 1$: $\mathfrak{h}$ is elementary indefinite of kind (C) and given by the data

\begin{align*}
  s_-, \sigma, s_+,
  \quad H(t) := b(t)\xi \xi T \xi T \quad \text{where} \quad b(t) := (t \circ \omega)'(t),
  \quad \dot{\sigma} := 0, \quad b_1 := 0, \quad d_0 := a_1, d_1 := 0.
\end{align*}

Case $\deg p > 1$: $\mathfrak{h}$ is elementary indefinite of kind (B) and given by the data

\begin{align*}
  s_-, \sigma, s_+,
  \quad H(t) := b(t)\xi \xi T \xi T \quad \text{where} \quad b(t) := (t \circ \omega)'(t),
  \quad \dot{\sigma} := \deg p - 2, \quad b_1 := -a_{\deg p}, \ldots, b_{\deg p + 1} := -a_2, \quad d_0 := a_1, d_1 := 0.
\end{align*}

We see from Proposition V.4.31 and Corollary V.5.6 that $\omega_\mathfrak{h} = \omega$.

b. Splitting-and-pasting.

Let $\omega$ be a maximal or finite maximal chain defined on

$$I = \bigcup_{j=0}^n (\sigma_j, \sigma_{j+1}) \cup \{\sigma_0, \sigma_{n+1}\},$$

and assume that $t \circ \omega$ and $(t \circ \omega|_J)^{-1}$ are locally absolutely continuous for any maximal interval contained in the domain of $\omega$. Choose a set $F = \{r_0, \ldots, r_{m+1}\}$ which is suitable for splitting $\omega$, cf. Lemma V.3.19, and which has the following additional properties:

(i) $F$ intersects each component of $I$.

(ii) $F$ contains all endpoints of indivisible intervals with infinite length.

(iii) If $\omega$ does not end with an indivisible interval towards $\sigma_{i+1}$, then $\max(F \cap [\sigma_0, \sigma_{i+1}])$ is not left endpoint of an indivisible interval of type $\phi_{i+1}$.

(iv) If $\omega$ does not start with an indivisible interval upwards from $\sigma_{i+1}$, then $\min(F \cap (\sigma_{i+1}, \sigma_{n+1}])$ is not right endpoint of an indivisible interval of type $\phi_{i+1}$.

The existence of a set $F$ with these properties is obvious.

Define chains $\omega_i$, $i = 0, \ldots, m$, as

$$\omega_i := \begin{cases} 
  \omega_{r_i \rightarrow r_{i+1}}, & (r_i, r_{i+1}) \text{ contains no singularity} \\
  \cup_{\phi(\sigma_j)} \omega_{r_i \rightarrow r_{i+1}}, & \sigma_j \in (r_i, r_{i+1}), \ (r_i, \sigma_j) \text{ not indivisible} \\
  \text{rev} \cup_{\phi(\sigma_j)} \omega_{r_i \rightarrow r_{i+1}}, & \sigma_j \in (r_i, r_{i+1}), \ (r_i, \sigma_j) \text{ indivisible}
\end{cases}$$

From the results we have established so far, we obtain the following corollary.

3.1 Corollary. For each chain $\omega_i$ there exists a general Hamiltonian $\mathfrak{h}_i$ such that $\omega_{\mathfrak{h}_i} = \omega_i$. 

31
Proof. If \( (r_i, r_{i+1}) \) contains no singularity, then \( \omega_{r_i, r_{i+1}} \in M_0' \cup M_0 \). Hence, by the first paragraph of previous subsection, there exists a positive definite Hamiltonian \( h_i \) with \( \omega_{h_i} = \omega_i \).

Assume next that \( \sigma_j \in (r_i, r_{i+1}) \) and that \( (r_i, \sigma_j) \) is not indivisible. By property \((ii)\) of the splitting set \( F \), this implies that sup\( (I_{reg} \cap [\sigma_0, \sigma_j]) = \sigma_j \). Similarly, if \( (\sigma_j, r_{j+1}) \) is not indivisible, then inf\( (I_{reg} \cap [\sigma_j, \sigma_{n+1}]) = \sigma_j \). Clearly, since we have rotated \( \omega_{r_i, r_{i+1}} \), by the angle \( \phi(\omega, \sigma_j) = \phi(\omega_{r_i, r_{i+1}}, \sigma_j) \), we have \( \phi(\omega_i, \sigma_j) = 0 \). Finally, property \((iii)\) of \( F \) ensures that \( \omega_i \) does not start with an indivisible interval of type \( 0 \). Altogether, \( \omega_i \) is a finite maximal chain defined on \( [r_i, \sigma_j] \cup (\sigma_j, r_{i+1}] \) which satisfies \((2.1)\)–\((2.3)\). Hence, by what we have proved in the previous section, there exists an elementary indefinite Hamiltonian \( h_i \) of kind \((A)\), such that \( \omega_{h_i} = \omega_i \).

