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# Quantum Fokker-Planck models: Limiting case in the Lindblad Condition

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In this article we study stationary states and the long time asymptotics for the quantum Fokker–Planck equation. We continue the investigation of an earlier work in which we derived convergence to a steady state if the Lindblad condition $D_{pp}D_{qq} - D_{pq}^2 \geq \gamma^2/4$ is satisfied with strict inequality. Here we extend our results to the limiting case that turns out to be more difficult because irreducibility of the quantum Markov semigroup does not follow from triviality of the generalized commutator with position and momentum operators.

*Keywords*: Quantum Markov Semigroups, Quantum Fokker Planck, steady state, large-time convergence.

## 1. Quantum Fokker–Planck model

This paper is concerned with the long-time asymptotics of quantum Fokker–Planck (QFP) models, a special type of open quantum systems that models the quantum mechanical charge-transport including diffusive effects, as needed, *e.g.*, in the description of quantum Brownian motion, quantum optics, and semiconductor device simulations. We shall consider two equivalent descriptions, the Wigner function formalism and the density matrix formalism. We continue our analysis that we commenced in [2].

In the quantum kinetic Wigner picture a quantum state is described by the real valued Wigner function $w(x, v, t)$, where $(x, v) \in \mathbb{R}^2$ denotes the position–velocity phase space. Its time evolution in a harmonic confinement potential $V_0(x) = \omega^2 \frac{x^2}{2}$ with $\omega > 0$ is given by the Wigner Fokker–Planck equation

$$\begin{aligned} \partial_t w &= \omega^2 x \partial_v w - v \partial_x w + Qw \,, \\ Qw &= 2\gamma \partial_v (vw) + D_{pp} \Delta_v w + D_{qq} \Delta_x w + 2D_{pq} \partial_v \partial_x w \,. \end{aligned} \tag{1}$$

The (real valued) diffusion constants $D_{pp}, D_{pq}, D_{qq}$ and the friction $\gamma > 0$ satisfy the Lindblad condition

$$\Delta := D_{pp} D_{qq} - D_{pq}^2 - \gamma^2/4 \geq 0 \,, \tag{2}$$

and $D_{pp}, D_{qq} \geq 0$. In fact (2) together with $\gamma > 0$ implies $D_{pp}, D_{qq} > 0$. We assume that the particle mass and $\hbar$ are scaled to 1. This equation has been partly derived in [7]. Well–posedness [3,4,6], the classical limit [5] and long time asymptotics for purely harmonic oscillator potential [17] have been studied. For some applications we refer the reader to [9,10]. More references can be found in [1] or [16].

This equation can be equivalently studied in the Heisenberg-picture. The corresponding evolution equation on the space of bounded operators is given by[2]

$$\frac{\mathrm{d}A_t}{\mathrm{d}t} = \mathcal{L}(A_t) \,,$$

subject to initial conditions $A_{t=0} = A_0$. The generator $\mathcal{L}$ of the evolution semigroup $\mathcal{T}$ is given by

$$\begin{aligned} \mathcal{L}(A) = {} & \frac{i}{2} \left[ p^2 + \omega^2 q^2 + 2V(q), A \right] + i\gamma \left\{ p, [q, A] \right\} \\ & - D_{qq}[p,[p,A]] - D_{pp}[q,[q,A]] + 2D_{pq}[q,[p,A]] \,, \quad A \in \mathscr{B}(\mathsf{h}) \,. \end{aligned}$$

It can be written in (generalised) GKSL form like

$$\mathcal{L}(A) = i[H, A] - \frac{1}{2} \sum_{\ell=1}^{2} \left( L_\ell^* L_\ell A - 2 L_\ell^* A L_\ell + A L_\ell^* L_\ell \right) \tag{3}$$

with the "adjusted" Hamiltonian

$$H = \frac{1}{2} \left( p^2 + \omega^2 q^2 + \gamma(pq + qp) \right) + V(q) \,,$$

and the Lindblad operators $L_1$ and $L_2$ given by

$$L_1 = \frac{-2D_{pq} + i\gamma}{\sqrt{2D_{pp}}} p + \sqrt{2D_{pp}}\, q \,, \qquad L_2 = \frac{2\sqrt{\Delta}}{\sqrt{2D_{pp}}}\, p \,. \tag{4}$$

