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SIMPLE ERROR ESTIMATORS FOR THE GALERKIN BEM FOR SOME HYPERSINGULAR INTEGRAL EQUATION IN 2D

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Abstract. A posteriori error estimation is an important tool for reliable and efficient Galerkin boundary element computations. For hypersingular integral equations in 2D with positive-order Sobolev space, we analyze the mathematical relation between the $h-h/2$-error estimator from [18], the two-level error estimator from [22], and the averaging error estimator from [7]. All of these a posteriori error estimators are simple in the following sense: First, the numerical analysis can be done within the same mathematical framework, namely localization techniques for the energy norm. Second, there is almost no implementational overhead for the realization. In particular, this is very much different to other a posteriori error estimators proposed in the literature. As model example serves the hypersingular integral equation associated with the 2D Laplacian, and numerical experiments underline the mathematical results.

1. Introduction

We consider a hypersingular integral equation

$$Wu = f \quad \text{on } \Gamma \tag{1.1}$$

for a relatively open and connected subset $\Gamma \subseteq \partial\Omega$ of the boundary of a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^2$. Here, W denotes the hypersingular integral operator which formally reads, e.g., for the Laplace operator

$$Wu(x) = \frac{1}{2\pi}\frac{\partial}{\partial n(x)} \int_\Gamma u(y)\frac{\partial}{\partial n(y)} \log|x-y|\, ds_y \quad \text{for } x \in \Gamma \tag{1.2}$$

with $\int_\Gamma ds$ the integration over the surface piece Γ and $n(x)$, for $x \in \Gamma$, the outer unit normal vector of Ω. Then, $\langle\!\langle u, v\rangle\!\rangle := \langle Wu, v\rangle$ defines a scalar product on a certain closed subspace $\mathcal{H}$ of $H^{1/2}(\Gamma)$, where $\langle\cdot,\cdot\rangle$ denotes the extended L^2-scalar product.

Based on a partition $\mathcal{T}_h = \{T_1, \dots, T_N\}$ of Γ, we consider the lowest-order Galerkin method with ansatz space $X_h := \mathcal{S}^1(\mathcal{T}_h) \cap \mathcal{H}$, which consists of continuous and $\mathcal{T}_h$-piecewise affine functions. We analyze different strategies for the a posteriori error control of $|||u-u_h|||$, where $u_h \in X_h$ denotes the Galerkin solution and where $|||\cdot|||$ denotes the energy norm induced by $\langle\!\langle\cdot,\cdot\rangle\!\rangle$. Altogether, thirteen different error estimators are derived and treated within one analytical framework. For the introduction, we only address some examples. The reader is refered to Section 6.2 for an overview of all error estimators under consideration. — A prior work [15] containing similar results, was concerned with weakly-singular integral equations with energy space $\widetilde{H}^{-1/2}(\Gamma)$, but the proofs therein do not simply apply to the present situation.

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First, we consider the $h-h/2$-strategy which has been proposed in [18] in the context of the weakly-singular integral equation with energy space $\widetilde{H}^{-1/2}(\Gamma)$. The canonical $h-h/2$-error estimator reads

$$\eta_H := |||u_{h/2} - u_h|||, \tag{1.3}$$

where $u_{h/2} \in X_{h/2} := \mathcal{S}^1(\mathcal{T}_{h/2}) \cap \mathcal{H}$ is the Galerkin solution with respect to the uniformly refined mesh $\mathcal{T}_{h/2}$. Whereas the energy norm $|||\cdot|||$ is non-local, an adaptive mesh-refinement can be steered by use of

$$\mu_H := \|h^{1/2}(u_{h/2} - u_h)'\|_{L^2(\Gamma)}, \tag{1.4}$$

where $h \in L^\infty(\Gamma)$ denotes the local mesh-size of $\mathcal{T}_h$ and where $(\cdot)'$ denotes the arc-length derivative. Second, we consider the two-level error estimator from [22]

$$\eta_T := \Big(\sum_{T_j \in \mathcal{T}_h} \frac{|\langle f - Wu_h\,, \varphi_j\rangle|^2}{|||\varphi_j|||^2} \Big)^{1/2}, \tag{1.5}$$

where $\varphi_j \in X_{h/2}$ denotes the hat-function with respect to the midpoint of an element $T_j \in \mathcal{T}_h$. Finally, we consider the averaging-based error estimators proposed in [7]

$$\eta_A := |||u_{h/2} - \overline{\mathbb{G}}_h u_{h/2}||| \quad \text{and} \quad \mu_A := \|h^{1/2}(u_{h/2} - \overline{\mathbb{G}}_h u_{h/2})'\|_{L^2(\Gamma)}, \tag{1.6}$$

where $\overline{\mathbb{G}}_h$ denotes the Galerkin projection onto the continuous and $\mathcal{T}_h$-piecewise quadratic functions $\overline{X}_h := \mathcal{S}^2(\mathcal{T}_h) \cap \mathcal{H}$. We stress that —unlike the error estimators η_H, μ_H, and η_T— the error estimators η_A and μ_A were introduced for a posteriori error estimation of the improved Galerkin error $|||u - u_{h/2}||| \le |||u - u_h|||$.

Our analytical results below can be briefly concluded as follows: First, we prove that all of the error estimators in this work are equivalent, i.e., for each two error estimators η and μ, there are constants $C_{\rm low}, C_{\rm high} > 0$ such that

$$C_{\rm low}^{-1}\, \mu \le \eta \le C_{\rm high}\, \mu. \tag{1.7}$$

Second, we consider efficiency and reliability of the error estimators η in the sense that

$$C_{\rm eff}^{-1}\, \eta \le |||u - u_h||| \le C_{\rm rel}\, \eta \tag{1.8}$$

with some constants $C_{\rm eff}, C_{\rm rel} > 0$. Whereas all introduced error estimators are efficient, reliability turns out to be equivalent to the saturation assumption

$$|||u - u_{h/2}||| \le q_S\, |||u - u_h||| \tag{1.9}$$

with some constant $q_S \in (0,1)$. To mention some further contributions of this work, we stress that $h-h/2$-based error estimators have not been considered in the context of hypersingular integral equations. Moreover, our analysis provides an alternative proof for the efficiency and reliability result for η_T from [20]. Finally, we prove that the error estimator

$$\widetilde{\mu}_A := \|h^{1/2}(u_{h/2}' - \overline{\Pi}_h u_{h/2}')\|_{L^2(\Gamma)} \tag{1.10}$$

with $\overline{\Pi}_h$ the L^2-projection onto the discontinuous $\mathcal{T}_h$-piecewise affine functions is, in fact, equivalent to the averaging error estimators η_A and μ_A. This gives a positive answer to an empirical observation from [7]. Throughout, our analysis is simple in the sense that it is only based on so-called localization techniques which allow to replace the energy norm $|||\cdot|||$ by a weighted H^1-seminorm $\|h^{1/2}(\cdot)'\|_{L^2(\Gamma)}$.

The content of the paper is organized as follows: Notations, preliminaries, and the localization arguments are collected in Section 2. Section 3 is concerned with error estimation by space enrichment, e.g., the $h - h/2$-error estimators. Section 4 treats the two-level error estimator η_T introduced and studied in [20, 22]. Section 5 considers error estimators based on local averaging which were proposed by [7, 8]. Section 6 addresses some implementational aspects. In particular, we show that the implementation of all error estimators under consideration is simple and straightforward. This is in sharp contrast to, e.g., residual-based error estimators [2, 3, 4, 5, 6, 9, 10, 16] which require, by others, well-adapted quadrature rules and a pointwise evaluation of the residual. Numerical experiments in Section 7 give empirical evidence that an adaptive mesh-refining strategy which is steered by the local contributions of one of the introduced error estimators, is much superior to uniform mesh-refinement.

2. Preliminaries

2.1. Fractional Order Sobolev Spaces. Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Given $0 < \alpha \leq 1$, the Sobolev space $H^\alpha(\partial\Omega)$ is the set of all real-valued functions on $\partial\Omega$ which are the traces of functions in $H^{\alpha+1/2}(\mathbb{R}^2)$ to $\partial\Omega$,

$$H^\alpha(\partial\Omega) := \{u|_{\partial\Omega} \,:\, u \in H^{\alpha+1/2}(\mathbb{R}^2)\}. \tag{2.1}$$

Moreover, it is consistent to define $H^0(\partial\Omega) := L^2(\partial\Omega)$ and to define Sobolev spaces of negative order by duality,

$$H^{-\alpha}(\partial\Omega) := H^\alpha(\partial\Omega)^*, \tag{2.2}$$

with corresponding norms and duality brackets $\langle\cdot\,,\cdot\rangle$ which extend the $L^2(\partial\Omega)$ scalar product. For the hypersingular integral equation on $\partial\Omega$, one considers the subspaces $H^\alpha_\star(\partial\Omega)$ to factor the constant functions out,

$$H^\alpha_\star(\partial\Omega) := \{u \in H^\alpha(\partial\Omega) \,:\, \langle 1\,, u\rangle = 0\}, \tag{2.3}$$

where 1 denotes the constant function. For a (relatively) open subset $\omega \subseteq \Gamma$ and $\alpha \geq 0$, we define the fractional order Sobolev space $H^\alpha(\omega)$ by extension

$$H^\alpha(\omega) := \{u|_\omega \,:\, u \in H^\alpha(\partial\Omega)\}, \tag{2.4}$$

where the norm of $u \in H^\alpha(\omega)$ is defined as the minimal norm of an extension, i.e.

$$\|u\|_{H^\alpha(\omega)} := \inf\{\|\widehat{u}\|_{H^\alpha(\partial\Omega)} \,:\, \widehat{u} \in H^\alpha(\partial\Omega) \text{ with } \widehat{u}|_\omega = u\}. \tag{2.5}$$

Furthermore, there are Sobolev spaces $\widetilde{H}^\alpha(\omega)$

$$\widetilde{H}^\alpha(\omega) := \{u \in H^\alpha(\partial\Omega) \,:\, \mathrm{supp}(u) \subseteq \overline{\omega}\} \tag{2.6}$$

associated with the usual $H^\alpha(\omega)$ norm. The corresponding spaces of negative order are again defined by duality

$$H^{-\alpha}(\omega) = \widetilde{H}^\alpha(\omega)^* \quad \text{and} \quad \widetilde{H}^{-\alpha}(\omega) = H^\alpha(\omega)^*. \tag{2.7}$$

Remark 1. *Note that $\widetilde{H}^\alpha(\partial\Omega) = H^\alpha(\partial\Omega)$. For $\omega \subsetneqq \partial\Omega$, there holds only $\widetilde{H}^\alpha(\omega) \subseteq H^\alpha(\omega)$ with $\|u\|_{H^\alpha(\omega)} \leq \|u\|_{\widetilde{H}^\alpha(\omega)}$ for all $u \in \widetilde{H}^\alpha(\omega)$.* □

Remark 2. *Note that according to Sobolev's inequality in 1D, each function $u \in H^\alpha(\omega)$ with $\alpha > 1/2$ is continuous. Moreover, each function $u \in H^1(\omega)$ is absolutely continuous, i.e. there holds the fundamental theorem of calculus with respect to the arclength-derivative.* $\square$

2.2. Hypersingular Integral Operator and Energy Norm. The analysis below only makes use of the following assumptions: Let

$$\mathcal{H} = \begin{cases} \widetilde{H}^{1/2}(\Gamma) := \{v \in H^{1/2}(\Gamma) \,:\, \mathrm{supp}(v) \subseteq \overline{\Gamma}\} & \text{in case of } \Gamma \subsetneqq \partial\Omega, \\ H^{1/2}_\star(\Gamma) := \{v \in H^{1/2}(\Gamma) \,:\, \int_\Gamma v\,ds = 0\} & \text{in case of } \Gamma = \partial\Omega. \end{cases} \tag{2.8}$$

We assume that $\langle\!\langle u\,,v\rangle\!\rangle := \langle Wu\,,v\rangle$ is a scalar product on $\mathcal{H}$ with corresponding norm $|\!|\!|\cdot|\!|\!| \sim \|\cdot\|_{H^{1/2}(\Gamma)}$ on $\mathcal{H}$. Moreover, for $\Gamma = \partial\Omega$, let $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ be a continuous and symmetric bilinear form on the entire space $H^{1/2}(\Gamma)$ and assume that $\langle\!\langle c\,,v\rangle\!\rangle = 0$ for all $c \in \mathbb{R}$ and $v \in H^{1/2}(\Gamma)$. This situation is met for several first-kind integral equations, which arise from elliptic PDEs. Besides the Laplace operator from the introduction, examples arise for the hypersingular integral equations associated with the Lamé and the Stokes problem. We stress that the dual space of $\mathcal{H}$ is given by

$$\mathcal{H}^* = \begin{cases} H^{-1/2}(\Gamma) & \text{in case of } \Gamma \subsetneqq \partial\Omega, \\ H^{-1/2}_\star(\Gamma) := \{g \in H^{-1/2}(\Gamma) \,:\, \langle g\,,1\rangle = 0\} & \text{in case of } \Gamma = \partial\Omega. \end{cases} \tag{2.9}$$

We then consider the variational formulation of (1.1)

$$\langle\!\langle u\,,v\rangle\!\rangle = \langle f\,,v\rangle \quad \text{for all } v \in \mathcal{H} \tag{2.10}$$

for a given right-hand side $f \in \mathcal{H}^*$. According to Riesz' theorem, there is a unique solution $u \in \mathcal{H}$ of (2.10). With a conforming discrete space $X_h \subset \mathcal{H}$, we consider the Galerkin method

$$\langle\!\langle u_h\,,v_h\rangle\!\rangle = \langle f\,,v_h\rangle \quad \text{for all } v_h \in X_h, \tag{2.11}$$

for which the Riesz theorem again provides a unique solution $u_h \in X_h$. We stress the Galerkin orthogonality

$$\langle\!\langle u - u_h\,,v_h\rangle\!\rangle = 0 \quad \text{for all } v_h \in X_h, \tag{2.12}$$

which in fact characterizes the discrete solution $u_h \in X_h$. In particular, there holds

$$|\!|\!|u - u_h|\!|\!| \le |\!|\!|u - v_h|\!|\!| \quad \text{for all } v_h \in X_h, \tag{2.13}$$

i.e. u_h is the best approximation of u with respect to X_h and the energy norm.