With the above notation, define general Hamiltonians \( h_i \) as

\[
\tilde{h}_i := \begin{cases} h_i, & (r_i, r_{i+1}) \text{ contains no singularity} \\ \cup_{\sigma_j \in (r_i, r_{i+1}), (r_i, \sigma_j) \text{ not indivisible}} \cup_{\sigma_j \in (r_i, r_{i+1})} \text{ rev} h_i, & (r_i, \sigma_j) \text{ indivisible} \end{cases}
\]

and set

\[
h_i := \bigcup_{i=0}^{m} \tilde{h}_i.
\]

By Lemma V.5.14 and Lemma V.5.15, we have \( \omega_{h_i} = \omega_{r_i, r_{i+1}}, \ i = 0, \ldots, m \). Lemma V.5.18 implies that \( \omega_{h_i} = \omega \).

In order to complete the proof of Theorem 1.5, it remains to note:

**3.2 Lemma.** Let \( \omega \) be a (finite) maximal chain. Then there exists a reparameterization \( \tilde{\omega} \) of \( \omega \), such that \( t \circ \tilde{\omega} \) and \( (t \circ \tilde{\omega})^{-1} \) are locally absolutely continuous for any maximal interval contained in the domain of \( \omega \).

Proof. We know from Proposition V.3.2, that if \( \omega \) is a maximal chain defined on \( \bigcup_{j=0}^{n=0} (\sigma_j, \sigma_{j+1}) \) and if we consider \( \omega \) to be continuously continued to \( [\sigma_0, \sigma_1] \cup \bigcup_{j=1}^{n} (\sigma_j, \sigma_{j+1}) \), then \( t \circ \omega \) maps \( [\sigma_0, \sigma_1] \cup \bigcup_{j=1}^{n} (\sigma_j, \sigma_{j+1}) \) bijectively onto \( [0, +\infty) \) and \( (\sigma_j, \sigma_{j+1}) \) bijectively onto \( (-\infty, +\infty) \) for \( i = 1, \ldots, n \). Hence, \( \beta : t \in (\sigma_i, \sigma_{i+1}) \mapsto i \cdot \pi + \)

\[32\]
arctan \circ \omega(t) sets up a bijection from \bigcup_{i=0}^{n}(\sigma_i, \sigma_{i+1}) onto \((0, \frac{\pi}{2}) \cup \bigcup_{i=1}^{n}(i\pi - \frac{\pi}{2}, i\pi + \frac{\pi}{2})\), and \(\tilde{\omega} := \omega \circ \beta^{-1}\) is a reparametrization of \(\omega\) such that
\[ t \circ \tilde{\omega}(s) = t \circ \omega \circ \beta^{-1}(s) = \tan(i \cdot \pi + \arctan(t \circ \omega \circ \beta^{-1}(s))) = \tan(s). \]

Therefore, \(t \circ \tilde{\omega}\) is locally absolutely continuous. The inverse of its restriction to a maximal interval contained in \((0, \frac{\pi}{2}) \cup \bigcup_{i=1}^{n}(i\pi - \frac{\pi}{2}, i\pi + \frac{\pi}{2})\) is \(\arctan\), and hence also locally absolutely continuous.

If \(\omega\) is a finite maximal chains, one can argue in the same way. \(\square\)

3.3 Remark. As we know from Corollary V.5.6 together with Remark V.3.22, the continuity condition on \(\omega\) is necessary in order that \(\omega = \omega_h\) for some general Hamiltonian \(h\).

4 Bijectivity modulo reparameterization

In this section we show that a general Hamiltonian \(h\) can, up to reparameterization, be fully recovered from the chain \(\omega_h\). This fact could, with considerable effort, be established by tracing back the constructions carried out in the previous section and in [KW/V]. However, we prefer to use a different, more elementary, method. This method will enable us to recover \(h\) from \(\omega_h\) as explicit as possible, and also, conversely, provide an algorithm how to compute \(\omega_h\) for a given general Hamiltonian \(h\).