Note that here we use the external potential $U(q) = \omega^2 q^2/2 + V(q)$. The harmonic oscillator potential is the simplest way of ensuring confinement to guarantee the existence of a non trivial steady state. $V(q)$ is a perturbation potential, assumed to be twice continuously differentiable and satisfy

$$|V'(x)| \leq g_V \left(1+|x|^2\right)^{\alpha/2} \ , \tag{5}$$

with $g_V > 0$ and $0 \leq \alpha < 1$ .

## 2. Previous results

In [2] we proved the existence of the minimal Quantum Markov semigroup (QMS) for the Lindbladian (3). We will only sketch the result here.
First note that all operators can be defined on the domain of the Number operator $N := (p^2 + q^2 - 1)/2$,

$$\mathrm{Dom}(N) = \left\{ u \in \mathsf{h} \,\middle|\, Nu \in \mathsf{h} \right\} = \left\{ u \in \mathsf{h} \,\middle|\, p^2 u,\ q^2 u \in \mathsf{h} \right\} .$$

For details on domain problems we refer to [2].
We consider the operator $G$, defined on $\mathrm{Dom}(N)$, by

$$\begin{aligned} G = -\frac{1}{2}\left(L_1^* L_1 + L_2^* L_2\right) - iH = -\left(D_{qq} + \frac{i}{2}\right) p^2 - \left(D_{pp} + \frac{i\omega^2}{2}\right) q^2 \\ + \left(D_{pq} - \frac{i\gamma}{2}\right)(pq + qp) + \frac{\gamma}{2} - iV(q) \,. \end{aligned} \tag{6}$$

It can be checked that the domain of the adjoint operator $G^*$ is again $\mathrm{Dom}(N)$. The operators $G$ and $G^*$ are dissipative and thus $G$ generates a strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ on $h$.

Since the *formal* mass preservation holds we can apply results from [12] to construct $\mathcal{T}$, the minimal QMS associated with $G$ and the $L_\ell$'s. Moreover applying results form [8] and [12] we proved the following theorem.

**Theorem 2.1.**[2] *Suppose that the potential $V$ is twice differentiable and satisfies the growth condition (5). Then the minimal semigroup associated with the closed extensions of the operators $G, L_1, L_2$ is Markov and admits a normal invariant state.*

Note that this also implies the existence of the predual semigroup $\mathcal{T}_*$ on $\mathcal{J}_1$, the set of positive trace–class operators (*i.e.* density metrices).

The next step in our analysis is the proof of irreducibility. This implies that any initial density matrix, in the evolution, gives a positive mass on any subspace of $\mathsf{h}$ and allows us to apply powerful convergence results.

A QMS $\mathcal{T}$ on $\mathcal{B}(\mathsf{h})$ is called *irreducible* if the only subharmonic projections[13] $\Pi$ in $\mathsf{h}$ (*i.e.* projections satisfying $\mathcal{T}_t(\Pi) \geq \Pi$ for all $t \geq 0$) are the trivial ones 0 or $\mathbb{1}$. If a projection $\Pi$ is subharmonic, the total mass of any normal state $\sigma$ with support in $\Pi$ (*i.e.* such that $\Pi\sigma\Pi = \Pi\sigma = \sigma\Pi = \sigma$), remains concentrated in $\Pi$ during the evolution. As an example, the support projection of a normal stationary state for a QMS is subharmonic.[13] Thus if a QMS is irreducible and has a normal invariant state, then its support projection must be $\mathbb{1}$, *i.e.* it must be *faithful*. Subharmonic projections are characterised by the following theorem.

