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Classical limit for linear and nonlinear Quantum Fokker-Planck systems

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Abstract

[1] We study the classical limit of some linear and nonlinear Quantum Fokker-Planck systems. In the nonlinear case we consider an Hartree-type potential. By the use of the Wigner transform and compactness methods, we prove the convergence of the system to a linear and nonlinear Vlasov Fokker-Planck equation respectively. The physical case with a Poisson coupling in three dimensions is included.

Subject classification: 81Q15, 82C10, 35Q40, 35S10, 47J35, 47H06, 47H20, 81V70, 81Q99, 81V99
Key words: classical limit, Fokker-Planck operator, Wigner measure, Hartree equation, quantum transport, open quantum system.

1 Introduction

The *Quantum Fokker-Planck (QFP) equation* is a dissipative evolution equation describing the motion of an ensemble of quantum particles, where the interactions with the external environment are modeled by a Fokker-Planck scattering term. In the Hartree approximation for the wave function, we can write the associated *von Neumann QFP equation* for the single-particle density matrix operator R_ϵ:

$$\begin{cases} \partial_t R_\epsilon &= \mathcal{L}(R_\epsilon), \qquad \mathcal{L}(R_\epsilon) := -\dfrac{i}{\epsilon}[H, R_\epsilon] + A(R_\epsilon), \quad t > 0, \\ R_\epsilon\big|_{t=0} &= R_{I\epsilon} \end{cases} \tag{1}$$

where $H := -\epsilon^2\Delta/2 + V$ is the Hamiltonian, and

$$A(R_\epsilon) := -\gamma[x, [\nabla, R_\epsilon]_+] + \epsilon^2 D_{qq}[\nabla, [\nabla, R_\epsilon]] - \frac{D_{pp}}{\epsilon^2}[x, [x, R_\epsilon]] + 2i\epsilon D_{pq}[x, [\nabla, R_\epsilon]]$$

is the Fokker-Planck operator (cf. [15]). The scaling parameter ϵ represents the reduced Planck constant. The effective electron mass is set to one. The non-negative Dekker coefficients $D_{pp}, D_{pq}, D_{qq}, \gamma$ defined in (7) model diffusive and dispersive mechanisms.

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Here $[B, C] = BC - CB$, $[B, C]_+ = BC + CB$ denote the commutator and the anti-commutator of the operators B and C, where we mean $BC = B \cdot C$ in case both B and C are vector valued. Finally x and V indicate multiplicative operators in $L^2(\mathbb{R}^N)$. The *density matrix operator* $R(t)$ at time t is a linear, non negative, hermitian, trace class operator on $L^2(\mathbb{R}^N)$. Since $R(t)$ is also Hilbert-Schmidt, it can be written in the integral form $(R(t)f)x = \int_{\mathbb{R}^d} \rho(t, x, y) f(y) dy, \ \forall f \in L^2(\mathbb{R}^N)$, where the *density matrix function* $\rho(t) = \rho(t, x, y) \in L^2(\mathbb{R}^N_x \times \mathbb{R}^N_y)$ represents the integral kernel. The corresponding PDE for the kernel ρ_ϵ of R_ϵ reads

$$\begin{cases} \partial_t \rho_\epsilon &= -\frac{i}{\epsilon} \left(H_x \rho_\epsilon - H_y \rho_\epsilon \right) + a(\rho_\epsilon), \qquad t > 0, \\ \rho_\epsilon \big|_{t=0} &= \rho_{I\epsilon}(x, y) \ \in L^2(\mathbb{R}^N_x \times \mathbb{R}^N_y), \end{cases} \tag{2}$$

where $H_x = -\epsilon^2 \frac{\Delta_x}{2} + V(t, x)$ and

$$\begin{aligned} a(\rho_\epsilon) := \quad & -\gamma\, (x - y) \cdot (\nabla_x - \nabla_y)\, \rho_\epsilon \, + \epsilon^2 D_{qq} \, |\nabla_x + \nabla_y|^2 \rho_\epsilon \\ & - \frac{D_{pp}}{\epsilon^2} \, |x - y|^2 \rho_\epsilon \, + 2\, i\, \epsilon\, D_{pq}\, (x - y) \cdot (\nabla_x + \nabla_y) \rho_\epsilon. \end{aligned}$$

The present work is concerned with the classical limit $\epsilon \to 0$ of (1) both with a linear and a nonlinear potential V. In the latter case a nonlinear self-consistent Hartree-type potential $V_\epsilon(t, x) = V_0(x) *_x \rho_\epsilon(t, x)$ will be considered (to be precise, the potential $V_\epsilon(t, x)$ depends linearly on ρ_ϵ, but it generates a nonlinearity in (1)), where $\rho_\epsilon(t, x) := \rho_\epsilon(t, x, x)$ is the particle density of the system, $*_x$ denotes the convolution in the x variable, and the given real valued function V_0 (interaction potential) can be either continuous or singular. In Section 2.1 we specify the considered potentials, which comprise the Coulomb interaction potentials $V_0 = \pm 1/|x|$ for $N = 3$. The analysis will be carried out at a kinetic level with the help of the Wigner transformation, first introduced by Wigner in [35]. If $R_\epsilon(t)$ and $\rho_\epsilon(t)$ are the solutions of (1) and (2) corresponding to the parameter ϵ, it can be shown that the rescaled *Wigner function* W_ϵ

$$W_\epsilon(t, x, \xi) = W_\epsilon[\rho_\epsilon](t, x, \xi) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i\xi \cdot y} \rho_\epsilon(t, x + \frac{\epsilon y}{2}, x - \frac{\epsilon y}{2}) \, dy \tag{3}$$

solves the *Wigner-Fokker-Planck (WFP) equation*

$$\begin{cases} \partial_t W_\epsilon \, + \, \xi \cdot \nabla_x W_\epsilon \, + \, \theta_\epsilon[V_\epsilon] W_\epsilon &= D_{pp} \, \Delta_\xi W_\epsilon \, + \, 2\, \gamma \, \operatorname{div}_\xi(\xi W_\epsilon) + \\ & \quad - 2\, \epsilon^2 D_{pq} \, \operatorname{div}_x(\nabla_\xi W_\epsilon) \, + \, \epsilon^2 D_{qq} \, \Delta_x W_\epsilon \\ W_\epsilon(t = 0, x, \xi) &= W_\epsilon[\rho_{I\epsilon}](x, \xi) \end{cases} \tag{4}$$

where $x, \ \xi \in \mathbb{R}^N$ are the phase-space variables, position and velocity, and $\theta_\epsilon[V_\epsilon]$ is the pseudo-differential operator

$$\theta_\epsilon[V_\epsilon] W_\epsilon(t, x, \xi) = \frac{i}{(2\pi)^N} \int_{\mathbb{R}^{2N}} \frac{1}{\epsilon} \Big(V_\epsilon \Big(x + \frac{\epsilon y}{2} \Big) - V_\epsilon \Big(x - \frac{\epsilon y}{2} \Big) \Big) W_\epsilon(t, x, \eta) e^{-i(\xi - \eta) \cdot y} d\eta dy.$$

As $\epsilon \to 0$ and under appropriate assumptions on the initial data $\{R_{I\epsilon}\}_\epsilon$ we shall show the convergence of a distributional solution of the Wigner-Fokker-Planck equation to a distributional solution of the following classical *Vlasov Fokker-Planck* (VFP) equation

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f - \nabla V \cdot \nabla_\xi f &= D_{pp} \Delta_\xi f + 2\gamma \ \mathrm{div}_\xi(\xi f) \\ f(t=0,x,\xi) &= f_I(x,\xi) \end{cases} \tag{5}$$

in a suitable topology, depending on the regularity of the potential. In the nonlinear case we shall obtain $\nabla V = \nabla V_0 *_x \rho_0(t,x)$ with $\rho_0(t,x) = \int_{\mathbb{R}^N} df(,\xi)$.
We shall follow the approach of Lions-Paul [26], who solved the classical limit for the Quantum-Vlasov system in $\mathbb{R}^N$ with several linear and nonlinear potentials. The Quantum-Vlasov (QV) equation is (1) with $A \equiv 0$ on the right hand side. Anyway, these two systems (QV and QFP) present remarkable differences, which generate the main difficulties in dealing with (1). The Quantum-Vlasov equation is a conservative quantum system, which arises from a Schrödinger system for the pure states (eventually coupled to a common Hartree-type potential) and hence can be treated both at the pure-state level and at the mixed-states level (by introducing the related density matrix). The Quantum Fokker-Planck equation is an open quantum system (i.e. a system interacting with an external quantum *environment*, cf. [13]) and admits only the mixed-state representation. Consequently, many properties, that are natural for Schrödinger systems, fail here: conservation of the occupation probabilities (i.e. eigenvalues of $R(t)$) and of the orthogonality of the pure states, and conservation of the total energy. Even the conservation of the positivity of the density matrix during the time evolution is not straightforward.
Hence, the main aim of the present work is to justify the validity of the argument of Lions-Paul even in this case, without the help of the Schrödinger formalism and to get the exact convergence of the nonlinear term.
The existence of a trace conserving and positivity preserving solution for (1) is obtained by rewriting (1) in Lindblad form (cf. [22]) and by constructing a conservative quantum dynamical semigroup as in [4]. This reformulation is possible by imposing the positivity for the matrix (cf. [15, 3])

$$\begin{pmatrix} \epsilon^2 D_{qq} & \epsilon^2 D_{pq} + \frac{i}{2}\epsilon\gamma \\ \epsilon^2 D_{pq} - \frac{i}{2}\epsilon\gamma & D_{pp} \end{pmatrix} \geq 0 \,. \tag{6}$$

The physical constants

$$D_{pp} = \eta k_B \tau, \ D_{qq} = \frac{\eta}{12 k_B \tau m^2}, \ D_{pq} = \frac{\eta \Omega}{12\pi k_B \tau m}, \ \gamma = \frac{\eta}{2m}. \tag{7}$$

represent the parameters of the quantum environment consisting of an heat bath of harmonic oscillators in thermal equilibrium. Here $\eta > 0$ is the coupling (damping) constant of the bath, k_B the Boltzmann constant, τ the temperature of the bath, m the effective mass and Ω the cut–off frequency of the reservoir oscillators.

By definition, if (6) holds for $\epsilon = 1$, then it holds for all $\epsilon \in (0, 1]$.
Condition (6) leads to distinguish two main regimes, which are kept until the limit procedure:

(H) Very high temperature model or Hypoelliptic case: $\gamma = D_{pq} = D_{qq} = 0$, $D_{pp} > 0$,

(E) Medium high temperature model with friction or Elliptic case: $\gamma > 0$.

As $\epsilon \to 0$, $(H) \to (5)$ with $D_{pp} \neq 0$ and $\gamma = 0$, while $(E) \to (5)$ with D_{pp}, $\gamma \neq 0$.
The mathematical derivation of the linear very-high-temperature model from the many-body dynamics has been recently performed in [10] via a Markovian approximation of the originally non–Markovian evolution of the electron in the oscillator bath. This paper also includes a critical review of the original Caldeira–Leggett master equation [9]. The medium-high-temperature model has been discussed in [16] and can be employed for a simple phenomenological modeling of the interaction with quantum environments.
As already mentioned, the solution to the Wigner-Fokker-Planck equation (4) comes from a Wigner transform of the kernel of the quantum dynamical semigroup solving (1). Note that in some cases (4) can be solved directly at the phase–space level (cf. [2, 3] for $N = 3$ with Poisson coupling and without external potentials). Concerning the Vlasov-Fokker-Planck equation (5), the limiting procedure will provide the existence of distributional solutions, which can be compared to the existing literature (e.g. [7] in the three dimensional case with Poisson coupling and without external potentials).
Finally, a wide body of literature testifies that the Wigner measure is a very useful mathematical tool to carry out several physical scaling limits: the classical limit of the Schrödinger equation for pure or mixed states [26, 28] sometimes combined with the mean-field limit [18]; homogenization in periodic and random media [5, 19, 27]; the high frequency limit of Helmholtz equation [11]; in geometrical optics and many other fields.
We observe that other techniques, such as WKB methods used for example in [20], are not applicable to our case. In the linear case, our result could be compared with the Weil formulation in [14]; further properties of the Fokker-Planck operator are in [21]. We finally notice that our analysis shows the convergence of the quantum Brownian motion to the classical Brownian motion [31].
The paper is organized as follows: in Section 2 we present existence results for the Cauchy problem (1) and consequently for the Wigner-Fokker-Planck equation (4). Then we study the uniform estimates for the solution. In Section 3 we identify a cluster point of the sequence $\{W_\epsilon\}_\epsilon$ by compactness methods in appropriate topologies and in Section 4 we study the classical limit with four different types of potential.