In order to show that \(h\) is determined by \(\omega_h\), we will see with the usual splitting-and-pasting procedure that it is enough to consider elementary indefinite Hamiltonians. The cases of kind (B) and kind (C) are then trivial; the core of the problem is – once more – to understand elementary indefinite Hamiltonians of kind (A). Matching this situation, the present section is divided into two subsections:

a. We immediately attack the hard part of the problem, and show how to recover an elementary indefinite Hamiltonian \(h\) of kind (A) from the chain \(\omega_h\). Our approach is based on a set of recurrence relations which relate the parameters of \(h\) with \(\omega(B(h))\), and which can easily be solved for the parameters of \(h\).

b. Here we carry out the ‘usual’ techniques in order to finish the proof of Theorem 1.6.

a. The recurrence relations.

We introduce a set of recurrence relations which involves some parameters. Let
\[ \Delta \in \mathbb{N}, \, \bar{\sigma} \in \mathbb{N}_0, \, \epsilon \in \{0,1\}, \]
\[ U_k, U_k^-, V_k, X_k \in \mathbb{C}, \, k \in \mathbb{N}, \]
\[ Y_k \in \mathbb{C}, \, k \in \{1, \ldots, \bar{\sigma}\}, \]
be given. For notational convenience, set \(Y_k := 0\) for \(k \in \mathbb{Z} \setminus \{1, \ldots, \bar{\sigma}\}\), and \(\alpha_k := 0\) for \(k \in \mathbb{Z} \setminus \mathbb{N}_0\).
Sequences \((\alpha_n)_{n \in \mathbb{N}_0}, (\beta_n)_{n \in \mathbb{N}_0},\) and \((\gamma_{n,t})_{n \in \mathbb{N}_0, l \in \mathbb{N}_0},\) can be defined from given initial values \(\alpha_0, \beta_0,\) and \(\gamma_{0,t}, l \in \mathbb{N}_0,\) by the recurrence relations \((n \in \mathbb{N})\)

\[
\begin{align*}
\alpha_n &= -\sum_{j=0}^{n-1} \left[ \alpha_j U_{n-j} + \beta_j V_{n-j} \right], \\
\beta_n &= \gamma_{n-1,0} - \sum_{j=0}^{n-1} \left[ \alpha_j X_{n-j} + \beta_j U_{n-j} \right] + \varepsilon U_n, \\
\gamma_{n,t} &= \gamma_{n-1,t+1} + \alpha_{n-\Delta Y_{\Delta+t-1}}, l \in \mathbb{N}_0.
\end{align*}
\]

(4.1)

Our interest in these recurrence relations origins in the following result. We will write \(a_2\) for the second component of a vector \(a \in \mathbb{C}^n\).

**4.1 Proposition.** Let \(\mathfrak{h}\) be an elementary indefinite Hamiltonian of kind (A) which is defined on \(I = [\sigma_-, \sigma+) \cup \{\sigma, s_+\}\) and given by the data \(H, d_0, \ldots, d_{2\Delta-1}, \delta, b_1, \ldots, b_{\bar{\delta}+1}\). Assume that \(b_{\bar{\delta}+1} = 0\). Write \(\omega(\mathfrak{B}(\mathfrak{h})) := (w_{ij}(z))_{i,j=1}^{2}\) and specify the parameters

\[
\Delta := \max \{\Delta(H|_{[\sigma_-, \sigma]}), \Delta(H|_{(\sigma, s_+)})\},
\]

\[
U_k := w_k(s_+), U^{-}_{k} := w_k(s_-), V_k := (B^k X \begin{pmatrix} 1 \\ 0 \end{pmatrix})(s_+), \quad k \in \mathbb{N},
\]

\[
Y_k := b_k, k \in \{1, \ldots, \bar{\delta}\}, \quad X_k := \begin{cases} d_{k-1} & k \in \{1, \ldots, 2\Delta-1\} \\ (w_{k-\Delta-1}, 0) & k \geq 2\Delta \end{cases}
\]

Let \((\alpha_n)_{n \in \mathbb{N}_0}, (\beta_n)_{n \in \mathbb{N}_0},\) and \((\gamma_{n,t})_{n \in \mathbb{N}_0, l \in \mathbb{N}_0},\) be the sequences defined by (4.1) using the specified parameters, \(\varepsilon := 0,\) and the initial values

\[
\alpha_0 := 1, \quad \beta_0 := 0, \quad \gamma_{0,t} := 0, \quad l \in \mathbb{N}_0.
\]

Let \((\hat{\alpha}_n)_{n \in \mathbb{N}_0}, (\hat{\beta}_n)_{n \in \mathbb{N}_0}, (\hat{\gamma}_{n,t})_{n \in \mathbb{N}_0, l \in \mathbb{N}_0}\) be defined from (4.1) using the same specified parameters, but \(\varepsilon := 1,\) and the initial values

\[
\hat{\alpha}_0 := 0, \quad \hat{\beta}_0 := 1, \quad \hat{\gamma}_{0,t} := 0, \quad l \in \mathbb{N}_0.
\]