**Theorem 2.2.**[13] *A projection $\Pi$ is subharmonic for the QMS associated with the operators $G, L_\ell$ if and only if its range $\mathcal{X}$ is an invariant subspace for all the operators $P_t$ of the contraction semigroup generated by $G$ (i.e. $\forall t \geq 0 : P_t\mathcal{X} \subseteq \mathcal{X}$) and $L_\ell\,(\mathcal{X} \cap Dom(G)) \subseteq \mathcal{X}$ for all $\ell$'s.*

The application to our model yields the following Theorem. A sketch of the proof will be given in the beginning of the next section.

**Theorem 2.3.**[2] *Suppose that $\Delta > 0$. Then the QMS $\mathcal{T}$ associated with (the closed extensions of) the operators $G, L_\ell$ given by (6) and (4) is irreducible and thus all normal invariant states are faithful.*

We denote by $\{\, H, L_1, L_1^*, L_2, L_2^* \,\}'$ the *generalized commutant*, *i.e.* the set of all operators that commute with $H$ as well as with $L_1$,$L_1^*$,$L_2$ and $L_2^*$. Now since the semigroup has a faithful invariant state a combination of result by Frigerio,[15] Fagnola and Rebolledo[11,14] gives (under some technical conditions that can be checked for our model[2]) a criterium for convergence towards the steady state. If $\{H, L_1, L_1^*, L_2, L_2^*\}' = \{L_1, L_1^*, L_2, L_2^*\}' = \mathbb{C}\mathbb{1}$ then $\mathcal{T}_{*t}(\sigma)$ converges as $t \to \infty$ towards a unique invariant state in the trace norm. From $\gamma > 0$ we conclude that $L_1$ and $L_1^*$ are linearly independent and thus $\{\, L_1, L_1^*, L_2, L_2^* \,\}'$ contains operators commuting with both $q$ and $p$. This yields $\mathbb{C}\mathbb{1} = \{\, L_1, L_1^*, L_2, L_2^* \,\}' \supseteq \{H, L_1, L_1^*, L_2, L_2^*\}'$ and leads to

**Corollary 2.1.** *Let $\gamma > 0$ and $V \in C^2(\mathbb{R})$ satisfy* (5)*. If the QMS associated with $G$ and $L_\ell$ is irreducible (by Thm. 2.3 this holds true if $\Delta > 0$) then it has a unique faithful normal invariant state $\rho$. Moreover, for all normal initial states $\sigma$, we have*

$$\lim_{t\to\infty} \mathcal{T}_{*t}(\sigma) = \rho$$

*in the trace norm.*

Note that in the limiting case $\Delta = 0$ the irreducibility can indeed fail:

**Proposition 2.1.**[2] *Let $V = 0$, $\Delta = 0$, and $0 < \gamma < \omega$. Under the conditions*

$$D_{pq} = -\gamma D_{qq} \quad \text{and} \quad D_{pp} = \omega^2 D_{qq}\,. \tag{7}$$

*the semigroup is not irreducible. It admits a steady state that is not faithful.*

## 3. Irreducibility for $\Delta = 0$

In this section we will show that the semigroup is irreducible if the conditions (7) are violated. In doing so we also extend our convergence result. The interesting case when conditions (7) hold but perturbation potential is different from zero is postponed to a later work. We conjecture that the semigroup becomes irreducible as soon as $V \neq 0$.

First we sketch the idea of the proof of irreducibility in the case $\Delta > 0$. By Theorem 2.2 a projection is subharmonic if its range $\mathcal{X}$ is invariant for $G$ as well as for $L_1$ and $L_2$. Since $L_1$ and $L_2$ are linearly independent if $\Delta > 0$ we know that $\mathcal{X}$ has to be invariant for $p$ and $q$. Thus it is also invariant for the creation and annihilation operators $a$ and $a^\dagger$. Now if the closed subspace $\mathcal{X}$ is nonzero it includes an eigenvector of the Number operator. Since it is invariant under both, the creation and the annihilation operator, it has to be the whole space. Now the only subharmonic projections are the trivial ones and the semigroup is irreducible. A precise proof becomes more involved due to domain problems and can be found in [2]. This proof breaks down if $\Delta = 0$ since in this case $L_2 = 0$. Thus we look for an operator that leaves $\mathcal{X}$ invariant and can replace $L_2$ in the above strategy.