2 Existence and Uniform estimates

2.1 Existence results

Let $X = \mathcal{J}_1^s$ be the Banach space of self-adjoint trace class operators on $L^2(\mathbb{R}^N)$ endowed with the trace norm. A *conservative Quantum Dynamical Semigroup (QDS)* on X is a strongly continuous semigroup $(S_t)_{t\geq 0}$ of bounded operators on X with $S_0 = I$, such that $R \geq 0$ implies $S_t(R) \geq 0$ (positivity preserving) and $\mathrm{Tr}(S_t(R)) = \mathrm{Tr}(R)$, $\forall R \in X$ (conservativity or Markov condition). The dynamics of a quantum mechanical system can be described by a quantum dynamical semigroup when the evolution operator $\mathcal{L}$ of the system is in Lindblad form (cf. [22, 12]), namely, calling $L := \sum_{j=1}^p L_j^* L_j$,

$$\partial_t R_\epsilon = \mathcal{L}(R_\epsilon) = -\frac{i}{\epsilon}[H_E, R_\epsilon] + \sum_{j=1}^p L_j R_\epsilon L_j^* - \frac{1}{2}(LR_\epsilon + R_\epsilon L), \tag{8}$$

where H_E is an extended self-adjoint Hamiltonian and L_j are some linear operators ("Lindblad operators"). If $D_{pq} = 0$, the equation (1) is in Lindblad form under the condition (6) by setting $H_E = H - i\epsilon\gamma[x, \nabla]_+/2$ and $L_j = \frac{2\sqrt{\gamma k_B \tau}}{\epsilon} x_j + \frac{\epsilon}{2}\sqrt{\frac{\gamma}{k_B\tau}}\,\partial_j$, with $p = N$ (cf. [16]). At the formal level, one can verify that such semigroup is also completely positive (cf. [12]), but we shall not use this property in the following.
While the linear problem can be analyzed in $X = \mathcal{J}_1^s$ (see Theorem 2.1 below), the non-linear problem with unbounded V_0 requires to restrict the QDS to some "energy" Banach spaces:

$$\begin{aligned} \mathcal{E}_{kin} &= \{R \in \mathcal{J}_1^s : \sqrt{1-\epsilon^2\Delta}\, R \sqrt{1-\epsilon^2\Delta} \in \mathcal{J}_1^s, \\ &\quad \|R\|_{\mathcal{E}_{kin}} := \|\sqrt{1-\epsilon^2\Delta}\, R\sqrt{1-\epsilon^2\Delta}\|_{\mathcal{J}_1}\}, \\ \text{or} \quad \mathcal{E} &= \{R \in \mathcal{J}_1^s : \sqrt{1-\epsilon^2\Delta+|x|^2}\, R\sqrt{1-\epsilon^2\Delta+|x|^2} \in \mathcal{J}_1^s, \\ &\quad \|R\|_{\mathcal{E}} := \|\sqrt{1-\epsilon^2\Delta+|x|^2}\, R\sqrt{1-\epsilon^2\Delta+|x|^2}\|_{\mathcal{J}_1}\}. \end{aligned}$$

Here $\sqrt{-\Delta}$ and $\sqrt{1-\epsilon^2\Delta}$ denote the pseudo-differential operators with Fourier symbols $|\xi|$ and $\sqrt{1+\epsilon^2|\xi|^2}$, while $\sqrt{1-\epsilon^2\Delta+|x|^2}$ is the square root of the strictly positive essentially self-adjoint operator $1-\epsilon^2\Delta+|x|^2$ having core $C_0^\infty(\mathbb{R}^N)$.
The space $\mathcal{E}$ has been introduced in [4] (with $\epsilon = 1$), where (8) has been investigated in the case of a general Lindblad operator of first order $L_j = \alpha_j \cdot x + \beta_j \cdot \nabla + \gamma_j$, $\alpha_j, \beta_j \in \mathbb{C}^N$, $\gamma_j \in \mathbb{C}$, associated either to a linear Hamiltonian or (in the space $\mathbb{R}^3$) to the nonlinear Hamiltonian $H_E = -\frac{\Delta}{2} + \frac{|x|^2}{2} + \frac{1}{|x|} *_x \rho + V_1(x) - i\mu[x, \nabla]_+$, for a bounded potential V_1 and $\mu \in \mathbb{R}$. In this section we are going to present an $N-$dimensional version of the result in [4], in order to provide the existence of solutions of (1) for a larger class of nonlinear potentials than the three dimensional Hartree $(1/|x|) *_x \rho$.
Here and in the following c and the expression $c(\cdot,\cdot,\cdot)$ will denote a positive constant, continuously dependent on the arguments in parenthesis and not explicitly dependent

on ϵ. The symbols $C_0,\ C_c,\ C_b,\ C_0^\infty$ denote the spaces of continuous functions which are, respectively, zero at infinity, with compact support, bounded, C^∞ with compact support. Finally, $\mathbb{R}^+ := [0, +\infty)$ and $L^1(\mathbb{R}^N)^+$ is the cone of non-negative L^1 functions. For the definition of the $p-$trace spaces $\mathcal{J}_p$ we refer to [30].
We first define the kinetic energy of R_ϵ as

$$E^{kin}(R_\epsilon) := \frac{\epsilon^2}{2}\mathrm{Tr}(\sqrt{-\Delta}R_\epsilon\sqrt{-\Delta})\,. \tag{9}$$

Since $R_\epsilon(t) \in \mathcal{J}_1^s$, we shall use its canonical spectral decomposition at each fixed time, which is unique up to multiplicity:

$$R_\epsilon(t) = \sum_{j\in\mathbb{N}} \lambda_{j\,\epsilon}(t)|\phi_{j\,\epsilon}(t)\rangle\langle\phi_{j\,\epsilon}(t)|\,, \quad \rho_\epsilon(t,x,y) = \sum_{j\in\mathbb{N}} \lambda_{j\,\epsilon}(t)\phi_{j\,\epsilon}(t,x)\overline{\phi}_{j\,\epsilon}(t,y), \tag{10}$$

where $\{\phi_{j\epsilon}(t)\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}^N)$ is an orthonormal basis of eigenvectors for $R_\epsilon(t)$ and $\{\lambda_{j\epsilon}(t)\}_{j\in\mathbb{N}} \subset l^1(\mathbb{N})$ are the corresponding eigenvalues, which are all non-negative if $R_\epsilon(t) \geq 0$. Accordingly, we can express the particle density ρ_ϵ, its L^1-norm and its gradient, the current j_ϵ and the kinetic energy $E^{kin}(R_\epsilon)$, for a fixed ϵ, as:

$$\rho_\epsilon(t,x) = \sum_{j\in\mathbb{N}} \lambda_{j\,\epsilon}(t)|\phi_{j\,\epsilon}(t,x)|^2\ ,\ \|\rho_\epsilon(t)\|_{L^1_x(\mathbb{R}^N)} = \sum_{j\in\mathbb{N}} \lambda_{j\,\epsilon}(t), \tag{11}$$

$$\nabla_x\rho_\epsilon(t,x) = 2\sum_{j\in\mathbb{N}} \lambda_{j\,\epsilon}(t)\ \mathrm{Re}(\nabla\phi_{j\,\epsilon}(t,x)\overline{\phi}_{j\,\epsilon}(t,x)), \tag{12}$$

$$j_\epsilon(t,x) = \epsilon\sum_{j\in\mathbb{N}} \lambda_{j\,\epsilon}(t)\ \mathrm{Im}(\nabla\phi_{j\,\epsilon}(t,x)\overline{\phi}_{j\,\epsilon}(t,x)), \tag{13}$$

$$E^{kin}(R_\epsilon(t)) = \frac{\epsilon^2}{2}\sum_{j\in\mathbb{N}} \lambda_{j\,\epsilon}(t)\|\nabla\phi_{j\,\epsilon}(t,x)\|^2_{L^2(\mathbb{R}^N)}\,. \tag{14}$$

Note that the *occupation probabilities* $\{\lambda_{j\,\epsilon}(t)\}_{j\in\mathbb{N}}$ are in general non constant in time for an open quantum system. Indeed, in contrast with a coherent Schrödinger system, an open quantum system can experience *decoherence*. In principle, its time evolution could carry an initial positive pure state into a combination of pure states. For a mathematical definition of decoherence see for example [29]. These facts motivate the requirements of positivity and trace conservation for the QDS. Since we do not have explicit information about the time evolution either of $\{\phi_{j\,\epsilon}(t)\}_j$ or of $\{\lambda_{j\,\epsilon}(t)\}_j$, the definitions (11)-(14) are of limited use for the well-posedness analysis. Anyway, for our purposes, the spectral decomposition enters only to perform some estimates at fixed times and for computations independent of the choice of basis. The correct definition for the density remains $\rho(t,x) = \rho(t, x+\frac{y}{2}, x-\frac{y}{2})\big|_{y=0}$, as pointed out in [26].
As mentioned in the introduction, four different hypothesis on the potential will be considered, two in the linear and two in the nonlinear ($V_\epsilon = V_0 * \rho_\epsilon$, with $V_0^\pm$ the positive, resp. negative, parts of V_0) case:

Regular linearity

$$V \in C^1(\mathbb{R}^N), \ |V(x)| \le c(1+|x|^2)\,. \tag{15}$$

Regular nonlinearity : $V_\epsilon = V_0 * \rho_\epsilon$ such that

$$V_0 \in C_b^1(\mathbb{R}^N)\,,\ V_0(x) = V_0(-x). \tag{16}$$

Singular linearity

$$V \in H^1_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\,. \tag{17}$$

Singular nonlinearity : $V_\epsilon = V_0 * \rho_\epsilon + V_1$ with V_1 satisfying:

$$\begin{aligned} &V_1 \in L^\infty(\mathbb{R}^N), \quad \nabla V_1 \in L^k(\mathbb{R}^N) + L^\infty(\mathbb{R}^N),\\ &\quad k = N \ \text{ for } \ N \ge 3,\ k > 2 \ \text{ for } \ N = 2,\ k = 2 \ \text{ for } \ N = 1, \end{aligned} \tag{18}$$

and with V_0 satisfying:

1. Condition on the interaction potential
   $V_0(x) = V_0(-x)$ and

$$\begin{aligned} \text{if } D_{qq} = 0:\quad & V_0 \in L^a(\mathbb{R}^N) + L^\infty(\mathbb{R}^N), \quad a = N/2 \text{ for } N \ge 3,\\ & \qquad\qquad a > 1 \text{ for } N = 2,\ a = 1 \text{ for } N = 1;\\ & V_0^- \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N),\ \text{with } p \ge \tfrac{N+4}{4},\ p \ne 2,\ \text{for } N \le 4,\\ & V_0^- \in L^{\frac{N}{2}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N), \qquad \text{for } N \ge 5.\\ \text{if } D_{qq} \ne 0:\quad & V_0 \in L^{\frac{2N^2+11N+16}{2(N+4)}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N), \quad \forall N \in \mathbb{N},\\ & \text{or also} \quad V_0 = \pm 1/|x| \qquad \text{for } N = 3. \end{aligned} \tag{19}$$

2. Condition on the gradient of the interaction potential

$$\nabla V_0 \in L^{\frac{2N+8}{N+8}}(\mathbb{R}^N) + L^p(\mathbb{R}^N), \ \text{ with } \frac{2N+8}{N+8} < p < \infty, \ \text{for all } N\,. \tag{20}$$

Note that for $N = 3$ the previous assumptions allow the Coulomb interaction potential $V_0 = \pm 1/|x|$ both for the repulsive and the attractive forces, and the related Hartree potential $V = \pm\frac{1}{|x|} *_x \rho$ solves the Poisson equation $\Delta V = \mp\rho$. The assumptions (19)-(20) come from the intersection of three types of constraints:

i) conditions for the local-in-time existence (25),

ii) a-priori estimates for the global-in-time existence in $\mathcal{E}_{kin}$ or $\mathcal{E}$, together with uniform-in-ϵ estimates for the kinetic energy (cf. Lemma 2.4.c),

iii) condition (20) is used to get a uniform bound for the L^2-norm of ∇V_ϵ (cf. Theorem 4.4 for application).

We observe that (19) are not all necessary conditions for Theorem 2.2, in sense that global existence holds under less restrictive hypothesis (but not all of them guarantee the uniform in ϵ estimates needed to apply the compactness argument of Section 4). Our aim is to provide a set of possible values for V_0, V_1 and the external potential, such as the classical limit can be performed.

We now introduce two existence theorems for (1). At this step we do not care about the $\epsilon-$dependence, and therefore we set $\epsilon = 1$. The first result holds in the space $\mathcal{J}_1^s$ for the Quantum Fokker-Planck system with an external quadratic potential and a nonlinearity with a bounded interaction potential.

Theorem 2.1. *Let $\epsilon = 1$. Let $R_I \geq 0$, $R_I \in \mathcal{J}_1^s$ and $V(t,x) = \alpha\frac{|x|^2}{2} + \beta \cdot x + V_1(x) + V_0(x) *_x \rho(t,x)$, V_1, $V_0 \in L^\infty(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^N$ and assume* (6)*. Then the evolution operator $\mathcal{L}$ of* (1) *generates on $\mathcal{J}_1^s$ a nonlinear conservative quantum dynamical semigroup S_t of contractions. This QDS yields the unique mild solution, in the sense of semigroups, for the Quantum Fokker-Planck equation, given by $S_t(R_I) \in C(\mathbb{R}^+;\, \mathcal{J}_1^s)$. Moreover, $\rho(t,x) \in C(\mathbb{R}^+;\, L^1(\mathbb{R}^N))$ and $V_0 *_x \rho(t,x) \in C(\mathbb{R}^+;\, L^\infty(\mathbb{R}^N))$.*

Proof. The existence of the QDS associated to the linear part of $\mathcal{L}$ with a quadratic potential is proven in [4], Theorem 3.9. One concludes by observing that the remaining term $[V_1 + V_0 *_x \rho,\, R]$ of $\mathcal{L}$ is a locally Lipschitz operator in $\mathcal{J}_1$ (cf. [24]). Further, let $B(t,s) = B^+(t,s) - B^-(t,s)$ be the decomposition in positive/negative parts for the $\mathcal{J}_1^s-$operator $R(t) - R(s)$ (cf. [30]) with particle density $b(t,s,x)$. The particle density becomes $\rho(t,x) - \rho(s,x) = b(t,s,x) = b^+(t,s,x) - b^-(t,s,x)$ and we get $\|\rho(t,x) - \rho(s,x)\|_{L^1} \leq \|B^+\|_{\mathcal{J}_1} + \|B^-\|_{\mathcal{J}_1} = \|R(t) - R(s)\|_{\mathcal{J}_1}$, which converges to zero for $t \to s$. The Young inequality implies the time-continuity for the self-consistent potential $V_0 *_x \rho(t)$ in $L^\infty(\mathbb{R}^N)$. The conservation of the trace and the positivity of $R(t)$ leads to the global-in-time existence in $\mathcal{J}_1^s$. □

We now consider an unbounded interaction potential V_0 and present an $N-$dimensional version of Theorem 5.3 in [4] for an appropriate class of interaction potentials. We also need to refine the energy estimate, in order to obtain $\epsilon-$independent estimates in section 2.2.