Then

\[
\begin{align*}
\frac{w_{12}(z)}{w_{22}(z)} &= \sum_{n=0}^{\infty} \beta_n z^n, \quad \frac{w_{12}(z)}{w_{22}(z)} - \sum_{n=0}^{\infty} (\alpha_n + \sum_{j=1}^{n-1} \hat{\alpha}_j U_{n-j}^{-} + U_{n}^{-}) z^n, \\
\frac{1}{w_{22}(z)} &= \sum_{n=0}^{\infty} \hat{\beta}_n z^n = \sum_{n=0}^{\infty} (\alpha_n + \sum_{j=0}^{n-1} \alpha_j U_{n-j}^{-}) z^n
\end{align*}
\]

(4.2)

The proof of this result is based on the following observation. For notational convenience set

\[p_j := w_j, j \geq \Delta, b_j := 0, j \notin \{1, \ldots, \bar{\delta} + 1\}.\]

**4.2 Lemma.** Let \(\mathfrak{h}\) be an elementary indefinite Hamiltonian of kind (A) with \(b_{\bar{\delta}+1} = 0\). Let \(x \in \mathcal{P}(\mathfrak{h})\) be of the form

\[
x = \sum_{j=0}^{n-1} \left[ \alpha_j p_{n-1-j} + \beta_j B^{n-1-j} \chi_{+} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] + \beta_j B^{n-1-j} \chi_{-} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{l=0}^{\Delta+\bar{\delta}-1} \gamma_{n-1,l} d_l,
\]

34
with some \( n \in \mathbb{N} \) and numbers \( \alpha_j, \beta_j, \beta'_j, j = 0, \ldots, n - 1, \) and \( \gamma_{n-l} \in \mathbb{C}, l = 0, \ldots, \Delta + \bar{\alpha} - 1. \) Moreover, let \( A := A(0, \frac{\pi}{2}) \) be the operator defined as in Corollary IV.5.6, and set \( \alpha_j := 0 \) for \( j < 0, \) and \( \gamma_{n, \Delta + \bar{\alpha}} := 0. \) Then

\[
A^{-1}x = \sum_{j=0}^{n} \left[ \alpha_j p_{n-j} + \beta_j B^{n-j} \chi_+ \left( \frac{1}{0} \right) + \beta'_j B^{n-j} \chi_- \left( \frac{1}{0} \right) \right] + \sum_{l=0}^{\Delta + \bar{\alpha} - 1} \gamma_{n,l} \delta_l,
\]

with

\[
\alpha_n := - \sum_{j=0}^{n-1} \left[ \alpha_j w_{n-j} (s_+)_2 + \beta_j (B^{n-j} \chi_+ \left( \frac{1}{0} \right)) (s_+)_2 \right],
\]
\[
\beta_n := \gamma_{n-1,0} - \sum_{j=0}^{n-1} \left[ \alpha_j d_{n-j-1} + \beta_j w_{n-j} (s_+)_2 - \beta'_j w_{n-j} (s_-)_2 \right], \quad (4.3)
\]
\[
\beta'_n := 0,
\]
\[
\gamma_{n,l} := \gamma_{n-1,l+1} + \alpha_{n-\Delta} b_{\Delta + \bar{\alpha} - l}, \quad l = 0, \ldots, \Delta + \bar{\alpha} - 1.
\]

Note here that, by Proposition IV.5.9, we have \( 0 \in \rho(A) \).

**Proof.** For any choice of numbers \( \alpha, \beta \) the element

\[
(\hat{y}; \hat{x}) := \sum_{j=0}^{n-1} \alpha_j (p_{n-j}; p_{n-j-1} + d_{n-j-1} \delta_0) + \alpha_{n-\Delta} (p_\Delta + b; p_{\Delta-1} + d_{\Delta-1} \delta_0) + \sum_{j=0}^{n-1} \beta_j (B^{n-j} \chi_+ \left( \frac{1}{0} \right); B^{n-j-1} \chi_+ \left( \frac{1}{0} \right) + w_{n-j} (s_+)_2 \delta_0) + \sum_{j=0}^{n-1} \beta'_j (B^{n-j} \chi_- \left( \frac{1}{0} \right); B^{n-j-1} \chi_- \left( \frac{1}{0} \right) - w_{n-j} (s_-)_2 \delta_0) + \sum_{l=1}^{\Delta + \bar{\alpha} - 1} \gamma_{n-1,l} (\delta_l; \delta_l) + \alpha (p_0; 0) + \beta \chi_+ \left( \frac{1}{0} \delta_0 \right)
\]

