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ESTIMATOR REDUCTION AND CONVERGENCE OF ADAPTIVE FEM AND BEM

M. AURADA, S. FERRAZ-LEITE, AND D. PRAETORIUS

Abstract. We propose a relaxed notion of convergence of adaptive finite element and boundary element schemes. Instead of asking for convergence of the error to zero, we only aim to prove estimator convergence in the sense that the adaptive algorithm drives the underlying error estimator to zero. We observe that certain error estimators satisfy an estimator reduction property which is sufficient for estimator convergence. The elementary analysis is only based on Dörfler marking and inverse estimates, but not on reliability and efficiency of the error estimator at hand. In particular, this covers certain adaptive algorithms in the context of FEM and BEM as well as heuristic strategies which are often successfully used to steer an adaptive anisotropic mesh-refinement. Our framework therefore contributes to understand adaptivity in FEM and BEM in a more general sense and gives a first mathematical justification for the proposed steering of anisotropic mesh-refinements.

1. Introduction

Usual discretization schemes like the finite element method (FEM) or the boundary element method (BEM) are based on a given triangulation $\mathcal{T}_\ell := \{T_1, \dots, T_N\}$ of the simulation domain and provide a numerical approximation u_ℓ of the exact solution u. Let $\rho_\ell := \big(\sum_{T\in\mathcal{T}_\ell} \rho_\ell(T)^2\big)^{1/2}$ be a *computable* a posteriori error estimator that associates some quantity $\rho_\ell(T)$ to each element $T \in \mathcal{T}_\ell$ which measures —at least heuristically— the local contribution of the error $\|u - u_\ell\|_T$ on T. These quantities may then be used to improve the triangulation $\mathcal{T}_\ell$ by local refinement. The common adaptive algorithm reads as follows:

Algorithm 1.1. *Fix $0 < \theta < 1$ and let $\mathcal{T}_\ell$ with $\ell = 0$ be the initial triangulation. For each $\ell = 0, 1, 2, \dots$ do:*

(i) *Compute discrete solution u_ℓ and error estimator ρ_ℓ.*
(ii) *Find minimal set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that*
$$\theta \sum_{T\in\mathcal{T}_\ell} \rho_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell} \rho_\ell(T)^2. \tag{1.1}$$
(iii) *Refine at least marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$.*
(iv) *Increase counter $\ell \mapsto \ell + 1$ an iterate.* □

Convergence of Adaptive FEM (AFEM). Convergence of this type of algorithms has first been proven in [D], where also the marking criterion (1.1) is introduced. The latter work considered the residual error estimator for a P1-FEM discretization of the Poisson problem, and it is assumed that the given volume data are sufficiently resolved on the initial mesh. In [MNS], the resolution of the data is included into the adaptive algorithm. The convergence analysis is based on reliability and the so-called *discrete local efficiency* of the residual error

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estimator, which relies on an *interior node property* for the local mesh-refinement. It is then shown that the error is contractive up to data oscillations. Optimality of this adaptive P1-FEM has first been shown by [S], where some *discrete local reliability* of the residual error estimator has been established and used.

In [CKNS], the authors prove convergence and optimality of adaptive FEM with Lagrange elements of fixed order for a certain class of linear elliptic PDEs, where Algorithm 1.1 is steered by the residual error estimator and where mesh-refinement is done by newest vertex bisection. Compared to [D, MNS, S], the analysis is improved in the sense that only reliability of the error estimator is used to show that a weighted sum of error and error estimator yields a contraction property. Whereas the preceding works [D, MNS, S] show that the marking criterion (1.1) is sufficient to prove convergence of AFEM, [CKNS, Lemma 5.9] states that (1.1) is already satisfied if error and data oscillations allow a strict error reduction. Said differently, the marking criterion (1.1) seems to be even necessary for optimal convergence behaviour of adaptive algorithms.

The work [MSV] considers a different approach for convergence. Instead of asking for convergence of the error to zero, the authors aim for a proof of the *estimator convergence*

$$\lim_{\ell\to\infty} \rho_\ell = 0. \tag{1.2}$$

Under *reliability* of ρ_ℓ, i.e. the error estimator provides (up to some constant) an upper bound for the error, (1.2) implies convergence of u_ℓ towards the exact solution u. The analysis of [MSV] is essentially based on the observation that adaptive mesh-refinement and conforming Galerkin schemes lead to a priori convergence

$$\lim_{\ell\to\infty} u_\ell = u_\infty \tag{1.3}$$

with some limit u_∞ which does not necessarily coincide with the exact solution u, see Lemma 1.3 below. Besides this, only *local efficiency* of the error estimator ρ_ℓ is used to verify (1.2) with the help of the Lebesgue dominated convergence theorem. Contrary to prior work [D, MNS, S, CKNS], the analytical framework of [MSV] covers various marking strategies instead of only (1.1) as well as different mesh-refining strategies instead of only newest vertex bisection (with or without interior node property).

Convergence of Adaptive BEM (ABEM). Only recently, a first convergence result for ABEM has been achieved. The reason for this is the non-locality of the involved boundary integral operators which leads to major difficulties in the numerical analysis of a posteriori error estimates. This is reflected by the fact that most error estimators in the context of BEM are so far only proven to be either reliable or efficient, cf. the discussion in [EFFP, EFGP, FOP]. Moreover, local properties of the error estimators like *local discrete reliability* or *local (discrete) efficiency* still remain mathematically open. In particular, this makes it impossible to prove contraction of the error with the techniques developed in [D, MNS]. Furthermore, the ideas of [MSV] cannot be applied either. First, the local efficiency of the error estimator remains open and seems to be a strong assumption in the context of boundary integral operators. Second, in case of weakly-singular integral equations, the exact solution u as well as the limit u_∞ of discrete solutions are not Lebesgue functions but distributions. This makes it impossible to use the Lebesgue dominated convergence theorem in the spirit of [MSV].

In [FOP], the technique of [CKNS] is applied to some $(h - h/2)$-error estimator proposed in [FP]. Under the saturation assumption, it is proven that a weighted sum of certain Galerkin errors and $(h-h/2)$-type error estimators satisfies a contraction property. We note that $(h - h/2)$-error estimators are always efficient, whereas reliability is equivalent to the saturation assumption. The latter can be proven for the P1-FEM of the Poisson problem [DN] and is used to prove convergence of $(h-h/2)$-based AFEM [FOP] but still remains open in the context of boundary element methods. Although the saturation assumption is empirically observed even for boundary element computations [FP], this makes the results of [FOP] in some sense mathematically unsatisfactory.

Estimator Reduction Implies Estimator Convergence. To overcome the dependence of the convergence result on the saturation assumption, we follow a different approach in this work: Instead of considering the error, we only ask for estimator convergence (1.2). From a conceptual point of view, this question is more natural since adaptive mesh-refinement is only based on the knowledge of the local contributions of ρ_ℓ. Moreover, this point of view allows the treatment of certain adaptive anisotropic mesh-refining strategies. Anisotropic meshes are in general necessary to resolve edge singularities effectively and to obtain optimal convergence results for BEM. To the best of our knowledge, there are —so far— no convergence results for adaptive Galerkin schemes with anisotropic mesh-refinement, even in the context of FEM.

Unlike [MSV], we restrict to the marking criterion (1.1) which, as has been pointed out before, seems to be necessary and sufficient for optimal convergence behaviour of adaptive schemes. Our analysis aims to provide an *estimator reduction* introduced in Equation (1.4) below for certain error estimators ρ_ℓ at hand. Compared to [MSV], our result is stronger in the sense that we prove —up to a zero sequence— a contraction property for the error estimator.

The two main observations of this paper are stated in the following elementary results, where the second is already contained in [MSV, Lemma 4.2], even in a more general formulation.

Proposition 1.2 (Estimator Reduction Implies Estimator Convergence). *Suppose that the sequence of error estimators $(\rho_\ell)_{\ell\in\mathbb{N}}$ satisfies some estimator reduction property*

$$\rho_{\ell+1} \le q\,\rho_\ell + \alpha_\ell \quad \text{for all } \ell \in \mathbb{N}_0 \tag{1.4}$$

with some fixed constant $0 < q < 1$ and some non-negative sequence $(\alpha_\ell)_{\ell\in\mathbb{N}}$ which satisfies $\lim_{\ell\to\infty} \alpha_\ell = 0$. Then, there holds the estimator convergence (1.2).

Proof. By induction on ℓ, the estimator reduction (1.4) implies

$$\rho_{\ell+1} \le q^{\ell+1}\rho_0 + \sum_{j=0}^{\ell} q^{\ell-j}\alpha_j \le q^{\ell+1}\rho_0 + \|(\alpha_n)\|_\infty \sum_{k=0}^{\ell} q^k$$

with $\|(\alpha_n)\|_\infty$ the supremum norm of the bounded sequence (α_n). In particular, the sequence (ρ_n) is bounded and $0 \le M := \limsup_{\ell\to\infty} \rho_\ell < \infty$ exists. Again, we apply (1.4) to see

$$M = \limsup_{\ell\to\infty} \rho_{\ell+1} \le q \limsup_{\ell\to\infty} \rho_\ell + \limsup_{\ell\to\infty} \alpha_\ell = q\,M.$$

With $0 < q < 1$, this yields $0 \le \liminf_{\ell\to\infty} \rho_\ell \le \limsup_{\ell\to\infty} \rho_\ell = 0$ and thus convergence (1.2). $\square$

The interpretation of the second observation is that Galerkin schemes with adaptive mesh-refinement always lead to a convergent sequence (u_ℓ) of discrete solutions (1.3). The limit u_∞, however, does not necessarily coincide with the continuous solution u.

