ASC Report No. 01/2009

Convergence of Adaptive Boundary Element Methods

Markus Aurada, Samuel Ferraz-Leite, Dirk Praetorius

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-02-5

Most recent ASC Reports

38/2008 *Roberta Bosi*
Classical Limit for Linear and Nonlinear Quantum Fokker-Planck Systems

37/2008 *Federico Bassitti, Lucia Ladelli, Daniel Matthes*
Central Limit Theorem for a Class of Ome-dimensional Equations

36/2008 *Markus Brunk, Ansgar Jüngel*
Self-Heating in a Coupled Thermo-Electric Circuit-Device Model

35/2008 *Ansgar Jüngel, Daniel Matthes*
Entropiemethoden für nichtlineare partielle Differentialgleichungen

34/2008 *Ansgar Jüngel, Josipa-Pina Milišić*
A Sixth-Order Nonlinear Parabolic Equation for Quantum Systems

33/2008 *Ansgar Jüngel, Ingrid Violet*
Mixed Entropy Estimates for the Porous-Medium Equation with Convection

31/2008 *Markus Brunk, Ansgar Jüngel*
Heating of Semiconductor Devices in Electric Circuits

31/2008 *Anton Arnold, Matthias Ehrhardt, Maike Schulte, Ivan Sofronov*
Discrete Transparent Boundary Conditions for the Schrödinger Equation on Circular Domains

30/2008 *Philipp Dörsek*
hp-Finite Element Method for Variational Inequalities

29/2008 *Bertram Düring, Daniel Matthes, Guiseppe Toscani*
A Boltzmann-type Approach to the Formation of Wealth Distribution Curves

Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

ISBN 978-3-902627-02-5

ASC
TU WIEN

CONVERGENCE OF ADAPTIVE BOUNDARY ELEMENT METHODS

M. AURADA, S. FERRAZ-LEITE, AND D. PRAETORIUS

Abstract. A posteriori error estimators and adaptive mesh-refinement have themselves proven to be an important tool for scientific computing. For error control in finite element methods (FEM), there is a broad variety of a posteriori error estimators available, and convergence as well as optimality of adaptive FEM is well-studied in the literature. This is in sharp contrast to the boundary element method (BEM). Although a posteriori error estimators and adaptive algorithms are also successfully applied to boundary element schemes, even convergence of adaptive BEM is hardly understood mathematically. In our contribution, we present and discuss recent mathematical results which give first positive answers for adaptive BEM.

1. Introduction

For two reasons, fast and accurate error estimation plays a key role in reliable and efficient scientific computing: First, one may want to check whether the solution of a numerical simulation is accurate enough. Second, if this is not the case, one aims to improve the discretization, e.g., by local refinement of the underlying mesh.

Both subjects are usually covered by so-called a posteriori error estimates and related adaptive mesh-refining algorithms. For error control in finite element methods (FEM), there is a broad variety of a posteriori error estimators available, see e.g. [1], and convergence as well as optimality of adaptive algorithms is well understood, cf. [5] and the references therein. This is in sharp contrast to the boundary element method (BEM), where only a few a posteriori error estimators have been proposed, cf. [3] for an overview. Moreover, even convergence of adaptive BEM is widely open, and first preliminary convergence results have only recently been obtained [2, 7].

2. Model problem

As BEM model problem, we use the simple-layer potential

$$Vu(x) = -\frac{1}{2\pi}\int_\Gamma \log|x-y|\,u(y)\,d\Gamma(y) \tag{1}$$

associated with the 2D Laplacian. Here, $\Gamma \subseteq \partial\Omega$ is an open and connected piece of the boundary $\partial\Omega$ of a Lipschitz domain $\Omega \subset \mathbb{R}^2$. Provided $\mathrm{diam}(\Omega) < 1$,

$$\langle\!\langle u\,,\,v\rangle\!\rangle := \int_\Gamma Vu(x)v(x)\,d\Gamma(x) \tag{2}$$

defines an equivalent scalar product on the Sobolev space $\mathcal{H} := \widetilde{H}^{-1/2}(\Gamma)$. For a given $\Phi \in \mathcal{H}^*$, the Lax-Milgram lemma thus proves the unique existence of (some unknown) $u \in \mathcal{H}$ with

$$\langle\!\langle u\,,\,v\rangle\!\rangle = \Phi(v) \quad \text{for all } v \in \mathcal{H}. \tag{3}$$

Date: January 05, 2009.