belongs to \( T(h) \) and satisfies \( \pi_{1,1} \Gamma(h)(\hat{y}; \hat{x}) = 0. \) By choosing \( \beta = \beta_n \) as defined in (4.3), we achieve that \( \hat{x} = x. \) By taking \( \alpha = \alpha_n, \) we achieve that \( \pi_{1,1} \Gamma(h)(\hat{y}; \hat{x}) = 0. \) Thus we have found an element in \( A(0, \frac{\pi}{2}) \) whose second component is equal to \( x, \) and conclude that \( A^{-1}x = \hat{y}. \) The asserted formulas for \( A^{-1}x \) and for \( \gamma_{n,l} \) follow by inspection of \( \hat{y}. \)

**Proof (of Proposition 4.1).** Let \( \gamma(z) \) and \( \varphi(z) \) be defined by

\[
\gamma(z) := (I + z(A - z)^{-1}) \left( \frac{1}{0} \right), \quad \varphi(z) := (I + z(A - z)^{-1}) p_0.
\]

Then, by the construction of \( \omega(\mathfrak{b}(h)), \) cf. (V.4.19), (V.4.20), and (V.4.21), we have

\[
[\gamma(z), \gamma(0)] = \frac{1}{z} w_{12}(z), \quad [\varphi(z), \varphi(0)] = \frac{1}{z} w_{21}(z), \quad [\gamma(z), \varphi(0)] = \frac{1}{z} w_{22}(z),
\]

35
Moreover, for $|z| < \|A^{-1}\|$, we have $I + z(A - z)^{-1} = \sum_{n=0}^{\infty} A^{-n}z^n$, and hence

\[
[\gamma(z), \gamma(0)] = \sum_{n=0}^{\infty} [A^{-n} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}] z^n, \quad [\varphi(z), \varphi(0)] = \sum_{n=0}^{\infty} [A^{-n}p_0, p_0] z^n,
\]

\[
[\varphi(z), \gamma(0)] = \sum_{n=0}^{\infty} [A^{-n}p_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix}] z^n, \quad [\gamma(z), \varphi(0)] = \sum_{n=0}^{\infty} [A^{-n} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, p_0] z^n.
\]

An inductive application of Lemma 4.2 gives

\[
\begin{align*}
A^{-n}p_0 &= \sum_{j=0}^{n} \left[ \alpha_j p_{n-j} + \beta_j B^{n-j} \chi(0) \right] + \Delta + \delta t \\
A^{-n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \sum_{j=0}^{n} \left[ \alpha_j p_{n-j} + \beta_j B^{n-j} \chi(0) \right] + \Delta + \delta t \\
A^{-n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \sum_{j=0}^{n} \left[ \alpha_j p_{n-j} + \beta_j B^{n-j} \chi(0) \right] + \Delta + \delta t
\end{align*}
\]

We compute

\[
\begin{align*}
[A^{-n}p_0, p_0] &= \\
&= \sum_{j=0}^{n} \left[ \alpha_j d_{n-j} + \beta_j w_{n-j+1}(s+)_2 \right] - \gamma_{n,0} = -\beta_{n+1}, \\
[A^{-n}p_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix}] &= \\
&= \sum_{j=0}^{n} \left[ \alpha_j (w_{n-j+1}(s+)_2 - w_{n-j+1}(s-)_2) + \beta_j (B^{n-j+1} \chi(0)(s+)_2) \right] = \\
&= -\alpha_{n+1} - \sum_{j=0}^{n} \alpha_j w_{n-j+1}(s-)_2, \\
[A^{-n} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, p_0] &= \\
&= \sum_{j=0}^{n} \left[ \alpha_j d_{n-j} + \beta_j w_{n-j+1}(s+)_2 \right] - \gamma_{n,0} - w_{n+1}(s+_2) = -\beta_{n+1}, \\
[A^{-n} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}] &= \\
&= \sum_{j=0}^{n} \left[ \alpha_j (w_{n-j+1}(s+)_2 - w_{n-j+1}(s-)_2) + \beta_j (B^{n-j+1} \chi(0)(s+)_2) \right] - \\
&- w_{n+1}(s-)_2 = -\alpha_{n+1} - \sum_{j=1}^{n} \alpha_j w_{n-j+1}(s-)_2 - w_{n+1}(s-)_2.
\end{align*}
\]

Putting these formulas together, yields the desired assertion.
For the treatment of inverse problems it is interesting to see how the equations (4.2) can be solved for \( \alpha, \omega \) present, e.g. certainly \( \ddot{o} \).