2.3. Galerkin Discretization. Let $\mathcal{T}_h$ be a partition of Γ, i.e. $\mathcal{T}_h = \{T_1, \dots, T_N\}$ is a finite set of pairwise disjoint, connected, and relatively open subsets of Γ such that $\overline{\Gamma} = \bigcup_{j=1}^N \overline{T}_j$. For the ease of presentation, we assume that the elements T_j are affine boundary pieces. We define the local mesh-width by

$$h \in L^\infty(\Gamma), \quad h|_{T_j} := h_j := \sup\{|x-y| \,:\, x, y \in T_j\}. \tag{2.14}$$

Moreover, the local mesh-ratio is given by

$$\kappa(\mathcal{T}_h) := \max\{h_j/h_k \,:\, T_j, T_k \in \mathcal{T}_h \text{ with } \overline{T}_j \cap \overline{T}_k \neq \emptyset\}, \tag{2.15}$$

i.e. by the maximal quotient of the lengths of two neighbouring elements. Refinement of an element $T_j \in \mathcal{T}_h$ means that T_j is split into two new elements of half length. Since the error estimates below depend on $\kappa(\mathcal{T}_h)$, our implementation always ensures that $\kappa(\mathcal{T}_h) \le 2\,\kappa(\mathcal{T}_0)$, where $\mathcal{T}_0$ denotes the initial mesh.

Throughout, $\mathcal{K}_h$ denotes the set of all nodes of a triangulation $\mathcal{T}_h$. Recall that $\#\mathcal{K}_h = \#\mathcal{T}_h$ for $\Gamma = \partial\Omega$, whereas $\#\mathcal{K}_h = \#\mathcal{T}_h + 1$ for $\Gamma \subsetneqq \partial\Omega$.

Let $\mathcal{P}^p(\mathcal{T}_h)$ denote the space of all $\mathcal{T}_h$-piecewise polynomials of degree $\le p$ with respect to the arclength and $\mathcal{S}^p(\mathcal{T}_h) := \mathcal{P}^p(\mathcal{T}_h) \cap C(\Gamma)$. The canonical discrete spaces X_h in (2.11) are given by

$$\widetilde{\mathcal{S}}^p(\mathcal{T}_h) := \mathcal{S}^p(\mathcal{T}_h) \cap \widetilde{H}^{1/2}(\Gamma) = \{v_h \in \mathcal{S}^p(\mathcal{T}_h)\,:\, v_h|_{\partial\Gamma} = 0\} \quad \text{for } \Gamma \subsetneqq \partial\Omega \tag{2.16}$$

and

$$\mathcal{S}^p_\star(\mathcal{T}_h) := \mathcal{S}^p(\mathcal{T}_h) \cap H^{1/2}_\star(\Gamma) = \{v_h \in \mathcal{S}^p(\mathcal{T}_h)\,:\, \textstyle\int_\Gamma v_h\,ds = 0\} \quad \text{for } \Gamma = \partial\Omega, \tag{2.17}$$

respectively. However, for implementational and even analytical reasons in Section 4, we shall consider a different space in case of $\Gamma = \partial\Omega$. We define

$$\mathcal{S}^p_0(\mathcal{T}_h) := \begin{cases} \widetilde{\mathcal{S}}^p(\mathcal{T}_h) & \text{in case of } \Gamma \subsetneqq \partial\Omega, \\ \{v_h \in \mathcal{S}^p(\mathcal{T}_h)\,:\, v_h(z_N) = 0\} & \text{in case of } \Gamma = \partial\Omega, \end{cases} \tag{2.18}$$

where $z_N \in \mathcal{K}_h$ is a fixed node of $\mathcal{T}_h$. Note that conformity $\mathcal{S}^p_0(\mathcal{T}_h) \subset \mathcal{H}$ holds, by definition, only for $\Gamma \subsetneqq \partial\Omega$. However, there holds the following elementary link between $\mathcal{S}^p_\star(\mathcal{T}_h)$ and $\mathcal{S}^p_0(\mathcal{T}_h)$ for $\Gamma = \partial\Omega$.

Lemma 2.1. *Assume that $\Gamma = \partial\Omega$. Then, $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ defines a scalar product on $\mathcal{S}^p_0(\mathcal{T}_h)$. Given $f \in \mathcal{H}^*$, there thus exists a unique Galerkin solution $u_h \in \mathcal{S}^p_0(\mathcal{T}_h)$ with respect to $\mathcal{S}^p_0(\mathcal{T}_h)$. Moreover, the Galerkin solution $u_h^\star \in \mathcal{S}^p_\star(\mathcal{T})$ with respect to $\mathcal{S}^p_\star(\mathcal{T}) \subset \mathcal{H}$ is given by $u_h^\star = u_h - |\Gamma|^{-1}\int_\Gamma u_h\,ds$. Finally, with $u \in \mathcal{H}$ the continuous solution of* (2.10)*, there holds $|\!|\!| u - u_h |\!|\!| = |\!|\!| u - u_h^\star |\!|\!|$ as well as* (2.12)*–*(2.13) *with the entire space $X_h = \mathcal{S}^p(\mathcal{T}_h)$ instead of only the subspace $\mathcal{S}^p_0(\mathcal{T}_h)$.*

Proof. Let $v_h \in \mathcal{S}^p_0(\mathcal{T}_h)$ with $\langle\!\langle v_h\,, v_h \rangle\!\rangle = 0$ and define $v_h^\star := v_h - \overline{v_h} \in \mathcal{S}^p_\star(\mathcal{T}_h)$, where $\overline{v}_h := |\Gamma|^{-1}\int_\Gamma v_h\,ds \in \mathbb{R}$. Then, we obtain $\langle\!\langle v_h^\star\,, v_h^\star \rangle\!\rangle = \langle\!\langle v_h\,, v_h \rangle\!\rangle - 2\langle\!\langle \overline{v}_h\,, v_h \rangle\!\rangle + \langle\!\langle \overline{v}_h\,, \overline{v}_h \rangle\!\rangle = 0$. Therefore, $v_h^\star = 0$ and v_h is constant. From $v_h(z_N) = 0$, we infer $v_h = 0$. Therefore, $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ is a scalar product on $\mathcal{S}^p_0(\mathcal{T}_h)$. Let $u_h \in \mathcal{S}^p_0(\mathcal{T}_h)$ be the associated Galerkin solution. For arbitrary $v_h^\star \in \mathcal{S}^p(\mathcal{T}_h)$, we define $v_h := v_h^\star - v_h^\star(z_N) \in \mathcal{S}^p_0(\mathcal{T}_h)$. As before, we obtain $\langle\!\langle u_h^\star\,, v_h^\star \rangle\!\rangle = \langle\!\langle u_h\,, v_h \rangle\!\rangle = \langle f\,, v_h \rangle = \langle f\,, v_h^\star \rangle$, where the final equality follows from $f \in \mathcal{H}^* = H^{-1/2}_\star(\Gamma)$. In particular, $u_h^\star \in \mathcal{S}^p_\star(\mathcal{T}_h)$ is the unique Galerkin solution with respect to $\mathcal{S}^p_\star(\mathcal{T}_h)$. The remaining claims follow by use of the same arguments. $\square$

The focus of this work is on the lowest-order Galerkin scheme for $p = 1$. We shall use the nodal basis $\{\phi_z\,:\, z \in \mathcal{K}_h\}$ of $\mathcal{S}^1(\mathcal{T}_h)$, where $\mathcal{K}_h$ denotes the set of nodes of $\mathcal{T}_h$ and where $\phi_z \in \mathcal{S}^1(\mathcal{T}_h)$, for $z \in \mathcal{K}_h$, denotes the corresponding hat function which satisfies $\phi_z(z') = \delta_{zz'}$ for all nodes $z' \in \mathcal{K}_h$.

2.4. Notational Conventions. If not stated otherwise, we use the following notation,

$$X_h := \mathcal{S}^1_0(\mathcal{T}_h), \quad X_{h/2} := \mathcal{S}^1_0(\mathcal{T}_{h/2}), \quad \text{and} \quad \overline{X}_h := \mathcal{S}^2_0(\mathcal{T}_h), \tag{2.19}$$

where the partition $\mathcal{T}_{h/2}$ is obtained from a uniform refinement of $\mathcal{T}_h$. Moreover, in Section 3 and 4, we use

$$\widehat{X}_h \in \{X_{h/2}, \overline{X}_h\}, \tag{2.20}$$

i.e. $\widehat{X}_h$ denotes either $X_{h/2}$ or $\overline{X}_h$. The Galerkin solutions with respect to X_h, $X_{h/2}$, $\overline{X}_h$, and $\widehat{X}_h$ are denoted by u_h, $u_{h/2}$, $\overline{u}_h$, and $\widehat{u}_h$, respectively. Throughout, Galerkin projections are denoted by $\mathbb{G}$. The indices indicate the corresponding space, e.g. $\overline{\mathbb{G}}_h$ denotes the Galerkin projection onto $\overline{X}_h$ and $\mathbb{G}_{h/2}$ denotes the Galerkin projection onto $X_{h/2}$.

Altogether, we shall introduce thirteen different error estimators below. Throughout, the notation of the a posteriori error estimators

$$\eta, \quad \widetilde{\eta}, \quad \mu, \qquad \text{and} \quad \widetilde{\mu} \tag{2.21}$$

uses the following convention: η denotes an error estimator which is based on the energy norm, whereas μ denotes an error estimator which is based on an $h^{1/2}$-weighted H^1-seminorm. Moreover, η and μ need the computation of a certain Galerkin projection $\mathbb{G}$, whereas $\widetilde{\eta}$ and $\widetilde{\mu}$ are based on simpler operators, c.f. the definition of the $h-h/2$-error estimators η_S in (3.1), $\widetilde{\eta}_S$ in (3.6), and μ_S as well as $\widetilde{\mu}_S$ in (3.7). The subscript indicates the type of error estimator, e.g. η_S shows that this error estimator is based on space enrichment.

2.5. Localization of $H^{1/2}$-Norm. The first lemma provides a localization of the energy norm for discrete functions $v_h \in \mathcal{S}^1(\mathcal{T}_h)$. Since $|||\cdot|||$ is an equivalent norm on the subspace $\mathcal{H}$ of $H^{1/2}(\Gamma)$, this localization is naturally given in terms of a mesh-size weighted H^1-seminorm.

Lemma 2.2. (i) *For any discrete function $v_h \in \mathcal{S}^p_0(\mathcal{T}_h)$ holds the inverse estimate*

$$\|h^{1/2}v_h'\|_{L^2(\Gamma)} \le C_{\rm inv}|||v_h||| \tag{2.22}$$

where the constant $C_{\rm inv} > 0$ depends only on Γ and the polynomial degree $p \ge 0$.
(ii) *For $p = 1$, the nodal interpolation operator $I_h : C(\overline{\Gamma}) \to \mathcal{S}^1(\mathcal{T}_h)$, $I_h v := \sum_{z\in\mathcal{K}_h} v(z)\phi_z$ and the L^2-orthogonal projection $\Pi_h : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_h)$ are related by*

$$(I_h v)' = \Pi_h v' \quad \textit{for all } v \in H^1(\Gamma). \tag{2.23}$$

(iii) *There holds the following approximation estimate*

$$|||v - I_h v||| \le C_{\rm apx}\|h^{1/2}(v - I_h v)'\|_{L^2(\Gamma)} \le C_{\rm apx}\|h^{1/2}v'\|_{L^2(\Gamma)} \quad \textit{for all } v \in H^1(\Gamma)\cap\mathcal{H}, \tag{2.24}$$

where the constant $C_{\rm apx} > 0$ depends only on Γ and $\kappa(\mathcal{T}_h)$.
(iv) *Moreover, the Galerkin projection $\mathbb{G}_h : \mathcal{H} \to \mathcal{S}^p_0(\mathcal{T}_h)$ satisfies*

$$|||v - \mathbb{G}_h v||| \le C_{\rm apx}\min\big\{\|h^{1/2}v'\|_{L^2(\Gamma)}, \|h^{1/2}(v - \mathbb{G}_h v)'\|_{L^2(\Gamma)}\big\} \quad \textit{for all } v \in H^1(\Gamma)\cap\mathcal{H} \tag{2.25}$$

with the constant $C_{\rm apx}$ from (iii).

Sketch of Proof. We only consider the case $\Gamma = \partial\Omega$ and stress that the simpler case $\Gamma \subsetneqq \partial\Omega$ follows along the same lines. The local inverse estimate (2.22) is proven in [7, Proposition 3.1] in the form

$$\|h^{1-\alpha}v_h'\|_{L^2(\Gamma)} \le \widetilde{C_{\rm inv}}\|v_h\|_{H^\alpha(\Gamma)} \quad \text{for all } v_h \in \mathcal{S}^p(\mathcal{T}_h),$$

where $\widetilde{C_{\rm inv}}$ depends only on Γ, $\alpha \ge 0$, and $p \in \mathbb{N}_0$. We only consider the case $\alpha = 1/2$. For $v_h \in \mathcal{S}^p_0(\mathcal{T}_h)$, we define $v_h^\star := v_h - |\Gamma|^{-1}\int_\Gamma v_h\,ds \in \mathcal{S}^p_\star(\mathcal{T})$. Now, norm equivalence on

$\mathcal{S}^p_\star(\mathcal{T}_h) \subset \mathcal{H} = H^{1/2}_\star(\Gamma)$ as well as $|||v_h||| = |||v_h^\star|||$ and $v_h' = (v_h^\star)'$ prove (2.22). To prove (2.23), it only remains to verify

$$\int_T (v - I_h v)' \, ds = 0 \quad \text{for all } T \in \mathcal{T}_h$$

which however follows from the fact that the nodal values of $v - I_h v$ are zero. The local approximation estimate from [2, Theorem 1] reads

$$\|v - I_h v\|_{H^\alpha(\Gamma)} \le \widetilde{C_{\rm apx}} \|h^{1-\alpha} v'\|_{L^2(\Gamma)} \quad \text{for all } v \in H^1(\Gamma), \tag{2.26}$$

where the constant $\widetilde{C_{\rm apx}}$ depends only on Γ, $\alpha \ge 0$, and on $\kappa(\mathcal{T}_h)$. For $\alpha = 1/2$, continuity of the bilinear form $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ yields $|||w||| \le C\, \|w\|_{H^{1/2}(\Gamma)}$ for all $w \in H^{1/2}(\Gamma)$. Therefore, we obtain

$$|||v - I_h v||| \le C_{\rm apx}\, \|h^{1/2} v'\|_{L^2(\Gamma)} \quad \text{for all } v \in H^1(\Gamma).$$