Theorem 2.2. *Let $\epsilon = 1$. Let $R_I \geq 0$, V_1 satisfying* (18) *and V_0 satisfying* (19)*. Assume* (6)*.*
a) If $R_I \in \mathcal{E}_{kin}$ and $V = V_0 * \rho + V_1$, then the Quantum-Fokker-Planck equation (1) *has a unique global mild solution $S_t(R_I) \in C(\mathbb{R}^+, \mathcal{E}_{kin})$, given by a nonlinear conservative QDS S_t defined on $\mathcal{E}_{kin}$. Further, $\rho(t,x) \in C(\mathbb{R}^+; L^q(\mathbb{R}^N))$ with $q \in [1, q_*]$ ($q_* = \frac{N}{N-2}$, $q_* < \infty$, $q_* = \infty$ for, resp., $N \geq 3, 2, 1$) and $V_0 *_x \rho(t,x) \in C(\mathbb{R}^+; L^\infty(\mathbb{R}^N))$.*
b) Moreover, if $R_I \in \mathcal{E}$ and $V = V_0 * \rho + \frac{|x|^2}{2} + V_1(x)$ hold, then (1) *has a unique global mild solution $S_t(R_I) \in C(\mathbb{R}^+, \mathcal{E})$.*

Proof. The proof relies on a perturbation method for evolution equations (cf. [24]).
Step 1: Linear case (i.e. $V_0 = 0$). Here we search the QDS generated by the linear evolution operator (1) in case $V = \alpha|x|^2/2$, with $\alpha = 0$ or 1, (this is equivalent to

take $\mathcal{L}$ in (8) with $H_E = -\Delta/2 + \alpha|x|^2/2 - i\gamma[x,\nabla]_+/2$, and with the appropriate L_j's). From Theorem 3.9 in [4] we know that a conservative QDS exists in $\mathcal{J}_1^s$. The aim is to prove that it is a QDS also in $\mathcal{E}_{kin}$ for $\alpha = 0$, and $\mathcal{E}$ for $\alpha = 1$. As in Lemma 4.3 and Proposition 4.4 of [4], one first writes the evolution for the kinetic and the quadratic potential energy of the linear system (if $\alpha = 1$):

$$\begin{aligned} \frac{d}{dt}E^{kin}(R) &= -\frac{i}{2}\,\mathrm{Tr}(-\nabla\,[V,\,R]\,\nabla) - 4\gamma E^{kin}(R) + ND_{pp}\mathrm{Tr}(R_I), \\ \frac{d}{dt}\mathrm{Tr}(\,\frac{|x|^2}{2}\,R) &= \frac{i}{2}\mathrm{Tr}(-\nabla\,[\,\frac{|x|^2}{2}\,,\,R]\,\nabla) + D_{qq}N\mathrm{Tr}(R_I). \end{aligned} \tag{21}$$

(the derivation of this system is a special case, for $V_0 = 0$, of the derivation of (30) below). Then one notes that

$$\begin{aligned} &|\tfrac{1}{2}\mathrm{Tr}\big(-\nabla\,[\,\tfrac{|x|^2}{2},R]\,\nabla\big)| = \tfrac{1}{2}|\mathrm{Tr}(-xR\nabla) + \overline{\mathrm{Tr}(xR\nabla)}| \le \textstyle\sum_{j\in\mathbb{N}} |\langle e_j, xR\nabla e_j\rangle_{L^2}| \\ &\le \textstyle\sum_{j\in\mathbb{N}} \|\sqrt{R}\nabla e_j\|_{L^2}\|\sqrt{R}xe_j\|_{L^2} \le E^{kin}(R) + \tfrac{1}{2}\mathrm{Tr}(xRx) \end{aligned} \tag{22}$$

where $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mathbb{R}^N)$. For $\Lambda := \sqrt{1-\Delta+|x|^2}$, one shows that each $R \in \mathcal{E}$ can be decomposed as $R = R_1 - R_2$, with $R_{1,\,2} = \Lambda^{-1}(\Lambda R\Lambda)^{\pm}\Lambda^{-1} \ge 0$, $R_1,\,R_2 \in \mathcal{E}$ and $(\Lambda R\Lambda)^{\pm}$ denoting the positive resp. negative part of $\Lambda R\Lambda \in \mathcal{J}_1^s$. Further, $\|R\|_{\mathcal{E}} = \|R_1\|_{\mathcal{E}} + \|R_2\|_{\mathcal{E}}$. Therefore one can always work with nonnegative R. Summing the two equations in (21), using (22), and applying Gronwall inequality, one gets $\mathrm{Tr}(\sqrt{-\Delta}R(t)\sqrt{-\Delta}) + \mathrm{Tr}(xR(t)x) \in C(\mathbb{R}^+,\mathbb{R})$ (as time dependent function). One concludes by applying the general Grümm Theorem (as in Proposition 4.4 of [4]) to prove that $R \in C(\mathbb{R}^+,\mathcal{E})$. We observe that without quadratic potential (i. e. $\alpha = 0$), the evolution of the kinetic energy provides directly an a-priori estimate in $\mathcal{E}_{kin}$, and therefore $R \in C(\mathbb{R}^+,\mathcal{E}_{kin})$.

Step 2: Local-in-time existence. *Case a) (i.e. $\alpha = 0$):* It is achieved if the perturbation operator $[V_0 *_x \rho + V_1,\,R]$ is locally Lipschitz in the space $\mathcal{E}_{kin}$. We now consider the nonlinear map $R \mapsto [V_R,\,R]$ in $\mathcal{E}_{kin}$ with $V_R = V_0 *_x \rho$, we set $\Lambda := \sqrt{1-\Delta}$ and prove an estimate of the form

$$\|\,[V_R,\,R]\,\|_{\mathcal{E}_{kin}} \le c\,\|R\|^2_{\mathcal{E}_{kin}}. \tag{23}$$

By density in $L^2(\mathbb{R}^N)$, let $f \in C_0^\infty(\mathbb{R}^N)$ and consider

$$\begin{aligned} \|\Lambda V_R\,R\Lambda\|_{\mathcal{J}_1} &\le \|\Lambda V_R\,\Lambda^{-1}\|_{\mathcal{B}(L^2)}\,\|\Lambda R\Lambda\|_{\mathcal{J}_1} \quad \text{where} \\ \|\Lambda V_R\,\Lambda^{-1}f\|^2_{L^2} &= \|\nabla(V_R\,\Lambda^{-1}f)\|^2_{L^2} + \|V_R\,\Lambda^{-1}f\|^2_{L^2}. \end{aligned} \tag{24}$$

The r.h.s. of the latter expression is bounded if $V_R \in L^\infty(\mathbb{R}^N)$ and if, by employing Hölder and Young inequality:

$$\begin{aligned} &\|V_R\|_{L^\infty} \le \|V_0\|_{L^a}\|\rho\|_{L^q}, \qquad 1 = 1/a + 1/q \\ &\|((\nabla V_R) + V_R\nabla)\Lambda^{-1}f\|_{L^2} \le \|V_0\|_{L^k}\|\nabla\rho\|_{L^p}\|\Lambda^{-1}f\|_{L^r} + \|V_R\|_{L^\infty}\|\nabla\Lambda^{-1}f\|_{L^2} \end{aligned}$$

where $\frac{3}{2} = \frac{1}{p} + \frac{1}{k} + \frac{1}{r}$, $1 \le a,q,k,p,r \le \infty$. We recall that $f \in L^2(\mathbb{R}^N)$ implies $\Lambda^{-1}f \in H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, due to the Sobolev embedding with $r = \frac{2N}{N-2}$ as optimal value for $N \geq 3$ (and $r = \infty$, $r < \infty$, for resp., $N = 1, 2$). Then one applies the decomposition $R = R_1 - R_2$ of Step 1 in the space $\mathcal{E}_{kin}$ (since $\Lambda = \sqrt{1-\Delta}$). The particle density of R can be splitted accordingly as $\rho(t,x) = \rho_1(t,x) - \rho_2(t,x)$. This decomposition let one estimate $\|\rho\|_{L^q}$ and $\|\nabla\rho\|_{L^p}$ in terms of the Lieb-Thirring-type inequalities of Lemma 5.4 (since this is possible only for positive operators). The best exponents are $q = \frac{N}{N-2}$ and $p = \frac{N}{N-1}$ for $N \geq 3$; $q < \infty$ and $p < 2$ for $N = 2$; $q = \infty$ and $p = 2$ for $N = 1$. Therefore, the assumptions

$$\begin{aligned} &V_0 \in L^a(\mathbb{R}^N) + L^\infty(\mathbb{R}^N), \\ &\quad a = N/2 \ \text{ for } \ N \geq 3, \ a > 1 \ \text{ for } \ N = 2, \ a = 1 \ \text{ for } \ N = 1\,, \end{aligned} \tag{25}$$

(with $k = a$) yield (23). This, together with the assumption (18) on the linear potential V_1, imply that the perturbation operator is locally Lipschitz in $\mathcal{E}_{kin}$. Hence, we have proved the local-in-time existence in $C([0, t_{max}), \mathcal{E}_{kin})$ for the nonlinear QDS.
Case b) (i.e. $\alpha = 1$): the existence in $\mathcal{E}$ is obtained for $\Lambda := \sqrt{1-\Delta+|x|^2}$ in the formulas above (with an extra term $\||x|V_R\,\Lambda^{-1}f\|^2_{L^2}$ to be added to r.h.s. of the 2nd equation in (24)). Finally, the time continuity of $\rho(t,x)$ in L^q follows from Lemma 5.4 after decomposing $B(t,s) = R(t) - R(s)$ in $B_{1,2}(t,s) \geq 0$ as in Step 1. This also leads to the time continuity of $V_0 *_x \rho(t)$ in $L^\infty(\mathbb{R}^N)$. The conservation of the positivity of the nonlinear semigroup can finally be shown as in Theorem 4.6.(d) of [4].

Step 3. The global-in-time existence in $\mathcal{E}_{kin}$ follows from the conservation of the trace $\mathrm{Tr}(R_I) = \mathrm{Tr}(R(t))$ (cf. Theorem 4.6.(c) of [4]) and an a-priori estimate for the kinetic energy

$$E^{kin}(R(t)) \leq c(T, \mathrm{Tr}(R_I), E^{kin}(R_I)), \quad \forall t \in [0,T].$$

In $\mathcal{E}$ we further need an estimate for the quadratic potential energy

$$\mathrm{Tr}(xRx) \leq c(T, \mathrm{Tr}(R_I), E^{kin}(R_I), \mathrm{Tr}(xR_Ix)), \quad \forall t \in [0,T].$$

(The computations are similar to those in Lemma 2.4.c). □

We now pass to the kinetic formulation of the problem and introduce some notations. We denote the Fourier transform of f and its conjugate by:

$$\mathcal{F}_{y\to\xi} f(x,\xi) = \int_{\mathbb{R}^N} e^{-i\xi\cdot y} f(x,y)\,dy\,, \qquad \overline{\mathcal{F}}_{y\to\xi} f(x,\xi) = \int_{\mathbb{R}^N} e^{i\xi\cdot y} f(x,y)\,dy\,.$$

The Wigner transform of an L^2-function ρ is

$$W[\rho](t,x,\xi) := \tfrac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i\xi\cdot y} \rho(t, x+\tfrac{y}{2}, x-\tfrac{y}{2}, t)\,dy\,,$$

with inverse $\rho(t,x,y) = \int_{\mathbb{R}^N} W(t, \frac{x+y}{2}, \xi)\, e^{i\xi\cdot(x-y)}\,d\xi\,.$
Its rescaled version (3) can be written as $W_\epsilon(t,x,\xi) := \frac{1}{\epsilon^N} W[\rho_\epsilon](t,x,\frac{\xi}{\epsilon})$.
It follows:

$$\|W_\epsilon\|_{L^2} = \frac{1}{(2\pi\epsilon)^{N/2}} \|\rho_\epsilon(x,y)\|_{L^2} = \frac{1}{(2\pi\epsilon)^{N/2}} \|R_\epsilon\|_{\mathcal{J}_2}\,. \tag{26}$$

The scaled *Hussimi transform* $W_\epsilon^H(t)$ of the Wigner function $W_\epsilon(t)$ is defined as

$$W_\epsilon^H(t,x,\xi) = W_\epsilon(t,x,\xi) *_x G_\epsilon(x) *_\xi G_\epsilon(\xi), \quad G_\epsilon(z) = (\pi\epsilon)^{-N/2} e^{-|z|^2/\epsilon} \tag{27}$$

and satisfies the following equalities based on the spectral decomposition (10)

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} W_\epsilon^H(t,x,\xi)\, d\xi dx = \mathrm{Tr}(R_\epsilon(t)), \int_{\mathbb{R}^N} W_\epsilon^H(t,x,\xi)\, d\xi = \rho_\epsilon(t,x) *_x G_\epsilon(x)\, a.e. \\ &\int_{\mathbb{R}^{2N}} |\xi|^2\, W_\epsilon^H(t,x,\xi)\, d\xi dx \;=\; \frac{\epsilon N}{2}\mathrm{Tr}(R_\epsilon(t)) \;+\; 2E^{kin}(R_\epsilon(t)) \\ &\int_{\mathbb{R}^{2N}} |x|^2\, W_\epsilon^H(t,x,\xi)\, d\xi dx \;=\; \frac{\epsilon N}{2}\mathrm{Tr}(R_\epsilon(t)) \;+\; \mathrm{Tr}(xR_\epsilon(t)x). \end{aligned} \tag{28}$$

The following existence theorem for the kinetic WFP equation (4) is straightforward, since the solution (3) is constructed by a Wigner transform of the solutions in Theorem 2.1 and 2.2. We recall the properties of the L^2 solution $W_\epsilon(t,x,\xi)$ and also that it solves (4) in the sense of the tempered distributions $\mathcal{S}'$, since in Section 4 we are going to work with Schwartz test functions (compare with Proposition II.1 [26]).

