ASC Report No. 15/2009

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Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-02-5

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ISBN 978-3-902627-02-5

CONVERGENCE OF ADAPTIVE BOUNDARY ELEMENT METHODS

CARSTEN CARSTENSEN⋆,† AND DIRK PRAETORIUS⋆

Abstract. In many applications, adaptive mesh-refinement is observed to be an efficient tool for the numerical solution of partial differential equations and integral equations. Convergence of adaptive schemes to the correct solution, however, is so far only understood for certain kind of differential equations. In general, it cannot be excluded that the adaptive algorithm computes a *convergent sequence* of discrete approximations with a limit which is *not* the correct solution. This work proposes a feedback loop which guarantees the convergence of the computed discrete approximations to the correct solution. Although stated for Symm's integral equation of the first kind, the main part of this work is written for a general audience in the context of weak forms as Riesz representations in Hilbert spaces. Numerical examples illustrate the adaptive strategies.

1. Symm's Integral Equation, Introduction, and Outline

1.1. Symm's Integral Equation of the First Kind with the Single-Layer Potential Operator. Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary $\partial\Omega$, and let $\Gamma \subset \partial\Omega$ be an open or closed surface. Suppose we are given the right-hand side f and an approximation ϕ_h for the unknown exact solution ϕ of Symm's integral equation of the first kind

$$V\phi = f \quad \text{in } \widetilde{H}^{1/2}(\Gamma) \tag{1.1}$$

for the single-layer potential (ds_y denotes surface integration on $\Gamma \subseteq \mathbb{R}^d$ with respect to the variable y) defined by

$$(V\phi)(x) = \int_\Gamma \phi(y)\kappa(x-y)\,ds_y \quad \text{for } x \in \Gamma \tag{1.2}$$

and interpreted in a weak sense for the kernel

$$\kappa(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & \text{for } d = 2, \\ +\frac{1}{4\pi}\,|x|^{-1} & \text{for } d = 3. \end{cases} \tag{1.3}$$

Date: May 21, 2009

⋆ The main part of this work has been written when the authors enjoyed the hospitality of the Hausdorff Institute of Mathematics at University of Bonn. The kind support and stimulating research environment within the semester program *Computational Mathematics* is thankfully acknowledged.
† Supported by the DFG Research Center MATHEON "Mathematics for key technologies" in Berlin and partly supported by the WCU program through KOSEF (R31-2008-000-10049-0).

It is well established that (provided $d = 3$ or $d = 2$ and Ω is compactly included in a unit ball) V defines a scalar product

$$(1.4) \qquad (\phi\,,\psi)_H := \int_\Gamma (V\phi)(x)\psi(x)\,ds_x$$

on the dual space $H := \widetilde{H}^{-1/2}(\Gamma)$ of the trace space $H^{1/2}(\Gamma)$ defined in Subsection 1.2, and the induced Hilbert norm $\|\cdot\|_H$ is an equivalent norm on H.

1.2. Fractional-Order Sobolev Spaces on Submanifolds. For any (relatively) open set $\omega \subseteq \partial\Omega$ and $0 \le \alpha \le 1$, we define Sobolev spaces of fractional order by

$$(1.5) \qquad \widetilde{H}^\alpha(\omega) = [L^2(\omega); H^1_0(\omega)]_\alpha \quad \text{and} \quad H^\alpha(\omega) = [L^2(\omega); H^1(\omega)]_\alpha$$

as complex interpolation $[X_0; X_1]_\alpha$ of X_0 and $X_1 \subseteq X_0$, cf. [1, 28] for details. The norm $\|\cdot\|_{H^1(\omega)}$ is given by the surface gradient ∇ as $\|u\|^2_{H^1(\omega)} = \|u\|^2_{L^2(\omega)} + \|\nabla u\|^2_{L^2(\omega)}$. The spaces $H^1(\omega)$ and $H^1_0(\omega)$ are defined as the respective completions of $\mathrm{Lip}(\omega)$ and $\{v \in \mathrm{Lip}(\omega) : v|_{\partial\omega} = 0\}$. Sobolev spaces with negative index are defined by duality,

$$(1.6) \qquad H^{-\alpha}(\Gamma) := \widetilde{H}^\alpha(\Gamma)^* \quad \text{and} \quad \widetilde{H}^{-\alpha}(\Gamma) := H^\alpha(\Gamma)^*$$

with corresponding norms and duality brackets (which extend the $L^2(\Gamma)$ scalar product)

$$(1.7) \qquad \langle\cdot\,,\cdot\rangle \quad \text{in } \widetilde{H}^{-\alpha}(\Gamma) \times H^\alpha(\Gamma).$$

1.3. A Posteriori BEM Error Control. A posteriori error estimators $\eta = \eta(\phi_h, f, \mathcal{T})$ are computable quantities in terms of the right-hand side f, a computed approximate solution ϕ_h, and the given underlying mesh $\mathcal{T} = \{T_1, \ldots, T_N\}$ which bound the exact error from below or above [so-called efficiency or reliability of η], see [5] for examples and some history of boundary element error control and [7, 19, 21, 24] for some update and the state of the art for Symm's integral equation.

The non-local character of the involved pseudodifferential operator V and the non-local Sobolev spaces [of functions on Γ] cause severe difficulties in the mathematical derivation of computable lower and upper error bounds for a discrete (known) approximation ϕ_h to the (unknown) exact solution ϕ. In particular, the discrete local efficiency of the error estimator is one key argument in the adaptive finite element convergence analysis [2, 17, 29, 30, 35, 36], which still remains open for boundary element methods.

The adaptation of the analysis of [12] to adaptive BEM required certain local properties of the involved integral operators to prove the crucial *estimator reduction*. Although observed experimentally, the mathematics of those properties is not understood.

For wavelet Galerkin BEM convergence and optimality of some adaptive schemes have recently been proven [16, 25], where optimality is based on monitoring dominant coefficients and a certain coarsening step. This paper is devoted towards some alternative efficient adaptive BEM algorithms with a first step of ensured convergence.

1.4. Convergence of Adaptive Algorithms to Some Function. This paper studies adaptive mesh-refining strategies for the numerical solution of differential and integral equations stated in the framework of the Riesz theorem: For any linear and continuous functional

$\Phi \in H^*$ on a real Hilbert space H, there is a unique $\phi \in H$ such that

$$(\phi\,,\psi)_H = \Phi(\psi) \quad \text{for all } \psi \in H, \tag{1.8}$$

and there holds

$$\|\phi\|_H = \|\Phi\|_{H^*} := \sup_{\substack{\psi \in H \\ \psi \neq 0}} \frac{\Phi(\psi)}{\|\psi\|_H}. \tag{1.9}$$

In practical applications, H is an infinite dimensional space and the unique solution ϕ of (1.8) is unknown. Instead one considers a sequence X_ℓ, for $\ell = 0, 1, 2, \dots$, of finite dimensional (and hence closed) subspaces. These spaces are usually obtained from certain mesh-refinements and hence nested, i.e.,

$$X_\ell \subseteq X_{\ell+1} \quad \text{for } \ell = 0, 1, 2, \dots \tag{1.10}$$

The application of the Riesz theorem to the spaces X_ℓ provides unique Galerkin solutions $\phi_\ell \in X_\ell$ characterized by

$$(\phi_\ell\,,\psi_\ell)_H = \Phi(\psi_\ell) \quad \text{for all } \psi_\ell \in X_\ell. \tag{1.11}$$

We thus have the Galerkin orthogonality

$$(\phi - \phi_\ell\,,\psi_\ell)_H = 0 \quad \text{for all } \psi_\ell \in X_\ell. \tag{1.12}$$

Said differently, $\phi_\ell = \Pi_\ell \phi$, where $\Pi_\ell : H \to X_\ell$ denotes the orthogonal projection onto X_ℓ.