We apply this estimate to $w := v - I_h v \in H^1(\Gamma) \cap \mathcal{H}$. From the projection property $I_h^2 = I_h$, we infer

$$|||v - I_h v||| = |||w - I_h w||| \le C_{\rm apx} \|h^{1/2} w'\|_{L^2(\Gamma)} = C_{\rm apx} \|h^{1/2}(v - I_h v)'\|_{L^2(\Gamma)}, \tag{2.27}$$

Note that the L^2-orthogonal projection Π_h onto $\mathcal{P}^0(\mathcal{T}_h)$ is even $\mathcal{T}_h$-elementwise the best approximation operator. This and (2.23) imply

$$\|(v - I_h v)'\|^2_{L^2(T_j)} = \|v' - \Pi_h v'\|^2_{L^2(T_j)} \le \|v'\|^2_{L^2(T_j)} \quad \text{for all } T_j \in \mathcal{T}_h.$$

Multiplying this estimate with the local mesh-width and summing over all elements, we see

$$\|h^{1/2}(v - I_h v)'\|_{L^2(\Gamma)} \le \|h^{1/2} v'\|_{L^2(\Gamma)}.$$

The combination of this and (2.27) concludes the proof of (2.24). It finally remains to prove (2.25). In a first step, we use the best approximation property of the Galerkin projection $\mathbb{G}_h$, cf. Lemma 2.1. This implies

$$|||v - \mathbb{G}_h v||| = \min_{v_h \in \mathcal{S}^p(\mathcal{T}_h)} |||v - v_h||| \le |||v - I_h v||| \le C_{\rm apx} \|h^{1/2} v'\|_{L^2(\Gamma)}.$$

The strengthened form (2.25) is again obtained by simple postprocessing: We consider $w := v - \mathbb{G}_h v \in H^1(\Gamma) \cap \mathcal{H}$ and use $\mathbb{G}_h^2 = \mathbb{G}_h$ to derive

$$|||v - \mathbb{G}_h v||| = |||w - \mathbb{G}_h w||| \le C_{\rm apx} \|h^{1/2} w'\|_{L^2(\Gamma)} = C_{\rm apx} \|h^{1/2}(v - \mathbb{G}_h v)'\|_{L^2(\Gamma)}.$$

Therefore, $|||v - \mathbb{G}_h v|||$ is even bounded by the minimal right-hand side. □

3. Error Estimation by Space Enrichment

Let $X_h \subset \widehat{X}_h$ be nested discrete subspaces of $\mathcal{H}$ with corresponding Galerkin solutions $u_h \in X_h$ and $\widehat{u}_h \in \widehat{X}_h$, respectively. We now use the difference of the two Galerkin solutions

$$\eta_S := |||\widehat{u}_h - u_h||| \tag{3.1}$$

to estimate the error $|||u - u_h|||$. The Galerkin orthogonality (2.12) for $\widehat{X}_h$ then yields

$$|||u - u_h|||^2 = |||u - \widehat{u}_h|||^2 + |||\widehat{u}_h - u_h|||^2 = |||u - \widehat{u}_h|||^2 + \eta_S^2$$

and thus $\eta_S \le |||u - u_h|||$. This proves efficiency of η_S with $C_{\rm eff} = 1$. The reliability of η_S is usually proven with the help of the *saturation assumption*

$$|||u - \widehat{u}_h||| \le q_S\, |||u - u_h||| \quad \text{with some uniform constant } q_S \in (0,1). \tag{3.2}$$

Under this assumption, we obtain $|||u - u_h|||^2 = |||u - \widehat{u}_h|||^2 + \eta_S^2 \le q_S^2\, |||u - u_h|||^2 + \eta_S^2$ and thus reliability

$$|||u - u_h||| \le \frac{1}{\sqrt{1 - q_S^2}}\, \eta_S. \tag{3.3}$$

The same arguments, in fact, imply that reliability of η_S yields the saturation assumption (3.2) with $q_S = (1 - C_{\rm rel}^{-2})^{1/2}$. We state these elementary observations in the following proposition.

Proposition 3.1. (i) *The error estimator η_S is always efficient with $C_{\rm eff} = 1$.*
(ii) *The saturation assumption* (3.2) *is equivalent to reliability of η_S with $C_{\rm rel} = (1 - q_S^2)^{-1/2}$.* □

Note that (3.2) always holds with $q_S = 1$ due to the best approximation property of Galerkin solutions. Therefore, the space $\widehat{X}_h$ has to be sufficiently larger than X_h to allow and guarantee (3.2) with uniform $q_S < 1$. In the following, we consider two canonical choices for the enriched space $\widehat{X}_h$: First, the $h - h/2$-strategy which has been proposed and studied in [18] in the context of the weakly singular integral equation with energy space $\widetilde{H}^{-1/2}(\Gamma)$. Let $\mathcal{T}_h = \{T_1, \dots, T_N\}$ be a partition of Γ and $\mathcal{T}_{h/2}$ be obtained from a uniform refinement of $\mathcal{T}_h$. We then may consider the discrete spaces

$$X_h := \mathcal{S}_0^1(\mathcal{T}_h) \quad \text{and} \quad \widehat{X}_h := \mathcal{S}_0^1(\mathcal{T}_{h/2}). \tag{3.4}$$

Alternatively, we might use the analogous $p - (p+1)$-strategy, where

$$X_h := \mathcal{S}_0^1(\mathcal{T}_h) \quad \text{and} \quad \widehat{X}_h := \mathcal{S}_0^2(\mathcal{T}_h). \tag{3.5}$$

For the finite element method and either of the choices (3.4)–(3.5), the saturation assumption (3.2) can be proven under some mild conditions on the local mesh refinement [14]. However, we stress that the saturation assumption — although observed in practice, cf. Section 7 below — has not been proven for the boundary element method, yet.

Moreover, for either of the two choices, the error estimator η_S suffers from two things: First, the energy norm $|||\cdot|||$ does not provide information, where the mesh $\mathcal{T}_h$ should be refined to decrease the error most efficiently. Second, we do not only have to compute the Galerkin approximation u_h with respect to X_h but even the computationally more expensive Galerkin solution $\widehat{u}_h$. A numerical algorithm clearly returns $\widehat{u}_h$, since this is a better approximation of the exact solution u than u_h.

Some kind of remedy is given by the following theorem: First, the nonlocal energy norm is replaced by an h-weighted H^1-seminorm. Second, we might replace u_h by the nodal interpolant $I_h\widehat{u}_h$ of the more accurate Galerkin solution $\widehat{u}_h$. Instead of solving a linear system with dense Galerkin matrix to obtain u_h, we thus only compute $I_h\widehat{u}_h$, which is done in real linear complexity.

Theorem 3.2. *Let X_h and $\widehat{X}_h$ be given by either* (3.4) *or* (3.5)*. Besides the error estimator η_S, we define*

$$\widetilde{\eta}_S := |||\widehat{u}_h - I_h\widehat{u}_h||| \tag{3.6}$$

as well as the h-weighted L^2-norm based error estimators

$$\mu_S := \|h^{1/2}(\widehat{u}_h - u_h)'\|_{L^2(\Gamma)} \quad \textit{and} \quad \widetilde{\mu}_S := \|h^{1/2}(\widehat{u}_h - I_h\widehat{u}_h)'\|_{L^2(\Gamma)}, \tag{3.7}$$

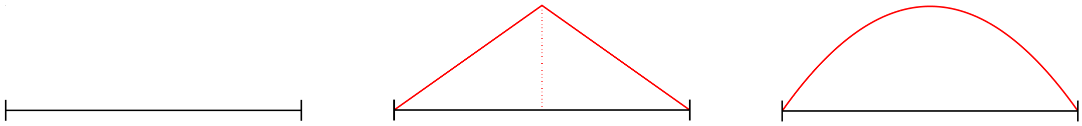

FIGURE 1. *The space $\widehat{X}_h$, which is either $\mathcal{S}_0^1(\mathcal{T}_{h/2})$ or $\mathcal{S}_0^2(\mathcal{T}_h)$ is decomposed into $X_h = \mathcal{S}_0^1(\mathcal{T}_h)$ and one-dimensional spaces $X_{h,j} := \mathrm{span}\{\varphi_j\}$. For each element $T_j \in \mathcal{T}_h$, we choose the additional function $\varphi_j \in \widehat{X}_h$ such that $\mathrm{supp}(\varphi_j) \subseteq T_j$ and such that the nodal interpolant satisfies $I_h\varphi_j = 0$.*

where I_h denotes the nodal interpolation operator. With the constants $C_{\rm inv}, C_{\rm apx} > 0$ of Lemma 2.2, there hold the equivalence estimates

$$\widetilde{\mu}_S \le \mu_S \le \sqrt{2}\, C_{\rm inv}\eta_S \quad \text{and} \quad \eta_S \le \widetilde{\eta}_S \le C_{\rm apx}\,\widetilde{\mu}_S. \tag{3.8}$$

Therefore, all error estimators are always efficient, and reliability holds under the saturation assumption (3.2).

Proof. Let $\mathbb{G}_h$ denote the Galerkin projection onto X_h. Note that $\mathbb{G}_h\widehat{u}_h = u_h$ according to $X_h \subset \widehat{X}_h$. Therefore, the best approximation property of the Galerkin projection and the approximation estimate (2.24) prove $\eta_S \le \widetilde{\eta}_S \le C_{\rm apx}\widetilde{\mu}_S$. The estimate $\mu_S \le \sqrt{2}\, C_{\rm inv}\eta_S$ follows from the inverse estimate (2.22) applied for $\widehat{X}_h$, where the additional factor $\sqrt{2}$ only arises in case of the $h - h/2$-strategy (3.4). Finally, recall that $(I_h v)' = \Pi_h v'$, where Π_h is the $\mathcal{T}_h$-elementwise L^2-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_h)$. This implies

$$h_{T_j}\|(\widehat{u}_h - I_h\widehat{u}_h)'\|^2_{L^2(T_j)} \le h_{T_j}\|(\widehat{u}_h - \mathbb{G}_h\widehat{u}_h)'\|^2_{L^2(T_j)} \quad \text{for all } T_j \in \mathcal{T}_h.$$

Summing these estimates over all elements $T_j \in \mathcal{T}_h$, we conclude $\widetilde{\mu}_S \le \mu_S$. □

4. Two-Level Error Estimation

Let $\mathcal{T}_h = \{T_1, \dots, T_N\}$ be a partition of Γ and $X_h \subset \widehat{X}_h$ be either given by (3.4) or by (3.5). Let $u_h \in X_h$ and $\widehat{u}_h \in \widehat{X}_h$ be the corresponding Galerkin solutions.

For each element $T_j \in \mathcal{T}_h$, we choose a function $\varphi_j \in \widehat{X}_h \backslash X_h$ with $\mathrm{supp}(\varphi_j) \subseteq T_j$ and nodal interpolant $I_h\varphi_j = 0$, cf. Figure 1. Let $X_{h,0} := X_h$ and $X_{h,j} := \mathrm{span}\{\varphi_j\}$ for $j = 1, \dots, N$. We denote with $\mathbb{G}_{h,j}$ the Galerkin projection onto $X_{h,j}$.

The following theorem has first been proven in [22] for uniform mesh-refinement in 2D and 3D. Their arguments were generalized for adaptive mesh-refinement in 2D in [20] and in [19] even in 3D. For the 2D case, we provide an alternative proof by means of the localization techniques for the energy norm.

Theorem 4.1. *There are constants $C_1, C_2 > 0$ which depend only on the constants $C_{\rm inv}, C_{\rm apx} > 0$ of Lemma 2.2 such that*

$$C_1^{-1}\,\eta_S \le \Big(\sum_{j=1}^N |||\mathbb{G}_{h,j}(\widehat{u}_h - u_h)|||^2\Big)^{1/2} \le C_2\,\eta_S, \tag{4.1}$$

where η_S denotes the error estimator from the previous section. In particular, with the refinement indicators $\eta_{T,j} := |||\mathbb{G}_{h,j}(\widehat{u}_h - u_h)|||$, the two-level error estimator $\eta_T := \big(\sum_{j=1}^N \eta_{T,j}^2\big)^{1/2}$

is equivalent to η_S. Therefore, η_T is always efficient, and reliability of η_T holds under the saturation assumption (3.2)*. Finally, $\eta_{T,j}$ can be written as*

$$\eta_{T,j} = \frac{|\langle\!\langle \widehat{u}_h - u_h\,,\varphi_j\rangle\!\rangle|}{|\!|\!| \varphi_j |\!|\!|} = \frac{|\langle f - W u_h\,,\varphi_j\rangle|}{|\!|\!| \varphi_j |\!|\!|} \quad \textit{for } j = 1,\ldots,N. \tag{4.2}$$

The proof of Theorem 4.1 needs the following three results.