Theorem 2.3. *Let $R_{I\,\epsilon} \in X$, $R_{I\,\epsilon} \geq 0$, and let $R_\epsilon(t) = S_t(R_{I\,\epsilon}) \in C(\mathbb{R}^+, X)$ be the mild solution of the Quantum Fokker-Planck equation* (1) *with $X = \mathcal{J}_1^s$, $\mathcal{E}_{kin}$ or $\mathcal{E}$ as in Theorem 2.1 and 2.2, with the linear and non-linear potentials $V_\epsilon(t,x)$, as in* (15)-(19). *Given $\rho_\epsilon(t,x,y)$ (the density matrix function of $R_\epsilon(t)$), the rescaled Wigner function $W_\epsilon(t,x,\xi)$ defined in* (3) *belongs to $C(\mathbb{R}^+, L^2(\mathbb{R}_x^N \times \mathbb{R}_\xi^N)) \cap C_b(\mathbb{R}^+ \times \mathbb{R}_x^N, \mathcal{F}L^1(\mathbb{R}_\xi^N)) \cap C_b(\mathbb{R}^+ \times \mathbb{R}_\xi^N, \mathcal{F}L^1(\mathbb{R}_x^N))$ ($g \in \mathcal{F}L^1(\mathbb{R}^N)$, if $\mathcal{F}^{-1}g \in L^1(\mathbb{R}^N)$) and solves the WFP equation* (4) *in $C_0^\infty(\mathbb{R}_t^+) \times \mathcal{S}'(\mathbb{R}_{x\xi}^{2N})$. Further, $W_\epsilon^H(t) \in L^1(\mathbb{R}_{x\xi}^{2N})^+$ and if $X = \mathcal{E}_{kin}$, then $|\xi|^2 W_\epsilon^H(t) \in L^1(\mathbb{R}_{x\xi}^{2N})$. In addition, for $X = \mathcal{E}$ we get $|x|^2 W_\epsilon^H(t) \in L^1(\mathbb{R}_{x\xi}^{2N})$ and $|x|^2 \rho_\epsilon(t) \in L^1(\mathbb{R}_x^N)^+$.*

2.2 Uniform Estimates

We know collect the estimates needed to identify a cluster point in the classical limit. The results $a) - b)$ hold for any choice of the potential (i.e. the assumptions (15) up to (19)). In point $c)$ we distinguish between estimates dependent or not on the L^2 norm of the Wigner function (see Section 4 for a motivation). The estimate becomes uniform in ϵ, when the related initial value quantities on the r.h.s. and the arguments of $c(\cdot,\cdot,\cdot)$ are uniformly bounded in ϵ.

Lemma 2.4. *Let $R_{I\epsilon} \in \mathcal{E}_{kin}$, $R_{I\epsilon} \geq 0$, and $R_{I\epsilon} \in \mathcal{E}$ (if in the presence of a quadratic potential), with $\epsilon \in (0,1]$. Then, the (nonnegative) global mild solution $R_\epsilon(t)$ of the Quantum-Fokker-Planck equation* (1), *defined in Theorem 2.1 and 2.2, satisfies the following estimates for $t \in [0,T]$ ($T < \infty$ arbitrary), and c not explicitly dependent on ϵ:*

$$\begin{aligned} a) \qquad & \|R_\epsilon(t)\|_{\mathcal{J}_1} &=\;& Tr(R_{I\epsilon}) \\ & Tr(R_\epsilon(t)^p)^{1/p} = \|R_\epsilon(t)\|_{\mathcal{J}_p} &\leq\;& Tr(R_{I\epsilon}), \quad \forall p \geq 1 \\ b) \qquad & Tr(R_\epsilon^2(t)) &\leq\;& Tr(R_{I\epsilon}^2)\, e^{N\gamma t} \\ & \|W_\epsilon(t)\|_{L^2(\mathbb{R}^{2N})} &\leq\;& \|W_{I\epsilon}\|_{L^2(\mathbb{R}^{2N})}\, e^{N\gamma t/2} \end{aligned}$$

c) Let V_1 satisfy (18) *and $V_\epsilon = V_0 *_x \rho_\epsilon + V_1 + \alpha|x|^2/2$ ($\alpha = 0, 1$). If $D_{qq} = \gamma = 0$, if V_0 satisfies* (25) *and $V_0^- \in L^\infty$, then*

$$E^{kin}(R_\epsilon(t)) \leq c\Big(T, Tr(R_{I\epsilon}), E^{kin}(R_{I\epsilon}), \alpha\, Tr(xR_{I\epsilon}x)\Big)\,;$$

and if $\alpha = 1$, then

$$Tr(xR_\epsilon(t)x) \leq c\Big(T, Tr(R_{I\epsilon}), Tr(xR_{I\epsilon}x), E^{kin}(R_{I\epsilon})\Big),$$

Otherwise, in the general case, we take V_0 as in (19) *and get*

$$E^{kin}(R_\epsilon(t)) \leq c\Big(T, Tr(R_{I\epsilon}), E^{kin}(R_{I\epsilon}), \alpha\, Tr(xR_{I\epsilon}x), \|W_{I\epsilon}\|_{L^2(\mathbb{R}^{2N})}\Big)\,;$$

and if $\alpha = 1$, then

$$Tr(xR_\epsilon(t)x) \leq c\Big(T, Tr(R_{I\epsilon}), Tr(xR_{I\epsilon}x), E^{kin}(R_{I\epsilon}), \|W_{I\epsilon}\|_{L^2(\mathbb{R}^{2N})}\Big).$$

Proof. *a*) The first relation is a consequence of the conservation of mass and positivity for the quantum dynamical semigroup of Section 2.1. This implies the second relation.

b) We first recall that the product of two $\mathcal{J}_2$−operators L and J has trace $\mathrm{Tr}(LJ) = \int_{\mathbb{R}^{2N}} \mathrm{kern}L(x,z)\, \mathrm{kern}J(z,x)\, dzdx$ (cf. [6]).
Then we multiply equation (1) by R_ϵ $(= R_\epsilon^*)$ and take traces:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\mathrm{Tr}(R_\epsilon^2) &= \gamma N \mathrm{Tr}(R_\epsilon^2) - \epsilon^2 D_{qq} \int |\nabla_z \rho_\epsilon(x,z) + \nabla_x \rho_\epsilon(x,z)|^2\, dxdz + \\
&\quad - \frac{D_{pp}}{\epsilon^2} \int |\rho_\epsilon(x,z)|^2 |x-z|^2\, dxdz + \\
&\quad -2\epsilon D_{pq}\, \mathrm{Im} \int \rho_\epsilon(x,z)(z-x)\cdot(\nabla_z \rho_\epsilon(z,x) + \nabla_x \rho_\epsilon(z,x))\, dxdz \\
&\leq \gamma N \mathrm{Tr}(R_\epsilon^2),
\end{aligned}$$

where in the last step we applied

$$-2\epsilon D_{pq}\, \mathrm{Im}\langle u, v\rangle_{L^2} \leq 2\epsilon D_{pq}(s\|u\|_{L^2}^2 + \frac{1}{s}\|v\|_{L^2}^2), \quad s = \frac{D_{pq}}{2\epsilon D_{qq}}$$

with $u = \overline{\rho_\epsilon(x,z)(z-x)}$, $v = \nabla_z \rho_\epsilon(z,x) + \nabla_x \rho_\epsilon(z,x)$ and from (6) we deduce $-(D_{pp}D_{qq} - \epsilon^2 D_{pq}^2) \leq 0$. One concludes with the Gronwall lemma to get the desired estimate for $\mathrm{Tr}(R_\epsilon^2)$. The estimate for $\|W_\epsilon\|_{L^2}$ follows directly from this result and from (26).

c) We now introduce the total energy

$$E^{tot}(R_\epsilon) = E^{kin}(R_\epsilon) + \frac{1}{2}\mathrm{Tr}(V_{R_\epsilon} R_\epsilon) + \mathrm{Tr}\big((V_1 + \alpha\frac{|x|^2}{2})R_\epsilon\big), \tag{29}$$

with $V_{R_\epsilon} = V_0 *_x \rho_\epsilon$ being the self-consistent potential with associated self-consistent potential energy $\frac{1}{2}\mathrm{Tr}(V_{R_\epsilon} R_\epsilon)$. Its time evolution is

$$\frac{d}{dt}E^{tot}(R_\epsilon) = \frac{\epsilon^2}{2}\mathrm{Tr}(\sqrt{-\Delta}\,\partial_t R_\epsilon\,\sqrt{-\Delta}) + \mathrm{Tr}(V_\epsilon\,\partial_t R_\epsilon)\,,$$

with $V_\epsilon = V_{R_\epsilon} + V_1 + \alpha|x|^2/2$ and where we used the relation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathrm{Tr}(V_{R_\epsilon}\,R_\epsilon) \quad &= \quad \mathrm{Tr}(V_{R_\epsilon}\,\partial_t R_\epsilon),\\ \text{for } V_0(x) = V_0(-x) \quad \text{and} \quad &\mathrm{Tr}(V_{R_\epsilon}\,R_\epsilon) = \int_{\mathbb{R}^{2N}} V_0(x-y)\rho_\epsilon(t,x)\rho_\epsilon(t,y)dxdy.\end{aligned}$$

Using (1) and after lengthy calculations which exploit the properties of the trace operator (cf. [30, 34]) we get:

$$\begin{aligned}\frac{d}{dt}E^{kin}(R_\epsilon) \quad &= \quad \frac{\epsilon^2}{2}\mathrm{Tr}\Big(-\nabla\,\big(-\frac{i}{\epsilon}\,[H,R_\epsilon]\,\big)\,\nabla\Big) + \frac{\epsilon^2}{2}\mathrm{Tr}\Big(-\nabla A(R_\epsilon)\nabla\Big)\\ &= \quad -\frac{i\epsilon}{2}\mathrm{Tr}(-\nabla[V_\epsilon,R_\epsilon]\nabla) - 4\gamma E^{kin}(R_\epsilon) + ND_{pp}\mathrm{Tr}(R_{I\epsilon}),\\ \mathrm{Tr}(V_\epsilon\,\partial_t R_\epsilon) \quad &= \quad \mathrm{Tr}\Big(-\frac{i}{\epsilon}V_\epsilon\,[H,R_\epsilon]\Big) + \mathrm{Tr}\Big(V_\epsilon A(R_\epsilon)\Big)\\ &= \quad \frac{i\epsilon}{2}\mathrm{Tr}(-\nabla[V_\epsilon,R_\epsilon]\nabla) + \epsilon^2 D_{qq}\mathrm{Tr}((\Delta V_\epsilon)R_\epsilon).\end{aligned} \tag{30}$$

We briefly explain how to arrive to this system. First of all, we use the equality $\mathrm{Tr}(\sqrt{-\Delta}R_\epsilon\sqrt{-\Delta}) = \mathrm{Tr}(-\nabla R_\epsilon\nabla) = \mathrm{Tr}(-\Delta R_\epsilon)$. The Hamiltonian contributions to the evolution of the kinetic and the potential energy are equal with opposite sign (cf. also [1] (3.30)):

$$\tfrac{\epsilon^2}{2}\mathrm{Tr}(-\nabla[H,R_\epsilon]\nabla) = \tfrac{\epsilon^2}{2}\mathrm{Tr}(-\nabla[V_\epsilon,R_\epsilon]\nabla) = -\mathrm{Tr}(V_\epsilon[-\tfrac{\epsilon^2}{2}\Delta,R_\epsilon]) = -\mathrm{Tr}(V_\epsilon[H,R_\epsilon]).$$

On the other hand, each of the Fokker-Planck contributions $(\epsilon^2/2)\mathrm{Tr}(-\nabla A(R_\epsilon)\nabla)$ and $\mathrm{Tr}(V_\epsilon A(R_\epsilon))$ can be divided into four parts (corresponding to the four coefficients). Let us compute for example the term $(-D_{pp}/2)\,\mathrm{Tr}(-\nabla[x,[x,R_\epsilon]]\nabla)$ in the evolution of E^{kin}: after some simplifications one gets

$$\begin{aligned}-\mathrm{Tr}(-\nabla[x,[x,R_\epsilon]]\nabla) = \quad &\mathrm{Tr}\big(|x|^2\nabla R_\epsilon\nabla - 2x(\nabla R_\epsilon\nabla)x + \nabla R_\epsilon\nabla|x|^2\big)\\ &+2\mathrm{Tr}\big(xR_\epsilon\nabla - R_\epsilon x\nabla + x\nabla R_\epsilon - \nabla R_\epsilon x\big).\end{aligned}$$

The first term on the r.h.s is clearly zero, while the second one equals $2N\mathrm{Tr}(R_\epsilon)$ (here, after a regularization of R_ϵ, cf. Appendix of [4], we computed the traces as $\mathrm{Tr}(R_\epsilon) = \int_{\mathbb{R}^N}\rho_\epsilon(x,x)dx$ and e.g. $\mathrm{Tr}(xR_\epsilon\nabla) = \int_{\mathbb{R}^N}\mathrm{kern}(xR_\epsilon\nabla)(x,x)\,dx$; see [6], Corollary 3.2). The sum of the two equations in (30) gives:

$$\frac{d}{dt}E^{tot}(R_\epsilon) = -4\gamma E^{tot}(R_\epsilon) + 4\gamma\mathrm{Tr}(V_\epsilon\,R_\epsilon) + ND_{pp}\mathrm{Tr}(R_{I\epsilon}) + \epsilon^2 D_{qq}\mathrm{Tr}((\Delta V_\epsilon)\,R_\epsilon). \tag{31}$$

In order to estimate this expression, we distinguish two subcases:

Case 1) $D_{qq} = D_{pq} = \gamma = 0$ *(Hypoelliptic case)*
Integrating (31) in time we get

$$E^{tot}(R_\epsilon(t)) = E^{tot}(R_{I\epsilon}) + N D_{pp}\, t\, \mathrm{Tr}(R_{I\epsilon}).$$

We write $\mathrm{Tr}(V_{R_\epsilon} R_\epsilon) = \mathrm{Tr}((V_0^+ *_x \rho_\epsilon)\, R_\epsilon) - \mathrm{Tr}((V_0^- *_x \rho_\epsilon)\, R_\epsilon)$.
If $V_0^- \in L^\infty(\mathbb{R}^N)$, then $\mathrm{Tr}((V_0^- *_x \rho_\epsilon) R_\epsilon) \le c\mathrm{Tr}(R_\epsilon)^2$. Consequently, on each compact time interval $[0,T]$, one gets the desired bound
$E^{kin}(R_\epsilon(t)) \le c\big(T, \mathrm{Tr}(R_{I\epsilon}), E^{kin}(R_{I\epsilon}), \alpha\mathrm{Tr}(xR_{I\epsilon}x)\big)$.
If else $V_0^- \notin L^\infty(\mathbb{R}^N)$, we must find a uniform bound of the negative part of the potential energy in terms of the kinetic energy, i.e. $\mathrm{Tr}((V_0^- *_x \rho_\epsilon)\, R_\epsilon) \le c(T, R_{I\epsilon})(E^{kin}(R_\epsilon))^\nu$ with $\nu \in (0,1)$ and $c(T, R_{I\epsilon})$ bounded uniformly in ϵ.
We first recall the inequality

$$\int_{\mathbb{R}^{2N}} U(x-y) f(x) f(y) dx dy \le \|U\|_{L^r(\mathbb{R}^N)} \|f\|^2_{L^q(\mathbb{R}^N)}, \quad \tfrac{2}{q} + \tfrac{1}{r} = 2,\ 1 \le q,\, r \le \infty,$$

and we use it together with Remark 5.5 to obtain

$$\mathrm{Tr}((V_0^- *_x \rho_\epsilon)\, R_\epsilon) \le \|V_0^-\|_{L^r} \|\rho_\epsilon\|^2_{L^q} \le c(T, \mathrm{Tr}(R_{I\epsilon}), \|W_{I\epsilon}\|_{L^2})\, (E^{kin}(R_\epsilon))^{2\alpha} \qquad (32)$$

with limit values (not attainable for $N = 4$) $(r,q) = (\frac{N+4}{4}, \frac{N+4}{N+2})$ for $N \le 3$ and $(r,q) = (\frac{N}{2}, \frac{N}{N-1})$ for $N \ge 4$ and with $2\alpha = 2(1-\theta)\frac{N}{N+4} \in (0,1)$ according to relation (52)-(53). In the case of the attractive 3-dimensional Coulomb potential $V_0 = -1/|x|$, we estimate alternatively:

$$|\mathrm{Tr}(V_{R_\epsilon} R_\epsilon)| = \|\nabla V_{R_\epsilon}\|^2_{L^2(\mathbb{R}^3)} \le c\|\rho_\epsilon\|^2_{L^{\frac{6}{5}}(\mathbb{R}^3)} \le c(T, \mathrm{Tr}(R_{I\epsilon}), \|W_{I\epsilon}\|_{L^2})\, (E^{kin}(R_\epsilon))^{\frac{6}{7}}.$$