Lemma 1.3 (A Priori Convergence of Adaptive Galerkin Schemes). *Suppose that $\mathcal{H}$ is a Hilbert space with norm $|||\cdot|||$ and X_ℓ is a sequence of nested closed subspaces, i.e. $X_\ell \subseteq X_{\ell+1}$. For fixed $u \in \mathcal{H}$, let $u_\ell \in X_\ell$ be the best approximation with respect to X_ℓ, i.e.*

$$|||u - u_\ell||| = \min_{v_\ell \in X_\ell} |||u - v_\ell|||. \tag{1.5}$$

Then, the limit $\lim_{\ell\to\infty} u_\ell \in \mathcal{H}$ exists. In particular, there holds $\lim_{\ell\to\infty} |||u_{\ell+1} - u_\ell||| = 0$.

Proof. Let X_∞ be the closure of $\bigcup_{\ell=0}^\infty X_\ell$ in $\mathcal{H}$. Then, X_∞ is a closed subspace of $\mathcal{H}$, and the best approximation $u_\infty \in X_\infty$ of u with respect to X_∞ exists. Best approximation in Hilbert spaces is realized in terms of the orthogonal projection so that the Pythagoras theorem reads

$$|||u - u_\ell|||^2 = |||u - u_\infty|||^2 + |||u_\infty - u_\ell|||^2.$$

In particular, u_ℓ is even the best approximation of u_∞ with respect to X_ℓ. Let $\varepsilon > 0$. Since $\bigcup_{\ell=0}^\infty X_\ell$ is dense in X_∞ and since the spaces X_ℓ are nested, we may choose some index ℓ_0 and some element $v_{\ell_0} \in X_{\ell_0}$ such that $|||u_\infty - v_{\ell_0}||| \le \varepsilon$. For $\ell \ge \ell_0$, the inclusion $X_{\ell_0} \subseteq X_\ell$ thus concludes $|||u_\infty - u_\ell||| = \min_{v_\ell \in X_\ell} |||u_\infty - v_\ell||| \le |||u_\infty - v_{\ell_0}||| \le \varepsilon$. $\square$

Remarks on Estimator Reduction. Below, the proofs of the estimator reduction (1.4) are only based on the marking criterion (1.1), use of the triangle inequality, and use of certain inverse estimates. In particular, the proofs will be independent of the Galerkin orthogonality. Moreover, the analysis is not restricted to the Hilbert space framework at all.

Throughout, we only use the a priori convergence of Galerkin schemes, and in this sense the idea of our work can be transfered to any numerical method which provides some a priori convergence of discrete solutions.

Outline of the Paper. The remaining content of the paper is organized as follows. In Section 2, we consider the Poisson problem and AFEM steered by the residual-based error estimator. The remaining Sections 3–4 treat Symm's integral equation as model problem for ABEM. Section 3 is concerned with isotropic mesh-refinement steered by $(h - h/2)$-based error estimators from [FP] and averaging on large patches introduced in [CP1, CP2]. Finally, Section 4 verifies the estimator reduction for an adaptive anisotropic mesh-refinement.

2. Residual-Based AFEM

We first consider an adaptive P1-FEM for the Poisson problem already treated in the seminal works [D, MNS, S]. In this context, the estimator reduction (1.4) was introduced and first proven, although not stated explicitly, in [CKNS]. Moreover, since the aim and scope of this work was on optimality of AFEM, the authors did not observe or mention that the estimator reduction already implies convergence of the adaptive algorithm in the sense of (1.2). Finally, the work [CKNS] treats a more general model problem and p-th order finite elements, but the Laplace equation with lowest-order elements might be an illustrative and simpler example.

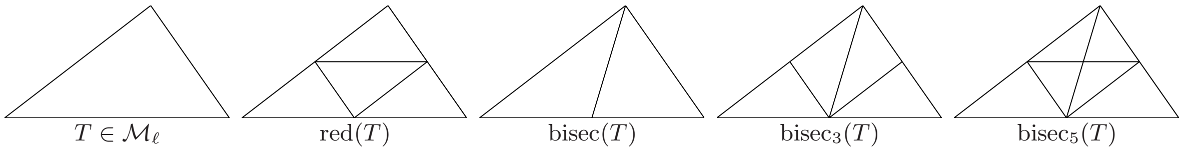

FIGURE 1. For marked elements $T \in \mathcal{M}_\ell$, all refinement rules based on newest vertex bisection and red-green-blue refinements guarantee $h_{\ell+1}|_T \le q\, h_\ell|_T$ with a uniform constant $q < 1$, which only depends on the initial mesh.

2.1. Model Problem. We consider the elliptic model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \partial_n u &= g \quad \text{on } \Gamma_N, \end{aligned} \tag{2.1}$$

with Ω a bounded Lipschitz domain in $\mathbb{R}^2$. The boundary Γ is split into a Dirichlet boundary Γ_D and a Neumann boundary Γ_N which satisfy $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ as well as $\Gamma_D \cap \Gamma_N = \emptyset$. Moreover, we assume that Γ_D has positive surface measure $|\Gamma_D| > 0$ so that (2.1) admits a unique weak solution. The energy scalar product of the weak formulation of (2.1) reads

$$\langle\!\langle u\,, v\rangle\!\rangle := \int_\Omega \nabla u \cdot \nabla v\, dx \quad \text{for all } u, v \in \mathcal{H} = \{u \in H^1(\Omega) \,:\, u|_{\Gamma_D} = 0\}. \tag{2.2}$$

In particular, the induced energy norm reads $|\!|\!| v |\!|\!| = \|\nabla v\|_{L^2(\Omega)}$. The weak formulation of (2.1) then reads

$$\langle\!\langle u\,, v\rangle\!\rangle = \int_\Omega f v\, dx + \int_{\Gamma_N} g v\, d\Gamma \quad \text{for all } v \in \mathcal{H} \tag{2.3}$$

and admits a unique solution $u \in \mathcal{H}$. We consider the lowest-order Galerkin scheme, where $\mathcal{T}_\ell$ is a regular triangulation of Ω into triangles and where $X_\ell = \{v_\ell \in \mathcal{S}^1(\mathcal{T}_\ell) \,:\, v_\ell|_{\Gamma_D} = 0\}$ with $\mathcal{S}^1(\mathcal{T}_\ell) = \{v_\ell \in C(\Omega) \,:\, \forall T \in \mathcal{T}_\ell \quad v_\ell|_T \text{ is affine}\}$. Then, the unique Galerkin solution $u_\ell \in X_\ell$ is determined by

$$\langle\!\langle u_\ell\,, v_\ell\rangle\!\rangle = \int_\Omega f v_\ell\, dx + \int_{\Gamma_N} g v_\ell\, d\Gamma \quad \text{for all } v_\ell \in X_\ell. \tag{2.4}$$

2.2. Mesh-Refinement and Local Mesh-Width. We assume that mesh-refinement is done in such a way that the created meshes $\mathcal{T}_\ell$ are uniformly shape-regular. This is ensured, for instance, by any mesh-refinement based on newest vertex bisection [NVB] or by the popular red-green-blue strategy [V96]. The local mesh-width $h_\ell \in L^\infty(\Omega)$ associated with $\mathcal{T}_\ell$ is defined in such a way that marked elements lead to a uniform decrease

$$h_{\ell+1}|_T \le q\, h_\ell|_T \quad \text{for all } T \in \mathcal{M}_\ell, \tag{2.5}$$

see Figure 1. If marked elements are red-refined into four similar elements or bisected with the help of bisec$_5(T)$, (2.5) is satisfied for $h_\ell|_T := \operatorname{diam}(T)$ and $q = 1/2$. In case of bisec$_3(T)$, an elementary calculation provides some $q < 1$, which only depends on the smallest angle of the triangles in the initial mesh $\mathcal{T}_0$. If marked elements are only bisected into two elements by bisec(T), (2.5) is satisfied for $h_\ell|_T := |T|^{1/2}$ and $q = 1/\sqrt{2}$.

In any case, we will simply write $h_T := h_\ell|_T$ for $T \in \mathcal{T}_\ell$.

2.3. Estimator Reduction for Residual Error Estimator. To steer Algorithm 1.1, we use the residual error estimator ρ_ℓ with local contributions

$$\rho_\ell(T)^2 = \|h_T f\|^2_{L^2(T)} + \|h_T^{1/2}[\partial_n u_\ell]\|^2_{L^2(\partial T\cap\Omega)} + \|h_T^{1/2}(g - \partial_n u_\ell)\|^2_{L^2(\partial T\cap\Gamma_N)}, \tag{2.6}$$

where $[\cdot]$ denotes the jump over an interior edge $E \in \mathcal{E}_\ell$ and $\mathcal{E}_\ell \subset \Omega$ denotes the set of all interior edges of triangulation $\mathcal{T}_\ell$. It is well-known that ρ_ℓ is reliable,

$$|||u - u_\ell||| \le C_{\rm rel}\,\rho_\ell. \tag{2.7}$$

and locally efficient up to oscillation terms,

$$\rho_\ell(T)^2 \le C_{\rm eff}^2 \left(\|\nabla(u-u_\ell)\|^2_{L^2(\omega_T)} + \|h_T(f - \Pi_\ell f)\|^2_{L^2(\omega_T)} + \|h_T^{1/2}(g - \Pi_\ell g)\|^2_{L^2(\partial T\cap\Gamma_N)}\right). \tag{2.8}$$

Here, $\omega_T = \bigcup\{T' \in \mathcal{T}_\ell \,:\, T\cap T' \neq \emptyset\}$ denotes the element patch of $T \in \mathcal{T}_\ell$, and Π_ℓ is the L^2-orthogonal projection onto the space of piecewise constant functions with respect to the underlying meshes $\mathcal{T}_\ell$ and $\mathcal{T}_\ell|_{\Gamma_N}$, respectively, cf. [V96].