Key words and phrases. boundary element method, a posteriori error estimate, adaptive mesh-refinement, convergence of adaptive algorithm.

To approximate u by the lowest-order Galerkin scheme, let $\mathcal{T}_\ell$ be a triangulation of Γ and $X_\ell = \mathcal{P}^0(\mathcal{T}_\ell) := \big\{ v_\ell : \Gamma \to \mathbb{R} \,:\, \forall T \in \mathcal{T}_\ell \quad v_\ell|_T \text{ is constant} \big\} \subset \mathcal{H}$. The (numerically computable) Galerkin solution $u_\ell \in X_\ell$ is the unique solution of

$$\langle\!\langle u_\ell \,,\, v_\ell \rangle\!\rangle = \Phi(v_\ell) \quad \text{for all } v_\ell \in X_\ell. \tag{4}$$

We aim to consider computable quantities η_ℓ which only depend on known and computed data (e.g., on u_ℓ and Φ) such that

$$C_{\mathrm{eff}}^{-1} \eta_\ell \le |\!|\!| u - u_\ell |\!|\!| \le C_{\mathrm{rel}} \eta_\ell. \tag{5}$$

Here, $|\!|\!| \cdot |\!|\!|$ denotes the energy norm induced by $\langle\!\langle \cdot \,,\, \cdot \rangle\!\rangle$. The lower and upper estimate are referred to as *efficiency* and *reliability* of the a posteriori error estimator η_ℓ, respectively.

3. $(h - h/2)$-based error estimators

The $(h-h/2)$-based strategy is a very basic and natural strategy to derive an a posteriori error estimator. Let $u_\ell \in X_\ell$ and $\widehat{u}_\ell \in \widehat{X}_\ell = \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$ be Galerkin solutions, where $\widehat{\mathcal{T}}_\ell$ is obtained by uniform refinement of $\mathcal{T}_\ell$. One then considers

$$\eta_\ell := |\!|\!| \widehat{u}_\ell - u_\ell |\!|\!| \tag{6}$$

to estimate $|\!|\!| u - u_\ell |\!|\!|$. By Galerkin orthogonality, η_ℓ is always efficient with known constant $C_{\mathrm{eff}} = 1$. Reliability of η_ℓ with $C_{\mathrm{rel}} = (1 - C_{\mathrm{sat}}^2)^{-1/2}$ follows from the saturation assumption

$$|\!|\!| u - \widehat{u}_\ell |\!|\!| \le C_{\mathrm{sat}} \, |\!|\!| u - u_\ell |\!|\!| \quad \text{with uniform } C_{\mathrm{sat}} \in (0,1). \tag{7}$$

Unlike the FEM, where (7) is proven for a sufficiently small mesh-size, the saturation assumption is open in the context of BEM but observed in practice, cf. [8] and the references therein.

Since the energy norm $|\!|\!| \cdot |\!|\!|$ is nonlocal, the error estimator η_ℓ does not provide information for a local mesh-refinement. By use of an inverse estimate and an approximation result, there holds

$$C_{\mathrm{apx}}^{-1} \eta_\ell \le \mu_\ell := \| h_\ell^{1/2} (\widehat{u}_\ell - u_\ell) \|_{L^2(\Gamma)} \le C_{\mathrm{inv}} \eta_\ell, \tag{8}$$

where $h_\ell \in L^\infty(\Gamma)$ denotes the local mesh-size defined by $h_\ell|_T := \mathrm{diam}(T)$ for $T \in \mathcal{T}_\ell$, cf. [8]. The local contributions

$$\mu_\ell(T) := \mathrm{diam}(T)^{1/2} \| \widehat{u}_\ell - u_\ell \|_{L^2(T)} \tag{9}$$

of μ_ℓ are then used to steer the following adaptive algorithm.