Case 2) $\gamma \neq 0$ *(Elliptic case):*
If $D_{qq} \neq 0$, then we bound the last term on the r.h.s. of (31)

$$\epsilon^2 D_{qq} |\mathrm{Tr}((\Delta V_{R_\epsilon}) R_\epsilon)| \le c\|V_0\|_{L^r} \|\epsilon\nabla\rho_\epsilon\|^2_{L^q} \le c(T, \mathrm{Tr}(R_{I\epsilon}), \|W_{I\epsilon}\|_{L^2})\, (E^{kin}(R_\epsilon))^{2\beta}$$

with $(r,q) = (\frac{2N^2+11N+16}{2(N+4)}, \frac{2N^2+11N+16}{2(N+3)(N+2)})$ as limit values and such that $2\beta = 2\,(1-\mu)\frac{N+2}{N+4} \in (0,1)$ according to relation (54). In the case of the attractive 3-dimensional Coulomb potential, we get:

$$\epsilon^2 D_{qq} \mathrm{Tr}((\Delta V_{R_\epsilon}) R_\epsilon) = c\epsilon^2 \|\rho_\epsilon\|^2_{L^2(\mathbb{R}^3)} \le c(T, \mathrm{Tr}(R_{I\epsilon}))\, E^{kin}(R_\epsilon)\,,$$

(cf. (53), (55)). This holds also for the repulsive 3-dimensional Coulomb potential (actually, in this case $\epsilon^2 D_{qq}\mathrm{Tr}((\Delta V_{R_\epsilon}) R_\epsilon) < 0$ and hence there is nothing to test). For V_1, we get $\epsilon^2 D_{qq} |\mathrm{Tr}((\Delta V_1) R_\epsilon)| = \epsilon^2 D_{qq} |\int_{\mathbb{R}^N} \nabla V_1 \cdot \nabla\rho_\epsilon dx| \le \epsilon c \|\nabla V_1\|_{L^k} \|\epsilon\nabla\rho_\epsilon\|_{L^r} \le c(\|R_\epsilon\|_{\mathcal{J}_1})\, E^{kin}(R_\epsilon)$.
Both for Cases 1 and 2, one ends up by collecting all the previous bounds and by applying the Gronwall inequality in (31) to the positive function $E(t) = E^{tot}(R_\epsilon) + c_E$, with c_E a positive constant chosen such that, by (32):
$0 \le -c(T, R_{I\epsilon})(E^{kin}(R_\epsilon))^{2\alpha} + E^{kin}(R_\epsilon) + c_E$. □

The next result shows that, under a uniform bound for $\|R_{I\epsilon}\|_{\mathcal{E}_{kin}}$, the sequence $\{\rho_\epsilon\}_{\epsilon\in(0,1]}$ is time equicontinuous in the weak-star norm of measures (called $\mathcal{M}(\mathbb{R}^N) - w*$ in the next section).

Lemma 2.5. *Under the hypothesis of Lemma 2.4, for all $\phi \in C_0(\mathbb{R}^N)$, there exists a constant c such that*

$$| \int_{\mathbb{R}^N} \phi(x) \left(\rho_\epsilon(t,x) - \rho_\epsilon(s,x)\right) dx \big| \leq c(|t-s|, \|R_{I\epsilon}\|_{\mathcal{E}_{kin}}, \|\phi\|_{L^\infty})$$

with $c(|t-s|) \to 0$ as $t \to s$. The estimate is uniform in ϵ in case $\|R_{I\epsilon}\|_{\mathcal{E}_{kin}} \leq c_1$, $\forall \epsilon \in (0,1]$.

Proof. By density and the positivity of the particle density, it is enough to consider $\phi \in C_0^\infty(\mathbb{R}^N)$, $\phi \geq 0$. In analogy to (21), we have

$$\frac{d}{dt}\mathrm{Tr}(\,\phi\, R_\epsilon) = \frac{i\epsilon}{2}\mathrm{Tr}(-\nabla\,[\,\phi\,,\,R_\epsilon]\,\nabla) + \epsilon^2 D_{qq}\mathrm{Tr}((\Delta_x\phi)R_\epsilon) := I_1 + I_2 \tag{33}$$

and, for all compact time intervals $[0,T]$,

$$\begin{aligned} |I_1| &\leq c\|\nabla\phi\|_{L^\infty(\mathbb{R}^N)}\, E^{kin}(R_\epsilon(t)) \leq c_1(T, \|R_{I\,\epsilon}\|_{\mathcal{E}_{kin}}),\\ |I_2| &\leq \epsilon c\|\nabla\phi\|_{L^\infty(\mathbb{R}^N)}\, \|\epsilon\nabla\rho_\epsilon\|_{L^1} \leq c_2(T, \|R_{I\,\epsilon}\|_{\mathcal{E}_{kin}}). \end{aligned}$$

where we applied (51) in the second expression. Recalling that $\mathrm{Tr}(\phi R_\epsilon) = \int_{\mathbb{R}^N} \phi(x)\,\rho_\epsilon(t,x)\,dx$, one obtains the desired result. □

3 The topology and convergence results

In [26] the following separable Banach algebra of test function has been introduced:

$$\mathcal{A} = \{\phi \in C_0(\mathbb{R}^N_x \times \mathbb{R}^N_\xi) : (\mathcal{F}_{\xi\to z}\phi)(x,z) \in L^1(\mathbb{R}^N_z; C_0(\mathbb{R}^N_x))\}$$

$$\|\phi\|_{\mathcal{A}} := \|\mathcal{F}_{\xi\to z}\phi\|_{L^1_z(C_x)} = \int_{\mathbb{R}^N} \sup_x |\mathcal{F}_{\xi\to z}\phi|(x,z)dz\,.$$

$\mathcal{A}'$ is the dual space of $\mathcal{A}$ and $\mathcal{A}' - w*$ denotes $\mathcal{A}'$ provided with the $w-*$ topology. $\mathcal{M}(\mathbb{R}^N)$ is the set of regular bounded signed measure on $\mathbb{R}^N$ (Radon measures). It represents the Banach dual space of $(C_0(\mathbb{R}^N); \|\,\|_{L^\infty})$. The norm of μ in $\mathcal{M}(\mathbb{R}^N)$ is the total variation $\|\mu\|_{\mathcal{M}} = |\mu|(\mathbb{R}^N)$. $\mathcal{M}(\mathbb{R}^N) - w*$ denotes the same space provided with the weak$-*$ topology $\sigma(\mathcal{M}, C_0(\mathbb{R}^N))$ (cf. [8, 32]). $\mathcal{M}^+(\mathbb{R}^N)$ represents the subspace of nonnegative bounded Radon measures. Moreover, $L^1(\mathbb{R}^N) \subset \mathcal{M}(\mathbb{R}^N)$.
The expression $\langle,\rangle$ denotes the Riesz duality bracket (e.g. $\langle f, g\rangle = \int \overline{f} g dx$ in L^2, $\int g df$ in $\mathcal{M}$).

Lemma 3.1. *Under the assumptions $R_{I\epsilon} \geq 0$, $Tr(R_{I\epsilon}) < c$ uniformly in $\epsilon \in (0,1]$, the sequence $\{W_\epsilon\}_{\epsilon\in(0,1]}$ of solutions of equation (4) is uniformly bounded in $C_b(\mathbb{R}^+, \mathcal{A}' - w*)$ and the sequence of Hussimi transforms $\{W^H_\epsilon\}_{\epsilon\in(0,1]}$ is uniformly bounded in $C_b(\mathbb{R}^+, \mathcal{M} - w*)$.*

Proof. Since $\{W_\epsilon\}_{\epsilon\in(0,1]} \subset C(\mathbb{R}^+; L^2(\mathbb{R}^N))$, one derives the time continuity in the $\mathcal{A}' - w*$ topology. Next, we show that $\|W_\epsilon(t)\|_{\mathcal{A}'} \leq \mathrm{Tr}(R_{I\epsilon})$, $\forall t \in \mathbb{R}^+$. Indeed, taken $t \geq 0$ and $\phi \in \mathcal{A}$,

$$\begin{aligned} |\langle W_\epsilon(t), \phi\rangle_{\mathcal{A}'\times\mathcal{A}}| &= |\langle \mathcal{F}_{\xi\to z} W_\epsilon(t), \mathcal{F}_{\xi\to z}\phi\rangle| \\ &\leq \|\overline{\mathcal{F}}_{\xi\to z} W_\epsilon(t)\|_{C_z(L^1_x)} \|\mathcal{F}_{\xi\to z}\phi\|_{L^1_z(C_x)} \leq \|\rho_\epsilon(t)\|_{L^1}\|\phi\|_{\mathcal{A}}. \end{aligned}$$

In the second inequality we have used the definition $\overline{\mathcal{F}}_{\xi\to z}W_\epsilon(t) = \rho_\epsilon(t, x+\frac{\epsilon z}{2}, x-\frac{\epsilon z}{2})$. Applying the decomposition (10) to $\rho(t,r,s)$ at time t fixed and then Cauchy-Schwartz inequality and Hölder inequality in the variable x, one obtains $\|\overline{\mathcal{F}}_{\xi\to z}W_\epsilon(t)\|_{C_z(L^1_x)} \leq \|\rho_\epsilon(t,x)\|_{L^1_x}$. The latter inequality, the hypothesis and the conservation of the charge density in Lemma 2.4.a) also imply $\|W_\epsilon(t)\|_{\mathcal{A}'} \leq c_I$. The properties of the sequence $\{W^H_\epsilon\}_{\epsilon\in(0,1]}$ follow consequently. □

Thanks to the uniform bounds of Lemma 2.4, we can identify a limit point of the sequence $\{W_\epsilon\}_{\epsilon\in(0,1]}$ and its related quantities. We first recall some definitions. A bounded sequence of nonnegative bounded measures on $\mu_j \in \mathcal{M}^+(\mathbb{R}^N)$ is *tight* if

$$\sup_j \int_{|x|>\lambda} d\mu_j \to 0, \quad \text{for } \lambda \to +\infty.$$

Moreover, $\mathcal{J}_1 = [Com(L^2(\mathbb{R}^{2N}))]'$ (the dual of the compact functionals on L^2) and $\mathcal{J}_2$ is an Hilbert space with internal product $\mathrm{Tr}(A^*B)$, cf. [30].
In the following Lemma one suppresses the time dependency, considered only as a parameter.

Lemma 3.2 ([26] Th. III.2). *Let $R_{I\epsilon} \geq 0$, $Tr(R_{I\epsilon}) < c$, uniformly in $\epsilon \in (0,1]$. Then for $\epsilon \to 0$, up to subsequences, the sequences below converge in the relative topologies:*

$$\begin{array}{llll} W_\epsilon & \to & \mu & \text{in } \mathcal{A}' - w* \\ W^H_\epsilon & \to & \mu^H & \text{in } \mathcal{M}^+(\mathbb{R}^{2N}) - w* \\ \rho_\epsilon(x,x) & \to & \rho_0(x,x) & \text{in } \mathcal{M}^+(\mathbb{R}^N) - w* \\ R_\epsilon & \to & R_0 & \text{in } \mathcal{J}_1 - w* \text{ and in } \mathcal{J}_2 - w \\ \rho_\epsilon(x,y) & \to & \sigma(x,y) & \text{in } L^2(\mathbb{R}^{2N}) - w. \end{array}$$

and the cluster points satisfy the following properties:

1. $\mu = \mu^H$ *and* $\mu \in \mathcal{M}^+(\mathbb{R}^{2N})$
2. $\mu \geq \sigma(x,x)\delta_0(\xi)$ *(δ_0 is the delta measure) and*

$$Tr(R_0) = \int_{\mathbb{R}^N} \sigma(x,x)dx \leq \int_{\mathbb{R}^{2N}} d\mu \leq \liminf_{\epsilon\to 0} \int_{\mathbb{R}^N} \rho_\epsilon(x,x)dx = Tr(R_\epsilon)$$

3. $\rho_0 \geq \int_{\mathbb{R}^N} d\mu(\cdot,\xi)$. *Further, let define the density matrix $\hat{R}_\epsilon$ with integral kernel*

$$(kern\hat{R}_\epsilon)(r,s) := \hat{\rho}_\epsilon(r,s) = \mathcal{F}_{x\to r}\overline{\mathcal{F}}_{y\to s}\rho_\epsilon(x,y).$$

If $(2\pi\epsilon)^{-N}\hat{\rho}_\epsilon(r/\epsilon, r/\epsilon)$ is tight in $\mathcal{M}^+(\mathbb{R}^N) - w*$, then $\rho_0 = \int_{\mathbb{R}^N} d\mu(\cdot,\xi)$.
If $(2\pi\epsilon)^{-N}\hat{\rho}_\epsilon(r/\epsilon, r/\epsilon) \to \mu_\xi$ in $\mathcal{M}^+(\mathbb{R}^N) - w*$, then $\mu_\xi \geq \int_{\mathbb{R}^N} d\mu(x,\cdot)$

4. The equality $\displaystyle\int_{\mathbb{R}^{2N}} d\mu = \lim_{\epsilon\to 0}\int_{\mathbb{R}^N} \rho_\epsilon(x,x)dx$
holds if and only if $(2\pi\epsilon)^{-N}\hat{\rho}_\epsilon(r/\epsilon, r/\epsilon)$ *and* $\rho_\epsilon(x,x)$ *are tight sequences in* $\mathcal{M}^+(\mathbb{R}^N)$*. Under these assumptions, we have*

$$\rho_\epsilon(x,y) \quad\to\quad \sigma(x,y) \;\; \textit{in } L^2(\mathbb{R}^{2N}) \textit{ if and only if } \mu = \sigma(x,x)\delta_0(\xi).$$

As already mentioned, in the case of an open quantum system, the spectral decomposition cannot give information on the time behaviour of the previous sequences. Therefore it is necessary to state the next result. It also includes a typical sufficient condition on $E^{kin}(R_{I\epsilon})$ leading to the equality $\rho_0 = \int_{\mathbb{R}^N} d\mu(\cdot,\xi)$, which is fundamental for the convergence of the nonlinear term of the WFP equation in the classical limit.