Lemma 1.1. *The limit $\phi_\infty := \lim_{\ell\to\infty} \phi_\ell$ exists in H and belongs to a subspace X_∞, defined as the closure of $\bigcup_{\ell=0}^\infty X_\ell$ in H.*

Proof. Note that X_∞ is a closed subspace of H and hence a Hilbert space. Moreover, X_∞ is separable. By Zorn's lemma, we thus find a (countable) orthonormal basis Σ as well as a partition of which into countably many finite sets Σ_ℓ such that $\bigcup_{j=0}^\ell \Sigma_j$ is an orthonormal basis of X_ℓ. Let $\phi_\infty := \Pi_\infty \phi \in X_\infty$ with the orthogonal projection $\Pi_\infty : H \to X_\infty$. Note that elementary functional analysis proves

$$\phi_\ell = \sum_{j=0}^{\ell} \sum_{\psi \in \Sigma_j} (\phi\,,\psi)_H \psi$$

for all $\ell \in \mathbb{N}_0$ and even in case $\ell = \infty$. In other words,

$$\|\phi_\infty - \phi_\ell\|_H^2 = \sum_{j=\ell+1}^{\infty} \sum_{\psi \in \Sigma_j} |(\phi_\ell\,,\psi)_H|^2 \xrightarrow{\ell\to\infty} 0.$$

This concludes the proof. $\square$

The lemma allows the following interpretation: For uniform mesh-refinement, there usually holds $X_\infty = H$ and thus $\phi = \phi_\infty$, i.e., we have convergence of the sequence of discrete solutions ϕ_ℓ from (1.11) towards the unique solution ϕ of (1.8). However, adaptive mesh-refinement may lead to $X_\infty \subsetneqq H$. In other words, the remaining question is whether the adaptive algorithm yields convergence with $\phi = \phi_\infty$ or $\phi \neq \phi_\infty$.

1.5. Adaptive Mesh-Refining Algorithm. The proposed solution procedure consists of the four steps

$$\texttt{SOLVE} \longrightarrow \texttt{UNIFORM REFINEMENT} \longrightarrow \texttt{ESTIMATE} \longrightarrow \texttt{ADAPTIVE COARSENING}.$$

In this context, we provide abstract algorithms which are proven to guarantee $\phi = \phi_\infty$. The assumptions made are very weak in the sense that we essentially only assume that there is a uniform refinement operation `unif` for discrete subspaces of H with the following properties.

- **Monotonicity:** For any discrete subspaces X and Y of H with $X \subseteq Y$ holds $X \subset \texttt{unif}(X) \subseteq \texttt{unif}(Y)$.
- **Density:** For a certain discrete subspace H_0 of H and any discrete subspace X_0 of H with $H_0 \subseteq X_0$ holds $\phi = \lim_{\ell\to\infty} \phi_\ell$ whenever $X_\ell := \texttt{unif}(X_{\ell-1})$ for the Galerkin solution ϕ_ℓ of (1.11).

Clearly, the mathematical proofs of optimality of the adaptive strategies, for instance with respect to the dimensions $\dim X_\ell$, are beyond the scope of this paper and can certainly not be proven in such a general framework.

Compared to the finite element method, the coarsening algorithm of [2] has been the first with proven optimal complexity before [36] proved the optimality for the standard AFEM. Moreover, in the context of wavelet methods, optimal adaptive algorithms are based on monitoring dominant coefficients, and results for linear [13] and nonlinear [14, 15] differential equations as well as recently for boundary integral equations [16, 25] have been achieved.

1.6. Outlook. Section 2 discusses the convergence of Galerkin schemes in Hilbert spaces as well as a first and second version of ABEM and its convergence. Section 3 gives details for the first-kind integral equation (1.1) with weakly-singular integral kernel associated with the Laplace equation. Numerical experiments for two benchmark examples conclude this paper.

2. Some Abstract Analytical Observations

2.1. Two-Level Error Estimator and Error Reduction Property. In the context of finite element methods, the first convergence results [17, 29] were based on the error reduction property

$$\|\phi - \phi_{\ell+1}\|_H \le q_{\mathrm{err}}\, \|\phi - \phi_\ell\|_H \tag{2.1}$$

with some constant $0 < q_{\mathrm{err}} < 1$. We first recall that (2.1) is equivalent to the reliability

$$C_{\mathrm{err}}^{-1}\, \|\phi - \phi_\ell\|_H \le \tau_\ell := \|\phi_{\ell+1} - \phi_\ell\|_H \tag{2.2}$$

of the two-level error estimator τ_ℓ with some constant $C_{\mathrm{err}} > 0$.

Lemma 2.1. (i) *Efficiency holds in the sense of* $\tau_\ell \le \|\phi - \phi_\ell\|_H$, *whence* $C_{\mathrm{err}} \ge 1$;
(ii) $C_{\mathrm{err}} = 1$ *is equivalent to* $\phi = \phi_{\ell+1}$;
(iii) *Reliability* (2.2) *with* $C_{\mathrm{err}} = (1 - q_{\mathrm{err}}^2)^{-1/2} > 1$ *is equivalent to the error reduction property* (2.1) *with* $q_{\mathrm{err}} = (1 - C_{\mathrm{err}}^{-2})^{1/2}$.

Proof. (i)&(ii). Let $\Pi_{\ell+1}$ denote the orthogonal projection onto $X_{\ell+1}$. Note that

$$\Pi_{\ell+1}(\phi - \phi_\ell) = \phi_{\ell+1} - \phi_\ell$$

so that $\tau_\ell \le \|\phi - \phi_\ell\|_H$. In particular, this implies $C_{\rm err} \ge 1$. Moreover, $C_{\rm err} = 1$ is equivalent to $0 = (1 - \Pi_{\ell+1})(\phi - \phi_\ell) = \phi - \phi_{\ell+1}$.
(iii). Since $X_\ell \subseteq X_{\ell+1}$, the Galerkin orthogonality reads

$$\|\phi - \phi_\ell\|_H^2 = \|\phi - \phi_{\ell+1}\|_H^2 + \|\phi_{\ell+1} - \phi_\ell\|_H^2 = \|\phi - \phi_{\ell+1}\|_H^2 + \tau_\ell^2.$$

Therefore, reliability of τ_ℓ implies error reduction

$$\|\phi - \phi_{\ell+1}\|_H^2 = \|\phi - \phi_\ell\|_H^2 - \tau_\ell^2 \le (1 - C_{\rm err}^{-2})\, \|\phi - \phi_\ell\|_H^2.$$