It is a bit more tricky to recover the number \( \ddot{o} \) from the knowledge of \( \Delta, \ddot{o}, \dot{w}_k(s_k)2 \) and \( (B^k\chi_{(\frac{1}{2})})(s_+)_2 \). An explicitly computable result is obtained only if an a priori bound for \( \ddot{o} \) is known. However, such an a priori bound is always present, e.g.

\[
\omega \in [2 \max \omega(\phi)] + 2.
\]

**4.3 Proposition.** Let \((\alpha_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}}, (\gamma_{n,l})_{n \in \mathbb{N}}, l \in \mathbb{N}_0, \) be defined by the recurrence (4.1) using some parameters \( \Delta, \ddot{o}, U_k, U_k^-, V_k, X_k, Y_k \) with \( Y_1 \neq 0 \) in case \( \ddot{o} > 0 \), the parameter \( \epsilon = 0 \) and the initial values

\[
\alpha_0 := 1, \quad \beta_0 := 0, \quad \gamma_{0,l} = 0, \quad l \in \mathbb{N}_0.
\]

Assume that the values \( \Delta, U_k, V_k \) for all \( k \in \mathbb{N}_0 \), \( X_n \) for \( n > 2\Delta \), and the sequences \((\alpha_n)_{n \in \mathbb{N}}\) and \((\beta_n)_{n \in \mathbb{N}}\), are known.

Then the remaining parameters \( \ddot{o}, X_n, n = 1, \ldots, 2\Delta, Y_n, n = 1, \ldots, \ddot{o}, \) and the sequence \( \gamma_{n,0}, n \in \mathbb{N}_0 \), can be computed as follows:

\[
\gamma_{n,0} = 0, \quad n = 1, \ldots, 2\Delta - 1,
\]

\[
X_n = -\beta_n - \sum_{j=1}^{n-1} [\alpha_j X_{n-j} + \beta_j U_{n-j}], \quad n = 1, \ldots, 2\Delta,
\]

\[
\gamma_{n,0} = \beta_{n+1} + \sum_{j=0}^{n-1} [\alpha_j X_{n-1-j} + \beta_j U_{n+1-j}], \quad n \geq 2\Delta.
\]

Define a sequence \( (Y_n)_{n \geq 2\Delta} \) recursively by

\[
Y_n = \gamma_{n,0} - \sum_{j=1}^{n-2\Delta} \alpha_j Y_{n-j}, \quad n \geq 2\Delta.
\]

Then

\[
\ddot{o} = \begin{cases} 
0 & \text{if } Y_n = 0 \text{ for all } n \\
\max \{n : Y_n = 0\} + 1 - 2\Delta & \text{if } Y_n \neq 0 \text{ for some } n
\end{cases}
\]

and

\[
Y_n = Y_{\ddot{o} - n + 2\Delta}, \quad n = 1, \ldots, \ddot{o}.
\]

**Proof.** First we show inductively that

\[
\gamma_{n,l} = \sum_{j=0}^{n-\Delta} \alpha_j Y_{j+\ddot{o}-(n-\Delta)-(l-\Delta)}, \quad n, l \in \mathbb{N}_0.
\]

For \( n = 0 \) this is obvious since \( \Delta \geq 1 \) and \( \gamma_{0,l} = 0, l \geq 0 \). By the recursive definition (4.1) we have for \( n \geq 1, l \in \mathbb{N}_0, \)

\[
\gamma_{n,l} = \gamma_{n-1,l+1} + \alpha_{n-\Delta} Y_{\ddot{o} + \ddot{o} - l} = \sum_{j=0}^{n-1-\Delta} \alpha_j Y_{j+\ddot{o}-(n-1-\Delta)-(l+1-\Delta)} + \alpha_{n-\Delta} Y_{\ddot{o} + \ddot{o} - l} = \sum_{j=0}^{n-\Delta} \alpha_j Y_{j+\ddot{o}-(n-\Delta)-(l-\Delta)}.
\]

37
For the proof of the present assertion we will repeatedly use the special case \( l = 0 \):

\[ \gamma_{n,0} = \sum_{j=0}^{n-\Delta} \alpha_j Y_{j+\bar{o}-n+2\Delta}, \ n \in \mathbb{N}_0. \]

If \( n < 2\Delta \), then \( \bar{o} + (2\Delta - n) > \bar{o} \), and hence \( Y_{j+\bar{o}+2\Delta-n} = 0 \), \( j = 0, \ldots, n - \Delta \). This yields (4.4).