Lemma 3.3. *Let* $R_{I\epsilon} \geq 0$, $Tr(R_{I\epsilon}) < c$ *and* $E^{kin}(R_{I\epsilon}) < c$, *(and* $Tr(xR_{I\epsilon}x) < c$, *with quadratic potential) uniformly in* $\epsilon \in (0,1]$.
Let $\{W_\epsilon(t,x,\xi)\}_{\epsilon\in(0,1]}$ *solutions of the Wigner-Fokker-Planck equation in sense of Theorem 2.3. If for* $\epsilon \to 0$, $T > 0$, *and up to subsequences,*

$$W_\epsilon(t,x,\xi) \;\to\; \mu(t,x,\xi) \quad \textit{in } C([0,T];\mathcal{A}' - w*)\,,$$

then

$$\begin{array}{lll} \rho_\epsilon(t,x) & \to \;\; \rho_0(t,x) = \int_{\mathbb{R}^N_\xi} d\mu(t) & \textit{in } C([0,T];\mathcal{M}^+(\mathbb{R}^N_x) - w*) \\ W^H_\epsilon(t,x,\xi) & \to \;\; \mu(t,x,\xi) & \textit{in } C([0,T];\mathcal{M}^+(\mathbb{R}^{2N}_x) - w*)\,. \end{array}$$

Proof. We first observe that the assumptions on $R_{I\epsilon}$ imply $E^{kin}(R_\epsilon(t)) < c(T)$ for $t \in [0,T]$, due to Lemma 2.4.c. Then, we prove that the time equicontinuity of the sequence $\{W_\epsilon\}_{\epsilon\in(0,1]}$ implies that of $\{W^H_\epsilon\}_{\epsilon\in(0,1]}$. Since W^H_ϵ and ρ_ϵ are nonnegative, it is enough to use nonnegative test functions $\psi \in S(\mathbb{R}^{2N})$, $\phi \in S(\mathbb{R}^N)$. By the definition (27), one gets

$$\langle (W_\epsilon(t) - W_\epsilon(s)) *_x G_\epsilon(x) *_\xi G_\epsilon(\xi), \psi\rangle = \langle W_\epsilon(t) - W_\epsilon(s), \psi *_x G_\epsilon(x) *_\xi G_\epsilon(\xi)\rangle\,.$$

Even if $\epsilon-$dependent, $\psi *_{x,v} G_\epsilon$ can be considered as an arbitrarily test function in $\mathcal{A}$ with bounded norm, since $\psi *_{x,v} G_\epsilon \to \psi$ in $\mathcal{A}$ as $\epsilon \to 0$ (cf. [26].Theorem III.1-1). Hence, the hypothesis implies that $\{W^H_\epsilon\}_\epsilon$ is time equicontinuous in $C([0,T];\mathcal{M}^+ - w*)$ and converges to $\mu(t)$ due to Lemma 5.2 and Lemma 3.2- 1. On the other hand, Lemma 2.5 and the hypothesis imply that the sequence $\{\rho_\epsilon\}_{\epsilon\in(0,1]}$ is time equicontinuous in $C([0,T];\mathcal{M}^+(\mathbb{R}^N_x) - w*)$. We now show that it converges to $\int_{\mathbb{R}^N_\xi} d\mu(t)$:

$$\begin{array}{l} |\langle \rho_\epsilon(t) - \int_{\mathbb{R}^N_\xi} d\mu(t),\, \phi\rangle| \leq |\langle \rho_\epsilon(t) - \rho_\epsilon(t) *_x G_\epsilon(x),\, \phi\rangle| \\ +|\langle \rho_\epsilon(t) *_x G_\epsilon(x) - \int_{\mathbb{R}^N_\xi} d\mu(t),\, \phi\rangle| := J_1 + J_2. \end{array}$$

Introducing the radial cut-off supported on the ball $B(0,2) \subset \mathbb{R}^N$:

$$\chi(x) \in C^\infty_0(\mathbb{R}^N),\; \chi \equiv 1 \text{ on } B(0,1), \chi \equiv 0 \text{ on } B(0,2)^c,\; 0 \leq \chi \leq 1\,, \tag{34}$$

and using $\|\mathcal{F}_{x\to z}\rho_\epsilon\|_{L^\infty} \le \|\rho_\epsilon\|_{L^1}$, we get

$$J_1 = |\langle \mathcal{F}_{x\to z}\rho_\epsilon(t),\, (\mathcal{F}_{x\to z}\phi)\,(1-e^{\epsilon|z|^2/4})\rangle| \le \|\rho_{I\epsilon}\|_{L^1}\int_{\mathbb{R}^N}|\mathcal{F}_{x\to z}\phi|(1-e^{\epsilon|z|^2/4})dz,$$
$$J_2 \le |\langle W^H_\epsilon(t)-\mu(t), \phi\chi(\tfrac{\xi}{\lambda})\,\rangle| + |\langle W^H_\epsilon(t)-\mu(t), \phi(1-\chi(\tfrac{\xi}{\lambda}))\,\rangle|\,.$$

By (28) and the hypothesis it follows: $0 \le \int_{\mathbb{R}^{2N}} \phi\,(1-\chi(\frac{\xi}{\lambda}))d\mu = \lim_{\epsilon\to 0}\int_{\mathbb{R}^{2N}} \phi\,(1-\chi(\frac{\xi}{\lambda}))W^H_\epsilon dxd\xi \le c\frac{1}{\lambda^2}\|\,|\xi|^2 W^H_\epsilon\|_{L^1(\mathbb{R}^{2N})} \le c/\lambda^2 \to 0$, for $\lambda\to+\infty$, unif. in ϵ. Therefore J_2 and of J_1 tend to zero as $\epsilon\to 0$, $\lambda\to+\infty$. □

4 Classical limit

The classical limit $\epsilon \to 0$ in equation (4) will be here investigated in four different cases. We rewrite (4) in distributional form

$$\int_0^{+\infty}\int_{\mathbb{R}^{2N}}\big(-\partial_t\varphi(t,x,\xi)-\xi\cdot\nabla_x\varphi(t,x,\xi)-\mathcal{P}_\epsilon\varphi(t,x,\xi)\big)W_\epsilon(t,x,\xi)+$$
$$+\varphi(t,x,\xi)\theta[V_\epsilon]W_\epsilon(t,x,\xi)\;dxd\xi dt-\int_{\mathbb{R}^{2N}}\varphi(0,x,\xi)W_{I\epsilon}(x,\xi)\;dxd\xi=0\,. \tag{35}$$

for the test functions $\varphi(t,x,\xi)=\psi(t)\phi(x,\xi)$ with $\psi\in C_0^\infty(\mathbb{R}^+)$ and

$$\phi\in\mathcal{T}:=\{\phi\in\mathcal{S}(\mathbb{R}^{2N})\mid\hat\phi\in C_0^\infty(\mathbb{R}^{2N})\}$$

where from now on we denote $\hat g(x,\xi):=\mathcal{F}_{\xi\to\eta}g(x,\eta)$. One notes that $\mathcal{T}$ is dense both in $\mathcal{A}$ and in $L^2(\mathbb{R}^{2N})$ (cf. [26]). Further,

$$\mathcal{P}_\epsilon := D_{pp}\Delta_\xi-2\gamma\xi\cdot\nabla_\xi-2\epsilon^2D_{pq}\nabla_x\cdot\nabla_\xi+\epsilon^2D_{qq}\Delta_x$$

is the formal adjoint of the Fokker-Planck operator. For $\epsilon\to 0$, we shall show that W_ϵ converges, in an appropriate topology, to a distributional solution f of the Vlasov Fokker-Planck equation:

$$\int_0^{+\infty}\int_{\mathbb{R}^{2N}}\big(-\partial_t\varphi(t,x,\xi)-\xi\cdot\nabla_x\varphi(t,x,\xi)-D_{pp}\varphi(t,x,\xi)+2\gamma\xi\cdot\nabla_\xi\varphi(t,x,\xi)+$$
$$+\nabla_\xi\varphi(t,x,\xi)\cdot\nabla_xV\,\big)\,df(t,x,\xi)-\int_{\mathbb{R}^{2N}}\varphi(0,x,\xi)\,df_I(x,\xi)=0 \tag{36}$$

where the positive bounded Radon mass $f_I\in\mathcal{M}^+(\mathbb{R}^{2N})$ is chosen such that

$$W_{I\epsilon}\to f_I\quad\text{in }\mathcal{A}'-w*,\quad\text{for }\epsilon\to 0. \tag{37}$$

The proof is typically divided into two steps: first the identification of a cluster point and second the passage to the limit $\epsilon\to 0$ in (35). Before proceeding, we denote the set U_ϕ as

$$U_\phi := \{z\in\mathbb{R}^N | z=x+\epsilon s\eta,\;(x,\eta)\in supp(\hat\phi),\;s\in[-1/2,1/2],\;\epsilon\in[0,1]\}\,.$$

We recall that, for $R_{I\epsilon} \geq 0$ there holds

$$\|R_{I\epsilon}\|_{\mathcal{E}_{kin}} = \mathrm{Tr}(R_{I\epsilon}) + \epsilon^2 \mathrm{Tr}(\sqrt{-\Delta} R_{I\epsilon} \sqrt{-\Delta})$$

and

$$\|R_{I\epsilon}\|_{\mathcal{E}} = \mathrm{Tr}(R_{I\epsilon}) + \epsilon^2 \mathrm{Tr}(\sqrt{-\Delta} R_{I\epsilon} \sqrt{-\Delta}) + \mathrm{Tr}(x R_{I\epsilon} x).$$

4.1 Regular potential

Here and in the following Section we consider the solution of the von Neumann Quantum Fokker-Planck equation found in Theorem 2.1 and 2.2 in its Wigner form, as presented in Theorem 2.3. We remember that the positive real constants c, c_I do not depend on ϵ.

4.1.1 Linear case

Theorem 4.1. *Let $\{W_\epsilon\}_{\epsilon\in(0,1]} \in C([0,T]; L^2(\mathbb{R}^{2N}))$ be solutions of the Cauchy problem* (4) *for the linear Wigner-Fokker-Planck equation in sense of Theorem 2.3, under the condition* (6) *and with V satisfying* (15)*. Let $\{W_{I\epsilon} = W_\epsilon[\rho_{I\epsilon}]\}_{\epsilon\in(0,1]}$ be the initial data with $Tr(R_{I\epsilon}) < c_I$ and c_I independent of ϵ. Then, up to extracting a subsequence, there exists $f \in C_b(\mathbb{R}^+; \mathcal{M}^+ - w*)$ such that for $\epsilon \to 0$, $\forall T > 0$,*

$$W_\epsilon(t, x, \xi) \quad \to \quad f(t, x, \xi) \quad \text{ in } \; C([0,T]; \mathcal{A}' - w*)$$

and f solves the Vlasov-Fokker-Planck equation (5) *in sense of distributions* (36) *with initial condition* (37)*.*

Proof. A limit function f is identified by the compactness argument in Lemma 5.1: the uniform boundedness of the sequence $\{W_\epsilon\}_{\epsilon\in(0,1]}$ in $C_b(\mathbb{R}^+; \mathcal{A}' - w*)$ has already been stated in Lemma 3.1. To test the time equicontinuity we consider $\phi \in \mathcal{T}$ and study the term $\langle \partial_t W_\epsilon, \phi \rangle_{\mathcal{T}' \times \mathcal{T}} = \langle \partial_t \hat{W}_\epsilon, \hat{\phi} \rangle_{\mathcal{D}' \times \mathcal{D}}$ and prove the following estimate:

$$\frac{d}{dt} \int \phi(x,\xi) W_\epsilon(t,x,\xi) dx d\xi \leq c\big(\|\phi\|_{\mathcal{A}}, c_1(\phi), \mathrm{Tr}(R_{I\epsilon}), \|\nabla V\|_{L^\infty(U_\phi)}\big). \tag{38}$$

We consider the equation (35) and estimate each term. The term with the pseudodifferential operator can be rewritten in this way (after a partial Fourier transform in ξ, and the Fundamental theorem of calculus)

$$\int \phi(x,\xi) \theta[V_\epsilon] W_\epsilon(t,x,\xi) \, dx \, d\xi$$
$$= (2\pi)^{-N} \int \hat{\phi}(x,\eta) i\eta \cdot \Big(\int_{-1/2}^{1/2} \nabla_x V(x + \epsilon s \eta) ds \Big) \overline{\hat{W}_\epsilon(t,x,\eta)} dx d\eta. \tag{39}$$