Conversely, the error reduction yields

$$\|\phi - \phi_\ell\|_H^2 = \|\phi - \phi_{\ell+1}\|_H^2 + \tau_\ell^2 \le q_{\rm err}^2 \|\phi - \phi_\ell\|_H^2 + \tau_\ell^2$$

and whence reliability of τ_ℓ with $C_{\rm err} = (1 - q_{\rm err}^2)^{-1/2}$. □

Note that the error reduction property with a uniform constant $q_{\rm err} < 1$ implies linear convergence of ϕ_ℓ towards $\phi = \phi_\infty$ with respect to the level ℓ. In the context of the finite element method, the error reduction property is obtained by use of a reliable error estimator

$$C_{\rm rel}^{-1}\, \|\phi - \phi_\ell\|_H \le \eta_\ell \tag{2.3}$$

and the discrete efficiency estimate

$$C_{\rm eff}^{-1}\, \eta_\ell \le \|\phi_{\ell+1} - \phi_\ell\|_H = \tau_\ell \tag{2.4}$$

proven locally with the help of some inner node property [17, 29]. Here, the reliability constant $C_{\rm rel} > 0$ and the efficiency constant $C_{\rm eff} > 0$ may depend on the right-hand side $\Phi \in H^*$, but not on ℓ, ϕ or ϕ_ℓ. The error estimator η_ℓ is a computable quantity that depends on Φ and ϕ_ℓ, but has to be independent of ϕ. The combination of the latter two estimates (2.3)–(2.4) yields reliability of the two-level estimator τ_ℓ and hence linear error reduction. For the sake of clarity, we have omitted the so-called oscillation terms, which usually arise in efficiency estimates for finite element methods.

In many applications, the discrete efficiency estimate (2.4) and even the usual efficiency estimate

$$C_{\rm eff}^{-1}\, \eta_\ell \le \|\phi - \phi_\ell\|_H \tag{2.5}$$

remains as an open question. This includes, for instance, a posteriori error estimates for boundary element methods, where estimators are usually either only proven to be efficient [8, 9, 19, 20, 21, 24, 26, 27, 31] or to be reliable [4, 6, 7, 10, 11]. The error estimators from [22, 23] and [33, 34] are, so far, the only a posteriori BEM error estimators which are proven to be reliable and efficient. The discrete efficiency estimate (2.4) is, however, open. A more detailed overview on a posteriori error estimation for the boundary element method can be found, e.g., in [5].

2.2. $(h - h/2)$-Error Estimator and Saturation Assumption. In usual adaptive algorithms, the space $X_{\ell+1}$ is obtained from X_ℓ by certain refinements. Consequently, the two-level error estimator from (2.2) cannot be used to obtain $X_{\ell+1}$. One remedy might be to compute the $(h - h/2)$-error estimator which is one standard strategy, e.g., in the context of ordinary differential equations. With the Galerkin solution $\widehat{\phi}_\ell \in \widehat{X}_\ell := \texttt{unif}(X_\ell)$, one considers

$$\widehat{\tau}_\ell := \|\widehat{\phi}_\ell - \phi_\ell\|_H. \tag{2.6}$$

With the same arguments as in Lemma 2.1, one proves that reliability of $\widehat{\tau}_\ell$

$$\|\phi - \phi_\ell\|_H \le C_{\rm sat}\, \widehat{\tau}_\ell \tag{2.7}$$

with some constant $C_{\rm sat} > 0$ is equivalent to the so-called saturation assumption

$$\|\phi - \widehat{\phi}_\ell\|_H \le q_{\rm sat}\, \|\phi - \phi_\ell\|_H \tag{2.8}$$

with some contraction constant $0 < q_{\rm sat} < 1$.

Lemma 2.2. (i) *Efficiency holds in the sense of $\widehat{\tau}_\ell \le \|\phi - \phi_\ell\|_H$, whence $C_{\rm sat} \ge 1$;*
(ii) $C_{\rm sat} = 1$ *is equivalent to $\phi = \widehat{\phi}_\ell$;*
(iii) *Reliability* (2.7) *with $C_{\rm sat} = (1 - q_{\rm sat}^2)^{-1/2} > 1$ is equivalent to the saturation assumption* (2.8) *with $q_{\rm sat} = (1 - C_{\rm sat}^{-2})^{1/2}$.* □

For the finite element method, the saturation assumption holds for model problems up to data oscillations [18]. In the context of the boundary element method, the saturation assumption still remains open. Numerical experiments from [24], however, indicate that (2.8) holds as well.

2.3. Convergence Control. This section contains the main observation for a feedback control to guarantee convergence of an adaptive algorithm. Note that by definition of the Galerkin scheme, $\phi = \phi_\ell$ implies $\phi = \phi_\ell = \phi_{\ell+k}$ for all $k \ge 0$. One may therefore define

$$\mu_\ell := \begin{cases} \tau_\ell / \|\phi - \phi_\ell\|_H & \text{provided } \phi \ne \phi_\ell, \\ 1 & \text{else} \end{cases} \tag{2.9}$$

with the two-level estimator $\tau_\ell = \|\phi_{\ell+1} - \phi_\ell\|_H \le \|\phi - \phi_\ell\|_H$. The following lemma characterizes convergence of ϕ_ℓ towards ϕ by means of μ_ℓ.

Lemma 2.3. *There holds $\phi = \phi_\infty$ if and only if $\prod_{\ell=0}^\infty (1 - \mu_\ell^2) = 0$.*

Proof. Fix $\ell \in \mathbb{N}$. According to the Pythagoras theorem, there holds

$$\|\phi - \phi_\ell\|_H^2 = \|(\phi - \phi_{\ell+1}) + (\phi_{\ell+1} - \phi_\ell)\|_H^2 = \|\phi - \phi_{\ell+1}\|_H^2 + \tau_\ell^2.$$

By definition of μ_ℓ, the last equation becomes

$$\|\phi - \phi_\ell\|_H^2 = \|\phi - \phi_{\ell+1}\|_H^2 + \mu_\ell^2\, \|\phi - \phi_\ell\|_H^2.$$

By induction, this yields

$$\|\phi - \phi_{\ell+1}\|_H^2 = (1 - \mu_\ell^2)\|\phi - \phi_\ell\|_H^2 = \|\phi - \phi_0\|_H^2 \prod_{j=0}^{\ell} (1 - \mu_j^2).$$

Note that $0 \le \mu_j \le 1$ so that the product on the right-hand side is decreasing and bounded from below as $\ell \to \infty$. In particular, the limit $\prod_{j=0}^\infty (1 - \mu_j^2)$ exists. Moreover, we thus infer that convergence $\phi = \phi_\infty$ is equivalent to $0 = \prod_{\ell=0}^\infty (1 - \mu_\ell^2)$. □

***Remark* 1.** The algorithms of the subsequent sections aim to ensure the reliability (2.2) of the two-level error estimator τ_ℓ. To be more precise: In each step, the algorithms check whether (2.2) can be guaranteed. Otherwise, the algorithms enforce one step of uniform mesh-refinement. $\square$

2.4. Adaptive Strategy Based on Reliable Error Estimator. Given a Galerkin solution ϕ_ℓ, we assume we can compute an error estimator η_ℓ which is reliable in the sense

$$\|\phi - \phi_\ell\|_H \le C_{\mathrm{rel}}\, \eta_\ell. \tag{2.10}$$

The adaptive algorithm then reads as follows. We stress that the precise adaptivity is hidden in step (d) below, which corresponds to some adaptive coarsening to construct $X_{\ell+1}$. For Symm's integral equation (1.1), we provide some possible realizations in Section 3.1 below.

Algorithm 2.4 (Main Loop, First Version). *Fix constants* $0 < \varrho < 1$, $\kappa_0 > 0$, *set* $\ell := 0$, $X_0 := H_0$. *For any* $\ell = 0, 1, 2, \dots$ *do* (a)–(d)*:*

(a) *Compute Galerkin solution* $\phi_\ell \in X_\ell$ *and corresponding error estimator* η_ℓ.