Essentially, the following observation is already stated in [CKNS, Corollary 3.4].

Theorem 2.1. *Let $0 < \theta < 1$ be a fixed constant and suppose that the indicators $\rho_\ell(T)$ from (2.6) are used in Algorithm 1.1. Then,*

$$\rho_{\ell+1} \le \big(1 - \theta(1-q)\big)^{1/2}\,\rho_\ell + C_{\rm shape}\,|||u_{\ell+1} - u_\ell|||, \tag{2.9}$$

where $0 < q < 1$ is the constant from (2.5). The constant $C_{\rm shape} > 0$ only depends on the shape regularity of $\mathcal{T}_{\ell+1}$ and thus remains bounded. In particular, there holds $\lim\limits_{\ell\to\infty}\rho_\ell = 0$.

Proof. First, the triangle inequality in the sequence space ℓ_2 proves

$$\begin{aligned}
\rho_{\ell+1} &= \Big(\sum_{T'\in\mathcal{T}_{\ell+1}} \|h_{T'} f\|^2_{L^2(T')} + \|h_{T'}^{1/2}[\partial_n u_{\ell+1}]\|^2_{L^2(\partial T'\cap\Omega)} + \|h_{T'}^{1/2}(g - \partial_n u_{\ell+1})\|^2_{L^2(\partial T'\cap\Gamma_N)}\Big)^{1/2} \\
&\le \Big(\sum_{T'\in\mathcal{T}_{\ell+1}} \|h_{T'} f\|^2_{L^2(T')} + \|h_{T'}^{1/2}[\partial_n u_\ell]\|^2_{L^2(\partial T'\cap\Omega)} + \|h_{T'}^{1/2}(g - \partial_n u_\ell)\|^2_{L^2(\partial T'\cap\Gamma_N)}\Big)^{1/2} \\
&\quad + \Big(\sum_{T'\in\mathcal{T}_{\ell+1}} \|h_{T'}^{1/2}[\partial_n(u_{\ell+1}-u_\ell)]\|^2_{L^2(\partial T'\cap\Omega)} + \|h_{T'}^{1/2}\partial_n(u_{\ell+1}-u_\ell)\|^2_{L^2(\partial T'\cap\Gamma_N)}\Big)^{1/2}.
\end{aligned}$$

Second, according to uniform shape regularity of the generated family $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$, there holds

$$\begin{aligned}
&\Big(\sum_{T'\in\mathcal{T}_{\ell+1}} \|h_{T'}^{1/2}[\partial_n(u_{\ell+1}-u_\ell)]\|^2_{L^2(\partial T'\cap\Omega)} + \|h_{T'}^{1/2}\partial_n(u_{\ell+1}-u_\ell)\|^2_{L^2(\partial T'\cap\Gamma_N)}\Big)^{1/2} \\
&\quad \le C_{\rm shape}\,\|\nabla(u_{\ell+1}-u_\ell)\|_{L^2(\Omega)} = C_{\rm shape}\,|||u_{\ell+1}-u_\ell|||.
\end{aligned}$$

Third, we define the set $\overline{\mathcal{M}}_\ell := \{T' \in \mathcal{T}_{\ell+1} \,:\, \exists T \in \mathcal{M}_\ell \quad T' \subseteq T\}$ containing all elements obtained by refinement of marked elements. Then, (2.5) implies

$$\begin{aligned}
&\sum_{T' \in \overline{\mathcal{M}}_\ell} \|h_{T'} f\|_{L^2(T')}^2 + \|h_{T'}^{1/2}[\partial_n u_\ell]\|_{L^2(\partial T' \cap \Omega)}^2 + \|h_{T'}^{1/2}(g - \partial_n u_\ell)\|_{L^2(\partial T' \cap \Gamma_N)}^2 \\
&\le q \sum_{T \in \mathcal{M}_\ell} \|h_T f\|_{L^2(T)}^2 + \|h_T^{1/2}[\partial_n u_\ell]\|_{L^2(\partial T \cap \Omega)}^2 + \|h_T^{1/2}(g - \partial_n u_\ell)\|_{L^2(\partial T \cap \Gamma_N)}^2. \\
&= q \sum_{T \in \mathcal{M}_\ell} \rho_\ell(T)^2.
\end{aligned}$$

Here, we have used that each element $T \in \mathcal{M}_\ell$ is the disjoint union of its sons $T' \in \overline{\mathcal{M}}_\ell$ and that the jump $[\partial_n u_\ell]$ is zero on all edges which lie inside a marked element $T \in \mathcal{M}_\ell$. Fourth, the same arguments prove

$$\begin{aligned}
&\sum_{T' \in \mathcal{T}_{\ell+1} \setminus \overline{\mathcal{M}}_\ell} \|h_{T'} f\|_{L^2(T')}^2 + \|h_{T'}^{1/2}[\partial_n u_\ell]\|_{L^2(\partial T' \cap \Omega)}^2 + \|h_{T'}^{1/2}(g - \partial_n u_\ell)\|_{L^2(\partial T' \cap \Gamma_N)}^2 \\
&\le \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell} \|h_T f\|_{L^2(T)}^2 + \|h_T^{1/2}[\partial_n u_\ell]\|_{L^2(\partial T \cap \Omega)}^2 + \|h_T^{1/2}(g - \partial_n u_\ell)\|_{L^2(\partial T \cap \Gamma_N)}^2 \\
&= \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell} \rho_\ell(T)^2.
\end{aligned}$$

Finally, we use the marking strategy (1.1) and $q < 1$ to derive

$$q \sum_{T \in \mathcal{M}_\ell} \rho_\ell(T)^2 + \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell} \rho_\ell(T)^2 = (q-1) \sum_{T \in \mathcal{M}_\ell} \rho_\ell(T)^2 + \sum_{T \in \mathcal{T}_\ell} \rho_\ell(T)^2 \le \big((1 - \theta(1-q)\big)\rho_\ell^2.$$

Combining the obtained estimates with $\mathcal{T}_{\ell+1} = \overline{\mathcal{M}}_\ell \cup (\mathcal{T}_{\ell+1} \setminus \overline{\mathcal{M}}_\ell)$, we conclude (2.9). □

3. Adaptive BEM with Isotropic Mesh-Refinements

3.1. Model Problem. Throughout, we consider the first-kind integral equation

$$(Vu)(x) := \int_\Gamma G(x,y) u(y) \, d\Gamma(y) = f(x) \quad \text{for } x \in \Gamma \tag{3.1}$$

with weakly-singular integral kernel

$$G(x,y) = -\frac{1}{2\pi} \log|x-y| \text{ for } d = 2 \quad \text{and} \quad G(x,y) = +\frac{1}{4\pi} \frac{1}{|x-y|} \text{ for } d = 3. \tag{3.2}$$

Here, Γ is an open piece of the boundary $\partial\Omega$ of a Lipschitz domain Ω in $\mathbb{R}^d$, and $d\Gamma$ denotes the integration along the arc or on the manifold for $d = 2, 3$, respectively. For $d = 2$, we additionally assume $\operatorname{diam}(\Omega) < 1$. Then, V is a symmetric and elliptic isomorphism between the fractional-order Sobolev spaces $\mathcal{H} := \widetilde{H}^{-1/2}(\Gamma)$ and its dual $H^{1/2}(\Gamma)$. For proofs and details, we refer to [M, SS]. Recall that $H^{1/2}(\Gamma)$ denotes the trace space

$$H^{1/2}(\Gamma) := \{F|_\Gamma \,:\, F \in H^1(\Omega)\} \tag{3.3}$$

which is associated with the (Hilbert) norm

$$\|f\|_{H^{1/2}(\Gamma)} := \inf\{\|F\|_{H^1(\Omega)} \,:\, F \in H^1(\Omega) \text{ with } F|_\Gamma = f\}. \tag{3.4}$$

Then, the energy space $\widetilde{H}^{-1/2}(\Gamma)$ is the algebraic-topological dual of $H^{1/2}(\Gamma)$ with respect to the extended L^2-scalar product $\langle \cdot\,, \cdot \rangle$. The energy scalar product is thus given by

$$\langle\!\langle u\,, v \rangle\!\rangle := \langle Vu\,, v\rangle \quad \text{for all } u, v \in \mathcal{H}, \tag{3.5}$$

and (3.1) is equivalently stated in the variational form

$$\langle\!\langle u\,, v \rangle\!\rangle = \langle f\,, v\rangle \quad \text{for all } v \in \mathcal{H}. \tag{3.6}$$

The induced energy norm $|\!|\!| v |\!|\!| := \langle\!\langle v\,, v \rangle\!\rangle^{1/2}$ defines an equivalent norm on $\mathcal{H}$.