By hypothesis $\nabla V \in C(\mathbb{R}^N)$ is bounded on the compact set U_ϕ, thus we have:

$$|\langle \theta[V_\epsilon] W_\epsilon, \phi \rangle| \leq (2\pi)^{-N} \mathrm{Tr}(R_{I\epsilon}) \; \|\nabla V\|_{L^\infty(U_\phi)} \; \| \sup_x |\hat{\phi}| |\eta| \|_{L^1_\eta} < c.$$

where $\| \sup_x |\hat{\phi}||\eta|\|_{L^1_\eta} \le c\|\phi\|_{\mathcal{A}}$ and $\|W_\epsilon(t)\|_{\mathcal{A}'} \le \mathrm{Tr}(R_{I\epsilon})$.
The remaining linear terms are also bounded

$$\begin{aligned} |\langle W_\epsilon, (\xi \cdot \nabla_x + \mathcal{P}_\epsilon)\phi\rangle| &\le 2c_1(\phi) \max(D_{pp}, \gamma, D_{pq}, D_{qq}) \, \mathrm{Tr}(R_{I\epsilon}) < c, \\ c_1(\phi) := \|\xi \cdot \nabla_x \phi\|_{\mathcal{A}} + \|\Delta_\xi \phi\|_{\mathcal{A}} &+ \|\xi \cdot \nabla_\xi \phi\|_{\mathcal{A}} + \|\nabla_x \cdot \nabla_\xi \phi\|_{\mathcal{A}} + \|\Delta_x \phi\|_{\mathcal{A}}. \end{aligned} \tag{40}$$

Therefore $\partial_t \hat{W}_\epsilon$ is bounded in $L^\infty((0,T);\mathcal{D}'(\mathbb{R}^{2N}))$, uniformly in ϵ, which is sufficient to prove the uniform time-equicontinuity of $\{W_\epsilon\}_{\epsilon\in(0,1]}$ in $\mathcal{A}' - w*$. Up to subsequences, a common accumulation function f for both sequences $\{W_\epsilon\}_\epsilon$ and $\{W_\epsilon^H\}_\epsilon$ is then identified on each time interval $[0,T]$ (cf. Lemma 3.3) and can be extended to $\mathbb{R}_t^+$ by a diagonalization process due to the uniform-in-time bound of Lemma 3.1. Further, f belongs to $C_b(\mathbb{R}^+; \mathcal{M}^+ - w*)$.
Passing to the limit $\epsilon \to 0$ *in* (35): Since ∇V is uniformly continuous on compact sets and $\hat{\phi}$ has compact support for $\phi \in \mathcal{T}$, we obtain

$$0 = \lim_{\epsilon\to 0} \|\hat{\phi}\, i\eta \cdot \int_{-1/2}^{1/2} \nabla_x V(x+\epsilon s\eta) ds - \hat{\phi}\, i\eta \cdot \nabla_x V(x)\|_{L^1_\eta(C_x(\mathrm{supp}\,\hat{\phi}))}.$$

By combining the strong convergence of this last term with the weak$-*$ convergence of $\{W_\epsilon\}_\epsilon$, one can compute the limit of the pseudodifferential term:

$$\begin{aligned} 0 &= \lim_{\epsilon\to 0} \sup_{t\in[0,T]} \frac{1}{(2\pi)^N} |\langle \hat{W}_\epsilon(t), \hat{\phi} i\eta \cdot \int_{-1/2}^{1/2} \nabla_x V(x+\epsilon s\eta) ds\rangle - \langle \hat{f}(t), \hat{\phi} i\eta \cdot \nabla_x V(x)\rangle| \\ &= \lim_{\epsilon\to 0} \sup_{t\in[0,T]} |\langle \theta[V_\epsilon] W_\epsilon, \phi\rangle - \langle f, \nabla_\xi \phi \cdot \nabla_x V\rangle| \end{aligned}$$

where $\hat{f}$ denotes the Fourier transform of the positive Radon mass f. The limit of the other linear terms in (35) is immediate in $\mathcal{A}' \times \mathcal{A}$:

$$0 = \lim_{\epsilon\to 0} \sup_{t\in[0,T]} |\langle W_\epsilon, (\xi\cdot\nabla_x + \mathcal{P}_\epsilon)\phi\rangle - \langle f, (\xi\cdot\nabla_x + D_{pp}\Delta_\xi - 2\gamma\xi\cdot\nabla_\xi)\phi\rangle|, \tag{41}$$

where the terms with D_{pq} and D_{qq} vanish, since the coefficients are $O(\epsilon^2)$. Finally, the limit f satisfies the initial condition $f(t=0) = f_I$ since $\langle W_{I\epsilon}, \phi\rangle \to \langle f_I, \phi\rangle$ in $\mathcal{A}' \times \mathcal{A}$. □

4.1.2 Nonlinear case

Note that in this case we need a uniform bound for the initial kinetic energies in order to get the equality $\rho_0 = \int_{\mathbb{R}^N} df(\cdot,\xi)$. For semplicity we consider only a self-consistent potentials, but other linear potentials can be added according to Theorem 4.1.

Theorem 4.2. *Let* $\{W_\epsilon\}_{\epsilon\in(0,1]} \in C([0,T]; L^2(\mathbb{R}^{2N}))$ *be solutions of the Cauchy problem* (4) *for the nonlinear Wigner-Fokker-Planck equation in sense of Theorem 2.3,* *under the condition* (6). *Let* $\{W_\epsilon[\rho_{I\epsilon}]\}_{\epsilon\in(0,1]}$ *be the initial data such that, uniformly in* ϵ,

$$Tr(R_{I\epsilon}) < c_I, \; \frac{\epsilon^2}{2} Tr(\sqrt{-\Delta} R_{I\epsilon} \sqrt{-\Delta}) < c_I.$$

Let $V_\epsilon = V_0 *_x \rho_\epsilon$ *satisfy* (16) *and one of the following*

$$i) \quad \nabla_x V_0 \in C_0(\mathbb{R}^N), \qquad ii) \quad Tr(x R_{I\epsilon} x) < c_I$$

Then, up to extracting a subsequence, there exists $f \in C_b(\mathbb{R}^+; \mathcal{M}^+ - w*)$, *with* $|\xi|^2 f \in L^\infty_{loc}(\mathbb{R}^+; \mathcal{M}^+)$, *such that for* $\epsilon \to 0$, $\forall T > 0$

$$W_\epsilon(t,x,\xi) \quad \to \quad f(t,x,\xi) \qquad in \;\; C([0,T]; \mathcal{A}' - w*).$$

f *solves the nonlinear Vlasov-Fokker-Planck equation* (5) *in sense of distributions* (36), *where the limit potential* $V = V_0 *_x \rho_0$ *is such that* $\rho_0(t,x) = \int_{\mathbb{R}^N_\xi} df(\cdot,\cdot,\xi)$ *and the initial condition is given by* (37).
In the case ii), there holds $|x|^2\rho_0 \in L^\infty_{loc}(\mathbb{R}^+; \mathcal{M}^+(\mathbb{R}^N))$ *and the convergence of the total mass:*

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^N} \rho_\epsilon(t,x) dx = \int_{\mathbb{R}^{2N}} df(t,x,v), \quad for\ all\ t \geq 0.$$

Proof. As in Theorem 4.1 we first test the time equicontinuity condition of Lemma 5.1 for $\{W_\epsilon\}_{\epsilon\in(0,1]}$. The only difference is given by the nonlinear term:

$$\langle \theta[V_\epsilon] W_\epsilon, \phi\rangle = (2\pi)^{-N} \int dx d\eta\, \hat{\phi}(x,\eta) i\eta \cdot \int_{-1/2}^{1/2} ds (\nabla_x V_0 *_x \rho_\epsilon)(t, x + \epsilon s\eta) \overline{\hat{W}_\epsilon(t,x,\eta)} \tag{42}$$

which, by $\|\nabla_x V_0 *_x \rho_\epsilon(t)\|_{L^\infty} \leq \|\nabla_x V_0\|_{L^\infty} \|\rho_\epsilon(t)\|_{L^1}$, by Lemma 2.4.a and the estimate in Lemma 3.1, is uniformly bounded:

$$|\langle \theta[V_\epsilon] W_\epsilon, \phi\rangle| \leq (2\pi)^{-N} \mathrm{Tr}(R_{I\epsilon})^2 \; \|\nabla V_0\|_{L^\infty(\mathbb{R}^N)} \; c \|\phi\|_{\mathcal{A}} < c\,.$$

By Lemma 5.1- 3.2 and up to subsequences, one concludes $W_\epsilon \to f$ in $C([0,T]; \mathcal{A}' - w*)$. Lemma 3.3 implies $W_\epsilon^H \to f$ in $C([0,T]; \mathcal{M}(\mathbb{R}^{2N}) - w*)$ and $\rho_\epsilon \to \rho_0 = \int_{\mathbb{R}^N_\xi} df(.,\xi)$ in $C([0,T]; \mathcal{M}(\mathbb{R}^N) - w*)$, with $\rho_0 \in C([0,T]; \mathcal{M}^+ - w*)$. Further, $f \in C_b(\mathbb{R}^+; \mathcal{M}^+ - w*)$ and $\int_{\mathbb{R}^{2N}} |\xi|^2 df(t) \leq c(T)$.
Passing to the limit $\epsilon \to 0$ *in* (35): The limit $\epsilon \to 0$ in the linear part of (35) can be consequently performed as in (41) and the terms with D_{pq} and D_{qq} vanish because the constants are $O(\epsilon^2)$. For the convergence of the nonlinear term we first prove that $\nabla V_0 *_x \rho_\epsilon$ tends to $\nabla V_0 *_x \rho_0$ uniformly and componentwise on compact sets of $\mathbb{R}^+_t \times \mathbb{R}^N_x$.
We study separately hypothesis i) and ii).
Part i) Let χ the radial cut-off (34) and $K_x \subset \mathbb{R}^N$ a compact set, $\lambda > 0$ a real number, $\langle a(\cdot), b(x-\cdot)\rangle := \int_{\mathbb{R}^N} b(x-y) da(y)$. Define $Z_\lambda(y) := \chi(\frac{y}{\lambda})\partial_{x_j} V_0(y) \in C_c(\mathbb{R}^N)$,

let $\cup_{n=1}^M \bar{B}(x_n,\mu)$ be a finite compact covering of K_x with balls of center $x_n \in K_x$ and radius μ. We have

$$\begin{aligned}
&\sup_{(t,x)\in[0,T]\times K_x} |\partial_{x_j} V_0 *_x \rho_\epsilon(t,x) - \partial_{x_j} V_0 *_x \rho_0(t,x)| \\
&\le \sup_{(t,x)\in[0,T]\times K_x} \Big\{ |Z_\lambda *_x (\rho_\epsilon(t,x) - \rho_0(t,x))| + |(\partial_{x_j} V_0 - Z_\lambda) *_x (\rho_\epsilon(t,x) - \rho_0(t,x))| \Big\} \\
&\le \sup_{t\in[0,T]} \sup_{n=1..M} \sup_{x\in \bar{B}(x_n,\mu)} |\langle \rho_\epsilon(t,\cdot) - \rho_0(t,\cdot), Z_\lambda(x-\cdot) - Z_\lambda(x_n-\cdot)\rangle| + \\
&\sup_{t\in[0,T]} \sup_{n=1..M} |\langle \rho_\epsilon(t,\cdot) - \rho_0(t,\cdot), Z_\lambda(x_n-\cdot)\rangle| + 2\mathrm{Tr}(R_{I\epsilon}) \sup_{|z|>\lambda} |(\partial_{x_j} V_0)(z)|. \qquad (43)
\end{aligned}$$

As $\epsilon \to 0$, the latter expression tends to zero. Indeed, $\forall \alpha$ small, $\forall \lambda$, there exists a radius μ s.t. $|Z_\lambda(x-y) - Z_\lambda(x_n - y)| \le \alpha$ in $\mathrm{supp}(Z_\lambda)$ with $|x - x_n| \le \mu$; hence the first term at the r.h.s. is arbitrarily small (unif. in t and ϵ since $\mathrm{Tr}(R_{I\epsilon}) < c$). The second term goes to zero since $\rho_\epsilon \to \rho_0$ in $C([0,T];\mathcal{M}(\mathbb{R}^N) - w*)$, while the third term on the r.h.s. is arbitrarily small as $\lambda \to +\infty$. Hence we get the desired convergence for $\partial_{x_j} V_0 *_x \rho_\epsilon$. Then, taken $x + s\epsilon\eta \in U_\phi$, with η fixed, we have

$$\begin{aligned}
&\sup_{(t,x)\in[0,T]\times K_x} \Big| \int_{-1/2}^{1/2} ds \Big(\partial_{x_j} V_0 *_x \rho_\epsilon(t, x+\epsilon s\eta) - \partial_{x_j} V_0 *_x \rho_0(t,x) \Big) \Big| \\
&\le \sup_{(t,x)\in[0,T]\times K_x} \int_{-1/2}^{1/2} \Big(|\partial_{x_j} V_0 *_x \rho_\epsilon(t,x+\epsilon s\eta) - \partial_{x_j} V_0 *_x \rho_0(t,x+\epsilon s\eta)| \\
&\qquad + |\partial_{x_j} V_0 *_x \rho_0(t,x+\epsilon s\eta) - \partial_{x_j} V_0 *_x \rho_0(t,x)| \Big) ds \\
&\le \sup_{(t,z)\in[0,T]\times U_\phi} |\partial_{x_j} V_0 *_x \rho_\epsilon(t,z) - \partial_{x_j} V_0 *_x \rho_0(t,z)| \\
&+ \int_{-1/2}^{1/2} ds \sup_{(t,x)\in[0,T]\times K_x} |\partial_{x_j} V_0 *_x \rho_0(t,x+\epsilon s\eta) - \partial_{x_j} V_0 *_x \rho_0(t,x)| := I_\epsilon + J_\epsilon.
\end{aligned}$$

As $\epsilon \to 0$, $I_\epsilon \to 0$ from (43). Analogously $J_\epsilon \to 0$, since $\rho_0(t, x+\epsilon y) \to \rho_0(t,x)$ in $C([0,T];\mathcal{M}(\mathbb{R}^N) - w*)$ as translation of a Radon measure (cf. [33], ch.IV.7, pg. 552). As in Theorem 4.1 we conclude that, $\forall T > 0$, (with $K := [0,T] \times \mathrm{supp}\,\hat{\phi}$)

$$\begin{aligned}
\lim_{\epsilon\to 0} \|\hat{\phi}\, i\eta \cdot \int_{-1/2}^{1/2} ds \Big(\nabla_x V_0 *_x \rho_\epsilon(t,x+\epsilon s\eta) - \nabla_x V_0 *_x \rho_0(t,x) \Big)\|_{L^1_\eta C_{x,t}(K)} &= 0 \\
\lim_{\epsilon\to 0} \sup_{t\in[0,T]} |\langle \theta[V_\epsilon] W_\epsilon, \phi\rangle - \langle f, \nabla_\xi \phi \cdot (\nabla_x V_0 *_x \rho_0)\rangle| &= 0,
\end{aligned}$$

with Cauchy initial condition $f(t=0) = f_I$.