(b) *Compute Galerkin solution* $\widehat{\phi}_\ell \in \widehat{X}_\ell := \mathtt{unif}(X_\ell)$ *and set* $\widehat{\tau}_\ell := \|\widehat{\phi}_\ell - \phi_\ell\|_H$.

(c) *If* $\widehat{\tau}_\ell < \kappa_\ell \eta_\ell$, *set* $X_{\ell+1} := \widehat{X}_\ell$ *and* $\kappa_{\ell+1} := \begin{cases} \kappa_\ell & \text{if } \widehat{\tau}_\ell = 0, \\ \widehat{\tau}_\ell/\eta_\ell & \text{else.} \end{cases}$

(d) *If* $\widehat{\tau}_\ell \ge \kappa_\ell \eta_\ell$, *choose* $X_{\ell+1}$ *with* $X_\ell \subset X_{\ell+1} \subseteq \widehat{X}_\ell$ *and* $\tau_\ell := \|\phi_{\ell+1} - \phi_\ell\|_H \ge \varrho\, \widehat{\tau}_\ell$ *and set* $\kappa_{\ell+1} := \kappa_\ell$.

***Remark* 2.** We stress that the reliability constant $C_{\mathrm{rel}} > 0$ is often unknown in practice. Moreover, the $(h-h/2)$-error estimator satisfies $\widehat{\tau}_\ell \le \|\phi - \phi_\ell\|_H \le C_{\mathrm{rel}}\eta_\ell$. Therefore, $\kappa_\ell > C_{\mathrm{rel}}$ excludes step (d) and leads to uniform mesh-refinement in step (c). Since uniform mesh-refinement is expected to be suboptimal, we decrease $\kappa_{\ell+1} = \widehat{\tau}_\ell/\eta_\ell < \kappa_\ell$ in case (c) and $\widehat{\tau}_\ell > 0$. $\square$

***Remark* 3.** Since $0 < \varrho < 1$, the estimate $\tau_\ell \ge \varrho\, \widehat{\tau}_\ell$ in (d) holds for the choice of $X_{\ell+1} := \widehat{X}_\ell$. However, uniform mesh-refinement usually leads to a suboptimal order of convergence with respect to the number of degrees of freedom $N_\ell := \dim X_\ell$. In practice, one aims to choose the space $X_{\ell+1}$ therefore with as low dimension as possible. We stress that Algorithm 2.4 does not include a precise statement of how to choose $X_{\ell+1}$. Possible constructions are the topic of subsequent sections. $\square$

***Remark* 4.** The essential idea of Algorithm 2.4 is that step (d) ensures

$$\|\phi - \phi_\ell\|_H \le C_{\mathrm{rel}}\eta_\ell \le C_{\mathrm{rel}}\kappa_\ell^{-1}\, \widehat{\tau}_\ell \le C_{\mathrm{rel}}\kappa_\ell^{-1}\varrho^{-1}\, \tau_\ell \tag{2.11}$$

which provides the reliability (2.2) of the two-level error estimator τ_ℓ on level ℓ with $C_{\mathrm{err}} = C_{\mathrm{rel}}\kappa_\ell^{-1}\varrho^{-1}$. Note that step (d) does, in particular, provide the reliability

$$\|\phi - \phi_\ell\|_H \le C_{\mathrm{rel}}\kappa_\ell^{-1}\, \widehat{\tau}_\ell \tag{2.12}$$

of the $(h - h/2)$-error estimator on level ℓ, whence the saturation assumption (2.8). — For the numerical experiments below, we choose $\varrho = 0.75$ and $\kappa_0 = 1$. $\square$

Theorem 2.5. *The sequence ϕ_ℓ of Galerkin solutions generated by Algorithm 2.4 converges to the unique solution ϕ of* (1.8).

Proof. Without loss of generality, we may assume that $\phi \neq \phi_\ell$ for all $\ell \in \mathbb{N}$, since otherwise $\phi = \phi_{\ell_0}$ for some $\ell_0 \in \mathbb{N}$ implies $\phi = \phi_{\ell_0} = \phi_\ell$ for all $\ell \geq \ell_0$.

First, we consider the case that there are infinitely many ℓ such that $\widehat{\tau}_\ell < \kappa_\ell \eta_\ell$ leads to step (c) in Algorithm 2.4. Choose the corresponding sequence (ℓ_k) of indices such that $X_{\ell_k+1} = \texttt{unif}(X_{\ell_k})$. Note that the monotonicity assumption on `unif` and $H_0 = X_0$ imply the inclusion $H_k \subseteq X_{\ell_k}$ with $H_k := \texttt{unif}(H_{k-1})$. Therefore, the best approximation property of ϕ_{ℓ_k} together with the density assumption on `unif` yield $\phi_{\ell_k} \to \phi$ as $k \to \infty$. Since the entire sequence ϕ_ℓ converges to ϕ_∞, we then conclude $\phi = \phi_\infty$.

Second, we assume that there are only finitely many ℓ such that $\widehat{\tau}_\ell < \kappa_\ell \eta_\ell$ leads to step (c) in Algorithm 2.4. Assume that (d) holds for all $\ell \geq \ell_0$, i.e., (c) holds at most for $\ell = 1, \dots, \ell_0 - 1$. For $\ell \geq \ell_0$, we have $\widehat{\tau}_\ell \geq \kappa_\ell \eta_\ell$. Equation (2.11) and $\phi \neq \phi_\ell$ then imply

$$\mu_\ell = \frac{\tau_\ell}{\|\phi - \phi_\ell\|_H} \geq C_{\rm rel}^{-1} \kappa_\ell \varrho > 0, \tag{2.13}$$

where $\kappa_\ell = \widehat{\tau}_{\ell_0}/\eta_{\ell_0} > 0$ is constant for $\ell \geq \ell_0$. From $0 \leq \mu_\ell \leq 1$, we therefore infer $\prod_{\ell=0}^\infty (1 - \mu_\ell^2) = 0$, and Lemma 2.3 concludes the proof. $\square$

2.5. Adaptive Strategy without Reliable Error Estimator. Under some circumstances, one may not want to use Algorithm 2.4. One of the reasons can be the following:

- There is no reliable error estimator η_ℓ at hand.
- The reliable error estimator η_ℓ is implementationally demanding.
- The reliable error estimator η_ℓ is certainly not efficient so that Algorithm 2.4 will lead to a suboptimal order of convergence caused by the overestimation of $\|\phi - \phi_\ell\|_H$ and hence of $\widehat{\tau}_\ell$.

In those case, one wants to use variants of the $(h - h/2)$-error estimator $\widehat{\tau}_\ell = \|\widehat{\phi}_\ell - \phi_\ell\|_H$ with the Galerkin solution $\widehat{\phi}_\ell \in \widehat{X}_\ell = \texttt{unif}(X_\ell)$. As has been noted above, $\widehat{\tau}_\ell$ is always an efficient error estimator, but reliability (2.7) is equivalent to the saturation assumption (2.8). Moreover, the decision whether $\widehat{\tau}_\ell$ can be guaranteed to be reliable, is the essential criterion in Algorithm 2.4. Consequently, $\eta_\ell := \widehat{\tau}_\ell$ cannot be used to steer Algorithm 2.4 reliably. One remedy might be the following variant of Algorithm 2.4, where we replace $\kappa_\ell \eta_\ell$ by a positive and monotonously decreasing sequence $(\lambda_\ell) \notin \ell^2$.