We consider the lowest-order Galerkin scheme, where $\mathcal{T}_\ell = \{T_1, \dots, T_N\}$ is a triangulation of Γ and where $X_\ell := \mathcal{P}^0(\mathcal{T}_\ell)$ denotes the space of all $\mathcal{T}_\ell$-piecewise constant functions on Γ. The Galerkin solution $u_\ell \in X_\ell$ with respect to X_ℓ is the unique solution of the variational form

$$\langle\!\langle u_\ell\,, v_\ell \rangle\!\rangle = \langle f\,, v_\ell\rangle \quad \text{for all } v_\ell \in X_\ell. \tag{3.7}$$

3.2. Mesh-Refinement and Local Mesh-Widths. Let $\mathcal{T}_\ell = \{T_1, \dots, T_N\}$ be a triangulation of Γ with associated mesh-size functions $h_\ell, \varrho_\ell \in L^\infty(\Gamma)$, where $h_\ell|_T := \mathrm{diam}(T)$ is the diameter of an element $T \in \mathcal{T}_\ell$ and where $\varrho_\ell|_T$ denotes the diameter of the largest inscribed circle in T. In this section, we consider isotropic mesh-refinement in the sense that any sequence $\mathcal{T}_\ell$ generated by the mesh-refinement rules satisfies

$$\sup_{\ell \in \mathbb{N}} \sigma(\mathcal{T}_\ell) < \infty, \quad \text{where} \quad \sigma(\mathcal{T}_\ell) := \max_{T \in \mathcal{T}_\ell} \frac{h_\ell|_T}{\varrho_\ell|_T} = \|h_\ell/\varrho_\ell\|_{L^\infty(\Gamma)}. \tag{3.8}$$

We consider $(h - h/2)$-based and averaging-based error estimators for BEM from [FP] and [CP1, CP2], respectively, where the local contributions are weighted by ϱ_ℓ. Therefore, the mesh-refinement now aims at a uniform reduction

$$\varrho_{\ell+1}|_T \le q \varrho_\ell|_T \quad \text{for all } T \in \mathcal{M}_\ell \quad \text{with some} \quad q < 1. \tag{3.9}$$

For the analysis, the mesh-refinement has to guarantee the inclusions

$$X_\ell \subseteq X_{\ell+1} \quad \text{and} \quad \widehat{X}_\ell \subseteq \widehat{X}_{\ell+1}. \tag{3.10}$$

Whereas the first inclusion is guaranteed for any mesh-refinement rule, the second inclusion for the uniformly refined meshes is crucial. We stress that the subsequently introduced mesh-refinement rules guarantee (3.10) even in the stronger form $X_\ell \subseteq X_{\ell+1} \subseteq \widehat{X}_\ell \subseteq \widehat{X}_{\ell+1}$.

For $d = 2$, we assume that the elements $T \in \mathcal{T}_\ell$ are affine line segments so that $\varrho_\ell = h_\ell$, i.e. $\sigma(\mathcal{T}_\ell) = 1$. When refined, an element T is bisected into two elements of half length so that (3.9) even holds in the form $\varrho_{\ell+1}|_T = q\varrho_\ell|_T$ with $q = 1/2$.

For $d = 3$, the triangulation $\mathcal{T}_\ell$ is either a regular triangulation consisting of flat triangles or an almost-regular triangulation consisting of flat rectangles with hanging nodes of order at most 1, cf. [FP].

First, if $\mathcal{T}_\ell$ is a regular triangulation into triangles, we note that all refinement rules based on newest vertex bisection satisfy the additional property (3.10), whereas red-green-blue refinement does not. We refer to [NVB] for the fact that newest vertex bisection based mesh-refinement only leads to finitely many similarity classes of triangles. In particular, $\sigma(\mathcal{T}_\ell)$ can be bounded uniformly by a constant that only depends on the initial mesh $\mathcal{T}_0$. Moreover, an elementary calculation proves that any newest vertex bisection based refinement rule guarantees (3.9), where $q < 1$ again only depends on $\mathcal{T}_0$.

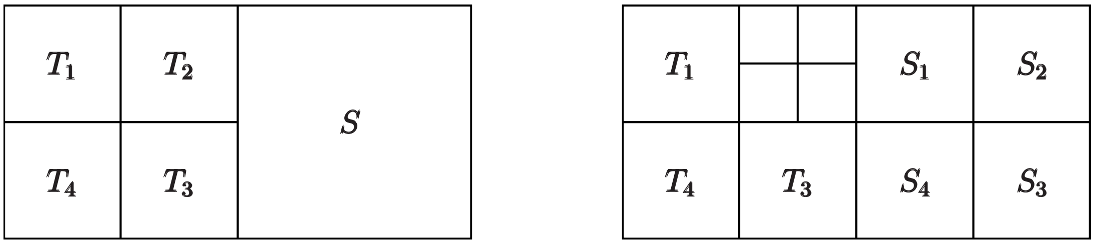

FIGURE 2. For isotropic mesh-refinement with rectangular elements, a marked element T is always refined uniformly into four new elements T_j. This isotropic refinement obviously yields $h_{\ell+1}|_T = \frac{1}{2}\,h_\ell|_T$ and $\varrho_{\ell+1}|_T = \frac{1}{2}\varrho_\ell|_T$ for the refined mesh-sizes. Moreover, one hanging node per edge is allowed (left). If, in the left configuration, element T_2 is marked for refinement, we mark element S for refinement as well (right).

Second, if $\mathcal{T}_\ell$ consists of rectangular elements, a marked element T is refined by use of the $\mathrm{unif}(T)$ rule shown in Figure 3. In particular, the shape-regularity constant $\sigma(\mathcal{T}_\ell) = \sigma(\mathcal{T}_0)$ does not change. As in 2D, (3.9) holds even in the form $\varrho_{\ell+1}|_T = q\varrho_\ell|_T$ with $q = 1/2$.

3.3. K-Mesh Property. The analysis of the error estimators under consideration depends on the uniform boundedness

$$\sup_{\ell\in\mathbb{N}} \kappa(\mathcal{T}_\ell) < \infty, \tag{3.11}$$

where the K-mesh constant $\kappa(\mathcal{T}_\ell) \geq 1$ is defined as follows:

- For any $T_j, T_k \in \mathcal{T}_\ell$ with $T_j \cap T_k \neq \emptyset$ holds $h_\ell|_{T_j}/h_\ell|_{T_k} \leq \kappa(\mathcal{T}_\ell)$ as well as $\varrho_\ell|_{T_j}/\varrho_\ell|_{T_k} \leq \kappa(\mathcal{T}_\ell)$, i.e. the local mesh-widths of neighbouring elements do not vary too rapidly.
- For any node $z \in \overline{\Gamma}$ of $\mathcal{T}_\ell$ holds $\#\{T \in \mathcal{T}_\ell \,:\, z \in T\} \leq \kappa(\mathcal{T}_\ell)$, i.e. each node does not belong to too many elements of $\mathcal{T}_\ell$.

We stress that (an upper bound (3.11) of) the K-mesh constant $\kappa(\mathcal{T}_\ell)$ enters the constants in the estimates from Theorem 3.1 and Theorem 3.4 below.

Note that for a sequence $\mathcal{T}_\ell$ of regular meshes consisting of triangles, the uniform shape-regularity (3.8) implies the uniform K-mesh property (3.11).

In order to ensure (3.11) in 2D, the refinement algorithm checks the mesh-size ratio of neighbouring elements: If $T_i \in \mathcal{T}_\ell$ is marked for refinement, any neighbour T_j with

$$h_\ell|_{T_j}/h_\ell|_{T_i} \geq 2 \tag{3.12}$$

is recursively marked as well. This guarantees $\kappa(\mathcal{T}_\ell) \leq 2\kappa(\mathcal{T}_0)$ for all generated meshes $\mathcal{T}_\ell$.

For meshes consisting of rectangular boundary elements in 3D, we naturally allow hanging nodes. However, to ensure the K-mesh property, we only allow one hanging node per edge, cf. Figure 2. This restriction automatically ensures $\varrho_\ell|_{T_i}/\varrho_\ell|_{T_j} \leq 2\kappa(\mathcal{T}_0)$ as well as $h_\ell|_{T_i}/h_\ell|_{T_j} \leq 2\kappa(\mathcal{T}_0)$ for neighbouring elements $T_i, T_j \in \mathcal{T}_\ell$ which share an edge. In particular, this implies $\kappa(\mathcal{T}_\ell) \leq 4\kappa(\mathcal{T}_0)$ for all generated meshes.

3.4. Estimator Reduction for $(h-h/2)$-Type Error Estimators. In the following, let $\widehat{\mathcal{T}}_\ell$ be the uniform refinement of $\mathcal{T}_\ell$. We denote by $\widehat{u}_\ell \in \widehat{X}_\ell := \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$ the corresponding Galerkin solution.

For the analysis below, we recall the inverse estimate

$$\|\varrho_\ell^{1/2} v_\ell\|_{L^2(\Gamma)} \le C_{\rm inv}\,|\!|\!| v_\ell |\!|\!| \quad \text{for all } v_\ell \in X_\ell \tag{3.13}$$

from [GHS, Theorem 3.6], where the constant $C_{\rm inv} > 0$ depends only on the K-mesh constant. Moreover, we recall the approximation estimate

$$|\!|\!| v - \Pi_\ell v |\!|\!| \le C_{\rm apx} \|h_\ell^{1/2}(v - \Pi_\ell v)\|_{L^2(\Gamma)} \le C_{\rm apx}\|h_\ell^{1/2} v\|_{L^2(\Gamma)} \tag{3.14}$$

proven in [CP1] where the constant $C_{\rm apx}$ depends only on Γ and Π_ℓ again denotes the L^2-orthogonal projection onto the space of piecewise constant functions with respect to the underlying mesh.