The relations (4.5) are now obvious from the second relation in (4.1), since \( \alpha_0 = 1 \) and \( \beta_0 = 0 \).

For \( n = 2\Delta \) we have \( \dot{Y}_{2\Delta} = \gamma_{2\Delta,0} = \dot{\bar{o}}. \) Let \( n > 2\Delta \), then inductively

\[ \dot{Y}_n = \gamma_{n,0} - \sum_{j=1}^{n-2\Delta} \alpha_j \dot{Y}_{n-j} = \dot{\bar{o}} - n + 2\Delta. \]

This shows that the relation (4.6) holds actually for all \( n \leq \bar{o} \). Since \( Y_1 \neq 0 \) in case \( \bar{o} > 0 \) and \( Y_n = 0 \) for all \( n < 1 \), the asserted formula for \( \bar{o} \) follows.

4.4 Corollary. Let \( \tilde{h} = (H, c, \bar{o}) \) and \( \tilde{h}' = (H', c', \bar{o}') \) be elementary indefinite Hamiltonians of kind (A), and assume that \( b_{i+1} = b'_{i+1} = 0 \). If \( \omega_{\bar{h}} = \omega_{\bar{h}'} \), then \( \tilde{h} = \tilde{h}' \).

Proof. The domains of the chains \( \omega_{\bar{h}} \) and \( \omega_{\bar{h}'} \) coincide, and thus \( I = I' \). Since \( \omega_{\bar{h}} \) and \( \omega_{\bar{h}'} \) are solutions of the respective differential equations (V.5.5), it follows that \( H = H' \). Thus also

\[ \Delta = \Delta', \ w_k = w'_k, \ B^k \chi_+ \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = (B')^k \chi_+ \left( \begin{array}{c} 1 \\ 0 \end{array} \right). \]

Since \( \omega(\mathcal{B}(\bar{h})) = \omega(\mathcal{B}(\bar{h}')) \), Proposition 4.1 combined with Proposition 4.3 yields that \( c = c' \) and \( \bar{o} = \bar{o}' \), i.e. \( \tilde{h} = \tilde{h}' \).

4.5 Lemma. Let \( \tilde{h} \) and \( \tilde{h}' \) be general Hamiltonians such that \( \omega_{\bar{h}} = \omega_{\bar{h}'} \), \( E = E' \), \( d_{i+1} = d'_{i+1} = 0 \) if \( H_{i-1} \) and \( H_i \) both end with an indivisible interval towards \( \sigma_i \), and \( b_{i+1} = b'_{i+1} = 0 \) for all other \( i \). Then \( \tilde{h} = \tilde{h}' \).

Proof. First note that equality of associated chains immediately implies \( n = n' \) and \( \sigma_i = \sigma'_i, \ i = 0, \ldots, n + 1 \). Hence the conditions \( d_{i+1} = d'_{i+1} = 0 \) and \( b_{i+1} = b'_{i+1} = 0 \), respectively, are meaningful. Also, since \( \omega_{\bar{h}} \) and \( \omega_{\bar{h}'} \) are solutions of the respective differential equations (V.5.5), it immediately follows that \( H = H' \).

Write \( E = E' = \{ s_0, \ldots, s_{N+1} \} \) with \( s_0 < \ldots < s_{N+1} \). Then, by Lemma V.5.4, we have

\[ \omega_{\bar{h}_{s_i\rightarrow s_{i+1}}} (t) = \omega_{\bar{h}}(s_i)^{-1} \omega_{\bar{h}}(t) = \omega_{\bar{h}'}(s_i)^{-1} \omega_{\bar{h}'}(t) = \omega_{\bar{h}_{s_i\rightarrow s_{i+1}}'} (t), \ t \in [s_i, s_{i+1}]. \]

If \( \theta_{s_i\rightarrow s_{i+1}} \) is positive definite, so is \( \theta'_{s_i\rightarrow s_{i+1}} \), and we have

\[ b_{s_i\rightarrow s_{i+1}} = H|_{(s_i, s_{i+1})} = H'|_{(s_i, s_{i+1})} = b'_{s_i\rightarrow s_{i+1}}. \]
Consider next the situation that \( h_{s_i \to s_{i+1}} \), and hence also \( h'_{s_i \to s_{i+1}} \), is indefinite, and let \( \sigma_j \) be the singularity contained in \( (s_i, s_{i+1}) \). Since \( H = H' \), we have \( \phi_j = \phi'_j \). Then \( \bigcirc_{\phi_i} h_{s_i \to s_{i+1}} \) and \( \bigcirc_{\phi_i} h'_{s_i \to s_{i+1}} \) are elementary indefinite, and are of the same kind. If they are of kind (B) or (C), we see from Proposition V.4.31 that \( \delta_j = \delta'_j \), \( b_{j\ell} = b'_{j\ell} \), \( \ell = 1, \ldots, \delta_j + 1 \), and \( d_{j0} = d'_{j0} \), i.e. \( \bigcirc_{\phi_i} h_{s_i \to s_{i+1}} = \bigcirc_{\phi_i} h'_{s_i \to s_{i+1}} \). If they are of kind (A), we refer to Corollary 4.4 to obtain