Algorithm 2.6 (Main Loop, Second Version). *Fix constants $0 < \varrho < 1$ and $0 < q < 1$ as well as a positive and monotonously decreasing sequence $(\sigma_\ell) \notin \ell^2$. Set $\ell := 0$, $X_0 := H_0$, $\lambda_0 := \sigma_0$. For any $\ell = 0, 1, 2, \dots$ do* (a)–(d)*:*

(a) *Compute Galerkin solution $\phi_\ell \in X_\ell$.*

(b) *Compute Galerkin solution $\widehat{\phi}_\ell \in \widehat{X}_\ell := \texttt{unif}(X_\ell)$ and set $\widehat{\tau}_\ell := \|\widehat{\phi}_\ell - \phi_\ell\|_H$.*

(c) *If $\widehat{\tau}_\ell < \lambda_\ell$, set $X_{\ell+1} := \widehat{X}_\ell$, $\lambda_{\ell+1} := \begin{cases} q \min\{\sigma_{\ell+1}, \lambda_\ell\} & \text{if } \widehat{\tau}_\ell = 0, \\ q \min\{\sigma_{\ell+1}, \widehat{\tau}_\ell\} & \text{else.} \end{cases}$*

(d) *If $\widehat{\tau}_\ell \geq \lambda_\ell$, choose $X_{\ell+1}$ with $X_\ell \subset X_{\ell+1} \subseteq \widehat{X}_\ell$ and $\tau_\ell := \|\phi_{\ell+1} - \phi_\ell\|_H \geq \varrho\, \widehat{\tau}_\ell$ and set $\lambda_{\ell+1} := \min\{\sigma_{\ell+1}, \lambda_\ell\}$.*

***Remark* 5.** In case of $\widehat{\tau}_\ell < \lambda_\ell$, we decrease $\lambda_{\ell+1} \le q\lambda_\ell$ in step (c). For sufficiently small $0 < q < 1$, we may then expect $\widehat{\tau}_{\ell+1} \ge \lambda_{\ell+1}$ for the next level $\ell + 1$, i.e., one uniform mesh-refinement on level ℓ causes at least one adaptive mesh-refinement on level $\ell + 1$. — In the numerical experiments below, $\varrho = 0.75$, $\sigma_\ell = \ell^{-1/2}$, and $q = 0.2$. □

Theorem 2.7. *The sequence ϕ_ℓ of Galerkin solutions generated by Algorithm 2.6 converges to the unique solution ϕ of* (1.8).

Proof. Arguing as in the proof of Theorem 2.5, we may assume that $\phi \neq \phi_\ell$ for all $\ell \in \mathbb{N}$ and that there are only finitely many ℓ with $\widehat{\tau}_\ell < \lambda_\ell$. In particular, we have $0 < \mu_\ell < 1$ for all $\ell \in \mathbb{N}$.

Assume that (d) holds for all $\ell \ge \ell_0$. For $\ell \ge \ell_0 + 1$, there holds $\widehat{\tau}_\ell \ge \lambda_\ell$, whence

$$\mu_\ell = \frac{\tau_\ell}{\|\phi - \phi_\ell\|_H} \ge \frac{\varrho\lambda_\ell}{\|\phi - \phi_\ell\|_H} \ge \frac{\varrho}{\|\phi - \phi_0\|_H}\,\lambda_\ell. \tag{2.14}$$

Since the sequence (σ_ℓ) decreases monotonously, mathematical induction proves

$$\lambda_\ell = \min\{\sigma_\ell, \lambda_{\ell-1}\} = \min\{\sigma_\ell, \lambda_{\ell_0}\}$$

for $\ell > \ell_0$. Consequently, $(\sigma_\ell) \notin \ell^2$ and $\lambda_{\ell_0} > 0$ yield $(\lambda_\ell) \notin \ell^2$, whence $\sum_{\ell=0}^\infty \mu_\ell^2 = \infty$ by (2.14). In particular, $0 < \mu_\ell^2 < 1$ shows

$$\log\Big(\prod_{\ell=0}^\infty (1 - \mu_\ell^2)\Big) = \sum_{\ell=0}^\infty \log(1 - \mu_\ell^2) \le -\sum_{\ell=0}^\infty \mu_\ell^2 = -\infty.$$

This yields $\prod_{\ell=0}^\infty (1 - \mu_\ell^2) = 0$ and Lemma 2.3 concludes the proof. □

***Remark* 6.** Recall that Algorithm 2.4 as well as Algorithm 2.6 aim to provide an error reduction $\|\phi - \phi_{\ell+1}\|_H \le q\,\|\phi - \phi_\ell\|_H$ with some $0 < q < 1$, whence convergence of the adaptive scheme. If this error reduction is obtained for k steps, we thus see $\|\phi - \phi_{\ell+k}\|_H \le q^k\|\phi - \phi_{\ell+k}\|_H$. As the geometric sequence (q^k) belongs to ℓ^2, its decrease is much faster than that of $(\lambda_\ell) \notin \ell^2$. Consequently, there holds $\widehat{\tau}_{\ell+k} \le \|\phi - \phi_{\ell+k}\|_H \le q^k\|\phi - \phi_\ell\|_H < \lambda_{\ell+k}$ for some k. Said differently, after a finite number of adaptive steps (d) for which the error reduction holds, Algorithm 2.6 will certainly lead to a uniform refinement in step (c). If Algorithm 2.6 performs in this sense, it will nevertheless lead to infinitely many uniform refinements. □

3. Application to Symm's Integral Equation

As a model problem serves Symm's integral equation of the first kind

$$V\phi(x) := -\frac{1}{2\pi}\int_\Gamma \log|x - y|\phi(y)\,ds_y = f(x) \quad \text{for } x \in \Gamma \tag{3.1}$$

on an open boundary piece $\Gamma \subseteq \partial\Omega$ of a bounded Lipschitz domain Ω in $\mathbb{R}^2$ with $\mathrm{diam}(\Omega) < 1$. With the scalar product from (1.4), the integral equation (3.1) is equivalently stated in the form (1.8) with $\Phi(\psi) := \langle f\,, \psi\rangle$. For given right-hand side $f \in H^{1/2}(\Gamma)$, we aim to approximate the (in general unknown) solution $\phi \in H := \widetilde{H}^{-1/2}(\Gamma)$ numerically. The lowest-order Galerkin scheme (1.11) with $\mathcal{T}$-piecewise constant ansatz and test functions $X_h := \mathcal{P}^0(\mathcal{T})$ reads: Seek $\phi_h \in \mathcal{P}^0(\mathcal{T})$ with

$$\int_{T_j} V\phi_h\,ds = \int_{T_j} f\,ds \quad \text{for all } T_j \in \mathcal{T}. \tag{3.2}$$

Here and throughout this paper, $\mathcal{T} = \{T_1, \dots, T_N\}$ is a partition of Γ into affine boundary pieces T_j with positive length $\operatorname{diam}(T_j) > 0$. For this discretization, the optimal order of convergence is $\mathcal{O}(h^{3/2})$ with respect to the maximal mesh-width $h := \max\{\operatorname{diam}(T) : T \in \mathcal{T}\}$ [32]. However, to observe this order of convergence numerically, the exact solution must satisfy $\phi \in H^1(\mathcal{T})$. This regularity is not met in general for domains with re-entrant corners, which lead to singularities of ϕ. Therefore, there is a need for a posteriori error control and related adaptive mesh-refinement which may lead to an optimal order of convergence $\mathcal{O}(N^{-3/2})$ with respect to the number $N = \#\mathcal{T}$ of elements.