Lemma 4.2. *For $\widehat{v}_h \in \widehat{X}_h$, there are coefficients $\lambda_j \in \mathbb{R}$ such that $\widehat{v}_h = I_h\widehat{v}_h + \sum_{j=1}^N \lambda_j\varphi_j$.*

Proof. Let $\mathcal{B} = \{\phi_1,\ldots,\phi_K\}$ denote the nodal basis of $\mathcal{S}^1(\mathcal{T}_h)$, where K denotes the number of nodes of $\mathcal{T}_h$, namely $K = N$ in case of $\Gamma = \partial\Omega$ and $K = N + 1$ in case of $\Gamma \subsetneqq \partial\Omega$. Clearly, $\widehat{\mathcal{B}} := \mathcal{B} \cup \{\varphi_1,\ldots,\varphi_N\}$ is a linearly independent subset of $\mathcal{S}^1(\mathcal{T}_{h/2})$ resp. $\mathcal{S}^2(\mathcal{T}_h)$. Moreover, it is an elementary observation that $\widehat{\mathcal{B}}$ contains $K + N = \#\mathcal{S}^1(\mathcal{T}_{h/2}) = \#\mathcal{S}^2(\mathcal{T}_h)$ elements. Therefore, $\widehat{\mathcal{B}}$ is a basis of $\mathcal{S}^1(\mathcal{T}_{h/2})$ resp. $\mathcal{S}^2(\mathcal{T}_h)$. For $\widehat{v}_h \in \widehat{X}_h$, we thus obtain $\widehat{v}_h = \sum_{k=1}^K \mu_k\phi_k + \sum_{j=1}^N \lambda_j\varphi_j$ with appropriate scalars $\mu_k, \lambda_j \in \mathbb{R}$. Note that, for each node z_ℓ of $\mathcal{T}_h$, there holds $\phi_k(z_\ell) = \delta_{k\ell}$ with Kronecker's delta as well as $\varphi_j(z_\ell) = 0$. This proves $\widehat{v}_h(z_\ell) = \mu_\ell$. Therefore, the first sum simplifies to $I_h\widehat{v}_h = \sum_{k=1}^K \mu_k\phi_k$. ☐

Lemma 4.3. *For any function $\widehat{v}_h \in \widehat{X}_h$ holds $|\!|\!|(1 - I_h)\widehat{v}_h|\!|\!| \le C_3\,|\!|\!|\widehat{v}_h|\!|\!|$, where the constant $C_3 > 0$ depends only on the constants $C_{\rm inv}, C_{\rm apx} > 0$ of Lemma 2.2.*

Proof. There holds $|\!|\!|(1 - I_h)\widehat{v}_h|\!|\!| \le C_{\rm apx}\|h^{1/2}\widehat{v}_h'\|_{L^2(\Gamma)} \le \sqrt{2}\,C_{\rm inv}C_{\rm apx}|\!|\!|\widehat{v}_h|\!|\!|$, where the pessimistic factor $\sqrt{2}$ arises in case of the $h - h/2$-strategy (3.4). ☐

Lemma 4.4. *For any functions $v_j \in X_{h,j}$ holds*

$$C_4^{-1}\Big(\sum_{j=1}^N |\!|\!|v_j|\!|\!|^2\Big)^{1/2} \le \Big|\!\Big|\!\Big|\sum_{j=1}^N v_j\Big|\!\Big|\!\Big| \le C_5\Big(\sum_{j=1}^N |\!|\!|v_j|\!|\!|^2\Big)^{1/2}, \tag{4.3}$$

where the constants $C_4, C_5 \ge 1$ depend only on the constants $C_{\rm inv}, C_{\rm apx} > 0$ of Lemma 2.2.

Proof. By choice of $X_{h,j} = \operatorname{span}\{\varphi_j\}$, there hold $v_j = (1 - I_h)v_j$ and $\operatorname{supp}(v_j) \subseteq T_j$. We thus infer

$$|\!|\!|v_j|\!|\!| = |\!|\!|(1 - I_h)v_j|\!|\!| \le C_{\rm apx}\|h^{1/2}v_j'\|_{L^2(\Gamma)} = C_{\rm apx}\|h^{1/2}v_j'\|_{L^2(T_j)}.$$

Summing these estimates over all $j = 1,\ldots,N$, we obtain

$$\sum_{j=1}^N |\!|\!|v_j|\!|\!|^2 \le C_{\rm apx}^2\sum_{j=1}^N \|h^{1/2}v_j'\|_{L^2(T_j)}^2 = C_{\rm apx}^2\Big\|h^{1/2}\sum_{j=1}^N v_j'\Big\|_{L^2(\Gamma)}^2 \le 2\,C_{\rm apx}^2C_{\rm inv}^2\Big|\!\Big|\!\Big|\sum_{j=1}^N v_j\Big|\!\Big|\!\Big|^2,$$

where we have used that the supports $\operatorname{supp}(v_j) \subseteq T_j$ are pairwise disjoint and the inverse estimate (2.22) for the function $\widehat{v}_h := \sum_{j=1}^N v_j$ in the final estimate. The converse inequality follows from the same type of arguments: With $\sum_{j=1}^N v_j = (1 - I_h)\sum_{j=1}^N v_j$, we estimate

$$\Big|\!\Big|\!\Big|\sum_{j=1}^N v_j\Big|\!\Big|\!\Big|^2 \le C_{\rm apx}^2\Big\|h^{1/2}\sum_{j=1}^N v_j'\Big\|_{L^2(\Gamma)}^2 = C_{\rm apx}^2\sum_{j=1}^N \|h^{1/2}v_j'\|_{L^2(T_j)}^2 \le 2\,C_{\rm inv}^2C_{\rm apx}^2\sum_{j=1}^N |\!|\!|v_j|\!|\!|^2.$$

This concludes the proof, where $C_4, C_5 \geq 1$ follows from the equalities in (4.3) if $v_j = 0$ for all $j = 2, \ldots, N$. $\square$

Proof of Theorem 4.1. We first prove

$$C_1^{-1} \, |||\widehat{v}_h||| \leq \Big(\sum_{j=0}^{N} |||\mathbb{G}_{h,j}\widehat{v}_h|||^2 \Big)^{1/2} \leq C_2 \, |||\widehat{v}_h||| \quad \text{for all } \widehat{v}_h \in \widehat{X}_h. \tag{4.4}$$

A triangle inequality $||| \sum_{j=0}^{N} \mathbb{G}_{h,j}\widehat{v}_h ||| \leq |||\mathbb{G}_{h,0}\widehat{v}_h||| + ||| \sum_{j=1}^{N} \mathbb{G}_{h,j}\widehat{v}_h ||| \leq \sqrt{2} \big(|||\mathbb{G}_{h,0}\widehat{v}_h|||^2 + ||| \sum_{j=1}^{N} \mathbb{G}_{h,j}\widehat{v}_h |||^2 \big)^{1/2}$ and Lemma 4.4 prove

$$\Big|\Big|\Big| \sum_{j=0}^{N} \mathbb{G}_{h,j}\widehat{v}_h \Big|\Big|\Big| \leq \sqrt{2}\, C_5 \Big(\sum_{j=0}^{N} |||\mathbb{G}_{h,j}\widehat{v}_h|||^2 \Big)^{1/2}.$$

Moreover, the symmetry of the Galerkin projection yields

$$\sum_{j=0}^{N} |||\mathbb{G}_{h,j}\widehat{v}_h|||^2 = \sum_{j=0}^{N} \langle\!\langle \mathbb{G}_{h,j}\widehat{v}_h \,, \widehat{v}_h \rangle\!\rangle \leq \Big|\Big|\Big| \sum_{j=0}^{N} \mathbb{G}_{h,j}\widehat{v}_h \Big|\Big|\Big| \, |||\widehat{v}_h|||.$$

The combination of the last two estimates proves the upper bound in (4.4)

$$\Big(\sum_{j=0}^{N} |||\mathbb{G}_{h,j}\widehat{v}_h|||^2 \Big)^{1/2} \leq \sqrt{2}\, C_5 \, |||\widehat{v}_h|||.$$

To prove the lower bound, we note that Lemma 4.2 implies $\widehat{v}_h = \sum_{j=0}^{N} \lambda_j \varphi_j$ with $\lambda_0 := 1$, $\varphi_0 := I_h \widehat{v}_h$, and appropriate coefficients $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$. Therefore, the Cauchy inequality proves

$$|||\widehat{v}_h|||^2 = \sum_{j=0}^{N} \langle\!\langle \widehat{v}_h \,, \lambda_j \varphi_j \rangle\!\rangle = \sum_{j=0}^{N} \langle\!\langle \mathbb{G}_{h,j}\widehat{v}_h \,, \lambda_j \varphi_j \rangle\!\rangle \leq \Big(\sum_{j=0}^{N} |||\mathbb{G}_{h,j}\widehat{v}_h|||^2 \Big)^{1/2} \Big(\sum_{j=0}^{N} |||\lambda_j \varphi_j||| \Big)^{1/2},$$

and it remains to dominate the second sum on the right-hand side by $|||\widehat{v}_h|||$. Lemma 4.4 proves $\big(\sum_{j=1}^{N} |||\lambda_j \varphi_j|||^2 \big)^{1/2} \leq C_4 \big|\big|\big| \sum_{j=1}^{N} \lambda_j \varphi_j \big|\big|\big| = C_4 |||(1 - I_h)\widehat{v}_h|||$, whence

$$\sum_{j=0}^{N} |||\lambda_j \varphi_j|||^2 \leq |||I_h \widehat{v}_h|||^2 + C_4^2 |||(1 - I_h)\widehat{v}_h|||^2 \leq 2|||\widehat{v}_h|||^2 + (2 + C_4^2)|||(1 - I_h)\widehat{v}_h|||^2.$$

Now, Lemma 4.3 provides

$$\Big(\sum_{j=0}^{N} |||\lambda_j \varphi_j|||^2 \Big)^{1/2} \leq \big(2 + (2 + C_4^2) C_3^2 \big)^{1/2} |||\widehat{v}_h|||$$

and thus concludes the proof of (4.4). Finally, we simply apply (4.4) for $\widehat{v}_h = \widehat{u}_h - u_h \in \widehat{X}_h$, where the term for $j = 0$ vanishes due to $\mathbb{G}_h \widehat{u}_h = u_h$. This proves (4.1), and it only remains to verify (4.2): The second equality $\langle\!\langle \widehat{u}_h - u_h \,, \varphi_j \rangle\!\rangle = \langle f - W u_h \,, \varphi_j \rangle$ follows from the Galerkin equation (2.10) for $\widehat{X}_h$ and the definition of the energy scalar product. The first equality in (4.2) follows from the explicit representation of the orthogonal projection $\mathbb{G}_{h,j}$ which reads

$$\mathbb{G}_{h,j}v = \frac{\langle\!\langle v\,,\varphi_j\rangle\!\rangle}{|||\varphi_j|||^2}\,\varphi_j \quad \text{for all } v \in \mathcal{H}.$$

This concludes the proof. $\square$

5. Error Estimation by Averaging on Large Patches

The estimators η_S and η_T from Section 3 and 4 are unsatisfactory in the sense that we have to compute the Galerkin data for $\widehat{X}_h$, but we only control the error with respect to X_h. This is different for the averaging error estimator discussed in this section, where $\widehat{X}_h = \mathcal{S}^1_0(\mathcal{T}_{h/2})$ and where we aim to control $|||u - \widehat{u}_h|||$: For a given partition $\mathcal{T}_h = \{T_1, \dots, T_N\}$ of Γ and $\mathcal{T}_{h/2}$ its uniform refinement, we use the spaces

$$X_{h/2} := \mathcal{S}^1_0(\mathcal{T}_{h/2}) \quad \text{and} \quad \overline{X}_h := \mathcal{S}^2_0(\mathcal{T}_h) \tag{5.1}$$

with corresponding Galerkin solutions $u_{h/2} \in X_{h/2}$ and $\overline{u}_h \in \overline{X}_h$, respectively. We consider the error estimator

$$\eta_A := |||u_{h/2} - \overline{\mathbb{G}}_h u_{h/2}|||, \tag{5.2}$$

where $\overline{\mathbb{G}}_h$ denotes the Galerkin projection onto $\overline{X}_h$. We stress, however, that because of $\overline{\mathbb{G}}_h$, this error estimator is computationally expensive. The following theorem is proven in [7, 8].

Theorem 5.1. *We define the constants*

$$q_A := \frac{|||u - \overline{u}_h|||}{|||u - u_{h/2}|||} \quad \textit{and} \quad \lambda_A := \max_{\overline{v}_h \in \overline{X}_h} \min_{v_{h/2} \in X_{h/2}} \frac{|||\overline{v}_h - v_{h/2}|||}{|||\overline{v}_h|||}. \tag{5.3}$$

Then, the error estimator η_A is efficient

$$\eta_A \le (1 + q_A)\, |||u - u_{h/2}|||. \tag{5.4}$$

Provided that $\lambda_A^2 + q_A^2 < 1$, there even holds reliability

$$|||u - u_{h/2}||| \le \frac{1}{\sqrt{1-\lambda_A^2} - q_A}\,\eta_A. \qquad \square \tag{5.5}$$

Remark 3. *Note that $\lambda_A^2 + q_A^2 < 1$ is a strong assumption which can hardly be checked in practice. If the exact solution u is sufficiently smooth or if the mesh is appropriately graded towards the singularities of u, there holds, however, $q_A \to 0$. Moreover, instead of $\mathcal{T}_{h/2}$, one might consider $\mathcal{T}_H$ obtained from ℓ uniform refinements of $\mathcal{T}_h$, i.e. $\|H/h\|_{L^\infty(\Gamma)} = 2^{-\ell}$. In this case, Lemma 2.2 proves*

$$\max_{\overline{v}_h \in \overline{X}_h} \min_{v_H \in X_H} \frac{|||\overline{v}_h - v_H|||}{|||\overline{v}_h|||} = \max_{\overline{v}_h \in \overline{X}_h} \frac{|||(1-\mathbb{G}_H)\overline{v}_h|||}{|||\overline{v}_h|||} \le 2^{-\ell/2}\, C_{\rm inv} C_{\rm apx} < 1$$

for ℓ sufficiently large. The numerical experiments in [7] *give experimental evidence that $\ell = 1$, i.e. $H = h/2$, is sufficient.* $\square$

The same arguments as in the proof of Theorem 3.2 apply to the localization of the averaging error estimator η_A. We stress that the estimates (5.7) as well as equivalence of

η_A and μ_A have already been proven in [7, Corollary 4.3, 4.4] and [8, Theorem 7.2]. In the preceding works, equivalence of η_A and $\widetilde{\mu}_A$, however, is only observed numerically. The proof of which as well as the equivalence of η_A and the $h-h/2$-error estimator η_{SH} is a new contribution.

Theorem 5.2. *Besides the averaging error estimator η_A, we define*

$$\mu_A := \|h^{1/2}(u_{h/2} - \overline{\mathbb{G}}_h u_{h/2})'\|_{L^2(\Gamma)} \quad \text{and} \quad \widetilde{\mu}_A := \|h^{1/2}(u'_{h/2} - \overline{\Pi}_h u'_{h/2})\|_{L^2(\Gamma)}, \tag{5.6}$$

where $\overline{\Pi}_h$ denotes the L^2-orthogonal projection onto the space $\mathcal{P}^1(\mathcal{T}_h)$ of $\mathcal{T}_h$-piecewise affine functions. Then, there holds

$$\widetilde{\mu}_A \le \mu_A \le \sqrt{2}\, C_{\rm inv}\, \eta_A. \tag{5.7}$$

Moreover, with the $h-h/2$-error estimators

$$\eta_{SH} := |||u_{h/2} - u_h||| \quad \text{and} \quad \widetilde{\mu}_{SH} := \|h^{1/2}(u_{h/2} - I_h u_{h/2})'\|_{L^2(\Gamma)} \tag{5.8}$$

there holds

$$\eta_A \le \eta_{SH} \le C_{\rm apx}\widetilde{\mu}_{SH} \quad \text{as well as} \quad \widetilde{\mu}_{SH} = 2\,\widetilde{\mu}_A. \tag{5.9}$$

In particular, all error estimators are equivalent.