\[
\bigcirc_{\phi_i} h_{s_i \to s_{i+1}} = \bigcirc_{\phi_i} h'_{s_i \to s_{i+1}}.
\]

Putting together all of these cases yields \( h = h' \).

It is now easy to finish the proof of Theorem 1.6.

**Proof (of Theorem 1.6).** Let general Hamiltonians \( h = (H, \sigma, \omega) \) and \( h' = (H', \sigma', \omega') \) be given.

Assume first that \( h \leftrightarrow h' \), and let \( \alpha : I' \to I \) be the increasing bijection such that \( \alpha \) and \( \alpha^{-1} \) are absolutely continuous and

\[
H'(t') = (H \circ \alpha)(t') \cdot \frac{d\alpha}{dt}(t'), \quad \text{for a.e. } t' \in I'.
\]

We see that \( \alpha(I'_{\text{reg}}) = I_{\text{reg}} \). Let \( x' \in I' \) and set \( x := \alpha(x') \). Inspecting the definition of a reparameterization, it follows that \( h \leftrightarrow h' \) implies \( h_{\gamma_x} \leftrightarrow h'_{\gamma_{x'}} \).

Together with Remark V.3.39 and Proposition V.4.7 we get

\[
\omega_{h'}(x') = \omega(\mathcal{B}(h'_{\gamma_{x'}})) = \omega(\mathcal{B}(h_{\gamma_x})) = \omega_h(x),
\]

for all \( x' \in (I'_{\text{reg}} \cup \{\sigma'_0\}) \setminus I'_{\text{sing}} \), cf. Definition V.5.3. If \( h \), and hence \( h' \), is regular, then this equality also holds for \( x' = \sigma'_{n+1} \). Since \( \omega_h(t) \) satisfies the differential equation (V.5.5) with \( H \), it follows from (4.7) that \( \omega_h \circ \alpha(t) \) satisfies (V.5.5) with \( H' \). The function \( \omega_{h'}(t) \) however satisfies the same differential equation. Since the set of all \( x' \) which satisfy (4.8) intersects every component of \( I' \cup \{\sigma'_0, \sigma'_{n+1}\} \), and \( \omega_{h'} \) coincides with \( \omega_h \circ \alpha \) on this set, it follows that \( \omega_{h'} = \omega_h \circ \alpha \) on all of \( I' \).

Conversely, assume that \( \omega_h \leftrightarrow \omega_{h'} \), and let \( \alpha : I' \to I \) be the increasing bijection with \( \omega_{h'} = \omega_h \circ \alpha \). Then \( t(\omega_{h'}) = t(\omega_h) \circ \alpha \), and we obtain with the help of Corollary V.5.8 in particular that \( \alpha \) and \( \alpha^{-1} \) are both absolutely continuous. Let \( h_1 \) be the general Hamiltonian defined by the requirement that \( h \sim_1 h_1 \) by means of the map \( \alpha : I' \to I \), cf. Remark V.3.38. Next, let \( h_2 \) be the general Hamiltonian defined by the requirement that \( h_1 \sim_3 h_2 \) and \( E_2 = E' \). Finally, let \( h_3 \) be the general Hamiltonian such that \( h_2 \sim_2 h_3 \) and such that \( d^{(3)}_{i1} = 0 \) if \( \sigma_i \) is of polynomial type, and \( b^{(3)}_{i, \delta_i+1} = 0 \) otherwise. By what we have already proved in the previous step, \( \omega_{h_3} = \omega_h \circ \alpha \). We see that \( h_3 \) and \( h' \) satisfy the hypothesis of Lemma 4.5, and hence \( h_3 = h' \). In particular, \( h \leftrightarrow h' \).

**References**


M. Kaltenbäck, H. Woracek
Institut für Analysis und Scientific Computing
Technische Universität Wien
Wiedner Hauptstr. 8–10/101
A–1040 Wien
AUSTRIA
email: michael.kaltenbaeck@tuwien.ac.at, harald.woracek@tuwien.ac.at

40