4. Adaptive mesh-refining algorithm

Based on the estimator μ_ℓ from Section 3 and on a fixed parameter $\theta \in (0,1)$, the usual adaptive algorithm reads as follows:

Algorithm 1. *For a given initial mesh $\mathcal{T}_\ell$ with $\ell = 0$ do:*

- (i) *Refine $\mathcal{T}_\ell$ uniformly to obtain $\widehat{\mathcal{T}}_\ell$.*
- (ii) *Compute discrete solutions u_ℓ and $\widehat{u}_\ell$.*
- (iii) *Find minimal set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that*

$$\theta \sum_{T \in \mathcal{T}_\ell} \mu_\ell(T)^2 \le \sum_{T \in \mathcal{M}_\ell} \mu_\ell(T)^2. \tag{10}$$

- (iv) *Refine at least marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$.*
- (v) *Increase counter $\ell \mapsto \ell + 1$ and iterate.* ■

Convergence of this type of algorithms has first been proven in [6], where also the marking criterion (10) is introduced. The latter work considered the residual error estimator for a P1-FEM discretization of the Poisson problem, and it is assumed that the given data are sufficiently resolved on the initial mesh. In [9], the resolution of the data is included into the adaptive algorithm. The convergence analysis is based on reliability and the so-called *discrete local efficiency* of the residual error estimator. The main idea of the convergence proof then is to show that the error is contractive up to the data oscillations. In [5], this has been weakened in the sense that it is proven that a weighted sum of error and error estimator yields a contraction property without requiring (discrete local) efficiency.

5. Convergence of $(\boldsymbol{h-h/2})$-steered adaptive BEM

Only recently, analogous results for adaptive BEM could be derived. A first convergence result for Algorithm 1 steered by μ_ℓ from Section 3, reads as follows [7]:

Theorem 2 (Ferraz-Leite, Ortner, Praetorius '08). *Provided that μ_ℓ is reliable and that marked elements are halved, there are constants $\kappa, \gamma \in (0,1)$ such that*

$$\Delta_\ell^2 := |||u-u_\ell|||^2 + |||u-\widehat{u}_\ell|||^2 + \gamma\,\mu_\ell^2 \tag{11}$$

satisfies

$$\Delta_{\ell+1} \le \kappa\,\Delta_\ell. \tag{12}$$

In particular, this implies convergence $u_\ell \to u$ as $\ell \to \infty$. ■

For the proof, it is crucial to observe that the error estimator μ_ℓ satisfies an *estimator reduction property*

$$\mu_{\ell+1} \le \varrho\,\mu_\ell + C\,|||\widehat{u}_{\ell+1} - \widehat{u}_\ell||| + C\,|||u_{\ell+1} - u_\ell||| \tag{13}$$

with some constants $\varrho \in (0,1)$ and $C > 0$. The verification of (13) follows by use of the Dörfler marking (10) and requires that the local contributions of μ_ℓ used for marking, have an h-weighting factor. The constant $\gamma \in (0,1)$ in the definition of Δ_ℓ is used to balance the constant $C > 0$ from (13) by use of the Galerkin orthogonality. This leads to $\Delta_{\ell+1}^2 \le |||u-u_\ell|||^2 + |||u-\widehat{u}_\ell|||^2 + \widetilde{\kappa}^2\,\gamma\,\mu_\ell^2$ for some $\widetilde{\kappa} \in (0,1)$. Using the reliability of μ_ℓ finally allows to prove (12).

Note that reliability of μ_ℓ, in fact, is equivalent to the saturation assumption (7) and thus, from a mathematical point of view, unsatisfactory. Moreover, from a conceptual point of view, Algorithm 1 has no knowledge on the Galerkin errors $|||u-u_\ell|||$ and $|||u-\widehat{u}_\ell|||$, but is only steered by the local contributions of μ_ℓ. From this point of view it seems to be natural to ask rather for convergence of $\mu_\ell \to 0$ as $\ell \to \infty$. This concept is followed in [2], and we stress that the following result —contrary to Theorem 2— is independent of the saturation assumption (7).