We recall the following main result from [FP]. Moreover, we stress that all mesh-refinement rules from Section 3.2 above guarantee uniform boundedness (3.8) of $\sigma(\mathcal{T}_\ell)$. This will be different for the anisotropic mesh-refinement discussed in Section 4, where $\sigma(\mathcal{T}_\ell)$ may tend to infinity as $\ell \to \infty$.

Theorem 3.1. *The error estimators*

$$\begin{aligned} \eta_\ell &= |\!|\!| \widehat{u}_\ell - u_\ell |\!|\!| & \widetilde{\eta}_\ell &= |\!|\!| (1-\Pi_\ell)\widehat{u}_\ell |\!|\!| \\ \mu_\ell &= \|\varrho_\ell^{1/2}(\widehat{u}_\ell - u_\ell)\|_{L^2(\Gamma)} & \widetilde{\mu}_\ell &= \|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{u}_\ell\|_{L^2(\Gamma)} \end{aligned} \tag{3.15}$$

satisfy the estimates

$$\widetilde{\mu}_\ell \le \mu_\ell \le \sqrt{2}\,C_{\rm inv}\,\eta_\ell \quad \textit{and} \quad \eta_\ell \le \widetilde{\eta}_\ell \le C_{\rm apx}\,\sigma(\mathcal{T}_\ell)^{1/2}\widetilde{\mu}_\ell, \tag{3.16}$$

where the constant $C_{\rm apx} > 0$ depends only on Γ. Moreover, η_ℓ, μ_ℓ, and $\widetilde{\mu}_\ell$ are always efficient in the sense that

$$\eta_\ell \le C_{\rm eff}\,|\!|\!| u - u_\ell |\!|\!| \tag{3.17}$$

with known efficiency constant $C_{\rm eff} = 1$. Finally, reliability of η_ℓ in the sense that

$$|\!|\!| u - u_\ell |\!|\!| \le C_{\rm rel}\,\eta_\ell \tag{3.18}$$

with some constant $C_{\rm rel} > 0$ is equivalent to the saturation assumption

$$|\!|\!| u - \widehat{u}_\ell |\!|\!| \le C_{\rm sat}\,|\!|\!| u - u_\ell |\!|\!| \tag{3.19}$$

with some constant $0 < C_{\rm sat} < 1$. □

Note that the error estimators $\rho_\ell \in \{\mu_\ell, \widetilde{\mu}_\ell\}$ can be employed to steer Algorithm 1.1 via

$$\mu_\ell(T) := \|\varrho_\ell^{1/2}(\widehat{u}_\ell - u_\ell)\|_{L^2(T)} \quad \text{and} \quad \widetilde{\mu}_\ell(T) := \|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{u}_\ell\|_{L^2(T)}. \tag{3.20}$$

In [FOP, Proof of Theorem 7] and [FOP, Proof of Theorem 8], we prove the following estimator reductions (3.21)–(3.22) for μ_ℓ and $\widetilde{\mu}_\ell$, respectively.

Theorem 3.2. *Let $0 < \theta < 1$ be a fixed constant and let $\mu_\ell(T)$ and $\widetilde{\mu}_\ell(T)$ be the indicators defined in* (3.20). *Let $0 < q < 1$ be the constant from* (3.9).

(i) *Suppose that we use the indicators $\rho_\ell(T) := \mu_\ell(T)$ in Algorithm 1.1. Then,*

$$\mu_{\ell+1} \le \big(1 - (1-q)\,\theta\big)^{1/2}\,\mu_\ell + C_{\rm mesh}\,\big(|\!|\!| \widehat{u}_{\ell+1} - \widehat{u}_\ell |\!|\!| + |\!|\!| u_{\ell+1} - u_\ell |\!|\!|\big) \quad \textit{for all } \ell \in \mathbb{N}_0. \tag{3.21}$$

(ii) *Suppose that we use the indicators* $\rho_\ell(T) := \widetilde{\mu}_\ell(T)$ *in Algorithm 1.1. Then,*

$$\widetilde{\mu}_{\ell+1} \le (1-\theta)^{1/2}\,\widetilde{\mu}_\ell + C_{\rm mesh}\,|||\widehat{u}_{\ell+1} - \widehat{u}_\ell||| \quad \textit{for all } \ell \in \mathbb{N}_0. \tag{3.22}$$

(iii) *The constant* $C_{\rm mesh} > 0$ *only depends on the chosen mesh-refinement and the initial mesh* $\mathcal{T}_0$*. The last two terms on the right-hand side of* (3.21) *as well as the last term on the right-hand side of* (3.22) *vanish as* $\ell \to \infty$*. In particular, Proposition 1.2 applies.*

Proof. For the convenience of the reader, we recall the proof of (3.21): The triangle inequality proves

$$\mu_{\ell+1} \le \|\varrho_{\ell+1}^{1/2}(\widehat{u}_\ell - u_\ell)\|_{L^2(\Gamma)} + \|\varrho_{\ell+1}^{1/2}((\widehat{u}_{\ell+1} - u_{\ell+1}) - (\widehat{u}_\ell - u_\ell))\|_{L^2(\Gamma)}$$

Note that the used mesh-refinement guarantees $(\widehat{u}_{\ell+1} - u_{\ell+1}) - (\widehat{u}_\ell - u_\ell) \in \widehat{X}_\ell$ as well as $q^{-1}\widehat{\varrho}_{\ell+1} \le \varrho_{\ell+1} \le C_{\rm refine}\widehat{\varrho}_{\ell+1}$ almost everywhere. The constant $C_{\rm refine}$ only depends on q and the chosen mesh-refinement. Therefore, the inverse estimate (3.13) gives

$$\begin{aligned}\|\varrho_{\ell+1}^{1/2}((\widehat{u}_{\ell+1} - u_{\ell+1}) - (\widehat{u}_\ell - u_\ell))\|_{L^2(\Gamma)} &\le C_{\rm refine}\,\|\widehat{\varrho}_{\ell+1}^{1/2}((\widehat{u}_{\ell+1} - \widehat{u}_\ell) - (u_{\ell+1} - u_\ell))\|_{L^2(\Gamma)}\\ &\le C_{\rm refine}\,C_{\rm inv}\,|||(\widehat{u}_{\ell+1} - \widehat{u}_\ell) - (u_{\ell+1} - u_\ell)|||.\end{aligned}$$

Isotropic mesh-refinement yields

$$\varrho_{\ell+1}|_T \le q\varrho_\ell|_T \quad \text{for all } T \in \mathcal{M}_\ell, \quad \text{as well as} \quad \varrho_{\ell+1}|_T \le \varrho_\ell|_T \quad \text{for all } T \in \mathcal{T}_\ell\backslash\mathcal{M}_\ell.$$

The marking strategy (1.1) gives

$$\begin{aligned}\|\varrho_{\ell+1}^{1/2}(\widehat{u}_\ell - u_\ell)\|_{L^2(\Gamma)}^2 &= \sum_{T\in\mathcal{M}_\ell}\|\varrho_{\ell+1}^{1/2}(\widehat{u}_\ell - u_\ell)\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{M}_\ell}\|\varrho_{\ell+1}^{1/2}(\widehat{u}_\ell - u_\ell)\|_{L^2(T)}^2\\ &\le q\sum_{T\in\mathcal{M}_\ell}\|\varrho_\ell^{1/2}(\widehat{u}_\ell - u_\ell)\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{M}_\ell}\|\varrho_\ell^{1/2}(\widehat{u}_\ell - u_\ell)\|_{L^2(T)}^2\\ &= (q-1)\sum_{T\in\mathcal{M}_\ell}\mu_\ell(T)^2 + \sum_{T\in\mathcal{T}_\ell}\mu_\ell(T)^2\\ &\le \big(1-(1-q)\,\theta\big)\,\mu_\ell^2.\end{aligned}$$

This concludes the proof of (3.21) with $C_{\rm mesh} = C_{\rm refine}C_{\rm inv}$. Lemma 1.3 proves that $\widehat{u}_\ell$ and u_ℓ converge to certain limits $\widehat{u}_\infty$ and u_∞, respectively. Consequently, the terms $|||\widehat{u}_{\ell+1} - \widehat{u}_\ell|||$ and $|||u_{\ell+1} - u_\ell|||$ vanish as $\ell \to \infty$. For the proof of (3.22), one only notes that there holds $\|\varrho_{\ell+1}^{1/2}(1 - \Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)} = 0$ for $T \in \mathcal{M}_\ell$. □

Under the saturation assumption (3.19), we can now prove that the adaptive algorithm leads to convergence $u_\infty = u$.