Proof. To prove (5.7), we recall the arguments of [8, Theorem 7.2]: Note that $\overline{\mathbb{G}}_h u_{h/2} \in \mathcal{S}^2_0(\mathcal{T}_h)$ and thus $(\overline{\mathbb{G}}_h u_{h/2})' \in \mathcal{P}^1(\mathcal{T}_h)$. Since $\overline{\Pi}_h$ is even the $\mathcal{T}_h$-elementwise orthogonal projection onto piecewise affine functions, this proves

$$h_{T_j}\|u'_{h/2} - \overline{\Pi}_h u'_{h/2}\|_{L^2(T_j)} \le h_{T_j}\|u'_{h/2} - (\overline{\mathbb{G}}_h u_{h/2})'\|_{L^2(T_j)} \quad \text{for all } T_j \in \mathcal{T},$$

whence $\widetilde{\mu}_A \le \mu_A$. The estimate $\mu_A \le \sqrt{2}\,C_{\rm inv}\eta_A$ follows from the inverse estimate applied to $\mathcal{S}^2_0(\mathcal{T}_{h/2})$. This concludes the verification of (5.7), and we proceed with the proof of (5.9): According to $u_h \in \mathcal{S}^1_0(\mathcal{T}_h) \subset \mathcal{S}^2_0(\mathcal{T}_h)$, the best approximation property of the Galerkin projection $\overline{\mathbb{G}}_h$ onto $\mathcal{S}^2_0(\mathcal{T}_h)$ implies $\eta_A \le \eta_{SH}$, and $\eta_{SH} \le C_{\rm apx}\widetilde{\mu}_{SH}$ has been proven in Theorem 3.2 above. It thus only remains to prove $\widetilde{\mu}_{SH} = 2\,\widetilde{\mu}_A$. Recall that $(I_h v)' = \Pi_h v'$, where Π_h denotes the L^2-orthogonal projection onto $\mathcal{T}_h$-piecewise constant functions. Therefore, we only need to show the $\mathcal{T}_h$-elementwise equality

$$\|u'_{h/2} - \Pi_h u'_{h/2}\|_{L^2(T_j)} = 2\,\|u'_{h/2} - \overline{\Pi}_h u'_{h/2}\|_{L^2(T_j)} \quad \text{for all } T_j \in \mathcal{T}.$$

This equality, however, has been verified for any $\psi_{h/2} \in \mathcal{P}^0(\mathcal{T}_{h/2})$ in [15, Proof of Theorem 5.5], for instance, for $\psi_{h/2} = u'_{h/2}$. ∎

6. Implementational Aspects

6.1. Computation of Galerkin Solutions. The entries of the Galerkin matrix are computed by use of

$$\langle Wu\,, v\rangle = \langle Vu'\,, v'\rangle \quad \text{for all } u, v \in H^1(\Gamma). \tag{6.1}$$

For $u_h, v_h \in X_h = \mathcal{S}^1_0(\mathcal{T}_h)$, the arc-length derivatives satisfy $u'_h, v'_h \in \mathcal{P}^0(\mathcal{T}_h)$. To compute the Galerkin matrix $\mathbf{A}$, one thus has to compute double integrals of the type $I_{jk} = \int_{T_j}\int_{T_k} \log|x-y|\,ds_y\,ds_x$, for two elements $T_j, T_k \in \mathcal{T}_h$ with lengths $h_j, h_k > 0$, respectively. Although these entries can be computed analytically [21], the analytic formulae appear to become numerically unstable for $\min\{h_j, h_k\} \ll \max\{h_j, h_k\}$ due to cancellation effects. We thus found that it is an issue of stability to use numerical quadrature for the outer integral for certain farfield entries: To be more precise, we ensure $h_k \geq h_j$ for the corresponding element-widths by changing the order of integration, if necessary. Provided that the Euclidean distance between T_j and T_k satisfies $\mathrm{dist}(T_j, T_k) \geq h_j$, we use a Gaussian quadrature rule for the outer integral over T_j, whereas the inner integral is computed analytically [21]. In case of $\mathrm{dist}(T_j, T_k) < h_j$, we use the analytic formulae of [21] to compute the double integral. By use of common techniques, it can be shown that this procedure leads to an approximate matrix $\widetilde{\mathbf{A}}$, such that the approximation error $\|\mathbf{A} - \widetilde{\mathbf{A}}\|_F$ tends to zero exponentially fast with the order p of the Gaussian quadrature. Throughout, we used $p = 16$.

If $\Gamma \subsetneqq \partial\Omega$ is an open boundary piece, there holds $N := \#\mathcal{T}_h = \#\mathcal{K}_h - 1$. Let $\mathcal{K}_h = \{z_0, \ldots, z_N\}$ with $z_0, z_N \in \partial\Gamma$. Then, a basis of $\mathcal{S}_0^1(\mathcal{T}_h)$ is given by $\mathcal{B} := \{\phi_1, \ldots, \phi_{N-1}\}$, where $\phi_k \in \mathcal{S}^1(\mathcal{T}_h)$ is the hat function associated with $z_k \in \mathcal{K}_h$.

If $\Gamma = \partial\Omega$ is closed, there holds $N := \#\mathcal{T}_h = \#\mathcal{K}_h$. With $\mathcal{K}_h = \{z_1, \ldots, z_N\}$, a basis of $\mathcal{S}_0^1(\mathcal{T}_h)$ is again given by $\mathcal{B} := \{\phi_1, \ldots, \phi_{N-1}\}$.

As basis of $\mathcal{S}_0^1(\mathcal{T}_{h/2})$ or $\mathcal{S}_0^2(\mathcal{T}_h)$, we use the hierarchical basis $\widehat{\mathcal{B}} := \{\phi_1, \ldots, \phi_{N-1}, \varphi_1, \ldots, \varphi_N\}$, where φ_j are the functions from Section 4, cf. Figure 1: In case of $\mathcal{S}_0^1(\mathcal{T}_{h/2})$, $\varphi_j \in \mathcal{S}_0^1(\mathcal{T}_{h/2})$ denotes the hat function with respect to the midpoint $m_j \in \mathcal{K}_{h/2}$ of some element $T_j \in \mathcal{T}_h$. In case of $\mathcal{S}_0^2(\mathcal{T}_h)$, $\varphi_j = \phi_k\phi_\ell$ denotes the element bubble associated with some element $T_j \in \mathcal{T}_h$ with nodes $z_k, z_\ell \in \mathcal{K}_h$.

6.2. Overview on Introduced Error Estimators. Let $\mathcal{T}_h = \{T_1, \ldots, T_N\}$ be a given triangulation of Γ and $\mathcal{T}_{h/2}$ be a uniform refinement of $\mathcal{T}_h$. Together with the spaces

$$X_h = \mathcal{S}_0^1(\mathcal{T}_h), \quad X_{h/2} = \mathcal{S}_0^1(\mathcal{T}_{h/2}), \quad \text{and} \quad \overline{X}_h = \mathcal{S}_0^2(\mathcal{T}_h)$$

and corresponding Galerkin solutions u_h, $u_{h/2}$, and $\overline{u}_h$, respectively, we have considered the following thirteen error estimators:

- $h - h/2$-based error estimators (Section 3 with $\widehat{X}_h = X_{h/2}$)

$$\eta_{SH} = |||u_{h/2} - u_h|||, \qquad \mu_{SH} = \|h^{1/2}(u_{h/2} - u_h)'\|_{L^2(\Gamma)},$$
$$\widetilde{\eta}_{SH} = |||u_{h/2} - I_h u_{h/2}|||, \qquad \widetilde{\mu}_{SH} = \|h^{1/2}(u_{h/2} - I_h u_{h/2})'\|_{L^2(\Gamma)},$$

- $h - h/2$-based two-level error estimator (Section 4 with $\widehat{X}_h = X_{h/2}$)

$$\eta_{TH} = \Big(\sum_{T\in\mathcal{T}_h} \eta_{TH,j}^2\Big)^{1/2} \quad \text{with} \quad \eta_{TH,j} = \frac{|\langle f - Wu_h\,, \varphi_j\rangle|}{|||\varphi_j|||},$$

- averaging-based error estimators (Section 5)

$$\eta_A = |||u_{h/2} - \overline{\mathbb{G}}_h u_{h/2}|||, \qquad \mu_A = \|h^{1/2}(u_{h/2} - \overline{\mathbb{G}}_h u_{h/2})'\|_{L^2(\Gamma)},$$
$$\widetilde{\mu}_A = \|h^{1/2}(u_{h/2}' - \overline{\Pi}_h u_{h/2}')\|_{L^2(\Gamma)},$$

- $p - (p+1)$-based error estimators (Section 3 with $\widehat{X}_h = \overline{X}_h$)

$$\eta_{SP} = |||\overline{u}_h - u_h|||, \qquad \mu_{SP} = \|h^{1/2}(\overline{u}_h - u_h)'\|_{L^2(\Gamma)},$$
$$\widetilde{\eta}_{SP} = |||\overline{u}_h - I_h\overline{u}_h|||, \qquad \widetilde{\mu}_{SP} = \|h^{1/2}(\overline{u}_h - I_h\overline{u}_h)'\|_{L^2(\Gamma)},$$

- $p-(p+1)$-based two-level error estimator (Section 4 with $\widehat{X}_h = \overline{X}_h$)

$$\eta_{TP} = \Big(\sum_{T\in\mathcal{T}_h} \eta_{TP,j}^2\Big)^{1/2} \quad \text{with} \quad \eta_{TP,j} = \frac{|\langle f - Wu_h\,,\overline{\varphi}_j\rangle|}{|\!|\!|\overline{\varphi}_j|\!|\!|}.$$

Here, I_h denotes the nodal interpolation operator, $\overline{\Pi}_h$ denotes the L^2-orthogonal projection onto the space $\mathcal{P}^1(\mathcal{T}_h)$, and $\overline{\mathbb{G}}_h$ denotes the Galerkin projection onto $\overline{X}_h$. The two-level basis functions $\varphi_j \in X_{h/2}$ and $\overline{\varphi}_j \in \overline{X}_h$ are visualized in Figure 1. Note that the eight estimators μ_{SH}, $\widetilde{\mu}_{SH}$, η_{TH}, μ_A, $\widetilde{\mu}_A$, μ_{SP}, $\widetilde{\mu}_{SP}$, and η_{TP} can be used for the marking step of the adaptive mesh-refining algorithm, whereas the other five global estimators η_{SH}, $\widetilde{\eta}_{SH}$, η_A, η_{SP}, and $\widetilde{\eta}_{SP}$ are only used for error estimation.

6.3. Implementation of Error Estimators. One major advantage of the error estimators under consideration is their easiness of implementation. Besides the Galerkin data, all expressions can be calculated analytically. This is a great advantage over, e.g., residual-based error estimators, where the implementation usually needs certain appropriate quadrature rules to compute the L^2-norm of the weakly-singular residual, cf. [2, 3, 4, 5, 6, 9, 10, 16]. In particular, the implementation of the error estimators from Section 3–5 does not need the finite-part integral representation of the hypersingular integral operator. The purpose of this section is to underline the simplicity of implementation.

6.3.1. *Estimators by Space Enrichment.* We assume that the data associated with X_h and $\widehat{X}_h$ are computed with respect to the basis $\mathcal{B}$ of X_h and the hierarchical basis $\widehat{\mathcal{B}}$ of $\widehat{X}_h$, introduced in Section 6.1. This leads to Galerkin matrices $\mathbf{A} \in \mathbb{R}^{(N-1)\times(N-1)}_{sym}$ and $\widehat{\mathbf{A}} \in \mathbb{R}^{(2N-1)\times(2N-1)}_{sym}$ as well as to right-hand side vectors $\mathbf{b} \in \mathbb{R}^{N-1}$ and $\widehat{\mathbf{b}} \in \mathbb{R}^{2N-1}$, with entries given by

$$\mathbf{A}_{k\ell} = \langle\!\langle \phi_k\,,\phi_\ell \rangle\!\rangle = \widehat{\mathbf{A}}_{k\ell} \quad \text{and} \quad \mathbf{b}_k = \langle f\,,\phi_k\rangle = \widehat{\mathbf{b}}_k$$

and

$$\widehat{\mathbf{A}}_{k,N-1+j} = \langle\!\langle \phi_k\,,\varphi_j \rangle\!\rangle = \widehat{\mathbf{A}}_{N-1+j,k}\,, \quad \widehat{\mathbf{A}}_{N-1+i,N-1+j} = \langle\!\langle \varphi_i\,,\varphi_j \rangle\!\rangle\,, \quad \text{and} \quad \widehat{\mathbf{b}}_{N-1+i} = \langle f\,,\varphi_i\rangle$$

for all indices $i,j = 1,\dots,N$ and $k,\ell = 1,\dots,N-1$. Throughout this section, we identify a vector $\mathbf{x} \in \mathbb{R}^{N-1}$ with its trivial extension $\mathbf{x} \in \mathbb{R}^{2N-1}$ if necessary.

Let $\mathbf{x} \in \mathbb{R}^{N-1}$ be the coefficient vector of the computed solution $u_h \in X_h$ with respect to $\mathcal{B}$, or with respect to $\widehat{\mathcal{B}}$ if we consider its trivial extension, i.e. $u_h = \sum_{k=1}^{N-1} \mathbf{x}_k\phi_k$ and $\mathbf{A}\mathbf{x} = \mathbf{b}$. Let $\widehat{\mathbf{x}} \in \mathbb{R}^{2N-1}$ be the coefficient vector of the computed solution $\widehat{u}_h \in \widehat{X}_h$ with respect to $\widehat{\mathcal{B}}$, i.e $\widehat{u}_h = \sum_{k=1}^{N-1} \widehat{\mathbf{x}}_k\phi_k + \sum_{j=1}^{N} \widehat{\mathbf{x}}_{N-1+j}\varphi_j$ and $\widehat{\mathbf{A}}\widehat{\mathbf{x}} = \widehat{\mathbf{b}}$. Note that the coefficient vector $\mathbf{y} \in \mathbb{R}^{N-1}$ of the nodal interpolation $I_h\widehat{u}_h$ is given by $\mathbf{y}_j := \widehat{\mathbf{x}}_j$ for $j = 1,\dots,N-1$.