However, both topics, a posteriori error estimation as well as adaptive mesh-refinement, are more involved for the boundary element method than for the finite element method. Whereas a certain number of error estimators has been introduced, e.g. in [4, 5, 6, 7, 10, 11, 22, 23, 33, 34], most of them are only proven to be reliable. Efficiency, also usually observed in practice, is only proven for quasi-uniform meshes [3]. Contrary, the error estimators of [8, 9, 19, 20, 21, 24, 26, 27, 31] are always efficient, whereas reliability of which crucially depends on the saturation assumption. Although the saturation assumption is experimentally observed in model examples, it is (to the best of the authors' knowledge) not guaranteed in the current literature on boundary element methods.

Unlike to the finite element method, the known a posteriori error estimators are not proven to satisfy a discrete efficiency estimate. This makes it impossible to prove the convergence of adaptive mesh-refining algorithms with the techniques developed in [17, 29, 30] for finite element schemes. However, the introduced mathematical framework allows to guarantee convergence of certain adaptive mesh-refining strategies.

In our setting, we have $X_\ell = \mathcal{P}^0(\mathcal{T}_\ell)$, and refinement of an element $T_j \in \mathcal{T}_\ell$ just means to split T_j into two disjoint boundary pieces of half length. Here and below, $T_j^{(1)}, T_j^{(2)} \in \mathcal{T}_{\ell+1}$ denote the (unique) elements obtained by refinement of an element $T_j \in \mathcal{T}_\ell$. The uniformly refined space $\widehat{X}_\ell = \texttt{unif}(X_\ell)$ reads $\widehat{X}_\ell = \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$, where the corresponding mesh $\widehat{\mathcal{T}}_\ell = \{T_1^{(1)}, T_1^{(2)}, \dots, T_N^{(1)}, T_N^{(2)}\}$ is obtained by uniform refinement of $\mathcal{T}_\ell$.

3.1. Adaptive Mesh-Refinement for Symm's Integral Equation. This subsection discusses two possible strategies to compute a mesh $\mathcal{T}_{\ell+1}$ and hence $X_{\ell+1} = \mathcal{P}^0(\mathcal{T}_{\ell+1})$ out of $\mathcal{T}_\ell$ and $\widehat{\mathcal{T}}_\ell$ which guarantee

$$X_\ell \subsetneqq X_{\ell+1} \subseteq \widehat{X}_\ell \tag{3.3}$$

as well as the criterion from step (d) of Algorithm 2.4 resp. Algorithm 2.6

$$\varrho\,\widehat{\tau}_\ell \le \tau_{\ell+1}. \tag{3.4}$$

Usually, adaptive mesh-refining strategies for $\mathcal{T}_\ell = \{T_1, \dots, T_N\}$ are based on refinement indicators $\eta_{\ell,1}, \dots, \eta_{\ell,N} \ge 0$ which are (somehow) related to a global error estimator η_ℓ, e.g.,

$$\eta_\ell = \Big(\sum_{j=1}^N \eta_{\ell,j}^2 \Big)^{1/2}. \tag{3.5}$$

The heuristic is to refine T_j with relatively large associated quantity $\eta_{\ell,j}$. In the numerical experiments below, we either use the local contributions

$$\eta_{\ell,j} := \mathrm{diam}(T_j)^{1/2} \|(f - V\phi_\ell)'\|_{L^2(T_j)} \tag{3.6}$$

of the weighted-residual error estimator from [4] or the local contributions of the $(h-h/2)$-based error estimator

$$\eta_{\ell,j} := \mathrm{diam}(T_j)^{1/2} \|\widehat{\phi}_\ell - \phi_\ell\|_{L^2(T_j)} \tag{3.7}$$

proposed in [24]. In case of (3.6), $(\cdot)'$ denotes the arclength derivative, and we assume $f \in H^1(\Gamma)$. Then, the error estimator η_ℓ from (3.5) is reliable [4], whereas efficiency remains open. Contrary to that, the error estimator η_ℓ based on the local quantities (3.7) is equivalent to the $(h-h/2)$-error estimator $\widehat{\tau}_\ell$, cf. [24]. In this case, η_ℓ is therefore efficient, whereas reliability is equivalent to the saturation assumption (2.8).

Algorithm 3.1 (Construction of $\mathcal{T}_{\ell+1}$ by Iterated Space-Enrichment). *In step (d) of Algorithm 2.4 resp. Algorithm 2.6, the mesh $\mathcal{T}_{\ell+1}$ is built from $\mathcal{T}_\ell = \{T_1, \dots, T_N\}$ as follows.*

(d.1) Compute refinement indicators $\eta_{\ell,1}, \dots, \eta_{\ell,N}$.
(d.2) Find a permutation π of $\{1, \dots, N\}$ such that $\eta_{\ell,\pi(1)} \ge \eta_{\ell,\pi(2)} \ge \dots \ge \eta_{\ell,\pi(N)}$.
(d.3) Choose minimal $k = 1, \dots, N$ such that the mesh

$$\mathcal{T}_{\ell+1} := \{T^{(1)}_{\pi(1)}, T^{(2)}_{\pi(1)}, \dots, T^{(1)}_{\pi(k)}, T^{(2)}_{\pi(k)}, T_{\pi(k+1)}, \dots, T_{\pi(N)}\}$$

and the corresponding Galerkin solution $\phi_{\ell+1} \in X_{\ell+1}$ satisfy (3.4). $\square$

According to Algorithm 3.1, we obtain $\mathcal{T}_{\ell+1}$ by refinement of the k elements $T_j \in \mathcal{T}_\ell$ with the largest refinement indicators $\eta_{\ell,j}$. Note that step (d.3) corresponds to a while-loop which is rather costly due to the iterated computation of Galerkin solutions.

To decrease the computational cost, we proceed as follows. For X_ℓ, we fix a numbering $\mathcal{T}_\ell = \{T_1, \dots, T_N\}$ and use the basis $\{\chi_1, \dots, \chi_N\}$ of characteristic functions of the elements $T_j \in \mathcal{T}_\ell$. By reordering the indices, we may assume that the permutation π from Algorithm 3.1 satisfies $\pi(j) = j$. For each T_j, let $\chi_j^{(1)}$ denote the characteristic function of the first child $T_j^{(1)} \in \widehat{\mathcal{T}}_\ell$. For the spaces

$$X_\ell^{(k)} := \mathcal{P}^0(\mathcal{T}_\ell^{(k)}), \quad \text{where} \quad \mathcal{T}_\ell^{(k)} := \{T_1^{(1)}, T_1^{(2)}, \dots, T_k^{(1)}, T_k^{(2)}, T_{k+1}, \dots, T_N\},$$

we use the two-level bases $\{\chi_1, \dots, \chi_N, \chi_1^{(1)}, \dots, \chi_k^{(1)}\}$. We then only need to compute the Galerkin matrix $\mathbf{A} \in \mathbb{R}^{2N\times 2N}_{sym}$ and its Cholesky factorization $\mathbf{A} = \mathbf{L}\mathbf{L}^T$ with respect to the basis $\{\chi_1, \dots, \chi_N, \chi_1^{(1)}, \dots, \chi_N^{(1)}\}$ of $\widehat{X}_\ell$. Note that the $(N+k)$-th minor $\mathbf{A}_{N+k}$ of $\mathbf{A}$ is the Galerkin matrix with respect to $X_\ell^{(k)}$, and its Cholesky factorization satisfies $\mathbf{A}_{N+k} = \mathbf{L}_{N+k}\mathbf{L}_{N+k}^T$. Therefore, the expensive computation of the Galerkin matrix and the Cholesky factorization has only to be done once.