Theorem 3 (Aurada, Ferraz-Leite, Praetorius '08). *Provided that all marked elements are halved, there holds*

$$\lim_{\ell\to\infty} \mu_\ell = 0 = \lim_{\ell\to\infty} \eta_\ell, \tag{14}$$

i.e. convergence of the $(h-h/2)$-error estimators to zero. ■

For the proof, the main observation is that adaptive Galerkin schemes lead to Cauchy sequences $(u_\ell)_{\ell\in\mathbb{N}}$ and $(\widehat{u}_\ell)_{\ell\in\mathbb{N}}$. Note that this does not imply convergence of u_ℓ or $\widehat{u}_\ell$ to the exact solution $u \in \mathcal{H}$, but only to certain limits $u_\infty, \widehat{u}_\infty \in \mathcal{H}$, which may not even coincide. With this *a priori convergence of adaptive BEM*, the estimator reduction (13) may be written in Landau small-$\mathcal{O}$ notation

$$\mu_{\ell+1} \le \varrho\, \mu_\ell + \mathcal{O}(\ell), \tag{15}$$

and an inductive argument thus proves convergence of μ_ℓ to zero.

Finally, the saturation assumption (7) and Theorem 3 yield $|||u - u_\ell||| \le C_{\rm rel}\, \eta_\ell \to 0$, whence convergence of $u_\ell \to u$ as $\ell \to \infty$. Note that, in contrast to Theorem 2, the saturation assumption (7) now is only used in a second step.

6. Concluding remarks

The crucial step in the proofs of our convergence theorems is the estimator reduction (13), which is based on some h-weighting of the refinement indicators. Therefore, the analysis might carry over to adaptive algorithms steered by h-weighted residual error estimators or averaging error estimators, whereas the two-level error estimators and the Faermann error estimator seem to need further arguments. For the definition of these different error estimators, see [3, 4, 8] and the references therein.

Our analysis also applies to hypersingular integral equations and mixed formulations in 2D and 3D. For 3D, however, the proof in [7] —as well as the available a posteriori BEM error analysis— is restricted to the case of isotropic mesh-refinement, whereas anisotropic mesh-refinement is needed to resolve edge singularities efficiently. In addition, the new concept of convergence from [2] also seems to apply for certain anisotropic mesh-refining strategies as well as to adaptive FEM-BEM coupling.

Despite of convergence, even the question of optimal convergence rates of adaptive FEM is well-understood. Whereas prior works used an additional coarsening step, recent works prove optimality for Algorithm 1 steered by the residual error estimator. We refer to [5] and the references therein. The latter analysis, however, relies on a *discrete local reliability* of the error estimator, which remains open for adaptive Galerkin BEM. This will be a major topic for future research.

References

[1] Ainsworth, M. and Oden, J., *A posteriori error estimation in finite element analysis*, John Wiley & Sons, 2000.

[2] Aurada, M., Ferraz-Leite, S., and Praetorius, D., A new notion of convergence for adaptive boundary element methods, *work in progress*, 2009.

[3] Carstensen, C. and Faermann, B., Mathematical foundation of a posteriori error estimates and adaptive mesh-refining algorithms for boundary integral equations of the first kind, *Eng. Anal. Bound. Elem.*, 25, pp. 497–505, 2001.

[4] Carstensen, C. and Praetorius, D., Averaging techniques for the effective numerical solution of Symm's integral equation of the first kind, *SIAM J. Sci. Comput.*, 27, pp. 1226–1260, 2006.

[5] Cascon, J., Kreuzer, C., Nochetto, R., and Siebert, K., Quasi-optimal convergence rate for an adaptive finite element method, *SIAM J. Numer. Anal.*, 46, pp. 2524–2550, 2008.

[6] Dörfler, W., A convergent adaptive algorithm for Poisson's equation, *SIAM J. Numer. Anal.*, 33, pp. 1106–1124, 1996.

[7] Ferraz-Leite, S., Ortner, C., and Praetorius, D., Convergence of simple adaptive Galerkin schemes based on $h - h/2$ error estimators, *work in progress*, 2009.

[8] Ferraz-Leite, S. and Praetorius, D., Simple a posteriori error estimators for the h-version of the boundary element method, *Computing*, published online first, 2008.

[9] Morin, P., Nochetto, R., and Siebert, K., Data oscillation and convergence of adaptive FEM, *SIAM J. Numer. Anal.*, 38, pp. 466–488, 2000.

Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstrasse 8-10, A-1040 Wien, Austria
E-mail address: {Markus.Aurada,Samuel.Ferraz-Leite,Dirk.Praetorius}@tuwien.ac.at