With the introduced notation, the error estimators η_S and $\widetilde{\eta}_S$ simply read

$$\eta_S^2 = \langle\!\langle \widehat{u}_h - u_h\,,\widehat{u}_h - u_h \rangle\!\rangle = \big(\widehat{\mathbf{A}}(\widehat{\mathbf{x}} - \mathbf{x})\big)\cdot\big(\widehat{\mathbf{x}} - \mathbf{x}\big) \quad \text{and} \quad \widetilde{\eta}_S^2 = \big(\widehat{\mathbf{A}}(\widehat{\mathbf{x}} - \mathbf{y})\big)\cdot\big(\widehat{\mathbf{x}} - \mathbf{y}\big),$$

where the dot denotes the Euclidean scalar product in $\mathbb{R}^{2N-1}$. The computation of the error estimators μ_S and $\widetilde{\mu}_S$ is done elementwise

$$\mu_S^2 = \sum_{j=1}^{N} \mu_{S,j}^2 \quad \text{and} \quad \widetilde{\mu}_S^2 = \sum_{j=1}^{N} \widetilde{\mu}_{S,j}^2,$$

where the local refinement indicators are given by

$$\mu_{S,j}^2 := h_j \|(\widehat{u}_h - u_h)'\|_{L^2(T_j)}^2 \quad \text{and} \quad \widetilde{\mu}_{S,j}^2 := h_j \|(\widehat{u}_h - I_h \widehat{u}_h)'\|_{L^2(T_j)}^2$$

with $h_j > 0$ the length of $T_j \in \mathcal{T}_h$. Let $\overline{T}_j = \operatorname{conv}\{z_k, z_\ell\}$ with $z_k, z_\ell \in \mathcal{K}_h$. Note that the computation of the H^1-seminorm on T_j only involves the hat functions $\phi_k, \phi_\ell \in X_h$ as well as the hierarchical basis function φ_j. Let $\widehat{v}_h \in \{\widehat{u}_h - u_h,\, \widehat{u}_h - I_h\widehat{u}_h\}$ satisfy $\widehat{v}_h = \mathbf{z}_1^{(j)}\phi_k + \mathbf{z}_2^{(j)}\phi_\ell + \mathbf{z}_3^{(j)}\varphi_j$ on T_j. Then, $h_j\|\widehat{v}_h'\|_{L^2(T_j)}^2 = \mathbf{z}^{(j)} \cdot \mathbf{M}\mathbf{z}^{(j)}$ with the matrix

$$\mathbf{M} = h_j \begin{pmatrix} \|\phi_k'\|_{L^2(T_j)}^2 & \langle \phi_k' \,, \phi_\ell' \rangle_{L^2(T_j)} & \langle \phi_k' \,, \varphi_j' \rangle_{L^2(T_j)} \\ \langle \phi_k' \,, \phi_\ell' \rangle_{L^2(T_j)} & \|\phi_\ell'\|_{L^2(T_j)}^2 & \langle \phi_\ell' \,, \varphi_j' \rangle_{L^2(T_j)} \\ \langle \phi_k' \,, \varphi_j' \rangle_{L^2(T_j)} & \langle \phi_\ell' \,, \varphi_j' \rangle_{L^2(T_j)} & \|\varphi_j'\|_{L^2(T_j)}^2 \end{pmatrix} \in \mathbb{R}^{3\times 3}_{sym}.$$

Elementary calculations for $\widehat{X}_h = \mathcal{S}_0^1(\mathcal{T}_{h/2})$ and $\widehat{X}_h = \mathcal{S}_0^2(\mathcal{T}_h)$ yield

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \text{resp.} \quad \mathbf{M} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Note that the local coefficient vector reads $\mathbf{z}^{(j)} = (\widehat{\mathbf{x}}_k - \mathbf{x}_k, \widehat{\mathbf{x}}_\ell - \mathbf{x}_\ell, \widehat{\mathbf{x}}_{N-1+j})$ in case of $\widehat{v}_h = \widehat{u}_h - u_h$, whereas $\mathbf{z}^{(j)} = (0, 0, \widehat{\mathbf{x}}_{N-1+j})$ for $\widehat{v}_h = \widehat{u}_h - I_h\widehat{u}_h$. In particular, this results in $\widetilde{\mu}_{S,j} = 2\,|\widehat{\mathbf{x}}_{N-1+j}|$ for $\widehat{X}_h = \mathcal{S}_0^1(\mathcal{T}_{h/2})$ and $\widetilde{\mu}_{S,j} = \sqrt{1/3}\,|\widehat{\mathbf{x}}_{N-1+j}|$ for $\widehat{X}_h = \mathcal{S}_0^2(\mathcal{T}_h)$, respectively.

6.3.2. *Two-Level Error Estimators.* The computation of the local contributions of the two-level error estimator η_T simply reads

$$\eta_{T,j} = \frac{|\langle f \,, \varphi_j \rangle - \langle\!\langle u_h \,, \varphi_j \rangle\!\rangle|}{|\!|\!| \varphi_j |\!|\!|} = \frac{|\widehat{\mathbf{b}}_{N-1+j} - (\widehat{\mathbf{A}}\mathbf{x})_{N-1+j}|}{(\widehat{\mathbf{A}}_{N-1+j,N-1+j})^{1/2}} = \frac{|(\widehat{\mathbf{b}} - \widehat{\mathbf{A}}\mathbf{x})_{N-1+j}|}{(\widehat{\mathbf{A}}_{N-1+j,N-1+j})^{1/2}},$$

where we use the same notation as in Section 6.3.1.

6.3.3. *Averaging Error Estimators.* Contrary to the previous estimators, the implementational treatment of the averaging error estimators η_A and μ_A needs both refined spaces $\widehat{X}_h := \mathcal{S}_0^1(\mathcal{T}_{h/2})$ as well as $\overline{X}_h := \mathcal{S}_0^2(\mathcal{T}_h)$. With respect to the mesh $\mathcal{T}_h$, we denote by $\varphi_j \in \widehat{X}_h$ the hat functions for the element midpoints, whereas $\overline{\varphi}_j \in \overline{X}_h$ denote the element bubble functions. Using the same ideas as in Section 6.3.1, we obtain hierarchical bases $\widehat{\mathcal{B}}$ and $\overline{\mathcal{B}}$. This leads to Galerkin data $\widehat{\mathbf{A}}$, $\widehat{\mathbf{b}}$ with respect to $\widehat{X}_h$, and the coefficient vector of $\widehat{u}_h$ solves $\widehat{\mathbf{A}}\widehat{x} = \widehat{\mathbf{b}}$. To compute the Galerkin projection $\overline{\mathbb{G}}_h\widehat{u}_h \in \overline{X}_h$, we assemble the Galerkin matrix $\overline{\mathbf{A}}$ with respect to $\overline{\mathcal{B}}$ as well as the (non-symmetric) Galerkin-type matrix $\mathbf{B} \in \mathbb{R}^{(2N-1)\times(2N-1)}$ defined by

$$\mathbf{B}_{k\ell} = \langle\!\langle \phi_\ell \,, \phi_k \rangle\!\rangle,\ \mathbf{B}_{N-1+i,\ell} = \langle\!\langle \phi_\ell \,, \overline{\varphi}_i \rangle\!\rangle,\ \mathbf{B}_{k,N-1+j} = \langle\!\langle \varphi_j \,, \phi_k \rangle\!\rangle,\ \mathbf{B}_{N-1+i,N-1+j} = \langle\!\langle \varphi_j \,, \overline{\varphi}_i \rangle\!\rangle,$$

for $i, j = 1, \dots, N$ and $k, \ell = 1, \dots, N-1$. Note that all but the last block have already been assembled for either $\widehat{\mathbf{A}}$ or $\overline{\mathbf{A}}$. Then, the identities $\langle\!\langle \overline{\mathbb{G}}_h\widehat{u}_h \,, \phi_k \rangle\!\rangle = \langle\!\langle \widehat{u}_h \,, \phi_k \rangle\!\rangle$ and $\langle\!\langle \overline{\mathbb{G}}_h\widehat{u}_h \,, \overline{\varphi}_i \rangle\!\rangle = \langle\!\langle \widehat{u}_h \,, \overline{\varphi}_i \rangle\!\rangle$ prove that the coefficient vector $\overline{\mathbf{x}}$ of $\overline{\mathbb{G}}_h\widehat{u}_h$ solves $\overline{\mathbf{A}}\overline{\mathbf{x}} = \mathbf{B}\widehat{\mathbf{x}}$.

Due to the Galerkin orthogonality, there holds

$$\eta_A^2 = |\!|\!| \widehat{u}_h - \overline{\mathbb{G}}_h\widehat{u}_h |\!|\!|^2 = |\!|\!| \widehat{u}_h |\!|\!|^2 - |\!|\!| \overline{\mathbb{G}}_h\widehat{u}_h |\!|\!|^2 = \widehat{\mathbf{x}} \cdot \widehat{\mathbf{A}}\widehat{\mathbf{x}} - \overline{\mathbf{x}} \cdot \overline{\mathbf{A}}\overline{\mathbf{x}} = \widehat{\mathbf{x}} \cdot (\widehat{\mathbf{b}} - \mathbf{B}^T\overline{\mathbf{x}}).$$

Analogously to μ_S, the error estimator μ_A is computed elementwise by

$$\mu_A^2 = \sum_{T_j \in \mathcal{T}_h} \mu_{A,j}^2, \quad \text{where} \quad \mu_{A,j}^2 := h_j \|(\widehat{u}_h - \overline{\mathbb{G}}_h \widehat{u}_h)'\|_{L^2(T_j)}^2.$$

Let $T_j \in \mathcal{T}_h$ be a fixed element with nodes z_k, z_ℓ. Then, we have $\widehat{u}_h - \overline{\mathbb{G}}_h \widehat{u}_h = \mathbf{z}_1^{(j)} \phi_k + \mathbf{z}_2^{(j)} \phi_\ell + \mathbf{z}_3^{(j)} \varphi_j + \mathbf{z}_4^{(j)} \overline{\varphi}_j$ on T_j with the coefficient vector $\mathbf{z}^{(j)} = (\widehat{\mathbf{x}}_k - \overline{\mathbf{x}}_k, \widehat{\mathbf{x}}_\ell - \overline{\mathbf{x}}_\ell, \widehat{\mathbf{x}}_{N-1+j}, \overline{\mathbf{x}}_{N-1+j})$. Then, $\mu_{A,j}^2 = \mathbf{z}^{(j)} \cdot \widehat{\mathbf{M}} \mathbf{z}^{(j)}$ with the matrix

$$\widehat{\mathbf{M}} := h_j \begin{pmatrix} \|\phi_k'\|_{L^2(T_j)}^2 & \langle \phi_k', \phi_\ell' \rangle_{L^2(T_j)} & \langle \phi_k', \varphi_j' \rangle_{L^2(T_j)} & \langle \phi_k', \overline{\varphi}_j' \rangle_{L^2(T_j)} \\ \langle \phi_k', \phi_\ell' \rangle_{L^2(T_j)} & \|\phi_\ell'\|_{L^2(T_j)}^2 & \langle \phi_\ell', \varphi_j' \rangle_{L^2(T_j)} & \langle \phi_\ell', \overline{\varphi}_j' \rangle_{L^2(T_j)} \\ \langle \phi_k', \varphi_j' \rangle_{L^2(T_j)} & \langle \phi_\ell', \varphi_j' \rangle_{L^2(T_j)} & \|\varphi_j'\|_{L^2(T_j)}^2 & \langle \varphi_j', \overline{\varphi}_j' \rangle_{L^2(T_j)} \\ \langle \phi_k', \overline{\varphi}_j' \rangle_{L^2(T_j)} & \langle \phi_\ell', \overline{\varphi}_j' \rangle_{L^2(T_j)} & \langle \varphi_j', \overline{\varphi}_j' \rangle_{L^2(T_j)} & \|\varphi_j'\|_{L^2(T_j)}^2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{pmatrix}.$$

Finally, it remains to compute the error estimator

$$\widetilde{\mu}_A^2 = \sum_{T_j \in \mathcal{T}_h} \widetilde{\mu}_{A,j}^2, \quad \text{where} \quad \widetilde{\mu}_{A,j}^2 = h_j \|\widehat{u}_h - \overline{\Pi}_h \widehat{u}_h\|_{L^2(T_j)}^2.$$

By use of the local relation of $\widetilde{\mu}_{A,j}$ and the $h - h/2$-based error estimator $\widetilde{\mu}_{SH,j}$, we obtain

$$\widetilde{\mu}_{A,j}^2 = \frac{1}{4} \widetilde{\mu}_{SH,j}^2 = |\widehat{\mathbf{x}}_{N-1+j}|^2,$$

cf. the proof of Theorem 5.2 and Section 6.3.1 above.

6.4. Galerkin Errors and Experimental Saturation Constants. Throughout, the Galerkin errors are computed by use of the Galerkin orthogonality

$$|||u - u_h|||^2 = |||u|||^2 - |||u_h|||^2. \tag{6.2}$$

The squared energy norm of a Galerkin solution u_h reads $|||u_h|||^2 = \mathbf{A}\mathbf{x} \cdot \mathbf{x}$ with the Galerkin matrix $\mathbf{A}$ and the coefficient vector $\mathbf{x}$ corresponding to u_h. If the exact solution $u \in \mathcal{H}$ is unknown, the energy $|||u|||^2$ is extrapolated by Aitkin's Δ^2-method as follows: For a sequence $\mathcal{T}_h^{(k)}$ of uniformly refined meshes, we compute the sequence of energies $E_k = |||u_h^{(k)}|||^2$, where $u_h^{(k)}$ denotes the discrete solution corresponding to the triangulation $\mathcal{T}_h^{(k)}$. We found that Δ^2-extrapolation of the sequence E_k then yields a sufficiently accurate approximation of $|||u|||^2$.

In particular, (6.2) allows to compute the experimental saturation constants

$$q_{SH} := \frac{|||u - u_{h/2}|||}{|||u - u_h|||}, \quad q_{SP} := \frac{|||u - \overline{u}_h|||}{|||u - u_h|||}, \quad \text{and} \quad q_A := \frac{|||u - \overline{u}_h|||}{|||u - u_{h/2}|||} = q_{SP}/q_{SH}. \tag{6.3}$$

The computation of the constant λ_A from (5.3) leads to a generalized eigenvalue problem

$$\lambda_A^2 = \max_{\overline{v}_h \in \overline{X}_h} \frac{|||\overline{v}_h - \mathbb{G}_{h/2} \overline{v}_h|||^2}{|||\overline{v}_h|||^2} = \max_{\overline{v}_h \in \overline{X}_h} \frac{|||\overline{v}_h|||^2 - |||\mathbb{G}_{h/2} \overline{v}_h|||^2}{|||\overline{v}_h|||^2}, \tag{6.4}$$

which is solved by use of the Matlab function `eig`. We refer to [15, Section 6.2] for details.