The first approach stated in Algorithm 3.1, somehow leads to a minimal increase of elements to ensure (3.4). Alternatively, we may try to use the usual h-refinement strategy. However, we enforce (3.4) to hold: In a correction step, we choose $\mathcal{T}_{\ell+1}$ as the uniform refinement of $\mathcal{T}_\ell$ if indicator-based refinement did not lead to sufficiently large $\tau_\ell = \|\phi_{\ell+1} - \phi_\ell\|_H$. The following realization of step (d) can thus be understood as a feedback-loop for h-adaptive algorithms.

Algorithm 3.2 (Bulk-Criterion Based Construction of $\mathcal{T}_{\ell+1}$). *Given $0 \le \theta \le 1$ in step (d) of Algorithm 2.4 resp. Algorithm 2.6, the mesh $\mathcal{T}_{\ell+1}$ is built from $\mathcal{T}_\ell = \{T_1, \dots, T_N\}$ as follows.*

- *(d.1) Compute refinement indicators $\eta_{\ell,1}, \dots, \eta_{\ell,N}$.*
- *(d.2) Mark an element $T_j \in \mathcal{T}_\ell$ for refinement provided $\eta_{\ell,j} \ge \theta \max\{\eta_{\ell,k} : k = 1, \dots, N\}$.*
- *(d.3) Generate the mesh $\mathcal{T}_{\ell+1}$ from $\mathcal{T}_\ell$ by refinement of the marked elements.*
- *(d.4) If (3.4) fails, set $\mathcal{T}_{\ell+1} := \widehat{\mathcal{T}}_\ell$.* □

We stress that the choice of $\theta = 0$ in step (d.2) yields uniform mesh-refinement, whereas $\theta > 0$ yields an adaptive mesh-refinement. For the numerical experiments below, we choose $\theta = 0.5$ in case of adaptive mesh-refinement. Finally, note that the bulk criterion in (d.2) can be replaced by any other marking strategy such as the ℓ^2-criterion due to Dörfler [17] with the minimal set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that

$$(1-\theta^2)\,\eta_\ell^2 = (1-\theta^2) \sum_{T_j \in \mathcal{T}_\ell} \eta_{\ell,j}^2 \le \sum_{T_j \in \mathcal{M}_\ell} \eta_{\ell,j}^2. \tag{3.8}$$

In the numerical experiments below, we compare the following six mesh-refining strategies.

- **Uniform** mesh-refinement, which is guaranteed to converge. Throughout, we observe, however, poor convergence rates which are due to generic singularities of the exact solution.
- **Standard adaptive** mesh-refinement, where we use the Dörfler marking (3.8) with $\theta = 0.5$ for the weighted-residual estimator (3.6) and where we neglect the feedback control. This corresponds formally to the choice of $\kappa_0 = 0 = \varrho$ in Algorithm 2.4. Note that, so far, this algorithm is not proven to converge mathematically.
- Adaptive mesh-refinement based on **Alg**orithm **2.4** and **Alg**orithm **3.1** for the weighted-residual error estimator (3.6) with $\varrho = 0.75$, $\kappa_0 = 1$.
- Adaptive mesh-refinement based on **Alg**orithm **2.4** and **Alg**orithm **3.2** for the weighted-residual error estimator (3.6) with $\varrho = 0.75$, $\kappa_0 = 1$, $\theta = 0.5$.
- Adaptive mesh-refinement based on **Alg**orithm **2.6** and **Alg**orithm **3.1** for the $(h-h/2)$-based indicators (3.7) with $\varrho = 0.75$, $\sigma_\ell = \ell^{-1/2}$, $q = 0.2$.
- Adaptive mesh-refinement based on **Alg**orithm **2.6** and **Alg**orithm **3.2** for the $(h-h/2)$-based indicators (3.7) with $\varrho = 0.75$, $\sigma_\ell = \ell^{-1/2}$, $q = 0.2$, $\theta = 0.5$.

3.2. Symm's Integral Equation on a Slit. Symm's integral equation on a slit,

$$V\phi = 1 \quad \text{on } \Gamma = (-1,1) \times \{0\},$$

allows the exact solution $\phi \in H^{-\varepsilon}(\Gamma) \setminus L^2(\Gamma)$ for any $\varepsilon > 0$,

$$\phi(x,0) = -2(1-x^2)^{-1/2} \quad \text{for all } -1 < x < 1, \tag{3.9}$$

with singularities at the tips $x = \pm 1$. The error with respect to the energy norm is computed with help of the Galerkin orthogonality

$$\|\phi - \phi_\ell\|_H^2 = \|\phi\|_H^2 - \|\phi_\ell\|_H^2 = \pi - \|\phi_\ell\|_H^2 \tag{3.10}$$

with the continuous energy $\|\phi\|_H^2 = \pi$.

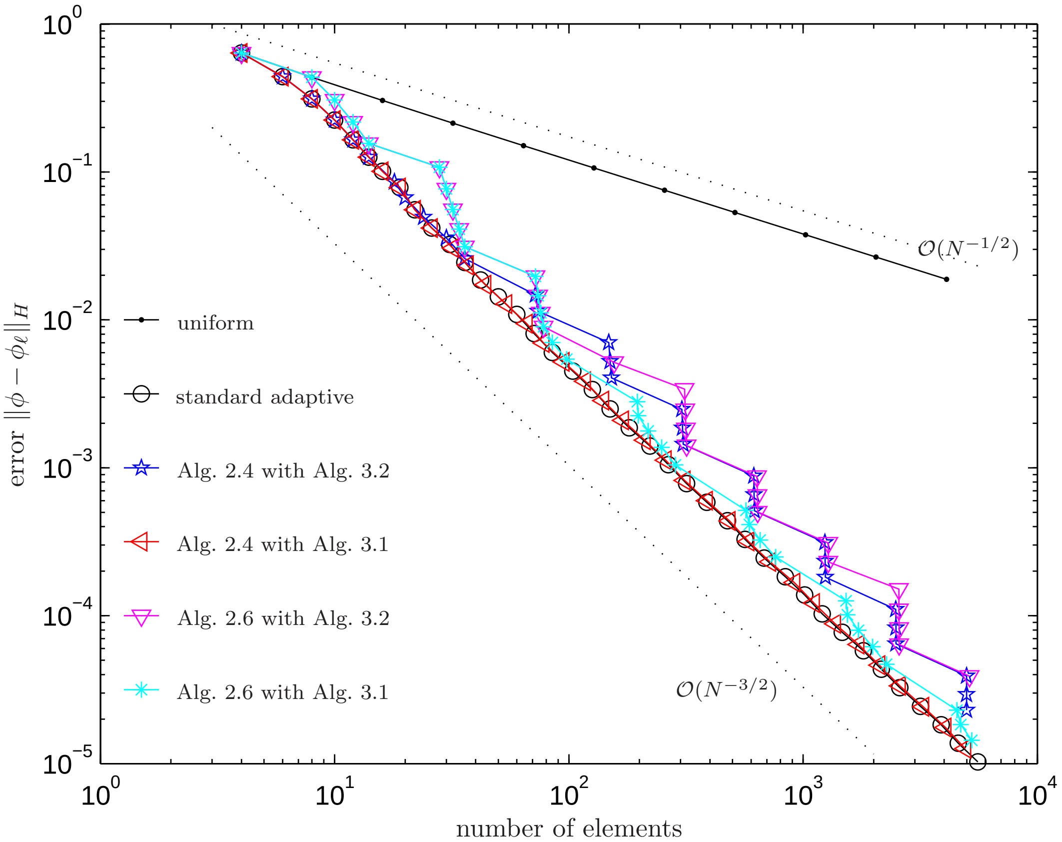

FIGURE 3.2.1. Convergence history of error $\|\phi - \phi_\ell\|_H$ in Slit Problem 3.2 for six different mesh-refining strategies of Subsection 3.1.