Since $\mathcal{X}$ is $G$ and $L_1$ invariant, the most natural choice for such an operator should be a polynomial in (the non-commuting) $G$ and $L_1$. We do all calculations on $\mathcal{C}_c^\infty$ disregarding commutator domains, *i.e.* understanding $[\cdot\,,\cdot]$ as $\overline{[\cdot\,,\cdot]}$. All operators can be extended to $\mathrm{Dom}(N)$ as in Ref. [2].

**Lemma 3.1.** *Let $\Delta = 0$. The following identities hold on $\mathcal{C}_c^\infty$:*

$$[G, L_1] = B + \frac{-2D_{pq} + i\gamma}{\sqrt{2D_{pp}}}\, V'(q) \tag{8}$$

*where*

$$B = \frac{2\gamma(-2D_{pq} + i\gamma) - 2D_{pp}}{\sqrt{2D_{pp}}} p + \omega^2 \frac{-2D_{pq} + i\gamma}{\sqrt{2D_{pp}}}\, q.$$

*The operator $B$ is linearly dependent of $L_1$ if and only if the identities (7) hold. In this case*

$$B = \omega^2 \frac{-2D_{pq} + i\gamma}{2D_{pp}}\, L_1. \tag{9}$$

**Proof.** Since $L_2 = 0$ we have $G = -\frac{1}{2}L_1^* L_1 - iH$. A straightforward but rather lengthy calculation using the CCR $[q,p] = i$ leads to formula (8).
Two operators $xp + yq$, $zp + wq$ (with $x, y, z, w \in \mathbb{C} - \{0\}$) are linearly *dependent* if and only if $x/z = y/w$. Therefore $B$ and $L_1$ are linearly dependent if and only if

$$\frac{2\gamma(-2D_{pq} + i\gamma) - 2D_{pp}}{-2D_{pq} + i\gamma} = \omega^2 \frac{(-2D_{pq} + i\gamma)}{2D_{pp}} \tag{10}$$

Clearly $\Delta = 0$ is equivalent to $4D_{pp}D_{qq} = (-2D_{pq} + i\gamma)(-2D_{pq} - i\gamma)$, *i.e.*

$$\frac{2D_{pp}}{-2D_{pq} + i\gamma} = \frac{-2D_{pq} - i\gamma}{2D_{qq}}.$$

Therefore (10) can be written in the form

$$2\gamma + \frac{2D_{pq} + i\gamma}{2D_{qq}} = \omega^2 \frac{(-2D_{pq} + i\gamma)}{2D_{pp}}.$$

The imaginary part of the left and right-hand side coincide if and only if $D_{pp} = \omega^2 D_{qq}$. Then the real parts coincide if and only if $D_{pq} = -\gamma D_{qq}$.

Now, if the identities $D_{pp} = \omega^2 D_{qq}$ and $D_{pq} = -\gamma D_{qq}$ hold, then we can write $B$ as

$$B = \frac{-2D_{pq} + i\gamma}{\sqrt{2D_{pp}}} \left( \left( 2\gamma - \frac{2D_{pp}}{-2D_{pq} + i\gamma} \right) p + \omega^2 q \right).$$

Writing

$$2\gamma - \frac{2D_{pp}}{-2D_{pq} + i\gamma} = 2\gamma + \frac{2D_{pq} + i\gamma}{2D_{qq}} = \omega^2 \frac{-2D_{pq} + i\gamma}{2D_{pp}}$$

we find the identity (9). □

**Theorem 3.1.** *Let $\Delta = 0$ and $D_{pq} \neq -\gamma D_{qq}$. Moreover assume that $V$ is twice continuously differentiable with $V''$ bounded.*
*The QMS $\mathcal{T}$ associated with (the closed extensions of) the operators $G, L_1$ given by (6) and (4) is irreducible.*