6.5. Adaptive Algorithm. In the numerical experiments below, we compare uniform mesh-refinement with some indicator-steered mesh-refinement. For element marking, we use some local error estimation strategy ϱ in the sense that

$$\varrho^2 = \sum_{T_j \in \mathcal{T}_h} \varrho_j^2.$$

Then, the adaptive algorithm reads as follows:

Algorithm 6.1. Input: *Initial mesh* $\mathcal{T}_h$, *parameter* $0 < \theta \leq 1$, *error estimation strategy* ϱ.

(i) *Compute Galerkin solution.*

(ii) *Compute refinement indicators* ϱ_j *for all* $T_j \in \mathcal{T}_h = \{T_1, \dots, T_N\}$.

(iii) *Choose the minimal set* $\mathcal{M}_h \subseteq \mathcal{T}_h$ *such that*

$$\theta \sum_{T_j \in \mathcal{T}_h} \varrho_j^2 \leq \sum_{T_j \in \mathcal{M}_h} \varrho_j^2. \tag{6.5}$$

(iv) *Refine at least all marked elements* $T_j \in \mathcal{M}_h$, *generate a new mesh* $\mathcal{T}_h$, *and goto* (i).

Output: *Sequence of certain Galerkin approximations and error estimators* ϱ. □

Possible choices for ϱ are the error estimators μ_S, $\widetilde{\mu}_S$, η_T, μ_A, and $\widetilde{\mu}_A$. The marking criterion (6.5) was introduced in [13] and it is nowadays used to prove convergence and optimality of adaptive FEM [11]. Convergence of adaptive BEM is widely open. The recent work [17] shows that the saturation assumption yields convergence of the μ_{SH}- and $\widetilde{\mu}_{SH}$-steered adaptive algorithms for the $h - h/2$-strategy in the sense that the (computed) discrete solutions converge towards the (unknown) exact solution. Their arguments apply to the $p - (p+1)$-strategy without (other but notational) modifications. Convergence for the $\widetilde{\mu}_A$-based adaptive BEM immediately follows from the identity $2\widetilde{\mu}_{A,j} = \widetilde{\mu}_{S,j}$ proven in Theorem 5.2 above. Even if the saturation assumption fails to hold in general, the new concept of *estimator reduction* proves that Algorithm 6.1 steered by μ_{SH}, $\widetilde{\mu}_{SH}$, μ_{SP}, $\widetilde{\mu}_{SP}$, μ_A, or $\widetilde{\mu}_A$ drives the respective error estimator to zero [1]. Only convergence of the η_{TH}-steered and η_{TP}-steered adaptive algorithms remains in this sense mathematically open.

Note that the local mesh-ratio $\kappa(\mathcal{T}_h)$ enters critically in our localization estimate from Lemma 2.2 in the sense that $C_{\rm apx}$ may tend to infinity together with $\kappa(\mathcal{T}_h)$. Therefore, we additionally refine elements in step (iv) of Algorithm 6.1 to ensure that $\kappa(\mathcal{T}_h)$ stays bounded.

7. Numerical experiments

We consider three numerical examples for the hypersingular integral equation (1.1) for the Laplace operator on different domains. We compare uniform mesh-refinement with an indicator-based adaptive mesh-refinement. For adaptive mesh-refinement, we use Algorithm 6.1 steered by the local contributions of μ_{SH} with $\theta = 0.5$.

7.1. Slit Problem. In our first experiment, we consider

$$Wu = 1 \quad \text{on } \Gamma = (-1,1) \times \{0\}. \tag{7.1}$$

The exact solution u of (7.1) is known and reads

$$u(x,0) = 2\sqrt{1-x^2} \quad \text{for all } -1 < x < 1.$$

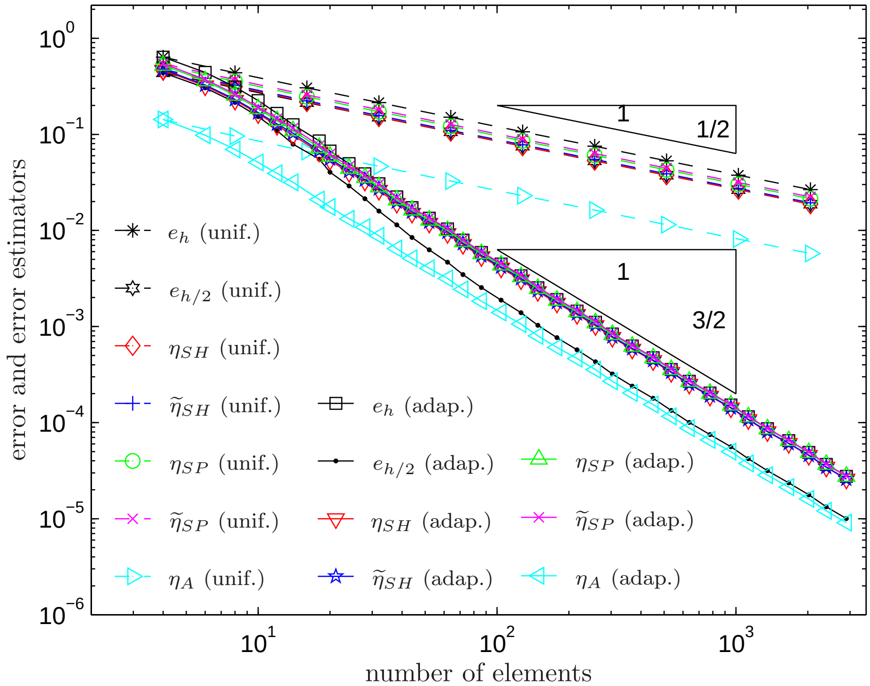

FIGURE 2. *Galerkin errors $e_h = |||u-u_h|||$ and $e_{h/2} = |||u-u_{h/2}|||$ as well as five global error estimators in Slit Problem 7.1 for uniform (unif.) and μ_{SH}-adaptive mesh-refinement (adap.). We observe an optimal grading of the adaptive meshes in the sense that the optimal order $\mathcal{O}(N^{-3/2})$ of convergence is observed for the Galerkin errors. Moreover, we stress that $e_h = |||u-u_h|||$ is accurately estimated by η_{SH}, $\widetilde{\eta}_{SH}$, η_{SP}, and $\widetilde{\eta}_{SP}$, whereas η_A is asymptotically accurate to estimate $e_{h/2} = |||u-u_{h/2}|||$.*

Direct computation yields $|||u|||^2 = \pi$. The uniform initial mesh consists of four elements. Figure 2 shows the curves of the errors $|||u-u_h|||$ and $|||u-u_{h/2}|||$ as well as the five global error estimators for both, uniform and adaptive mesh-refinement. We plot the experimental results over the number of elements N, where both axes are scaled logarithmically. Therefore, a straight line g with a slope $-\alpha$ corresponds to a dependence $g = \mathcal{O}(N^{-\alpha})$. Whereas uniform mesh-refinement leads to a poor order of convergence $\mathcal{O}(N^{-1/2})$, the adaptive strategy leads to $\mathcal{O}(N^{-3/2})$ which is optimal for the lowest-order boundary element discretization. For the adaptive strategy, Figure 3 visualizes the remaining eight error estimators.

Note that our theory is confirmed in the sense that, for one fixed mesh-refining strategy, the curves of all error estimators are parallel, i.e., all error estimators are equivalent. Moreover, we observe that the estimator curves are parallel to the error curves. By others, this gives empirical evidence for the saturation assumption.

Figure 4 shows the experimental efficiency constants. We note that the constant for the non-local estimators tends to 1, which is also observed in Figure 2. Finally, the experimental saturation constants are plotted in Figure 5. We empirically confirm the theoretical assumptions that these constants are < 1.

We stress that Algorithm 6.1 steered with any $\varrho \in \{\widetilde{\mu}_{SH}, \mu_{SP}, \widetilde{\mu}_{SP}, \eta_{TH}, \eta_{TP}, \mu_A, \widetilde{\mu}_A\}$ instead of $\varrho = \mu_{SH}$ leads to the same qualitative behaviour (not displayed).

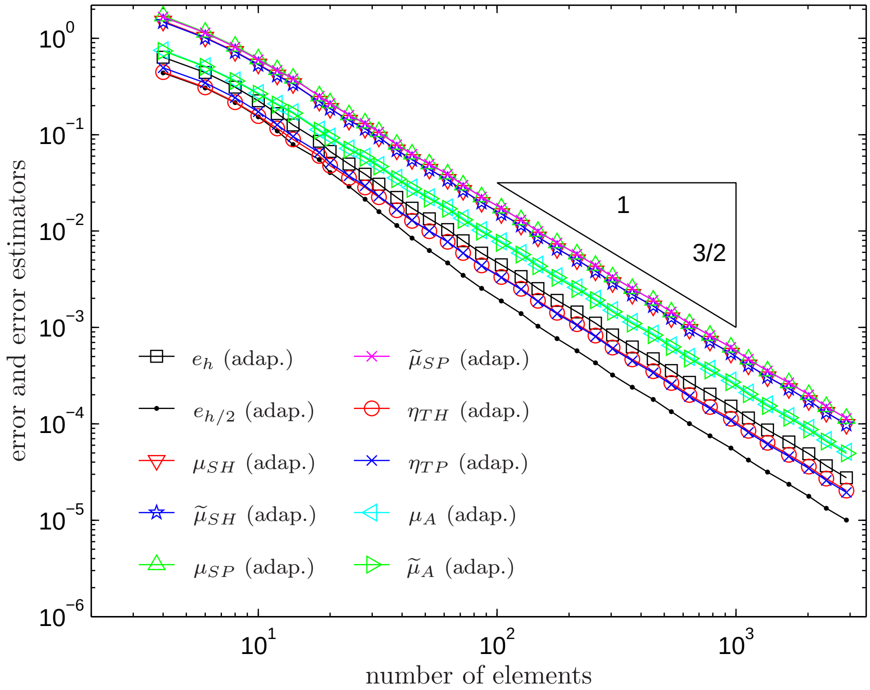

FIGURE 3. *Galerkin errors $e_h = |||u - u_h|||$ and $e_{h/2} = |||u - u_{h/2}|||$ and local error estimators in Slit Problem 7.1 for μ_{SH}-adaptive mesh-refinement.*

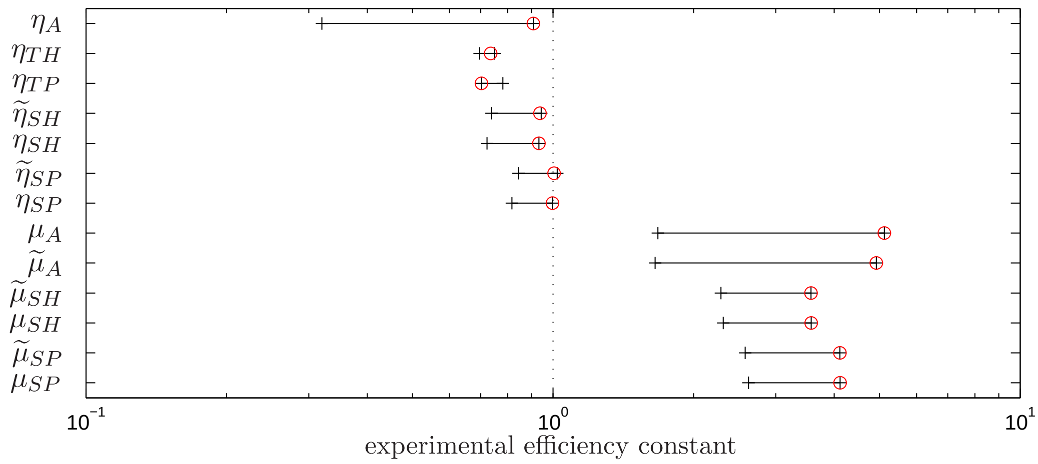

FIGURE 4. *Experimental efficiency constants $C_{\rm eff}$ of error estimators for μ_{SH}-adaptive mesh-refinement in Slit Problem 7.1: Note that the intervals indicate the range of the efficiency constant $C_{\rm eff}$, whereas the circle indicates the precise value of $C_{\rm eff}$ in the final step of the adaptive computation. For the $h - h/2$-based and $p - (p+1)$-based error estimators, we consider the ratio of error estimator and $|||u - u_h|||$, e.g., $C_{\rm eff} = \eta_{SH}/|||u - u_h|||$. Since the averaging error estimators are expected to estimate $|||u - u_{h/2}|||$, we consider the ratio of estimator and $|||u - u_{h/2}|||$ instead, e.g., $C_{\rm eff} = \eta_A/|||u - u_{h/2}|||$.*

7.2. Angle Problem. In the second experiment, we consider (1.1) with right-hand side

$$f(x,y) = (x+1)(y-1) \quad \text{on } \Gamma, \tag{7.2}$$

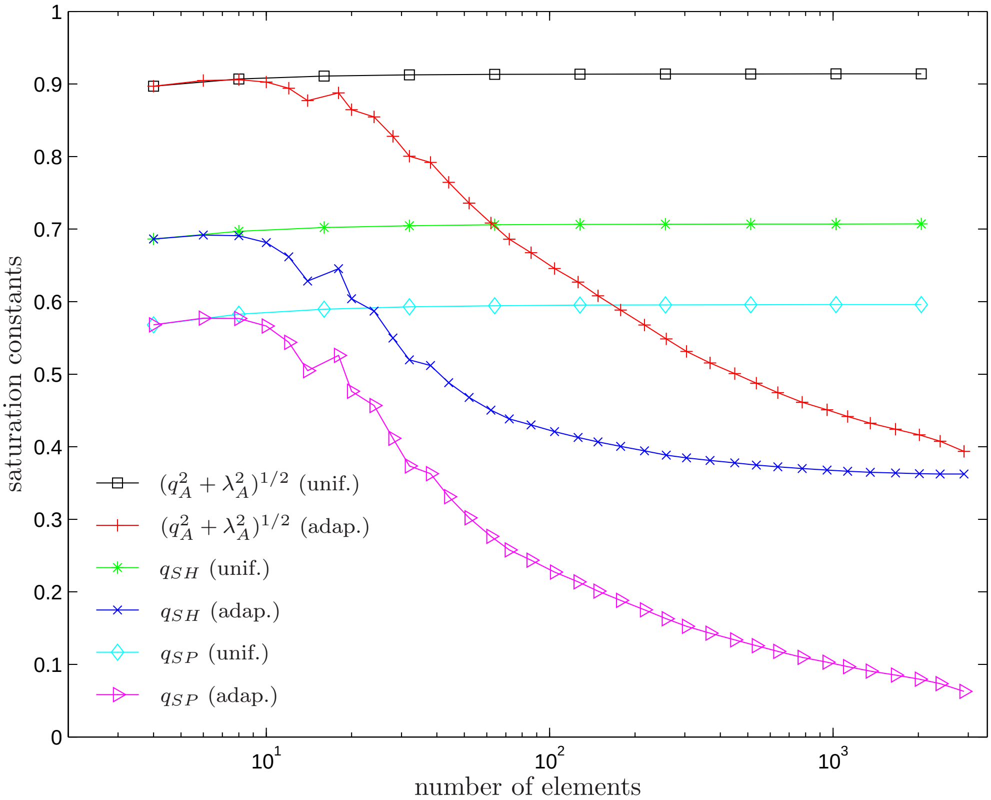

FIGURE 5. *All experimental saturation constants (cf. Section 6.4) in Slit Problem 7.1 are uniformly bounded < 1, which yields reliability of all error estimators. We stress that all constants appear to depend on the smoothness of the unknown solution u in the sense that they are improved in case of μ_{SH}-adaptive mesh-refinement.*

on an angle domain $\Gamma = (-1,0) \times \{0\} \cup \{0\} \times (0,1)$. Since the exact solution u of (1.1) is unknown, the energy $|||u|||^2 = 1.324965092745831$ is obtained by extrapolation, cf. Section 6.1.