Corollary 3.3. *Let* $0 < \theta < 1$ *be a fixed constant and suppose that we use either* μ_ℓ *or* $\widetilde{\mu}_\ell$ *for marking in Algorithm 1.1. Assume that the saturation assumption* (3.19) *is valid, at least for infinitely many steps* ℓ *of the adaptive algorithm. Then, there holds*

$$\lim_{\ell\to\infty}\mu_\ell = \lim_{\ell\to\infty}\widetilde{\mu}_\ell = \lim_{\ell\to\infty}|||u - u_\ell||| = 0. \tag{3.23}$$

Proof. Recall that the saturation assumption (3.19) is equivalent to the reliability (3.18) of the $(h - h/2)$-error estimator η_ℓ, cf. Theorem 3.1. Moreover, the mesh-refining strategy in this section is isotropic. Therefore, μ_ℓ as well as $\widetilde{\mu}_\ell$ are equivalent to η_ℓ, cf. (3.16). Finally, convergence of the estimator, e.g., $\lim_{\ell\to\infty} \mu_\ell = 0$, implies $|||u - u_\ell||| \le C_{\rm rel}\, \eta_\ell \to 0$ as $\ell \to \infty$. □

Remark 1. *In* [FOP, Theorem 8]*, we prove the following result: Suppose that we use the indicators $\mu_\ell(T)$ for marking in Algorithm 1.1. Under the saturation assumption* (3.19)*, there are constants $\gamma, \kappa \in (0,1)$ such that $\Delta_\ell := |||u-u_\ell|||^2 + |||u-\widehat{u}_\ell|||^2 + \gamma\, \mu_\ell(T)^2 \ge 0$ satisfies $\Delta_{\ell+1} \le \kappa\, \Delta_\ell$. In particular, one obtains convergence $\Delta_\ell \xrightarrow{\ell\to\infty} 0$.— The same result holds for μ_ℓ replaced by $\widetilde{\mu}_\ell$, cf.* [FOP, Theorem 7]*. Although the results in* [FOP] *are stronger, so are the assumptions, i.e., uniform saturation assumption* (3.19) *for all steps $\ell = 0, 1, 2, \ldots$. Contrary to those results, we now have decoupled the convergence of the error estimator in Theorem 3.2 from the convergence of the error $\lim_{\ell\to\infty} |||u - u_\ell||| = 0$.* □

3.5. Estimator Reduction for Averaging Error Estimators. In this section, we additionally consider the space $X_\ell^{(1)} := \mathcal{P}^1(\mathcal{T}_\ell)$ of all $\mathcal{T}_\ell$-piecewise affine, but not necessarily continuous functions. Let $\mathbb{G}_\ell^{(1)}$ and $\Pi_\ell^{(1)}$ denote the Galerkin and L^2-projections onto $X_\ell^{(1)}$. The work [CP1] proposes to use averaging on large patches for a posteriori error estimation, i.e., the error $|||u - \widehat{u}_\ell|||$ is measured by use of $\mathbb{G}_\ell^{(1)}\widehat{u}_\ell$ or $\Pi_\ell^{(1)}\widehat{u}_\ell$. We recall the following main result from [CP1, CP2].

Theorem 3.4. *The error estimators*

$$\begin{array}{rclrcl} \alpha_\ell &=& |||(1-\mathbb{G}_\ell^{(1)})\widehat{u}_\ell||| & \qquad \widetilde{\alpha}_\ell &=& |||(1-\Pi_\ell^{(1)})\widehat{u}_\ell||| \\ \beta_\ell &=& \|\varrho_\ell^{1/2}(1-\mathbb{G}_\ell^{(1)})\widehat{u}_\ell\|_{L^2(\Gamma)} & \qquad \widetilde{\beta}_\ell &=& \|\varrho_\ell^{1/2}(1-\Pi_\ell^{(1)})\widehat{u}_\ell\|_{L^2(\Gamma)} \end{array} \tag{3.24}$$

satisfy the estimates

$$\widetilde{\beta}_\ell \le \beta_\ell \le \sqrt{2}\, C_{\rm inv}\, \alpha_\ell, \quad \alpha_\ell \le |||u - u_\ell|||, \quad \text{and} \quad \alpha_\ell \le \widetilde{\alpha}_\ell \le C_{\rm apx}\, \sigma(\mathcal{T}_\ell)^{1/2} \widetilde{\beta}_\ell. \tag{3.25}$$

The error estimators $\widetilde{\beta}_\ell$, β_ℓ, and α_ℓ are, in particular, efficient to estimate $|||u - u_\ell|||$. Moreover, there holds

$$\alpha_\ell \le |||u - \widehat{u}_\ell||| + |||(1-\mathbb{G}_\ell^{(1)})u|||, \tag{3.26}$$

which is understood as efficiency of $\widetilde{\beta}_\ell$, β_ℓ, and α_ℓ with respect to $|||u - \widehat{u}_\ell|||$, up to terms of higher order. Let $\widehat{\mathbb{G}}_\ell$ denote the Galerkin projection onto $\widehat{X}_\ell$. Provided that

$$q_\ell := |||(1-\widehat{\mathbb{G}}_\ell)\mathbb{G}_\ell^{(1)} : \mathcal{H} \to \mathcal{H}||| = \max_{v_\ell^{(1)} \in X_\ell^{(1)}\setminus\{0\}} \min_{\widehat{v}_\ell \in \widehat{X}_\ell} \frac{|||v_\ell^{(1)} - \widehat{v}_\ell|||}{|||v_\ell^{(1)}|||} < 1, \tag{3.27}$$

there even holds

$$|||u - \widehat{u}_\ell||| \le (1 - q_\ell^2)^{-1/2}\big(\alpha_\ell + |||(1-\mathbb{G}_\ell^{(1)})u|||\big), \tag{3.28}$$

which is interpreted as reliability of α_ℓ and $\widetilde{\alpha}_\ell$ with respect to $|||u - \widehat{u}_\ell|||$, up to terms of higher order. □

Following the lines of proof of Theorem 3.2, we obtain the following estimator reductions (3.29)–(3.30) for β_ℓ and $\widetilde{\beta}_\ell$, respectively.

Theorem 3.5. *Let $0 < \theta < 1$ be a fixed constant and let $\beta_\ell(T)$ and $\widetilde{\beta}_\ell(T)$ be the indicators defined in* (3.24)*. Let $0 < q < 1$ be the constant from* (3.9)*.*

(i) *Suppose that we use the indicators $\rho_\ell(T) := \beta_\ell(T)$ in Algorithm 1.1. Then,*

$$\begin{aligned}\beta_{\ell+1} &\le \big(1-(1-q)\,\theta\big)^{1/2}\,\beta_\ell \\ &\quad + C_{\rm mesh}\,(|||\widehat{u}_{\ell+1}-\widehat{u}_\ell||| + |||\mathbb{G}^{(1)}_{\ell+1}\widehat{u}_{\ell+1} - \mathbb{G}^{(1)}_\ell\widehat{u}_\ell|||) \quad \text{for all } \ell\in\mathbb{N}_0.\end{aligned} \tag{3.29}$$

(ii) *Suppose that we use the indicators $\rho_\ell(T) := \widetilde{\beta}_\ell(T)$ in Algorithm 1.1. Then,*

$$\widetilde{\beta}_{\ell+1} \le (1-\theta)^{1/2}\,\widetilde{\beta}_\ell + C_{\rm mesh}\,|||\widehat{u}_{\ell+1}-\widehat{u}_\ell||| \quad \text{for all } \ell\in\mathbb{N}_0. \tag{3.30}$$

(iii) *The constant $C_{\rm mesh} > 0$ only depends on the chosen mesh-refinement and the initial mesh $\mathcal{T}_0$. The last two terms on the right-hand side of* (3.29) *as well as the last term on the right-hand side of* (3.30) *vanish as $\ell\to\infty$. In particular, Proposition 1.2 applies.*

Proof. The proofs of (i) and (ii) follow along the same lines as in Theorem 3.2. To verify (iii), note that Lemma 1.3 proves convergence $\widehat{u}_\infty := \lim_\ell \widehat{u}_\ell$, whence $|||\widehat{u}_{\ell+1}-\widehat{u}_\ell||| \to 0$, and Proposition 1.2 applies to $\widetilde{\beta}_\ell$. Moreover, $\widehat{u}^{(1)}_\ell := \mathbb{G}^{(1)}_\ell\widehat{u}_\infty \in X^{(1)}_\ell$ is the best approximation of $\widehat{u}_\infty$ with respect to $X^{(1)}_\ell$. Therefore, Lemma 1.3 applies and proves that the limit $\widehat{u}^{(1)}_\infty := \lim_\ell \widehat{u}^{(1)}_\ell$ exists. Finally, a triangle inequality and stability of the Galerkin projection $\mathbb{G}^{(1)}_\ell$ yield

$$\begin{aligned}|||\mathbb{G}^{(1)}_{\ell+1}\widehat{u}_{\ell+1} - \mathbb{G}^{(1)}_\ell\widehat{u}_\ell||| &\le |||\mathbb{G}^{(1)}_{\ell+1}\widehat{u}_{\ell+1} - \mathbb{G}^{(1)}_{\ell+1}\widehat{u}_\infty||| + |||\mathbb{G}^{(1)}_{\ell+1}\widehat{u}_\infty - \mathbb{G}^{(1)}_\ell\widehat{u}_\infty||| + |||\mathbb{G}^{(1)}_\ell\widehat{u}_\infty - \mathbb{G}^{(1)}_\ell\widehat{u}_\ell||| \\ &\le |||\widehat{u}_{\ell+1}-\widehat{u}_\infty||| + |||\widehat{u}^{(1)}_{\ell+1}-\widehat{u}^{(1)}_\ell||| + |||\widehat{u}_\infty-\widehat{u}_\ell||| \xrightarrow{\ell\to\infty} 0.\end{aligned}$$

Consequently, Proposition 1.2 also applies to the error estimator β_ℓ. □

Remark 2. *Note that the last term in* (3.29) *reads $|||(\mathbb{G}^{(1)}_{\ell+1}\widehat{\mathbb{G}}_{\ell+1} - \mathbb{G}^{(1)}_\ell\widehat{\mathbb{G}}_\ell)u|||$. Since the spaces $X^{(1)}_\ell$ and $\widehat{X}_\ell$ are not nested, the operator $\mathbb{G}^{(1)}_\ell\widehat{\mathbb{G}}_\ell$ is* not *a Galerkin projection. This prevents to use the arguments of* [FOP] *to prove some contraction property for the (weighted) sum of error and β_ℓ — provided that β_ℓ is reliable. Instead, our new argument applies directly and proves $\lim\limits_{\ell\to\infty}\beta_\ell = 0$.* □

4. Adaptive BEM with Anisotropic Mesh-Refinement

We consider the model problem of Section 3. Since isotropic mesh-refinement does usually not recover the optimal order of convergence in 3D BEM computations, we extend the refinement strategy in Algorithm 1.1. For this purpose, we restrict to rectangular boundary elements. We use a strategy introduced in [FP, Section 4.5] for the $(h-h/2)$-based estimators μ_ℓ and $\widetilde{\mu}_\ell$ to decide whether a marked rectangle $T\in\mathcal{T}_\ell$ is refined isotropically into four rectangles or anisotropically into two rectangles, respectively, cf. Figure 3. For $\widetilde{\mu}_\ell$, we prove that this strategy yields the estimator reduction. Finally, we extend these ideas to prove the estimator reduction for some anisotropic mesh-refinement steered by $\widetilde{\beta}_\ell$.