**Proof.** We only point out the difference with respect to the proof of in the case $\Delta > 0$ in [2]. The proof will proceed in tree steps. First we show that the range $\mathcal{X}$ of a subharmonic protection has to be invariant under the multiplication operator $V''(q)$. In step two we use this to show that $\mathcal{X}$ has to be invariant under an operator of the form $q(1 + zV''(q))$ for some $z \in \mathbb{C}$ with $\Im(z) \neq 0$. In step tree we conclude by a technical argument that this ensures invariance of $\mathcal{X}$ under multiplication by $q$ and complete the proof.

Step 1: The subspace $\mathcal{X}$ has to be invariant under the double commutator $[\,[G, L_1], L_1\,]$ *i.e.*, more precisely

$$[\,[G, L_1], L_1\,]\,(\mathcal{X} \cap \mathrm{Dom}(N^n)) \subseteq \mathcal{X} \cap \mathrm{Dom}(N^{n+2})$$

for all $n \geq 0$. A straightforward computation shows that

$$[\,[G, L_1], L_1\,] = z\mathbb{1} + i\frac{(-2D_{pq} + i\gamma)^2}{2D_{pp}}V''(q)$$

for some $z \in \mathbb{C}$. Therefore, by the density of $\mathcal{X} \cap \mathrm{Dom}(N^2)$ in $\mathcal{X}$, and boundedness of the self-adjoint multiplication operator $V''(q)$, we have

$$V''(q)(\mathcal{X}) \subseteq \mathcal{X}.$$

Step 2: We first calculate the commutator $[G, [G, L_1]]$.
To shorten the notation we set $\alpha := \frac{-2D_{pq}+i\gamma}{\sqrt{2D_{pp}}}$. With this abbreviation $\Delta = 0$ becomes $|\alpha|^2 = \alpha\overline{\alpha} = 2D_{qq}$, and we have

$$[G, L_1] = (2\gamma\alpha - \sqrt{2D_{pp}})p + \omega^2\alpha q + \alpha V'(q)\ .$$

Straightforward calculations yield

$$\begin{aligned}[G, [G, L_1]] = &\Big[(\alpha\omega^2(i\alpha\overline{\alpha} - 1) - i\alpha\sqrt{2D_{pp}}(2\gamma\alpha - \sqrt{2D_{pp}})\Big]\, p +\\ &\Big[i\sqrt{2D_{pp}}\alpha^2\omega^2 - 2i(D_{pp} + i\omega^2/2)(2\gamma\alpha - \sqrt{2D_{pp}})\Big]\, q +\\ i(\alpha\overline{\alpha}/2 + i/2)\alpha\{p, V''(q)\} &+ (2\gamma\alpha - \sqrt{2D_{pp}})V'(q) + i\sqrt{2D_{pp}}\alpha^2 V''(q)q\ ,\end{aligned}$$

where $\{p, V''(q)\}$ denotes the anticommutator.
Note that $\mathcal{X}$ is invariant under $L_1$ and by Step 1 also under the multiplication operator $V''(q)$. Thus it has to be invariant under the anticommutator

$$\{L_1, V''(q)\} = \alpha\{p, V''(q)\} + 2\sqrt{2D_{pp}}qV''(q)\ .$$

We can remove the term proportional to $\{p, V''\}$ from the double commutator by adding a suitable multiple of $\{L_1, V''\}$. The term proportional to $V'$ can be eliminated by a multiple of $[G, L_1]$ and finally we use $L_1$ to cancel the term with the momentum operator. Doing the tedious algebra leads to

$$\begin{aligned}&[G, [G, L_1]] + c_1\{L_1, V''\} + c_2[G, L_1] + c_3 L_1 =\\ &(-2\gamma\alpha + \sqrt{2D_{pp}})\left[\left(\omega^2 + (-2\gamma\alpha + \sqrt{2D_{pp}})\sqrt{2D_{pp}}/\alpha^2\right) q + qV''(q)\right]\ ,\end{aligned}$$