The uniform initial mesh consists of four elements. Since the numerical observations are similar to those in Section 7.1, Figure 6 only shoes the Galerkin errors and the error estimators μ_{SH}, η_{SH}, and η_A. We observe that the adaptively generated meshes (not displayed) show some refinement towards both tips as well as to the interior angle at the point $(0,0)$.

7.3. L-Shape Problem. In our last experiment, we consider (1.1) with right-hand side

$$f(x,y) = x - c \quad \text{on } \Gamma, \tag{7.3}$$

where $\Gamma = \partial\Omega$ is the boundary of the L-shaped domain $\Omega = (-1,1)^2 \backslash [0,1]^2$. The constant $c = -1/8$ is chosen such that $\int_\Gamma f\, ds = 0$. The uniform initial mesh consists of 8 elements. The exact solution u of (1.1) is unknown, and we use the extrapolated value $|||u|||^2 = 12.95241880428523$.

Again, the outcome of our numerical experiments is similar to those of Section 7.1, and Figure 7 only shows Galerkin errors and some error estimators for uniform and adaptive mesh-refinement.

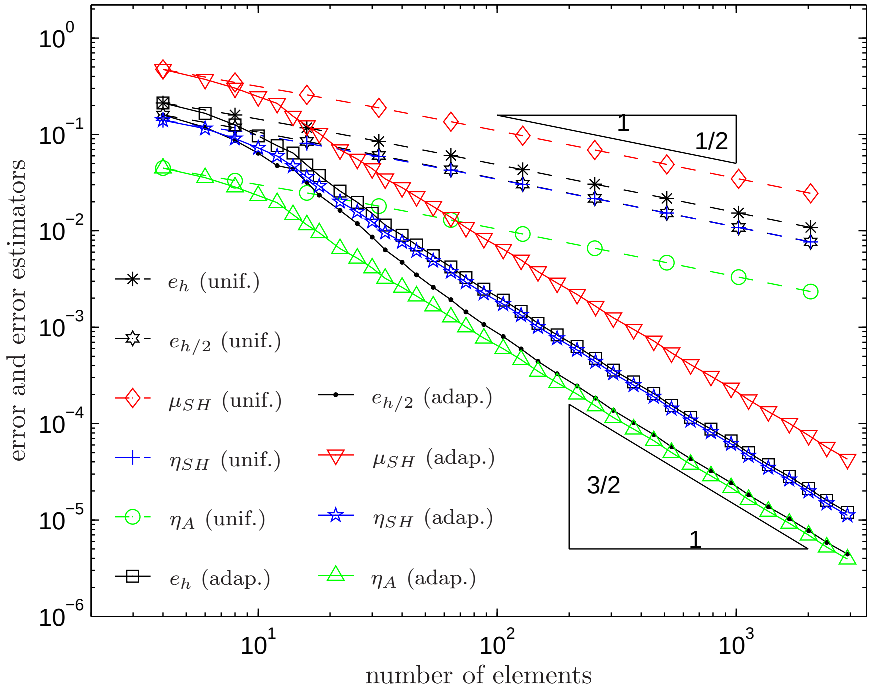

FIGURE 6. *Galerkin errors* $e_h = |||u - u_h|||$ *and* $e_{h/2} = |||u - u_{h/2}|||$ *as well as some error estimators in Problem 7.2 for uniform (unif.) and* μ_{SH}*-adaptive mesh-refinement (adap.).*

Figure 8 shows some adaptively generated meshes. We observe a strong refinement towards all five corners of Γ, and particularly to the reentrant corner at the point $(0, 0)$ between element T_4 and T_5 of Figure 8.

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References

[1] M. Aurada, S. Ferraz-Leite, D. Praetorius: *Estimator Reduction and Convergence of Adaptive FEM and BEM*, work in progress (2009).

[2] C. Carstensen: *An A Posteriori Error Estimate for a First-Kind Integral Equation*, Math. Comp. 66 (1997), 139–155.

[3] C. Carstensen: *Efficiency of A Posteriori BEM Error Estimates for First Kind Integral Equation on Quasi-Uniform Meshes*, Math. Comp. 65 (1996), 69–84.

[4] C. Carstensen, B. Faermann: *Mathematical Foundation of A Posteriori Error Estimates and Adaptive Mesh-Refining Algorithms for Boundary Integral Equations of the First Kind*, Eng. Anal. Bound. Elem. 25 (2001), 497–509.

[5] C. Carstensen, S. Funken, E. Stephan: *A Posteriori Error Estimates for h-p-Boundary Element Methods*, Appl. Anal. 61 (1996), 233–253.

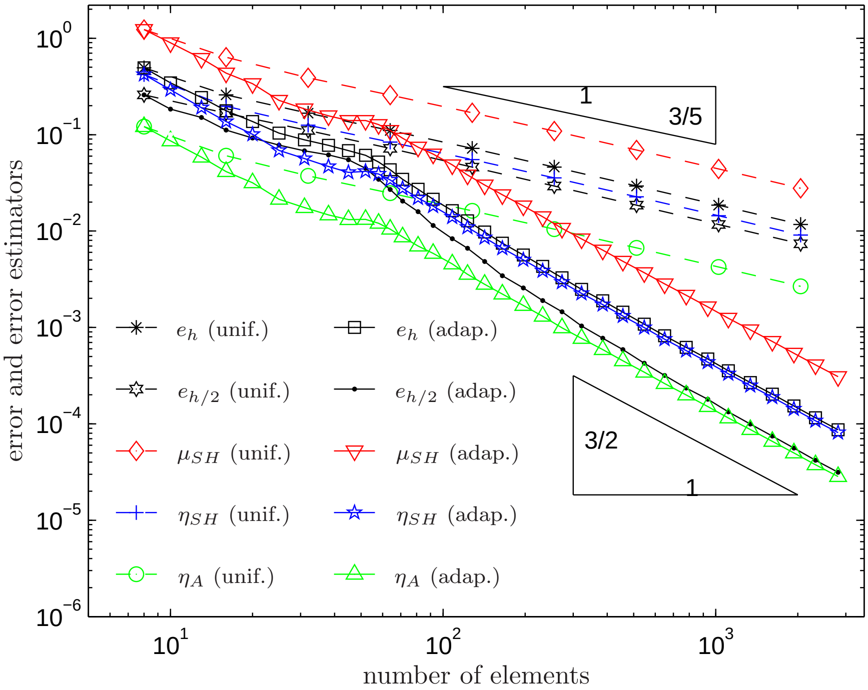

FIGURE 7. *Galerkin errors $e_h = |||u - u_h|||$ and $e_{h/2} = |||u - u_{h/2}|||$ as well as some error estimators in L-Shape Problem 7.3 for uniform (unif.) and μ_{SH}-adaptive mesh-refinement (adap.).*

[6] C. Carstensen, M. Maischak, D. Praetorius, E. Stephan: *Residual-Based A Posteriori Error Estimate for Hypersingular Equation on Surfaces*, Numer. Math. 97 (2004), 397–425.

[7] C. Carstensen, D. Praetorius: *Averaging Techniques for the A Posteriori BEM Error Control for a Hypersingular Integral Equation in Two Dimensions*, SIAM J. Sci. Comput. 29 (2007), 782–810.

[8] C. Carstensen, D. Praetorius: *Averaging Techniques for A Posteriori Error Control in Finite Element and Boundary Element Analysis*, Comput. Mech. 29 (2007), 29–59.

[9] C. Carstensen, E. Stephan: *A Posteriori Error Estimates for Boundary Element Methods*, Math. Comp. 64 (1995), 483–500.

[10] C. Carstensen, E. Stephan: *Adaptive Boundary Element Methods for Some First Kind Integral Equations*, SIAM J. Numer. Anal. 33 (1996), 1106–1124.

[11] J. Cascon, C. Kreuzer, R. Nochetto, K. Siebert: *Quasi-Optimal Convergence Rate for an Adaptive Finite Element Method*, SIAM J. Numer. Anal. 46 (2008), 2524–2550.

[12] M. Costabel: *Boundary Integral Operators on Lipschitz Domains: Elementary Results*, SIAM J. Math. Anal. 19 (1988), 613–626.

[13] W. Doerfler: *A Convergent Adaptive Algorithm for Poisson's Equation*, SIAM J. Numer. Anal. 33 (1996), 1106–1124.

[14] W. Doerfler, R. Nochetto: *Small Data Oscillation Implies the Saturation Assumption*, Numer. Math. 91 (2002), 1–12.

[15] C. Erath, S. Ferraz-Leite, S. Funken, D. Praetorius: *Energy Norm Based A Posteriori Error Estimation for Boundary Element Methods in Two Dimensions*, Appl. Numer. Math., in press (2009).

[16] B. Faermann: *Local A Posteriori Error Indicators for the Galerkin Discretization of Boundary Integral Equations*, Numer. Math. 79 (1998), 43–76.

[17] S. Ferraz-Leite, C. Ortner, D. Praetorius: *Convergence of Simple Adaptive Galerkin Schemes Based on $h - h/2$ Error Estimators*, ASC Report 06/2009, Institute for Analysis and Scientific Computing, Vienna University of Technology (2009).

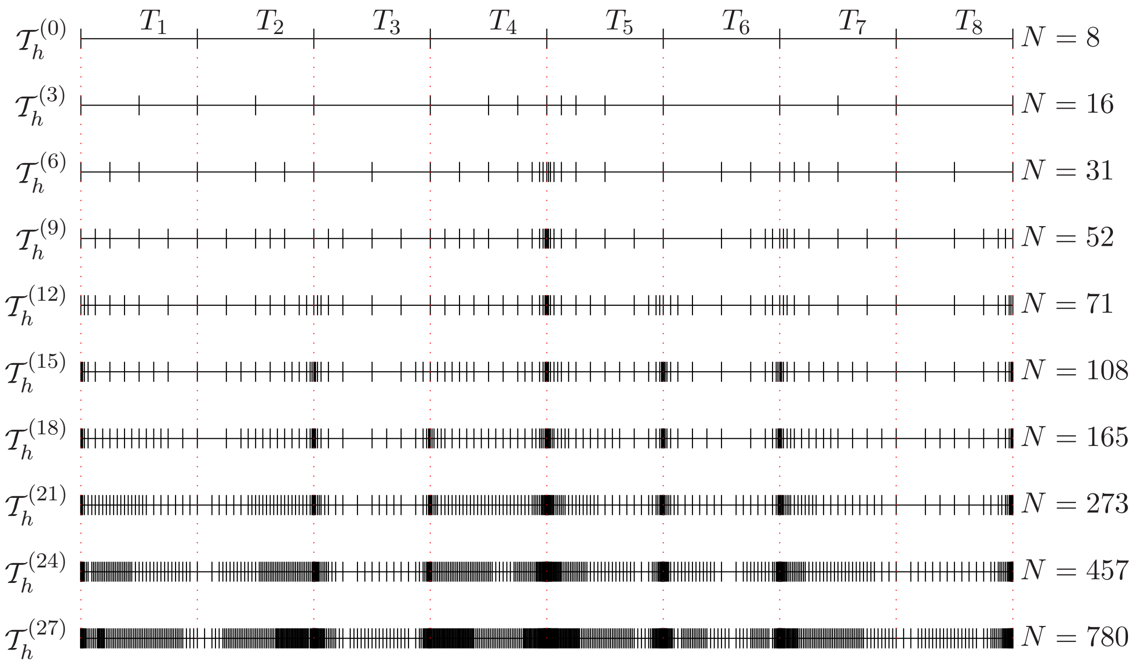

FIGURE 8. *Sequence of adaptive mesh-refinement in L-Shape Problem 7.3, plotted over the arclength $s = 0, \ldots, 8$, where arclength $s = 4$ corresponds to the reentrant corner. The sequence shows the refinement of each T_j of the initial mesh. We observe a strong refinement towards all five corners of Γ.*

[18] S. Ferraz-Leite, D. Praetorius: *Simple A Posteriori Error Estimators for the h-Version of the Boundary Element Method*, Computing 83 (2008), 135–162.

[19] N. Heuer: *An hp-Adaptive Refinement Strategy for Hypersingular Operators on Surfaces*, Numer. Methods Partial Differential Equations 18 (2002), 396–419.

[20] N. Heuer, M. Mellado, E. Stephan: *hp-Adaptive Two-Level Methods for Boundary Integral Equations on Curves*, Computing 67 (2001), 305–334.

[21] M. Maischak: *The Analytical Computation of the Galerkin Elements for the Laplace, Lamé and Helmholtz Equation in 2D BEM*, Technical Report, Institute for Applied Mathematics, University of Hannover (1999).

[22] M. Maischak, P. Mund, E. Stephan: *Adaptive Multilevel BEM for Acoustic Scattering*, Comput. Methods Appl. Mech. Engrg. 150 (1997), 351–367.

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