4.1. $(h-h/2)$-Error Estimator. Let $T_1,\dots,T_4\in\widehat{\mathcal{T}}_\ell$ denote the four son-elements of a marked coarse-mesh rectangle $T\in\mathcal{T}_\ell$, where we use the same numbering as for the isotropic

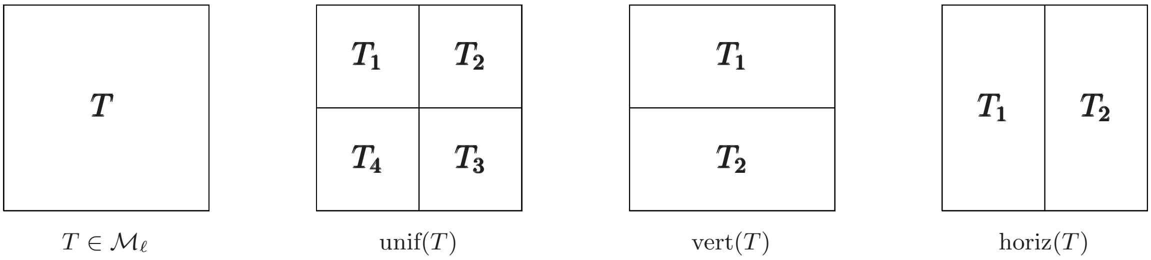

FIGURE 3. The extended Algorithm 1.1 in Section 4 gives a criterion whether a marked rectangle $T \in \mathcal{M}_\ell$ (left) is refined isotropically into four elements $T_1, \ldots, T_4$ or anisotropically into two elements T_1 and T_2. In the latter case, the algorithm decides whether vertical or horizontal refinement seems to be more appropriate.

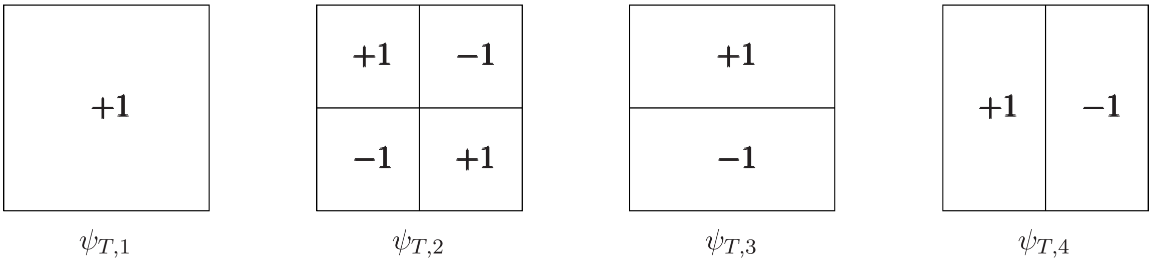

FIGURE 4. For each rectangle $T \in \mathcal{T}_\ell$, we introduce four $\widehat{\mathcal{T}}_\ell$-piecewise constant functions $\psi_{T,j} \in \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$, which are extended by zero to $\Gamma \setminus T$.

refinement of Figure 3. We consider the four piecewise constant functions $\psi_{T,j} \in \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$ from Figure 4 and observe that $\{\psi_{T,1}, \ldots, \psi_{T,4}\}$ is an L^2-orthogonal basis of $\mathcal{P}^0(\{T_1, \ldots, T_4\})$. Therefore, the already computed $\widehat{u}_\ell|_T \in \mathcal{P}^0(\{T_1, \ldots, T_4\})$ can be written in the form

$$\widehat{u}_\ell|_T = \sum_{j=1}^{4} c_{T,j}\psi_{T,j} \quad \text{with the Fourier coefficients} \quad c_{T,j} = \frac{(\psi_{T,j}\ ,\widehat{u}_\ell)_{L^2(T)}}{\|\psi_{T,j}\|_{L^2(T)}^2}. \tag{4.1}$$

The decision whether isotropic or anisotropic refinement is more appropriate, is now done as follows: Let $0 < \tau < 1$ be an additional parameter. We assume that $T \in \mathcal{M}_\ell$ is marked for refinement.

- If $c_{T,2}^2 + c_{T,3}^2 \leq \frac{\tau}{1-\tau}\, c_{T,4}^2$, we use horizontal refinement to create two sons $T_1, T_2 \in \mathcal{T}_{\ell+1}$.
- If $c_{T,2}^2 + c_{T,4}^2 \leq \frac{\tau}{1-\tau}\, c_{T,3}^2$, we use vertical refinement to create two sons $T_1, T_2 \in \mathcal{T}_{\ell+1}$.
- Otherwise, $T \in \mathcal{M}_\ell$ is refined isotropically into four sons $T_1, \ldots, T_4 \in \mathcal{T}_{\ell+1}$.

In order to ensure the uniform boundedness of the K-mesh constant $\kappa(\mathcal{T}_\ell)$ we additionally check the mesh-size ratio ϱ_ℓ of neighbouring elements and possibly mark additional elements as it is done in 2D for the mesh-size ratio with respect to h_ℓ.

Remark 3. *In* [FP, Section 4.5], *we use the above stated refinement strategy with* $\widetilde{\tau} := \tau/(1-\tau) = 1/2$ *which is equivalent to the choice* $\tau = 1/3$. □

Remark 4. *In addition to* [FP]*, we stress the following observation: For $\tau < 1/2$, the proposed criterion cannot mark $T \in \mathcal{M}_\ell$ for both, horizontal and vertical refinement. To see this, we argue by contradiction and assume that there holds $c_{T,2}^2 + c_{T,3}^2 \le \frac{\tau}{1-\tau}\, c_{T,4}^2$ as well as $c_{T,2}^2 + c_{T,4}^2 \le \frac{\tau}{1-\tau}\, c_{T,3}^2$ for some $T \in \mathcal{M}_\ell$. Note that this is equivalent to*

$$c_{T,2}^2 + c_{T,3}^2 \le \tau\,(c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2) \quad \textit{and} \quad c_{T,2}^2 + c_{T,4}^2 \le \tau\,(c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2).$$

Now, $\tau < 1/2$ yields

$$c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2 \le \tau\,(c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2) + \frac{1}{2}(c_{T,3}^2 + c_{T,4}^2) < c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2,$$

from which we infer $c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2 = 0$. This however implies $\widehat{u}_\ell|_T = (\Pi_\ell \widehat{u}_\ell)|_T$. Then, $\widetilde{\mu}_\ell(T) = 0$ contradicts $T \in \mathcal{M}_\ell$ according to the minimality of $\mathcal{M}_\ell$. □

Remark 5. *Note that $\|\psi_{T,j}\|_{L^2(T)}^2 = |T|$ so that the denominator in* (4.1) *can be neglected for the implementation.* □

We now prove that this anisotropic mesh-refining strategy yields the estimator reduction for the error estimator $\widetilde{\mu}_\ell$.

Theorem 4.1. *Let $0 < \theta < 1$ and $0 < \tau < 1$ be fixed constants. Suppose that we use the indicators $\rho_\ell(T) := \widetilde{\mu}_\ell(T)$ defined in* (3.20) *for marking in Algorithm 1.1 and the above described heuristics to decide the type of refinement. Then,*

$$\widetilde{\mu}_{\ell+1} \le \big(1 - \theta(1-\tau)\big)^{1/2}\, \widetilde{\mu}_\ell + \sqrt{2}\, C_{\rm inv}\, |||\widehat{u}_{\ell+1} - \widehat{u}_\ell||| \quad \textit{for all } \ell \in \mathbb{N}_0, \tag{4.2}$$

and the second term on the right-hand side vanishes as $\ell \to \infty$. In particular, Proposition 1.2 applies and proves convergence $\lim\limits_{\ell\to\infty} \widetilde{\mu}_\ell = 0$.