Figure 3.2.1 plots the experimental errors $\|\phi - \phi_\ell\|_H$ for the six mesh-refining strategies described in Section 3.1 above. As predicted by theory, uniform mesh-refinement leads to a poor convergence rate $\|\phi - \phi_\ell\|_H = \mathcal{O}(h^{1/2})$ with respect to the uniform mesh-size h. In some sense, this is cured by the proposed adaptive strategies. Figure 3.2.1 shows that the standard adaptive strategy leads to the optimal order of convergence $\mathcal{O}(N^{-3/2})$ with respect to the number of elements. Moreover, we empirically observe $\tau_\ell/\widehat{\tau}_\ell \geq 0.7$. Almost the same convergence behaviour is obtained for adaptive mesh-refinement steered by Algorithm 2.4 and Algorithm 3.1.

The combination of Algorithm 2.4 and Algorithm 3.2 leads to a sequence with $\tau_\ell/\widehat{\tau}_\ell \geq 0.7$, where these quotients turn out to satisfy $\tau_\ell/\widehat{\tau}_\ell < 0.75 = \varrho$ for certain steps ℓ. In these steps, the feedback loop of Algorithm 3.2 enforces uniform mesh-refinement visible in the behaviour of the corresponding error curve. We found that a lower choice $\varrho = 0.5$ leads to the same behaviour as for the standard adaptive algorithm since the critical criterion (3.4) is always satisfied (not displayed here).

For the $(h - h/2)$-steered Algorithm 2.6, we essentially observe the same behaviour as for Algorithm 2.4. We recall, however, that we have to expect certain uniform mesh-refinements after a fixed number of adaptive mesh-refinement steps. This is in fact deducible in the sense that step (c) of Algorithm 2.6 empirically leads to one step of uniform mesh-refinement after 4 to 5 steps of adaptive mesh-refinement.

3.3. Symm's Integral Equation for Dirichlet Problem. Symm's integral equation

$$V\phi = (K + 1/2)g \tag{3.11}$$

with $g(x) = r^{2/3}\cos(2\varphi/3)$ on the L-shaped domain $\Omega \subset \mathbb{R}^2$ with diameter $\operatorname{diam}(\Omega) = 1/2$ and a re-entrant corner at $(0,0) \in \mathbb{R}^2$ with polar coordinates (r, φ) of $x \in \Gamma$ involves the double-layer potential

$$Kg(x) := -\frac{1}{2\pi}\oint_\Gamma \frac{\partial}{\partial n_y} \log|x - y|\, g(y)\, ds_y \quad \text{for } x \in \Gamma. \tag{3.12}$$

The unique solution of (3.11) is the normal derivative $\phi = \partial u/\partial n$ of the solution $u(x) = r^{2/3}\cos(2\varphi/3)$ of the Dirichlet problem

$$-\Delta u = 0 \text{ in } \Omega \quad \text{with} \quad u = g \text{ on } \Gamma = \partial\Omega. \tag{3.13}$$

The numerical results for the error $\|\phi - \phi_\ell\|_H$ are displayed in Figure 3.3.1, and we observe the same behaviour as in Slit Problem 3.2 except, of course, the convergence speed $N^{-2/3}$ for uniform mesh-refinement.

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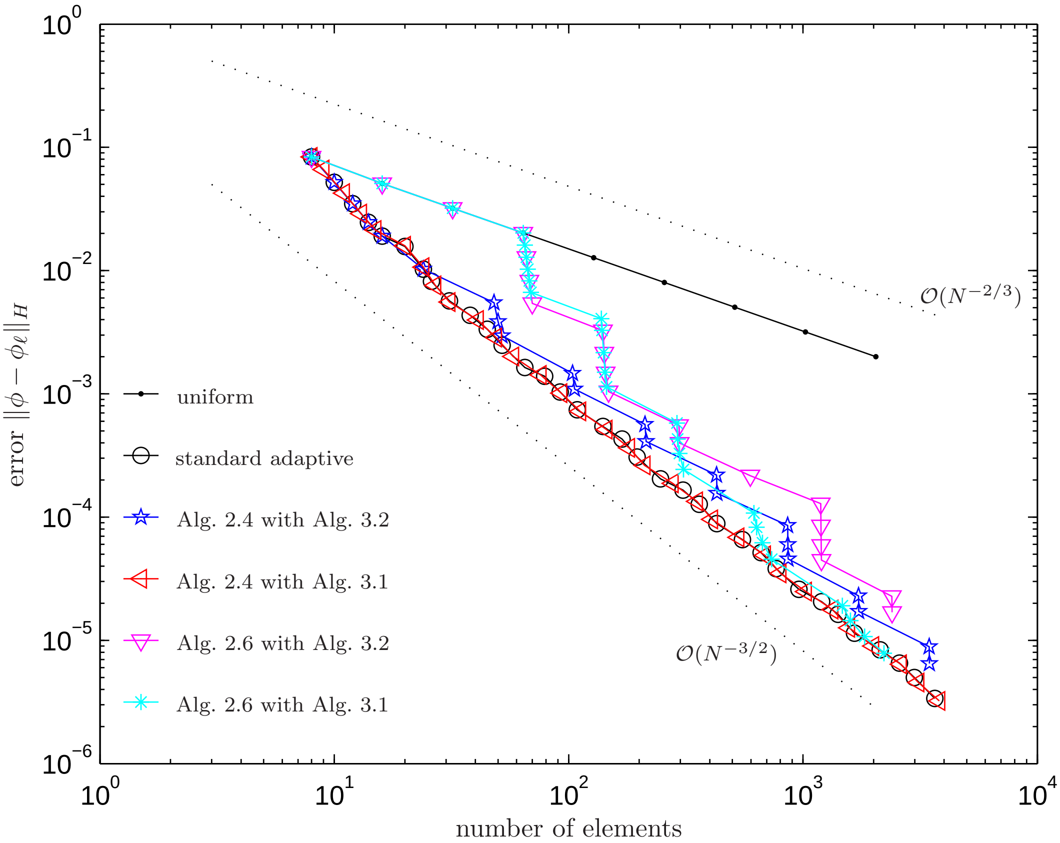

FIGURE 3.3.1. Convergence history of error $\|\phi-\phi_\ell\|_H$ in Dirichlet Problem 3.3 for six different mesh-refining strategies of Subsection 3.1.

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