for explicit constants $c_1$, $c_2$, $c_3 \in \mathbb{C}$.
Since $\mathcal{X}$ is invariant for all operators on the left hand side of the above

equation (and the coefficient has absolute value different from zero) it is also invariant for $q(y+V'')$ with $y=\omega^2+(-2\gamma\alpha+\sqrt{2D_{pp}})\sqrt{2D_{pp}}/\alpha^2$. The real and imaginary parts of $y$ are given by

$$\Re(y) = \frac{2D_{pp}}{(4D_{pq}^2+\gamma^2)^2}\left[4\gamma D_{pq}(4D_{pq}^2+\gamma^2)+2D_{pp}(4D_{pq}^2-\gamma^2)\right]+\omega^2$$
$$\Im(y) = \frac{2D_{pp}}{(4D_{pq}^2+\gamma^2)^2}\left[2\gamma^2(4D_{pq}^2+\gamma^2)+8\gamma D_{pq}D_{pp}\right] \ .$$

Note that $\Im(y)=0$ if and only if $D_{pq}=-\gamma D_{qq}$, as can be seen by using $\Delta=0$ in the equation above. The condition for the real part to be zero, $D_{pq}/D_{qq} = -\gamma \pm \sqrt{\gamma^2-\omega^2+\gamma^2/(4D_{qq}^2)}$, is more difficult to see but direct calculations yield that when $\Im(y)=0$, then $\Re(y)=0$ if and only if $D_{pp}=\omega^2 D_{qq}$. Thus $|y|$ is zero exactly if $B$ and $L_1$ are linearly dependent. Since from our assumptions $D_{pq}\neq -\gamma D_{qq}$ we can invert $y$ and see that $\mathcal{X}$ is invariant for an operator $q(1+zV'')$ with $\Im(z)=-\Im(y)/|y|^2\neq 0$.

Step 3: Note that

$$|1+zV''(x)|^2 = (1+\Re(z)V''(x))^2+(\Im(z))^2(V''(x))^2$$

and $1+zV''(x)$ is non-zero for all $x\in\mathbb{R}$ because there is no $x$ such that $1+\Re(z)V''(x)=0=V''(x)$ (recall $\Im(z)\neq 0$). Moreover, for the same reason there is no sequence $(x_n)_{n\geq 1}$ of real numbers such that $1+\Re(z)V''(x_n)$ and $V''(x_n)$ both vanish as $n$ goes to infinity. It follows that

$$\inf_{x\in\mathbb{R}} |1+zV''(x)|^2 > 0.$$

and $1+zV''$ has a bounded inverse. This is given by spectral calculus of normal operators by

$$(1+zV'')^{-1} = \int_0^\infty \mathrm{e}^{-t(1+zV'')}\mathrm{d}t$$

and, since $\mathcal{X}$ is invariant under all powers $(1+zV'')^n$, it is invariant under $\mathrm{e}^{-t(1+zV'')}$ and also under the resolvent operator $(1+zV'')^{-1}$.
Now, for all $u\in\mathcal{X}\cap\mathrm{Dom}(G)$ we have $(1+zV''(q))^{-1}u=v\in\mathcal{X}\cap\mathrm{Dom}(q^2)$ and thus

$$qu = (q(1+zV''(q)))\,(1+zV''(q))^{-1}u = (q(1+zV''(q)))\,v\in\mathcal{X}.$$

It follows that $\mathcal{X}$ is $q$–invariant. Since $\mathcal{X}$ is also $L_1$ invariant it has to be $p$ invariant. Thus it is invariant under the creation operator $a=(q+ip)/\sqrt{2}$ and the annihilation operator $a^\dagger=(q-ip)/\sqrt{2}$ and $\mathcal{X}$ has to be either zero or coincide with the whole space (see [2]). □

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