Proof. Note that, for rectangular elements, $\widehat{\varrho}_{\ell+1} = \varrho_{\ell+1}/2$. We proceed as in the proof of Theorem 3.2. The triangle inequality and the inverse estimate (3.13) prove

$$\begin{aligned}
\widetilde{\mu}_{\ell+1} &= \|\varrho_{\ell+1}^{1/2}(1 - \Pi_{\ell+1})\widehat{u}_{\ell+1}\|_{L^2(\Gamma)} \\
&\le \|\varrho_{\ell+1}^{1/2}(1 - \Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(\Gamma)} + \|\varrho_{\ell+1}^{1/2}(1 - \Pi_{\ell+1})(\widehat{u}_{\ell+1} - \widehat{u}_\ell)\|_{L^2(\Gamma)} \\
&\le \|\varrho_{\ell+1}^{1/2}(1 - \Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(\Gamma)} + \|\varrho_{\ell+1}^{1/2}(\widehat{u}_{\ell+1} - \widehat{u}_\ell)\|_{L^2(\Gamma)} \\
&\le \|\varrho_{\ell+1}^{1/2}(1 - \Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(\Gamma)} + \sqrt{2}\, C_{\rm inv}\, |||\widehat{u}_{\ell+1} - \widehat{u}_\ell|||,
\end{aligned}$$

where we have additionally used that $\Pi_{\ell+1}$ is even the $\mathcal{T}_{\ell+1}$-elementwise L^2-orthogonal projection. Now, let $T \in \mathcal{M}_\ell$ be a marked element.

- If T is refined isotropically, there holds
$$\|(1 - \Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)} = 0.$$
- If T is refined by horizontal refinement, there holds
$$\|(1 - \Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)}^2 = |T|\,\big(c_{T,2}^2 + c_{T,3}^2\big).$$

  Moreover, the proposed mesh-refinement yields $c_{T,2}^2 + c_{T,3}^2 \le \frac{\tau}{1-\tau}\, c_{T,4}^2$, which is equivalent to
$$|T|\,\big(c_{T,2}^2 + c_{T,3}^2\big) \le \tau\,|T|\,\big(c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2\big) = \tau\,\|(1 - \Pi_\ell)\widehat{u}_\ell\|_{L^2(T)}^2.$$

- If T is refined by vertical refinement, there holds

$$\|(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)}^2 = |T|\,\big(c_{T,2}^2 + c_{T,4}^2\big),$$

  and the mesh-refinement strategy yields $c_{T,2}^2 + c_{T,4}^2 \le \frac{\tau}{1-\tau}\,c_{T,3}^2$. This again leads to

$$|T|\,\big(c_{T,2}^2 + c_{T,4}^2\big) \le \tau\,|T|\,\big(c_{T,2}^2 + c_{T,3}^2 + c_{T,4}^2\big) = \tau\,\|(1-\Pi_\ell)\widehat{u}_\ell\|_{L^2(T)}^2.$$

Since $\varrho_{\ell+1}|_T \in \mathbb{R}$ is constant, we thus obtain in any case

$$\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)}^2 \le \tau\,\|\varrho_{\ell+1}^{1/2}(1-\Pi_\ell)\widehat{u}_\ell\|_{L^2(T)}^2 \le \tau\,\widetilde{\mu}_\ell(T)^2 \quad \text{for all } T \in \mathcal{M}_\ell.$$

Moreover, there clearly holds

$$\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)}^2 \le \|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{u}_\ell\|_{L^2(T)}^2 = \widetilde{\mu}_\ell(T)^2 \quad \text{for all } T \in \mathcal{T}_\ell\backslash\mathcal{M}_\ell.$$

Together with the marking strategy (1.1), this implies

$$\begin{aligned}
\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(\Gamma)}^2 &= \sum_{T\in\mathcal{M}_\ell} \|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{M}_\ell} \|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{u}_\ell\|_{L^2(T)}^2 \\
&\le \tau \sum_{T\in\mathcal{M}_\ell} \widetilde{\mu}_\ell(T)^2 + \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{M}_\ell} \widetilde{\mu}_\ell(T)^2 \\
&= -(1-\tau) \sum_{T\in\mathcal{M}_\ell} \widetilde{\mu}_\ell(T)^2 + \sum_{T\in\mathcal{T}_\ell} \widetilde{\mu}_\ell(T)^2 \\
&\le \big(1-\theta(1-\tau)\big)\,\widetilde{\mu}_\ell^2
\end{aligned}$$

and concludes the proof. □

4.2. Averaging Error Estimator. The ideas of the previous section can be generalized to anisotropic mesh-refinement steered by the averaging estimator $\widetilde{\beta}_\ell$ from (3.24). For $T \in \mathcal{M}_\ell$, let $\Pi^{(1)}_{\mathrm{unif}(T)}$, $\Pi^{(1)}_{\mathrm{vert}(T)}$, and $\Pi^{(1)}_{\mathrm{horiz}(T)}$ denote the L^2-orthogonal projections onto $\mathcal{P}^1(\mathrm{unif}(T))$, $\mathcal{P}^1(\mathrm{vert}(T))$, and $\mathcal{P}^1(\mathrm{horiz}(T))$, respectively, cf. Figure 3. As before, let $0 < \tau < 1$ be an additional parameter and assume that $T \in \mathcal{T}_\ell$ is marked for refinement.

- If $\|\varrho_\ell^{1/2}(1-\Pi^{(1)}_{\mathrm{horiz}(T)})\widehat{u}_\ell\|_{L^2(T)}^2 \le \tau\widetilde{\beta}_\ell(T)^2$, we use horizontal refinement to create two sons $T_1, T_2 \in \mathcal{T}_{\ell+1}$.
- If $\|\varrho_\ell^{1/2}(1-\Pi^{(1)}_{\mathrm{vert}(T)})\widehat{u}_\ell\|_{L^2(T)}^2 \le \tau\widetilde{\beta}_\ell(T)^2$, we use vertical refinement to create two sons $T_1, T_2 \in \mathcal{T}_{\ell+1}$.
- Otherwise, $T \in \mathcal{M}_\ell$ is refined isotropically into four sons $T_1, \dots, T_4 \in \mathcal{T}_{\ell+1}$.

As in the previous section, we check the mesh-size ratio ϱ_ℓ of neighbouring elements and possibly mark them for refinement in order to ensure the uniform boundedness of the K-mesh constant $\kappa(\mathcal{T}_\ell)$.

Theorem 4.2. *Let $0 < \theta < 1$ and $0 < \tau < 1$ be fixed constants. Suppose that we use the indicators $\rho_\ell(T) := \widetilde{\beta}_\ell(T)$ defined in (3.24) in Algorithm 1.1. Then,*

$$\widetilde{\beta}_{\ell+1} \le \big(1-\theta(1-\tau)\big)^{1/2}\,\widetilde{\beta}_\ell + \sqrt{2}\,C_{\mathrm{inv}}\,|||\widehat{u}_{\ell+1} - \widehat{u}_\ell||| \quad \text{for all } \ell \in \mathbb{N}_0, \tag{4.3}$$

and the second term on the right-hand side vanishes as $\ell \to \infty$. In particular, Proposition 1.2 applies and proves convergence $\lim_{\ell\to\infty} \widetilde{\beta}_\ell = 0$.

Proof. We follow the lines of the proof of Theorem 4.1. As above, the triangle inequality and the inverse estimate (3.13) prove

$$\widetilde{\beta}_{\ell+1} \le \|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1}^{(1)})\widehat{u}_\ell\|_{L^2(\Gamma)} + \sqrt{2}\,C_{\rm inv}\,|||\widehat{u}_{\ell+1}-\widehat{u}_\ell|||.$$

Now, let $T \in \mathcal{M}_\ell$ be a marked element.

- If T is refined isotropically, there holds
$$\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1}^{(1)})\widehat{u}_\ell\|_{L^2(T)} = 0.$$
- If T is refined by horizontal refinement, there holds
$$\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1}^{(1)})\widehat{u}_\ell\|_{L^2(T)}^2 \le \|\varrho_\ell^{1/2}(1-\Pi_{\mathrm{horiz}(T)}^{(1)})\widehat{u}_\ell\|_{L^2(T)}^2 \le \tau\widetilde{\beta}_\ell(T)^2.$$
- If T is refined by vertical refinement, there holds
$$\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1}^{(1)})\widehat{u}_\ell\|_{L^2(T)}^2 \le \|\varrho_\ell^{1/2}(1-\Pi_{\mathrm{vert}(T)}^{(1)})\widehat{u}_\ell\|_{L^2(T)}^2 \le \tau\widetilde{\beta}_\ell(T)^2.$$

In all cases, we thus obtain

$$\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1}^{(1)})\widehat{u}_\ell\|_{L^2(T)}^2 \le \tau\,\widetilde{\beta}_\ell(T)^2 \quad \text{for all } T \in \mathcal{M}_\ell.$$

Moreover, there clearly holds

$$\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1}^{(1)})\widehat{u}_\ell\|_{L^2(T)}^2 \le \widetilde{\beta}_\ell(T)^2 \quad \text{for all } T \in \mathcal{T}_\ell\backslash\mathcal{M}_\ell.$$

Together with the marking strategy (1.1), we again obtain

$$\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1}^{(1)})\widehat{u}_\ell\|_{L^2(\Gamma)}^2 \le \big(1-\theta(1-\tau)\big)\,\widetilde{\beta}_\ell^2$$

and conclude the proof. □

Remark 6. *We stress that, instead of first-order polynomials $p=1$ for the L^2-orthogonal projections $\Pi_*^{(p)}$, we may use arbitrary polynomial degree $p \ge 1$. In particular, the choice of $p=0$ leads to the criterion from* [FP] *discussed in Section 4.1. Therefore, the introduced anisotropic refinement rule can be seen as an extension of the original ideas, and the proof of the estimator reduction holds for arbitrary polynomial degree $p \ge 0$.* □

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