Part ii) In this case, Lemma 2.4.c provides a uniform bound for the second $x-$moment of the density ρ_ϵ on compact time intervals. The inequality (43) is then replaced by

$$|\partial_{x_j} V_0 *_x \rho_\epsilon(t,x) - \partial_{x_j} V_0 *_x \rho_0(t,x)| \leq \sup_{(t,x)\in[0,T]\times K_x} |\langle \rho_\epsilon(t,\cdot) - \rho_0(t,\cdot), Z_\lambda(x-\cdot)\rangle| + \|\partial_{x_j} V_0\|_{L^\infty(\mathbb{R}^N)} \frac{2}{\lambda^2} \sup_{(t,x)\in[0,T]\times K_x} \int_{\mathbb{R}^N} |x-y|^2 \rho_\epsilon(t,y)dy.$$

By letting $\epsilon \to 0$ and $\lambda \to +\infty$ in the latter expression, one gets $\nabla V_0 *_x \rho_\epsilon \to \nabla V_0 *_x \rho_0$, uniformly on compact sets $([0,T]\times K_x)^N$. Then one concludes the proof as in Part i). As a consequence of Lemma 3.2.4, the control of the second $x-$moment of ρ_ϵ implies also the convergence of the total mass. □

4.2 Irregular potential

Note that here and in the next Theorem we need a uniform bound for $\|W_{I\epsilon}\|_{L^2}$. This assumption is strong, since by (26) it implies that the eigenvalues of $R_{I\epsilon}$ are such that $\sum_{j\in\mathbb{N}} \lambda^2_{Ij_\epsilon} \to 0$ as $\epsilon \to 0$, even if the trace remains constant (e.g. $\sum_{j\in\mathbb{N}} \lambda_{Ij_\epsilon} = 1$). As many authors noticed, this fact prevents pure states, or finite combinations of pure states, from being I.C. in the classical limit.

4.2.1 Linear case

Theorem 4.3. *Let $\{W_\epsilon\}_{\epsilon\in(0,1]} \in C([0,T]; L^2(\mathbb{R}^{2N}))$ be solutions of the Cauchy problem* (4) *for the linear Wigner-Fokker-Planck equation in sense of Theorem 2.3, under the condition* (6) *and with V satisfying* (17). *Let $\{W_{I\epsilon} = W_\epsilon[\rho_{I\epsilon}]\}_{\epsilon\in(0,1]}$ be the initial data with*

$$Tr(R_{I\epsilon}) < c_I, \; \|W_{I\epsilon}\|_{L^2(\mathbb{R}^{2N})} < c_I,$$

unif. in ϵ. Then, up to extracting a subsequence, there exists $f \in C_b(\mathbb{R}^+; \mathcal{M}^+ - w*) \cap L^\infty_{loc}(\mathbb{R}^+; L^1 \cap L^2(\mathbb{R}^{2N}))$ such that for $\epsilon \to 0$, $\forall T > 0$

$$W_\epsilon(t,x,\xi) \quad \to \quad f(t,x,\xi) \qquad \textit{in } C([0,T]; \mathcal{A}' - w*) \textit{ and } L^\infty([0,T]; L^2(\mathbb{R}^{2N})) - w*$$

and with f solving the Vlasov-Fokker-Planck equation (5) *in sense of distributions* (36)*, with initial condition* (37)*.*
In the hypoelliptic case, $f \in C_b(\mathbb{R}^+; \mathcal{M}^+ - w*) \cap L^\infty(\mathbb{R}^+; L^1 \cap L^2(\mathbb{R}^{2N}))$.

Proof. As in Theorem 4.1, a limiting function $f \in C_b(\mathbb{R}^+; \mathcal{M}^+ - w*)$ is identified thanks to Lemma 5.1. In order to show the time equicontinuity, we estimate the pseudodifferential term (39) with the present potential:

$$|\langle \theta[V_\epsilon]W_\epsilon, \phi\rangle| \leq (2\pi)^{-N} \|W_\epsilon(t)\|_{L^2(\mathbb{R}^{2N})} \|\nabla V\|_{L^2(U_\phi)} \| \sup_x |\hat{\phi}(x,\eta)||\eta| \|_{L^2(\mathbb{R}^N_\eta)} < c(T) \qquad (44)$$

where $Q^\epsilon_\eta(V) := \frac{1}{\epsilon}\big(V\big(x+\frac{\epsilon\eta}{2}\big) - V\big(x-\frac{\epsilon\eta}{2}\big)\big) = \eta \cdot \int_{-1/2}^{1/2} \nabla_x V(x+\epsilon s\eta)ds$, as distribution. The remaining linear terms can be estimated as in (40). Thus, up to subsequences,

$W_\epsilon \to f$ in $C([0,T];\mathcal{A}' - w*)$ for all $T > 0$ and $f \in C_b(\mathbb{R}^+; \mathcal{M}^+ - w*)$ from Lemma 5.2 (applied to $\{W_\epsilon^H\}_\epsilon$) and Lemma 3.2.1.
On the other hand, the hypothesis and Lemma 2.4.b yield the uniform estimate $\|W_\epsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^{2N}))} < c(T)$. By extracting another subsequence, we get $W_\epsilon \to f$ in $L^\infty([0,T];L^2(\mathbb{R}^{2N})) - w*$ with the same limit as before since the functions $\mathcal{T}$ are dense in both topologies (in $\mathbb{R}^{2N}_{x,\xi}$). Consequently, $f \in L^\infty([0,T]; L^1 \cap L^2(\mathbb{R}^{2N}))$. In the hypoelliptic case, the uniform in time estimate $\|W_\epsilon\|_{L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^{2N}))} < c$ gives $f \in L^\infty(\mathbb{R}^+; L^1 \cap L^2(\mathbb{R}^N))$.
Passing to the limit $\epsilon \to 0$ in (35): The hypothesis $V \in H^1_{loc}(\mathbb{R}^N_x)$ implies

$$\lim_{\epsilon \to 0} \|(Q^\epsilon_\eta(V) - \eta \cdot \nabla V)\hat{\phi}\|_{L^2(\mathbb{R}^{2N})} = 0, \tag{45}$$

(one can see it after a regularization of V in H^1_{loc}) which, combined with the fact that $W_\epsilon \to f$ in $L^\infty([0,T];L^2(\mathbb{R}^{2N})) - w*$, implies the convergence of the pseudodifferential term, $\forall \psi \in C_0^\infty(\mathbb{R}^+)$,

$$\lim_{\epsilon \to 0} |\langle \theta_\epsilon[V] W_\epsilon, \phi\ \psi \rangle - \langle f, (\nabla_\xi \phi \cdot \nabla_x V)\ \psi \rangle| = 0. \tag{46}$$

The remaining linear part converges in sense of $L^\infty_t(L^2_{x\xi}) \times L^1_t(L^2_{x\xi})$ (and also in $\mathcal{A}' \times \mathcal{A}$, cf. (41)) and the terms with D_{pq} and D_{qq} vanish since the coefficients are $O(\epsilon^2)$. $\square$

4.2.2 Nonlinear case

For semplicity in the presentation we consider only $V_\epsilon = V_0 *_x \rho_\epsilon$. One can add a linear potential $V_1 + |x|^2/2$ under the additional condition $\mathrm{Tr}(x R_{I\epsilon} x) < c_I$.

Theorem 4.4. *Let $\{W_\epsilon\}_{\epsilon \in (0,1]} \in C([0,T]; L^2(\mathbb{R}^{2N}))$ be solutions of the Cauchy problem* (4) *for the nonlinear Wigner-Fokker-Planck equation in sense of Theorem 2.3, under the condition* (6). *Let $\{W_{I\epsilon} = W_\epsilon[\rho_{I\epsilon}]\}_{\epsilon \in (0,1]}$ be the initial data with*

$$Tr(R_{I\epsilon}) < c_I,\ \frac{\epsilon^2}{2} Tr(\sqrt{-\Delta} R_{I\epsilon} \sqrt{-\Delta}) < c_I,\ \|W_{I\epsilon}\|_{L^2(\mathbb{R}^{2N})} < c_I$$

Let $V_\epsilon = V_0 *_x \rho_\epsilon$ satisfy (19)-(20).
Then, up to extracting a subsequence, there exists $f \in C_b(\mathbb{R}^+; \mathcal{M}^+ - w*) \cap L^\infty_{loc}(\mathbb{R}^+; L^1 \cap L^2(\mathbb{R}^{2N}))$, with $|\xi|^2 f \in L^\infty_{loc}(\mathbb{R}^+; L^1)$, such that for $\epsilon \to 0$, $\forall T > 0$

$$W_\epsilon(t,x,\xi) \quad \to \quad f(t,x,\xi) \text{ in } C([0,T]; \mathcal{A}' - w*) \text{ and } L^\infty([0,T]; L^2(\mathbb{R}^{2N})) - w*$$

and with f solving the nonlinear Vlasov-Fokker-Planck equation (5) *in sense of distributions* (36), *where the limit potential $V = V_0 *_x \rho_0$ is such that $\rho_0(t,x) = \int_{\mathbb{R}^N_\xi} f d\xi$ and the initial condition is given by* (37).
In the hypoelliptic case, $f \in C_b(\mathbb{R}^+; \mathcal{M}^+ - w*) \cap L^\infty(\mathbb{R}^+; L^1 \cap L^2(\mathbb{R}^{2N}))$.

Proof. The Lieb-Thirring-type inequality (52) together with the hypothesis on the initial data and Lemma 2.4 give the uniform bounds:

$$\|\rho_\epsilon\|_{L^\infty(\mathbb{R}^+;L^1(\mathbb{R}^N_x))} < c, \quad \|\rho_\epsilon\|_{L^\infty([0,T];L^{(N+4)/(N+2)}(\mathbb{R}^N_x))} < c(T), \ \forall T > 0.$$

Further, the Young inequality and (20) yield the uniform estimates for the L^2-norm of $\nabla V_\epsilon(t)$, on compact time intervals:

$$\|\nabla V_0 *_x \rho_\epsilon(t)\|_{L^2(\mathbb{R}^N)} \leq c\|\rho_\epsilon(t)\|_{L^{\frac{N+4}{N+2}}(\mathbb{R}^N)} \leq c(T) \text{ , with } c \text{ independent of } \epsilon.$$

The analogous of (44) then holds and, as in Theorem 4.3, one derives $W_\epsilon \to f$ both in $C([0,T];\mathcal{A}' - w*)$ and in $L^\infty([0,T];L^2(\mathbb{R}^{2N})) - w*$, with $f \in C([0,T];\mathcal{M}^+ - w*) \cap L^\infty([0,T];L^1 \cap L^2(\mathbb{R}^{2N}))$. Further, $\int_{\mathbb{R}^{2N}} |\xi|^2 f(t)dxd\xi \leq c(T)$. By Cantor diagonalization and the trace conservation we also get $f \in C_b(\mathbb{R}^+;\mathcal{M}^+ - w*)$.
Passing to the limit $\epsilon \to 0$ *in* (35): As in Theorem 4.3 we get the convergence of the linear part, with the terms with D_{pq} and D_{qq} vanishing. The main difficulty is the limit of the nonlinear term. First step consists in verifying that $\nabla_x V_\epsilon = \nabla_x V_0 *_x \rho_\epsilon \to \nabla_x V_0 *_x \rho_0$ in $C([0,T];L^2_{loc}(\mathbb{R}^N))^N$, for all T bounded.
By (20) we have $\nabla_x V_0 = F_A + F_B \in L^{(2N+8)/(N+8)}(\mathbb{R}^N) + L^p(\mathbb{R}^N)$.
Let then $h_n \in C_c(\mathbb{R}^N)^N$ be a vector sequence such that $h_n \to F_A$ in $L^{(2N+8)/(N+8)}(\mathbb{R}^N)^N$, for $n \to \infty$ and with K_x a compact of $\mathbb{R}^N_x$. Consider $(j = 1, ..N)$

$$\begin{aligned} \|F_{A_j} *_x \rho_\epsilon(t) - F_{A_j} *_x \rho_0(t)\|_{L^2(K_x)} &\leq \|(h_n)_j *_x (\rho_\epsilon - \rho_0)(t)\|_{L^2(K_x)} \\ &+ \|(F_A - h_n)_j *_x (\rho_\epsilon - \rho_0)(t)\|_{L^2(K_x)} := I_\epsilon + J_\epsilon. \end{aligned} \tag{47}$$

From Lemma 3.3, as $\epsilon \to 0$ we get $\rho_\epsilon \to \rho_0 = \int f d\xi$ in $C([0,T];\mathcal{M} - w*)$ and thus $I_\epsilon \leq c \sup_{(t,x)\in[0,T]\times K_x} |(h_n)_j *_x (\rho_\epsilon - \rho_0)(t,x)| \to 0$ (as in (43)). Then note that, up to subsequences, $\rho_\epsilon \to \rho_0$ in $L^\infty([0,T];L^{(N+4)/(N+2)}(\mathbb{R}^N)) - w*$. Hence,

$$\begin{aligned} J_\epsilon &\leq \|(F_A - h_n)_j\|_{L^{(2N+8)/(N+8)}(\mathbb{R}^N)} \|\rho_\epsilon - \rho_0\|_{L^\infty([0,T];L^{(N+4)/(N+2)}(\mathbb{R}^N))} \\ &\leq c(T)\|(F_A - h_n)_j\|_{L^{(2N+8)/(N+8)}(\mathbb{R}^N)}. \end{aligned}$$

This last term is arbitrarily small for $n \to +\infty$ and independent of time. Then $J_\epsilon \to 0$, showing that $F_A *_x \rho_\epsilon \to F_A *_x \rho_0$ in $C([0,T];L^2(K_x))^N$. For the term $F_B *_x \rho_\epsilon$, one proceeds similarly to end up with the desired convergence for $\nabla_x V_0 *_x \rho_\epsilon$.
Last step of the procedure consists in carrying the limit in the non linear term. It is enough to verify

$$0 = \lim_{\epsilon\to 0} \|\hat{\phi} i\eta \cdot \int_{-1/2}^{1/2} \nabla_x V_0 *_x \rho_\epsilon(t, x + \epsilon s\eta)ds - \hat{\phi} i\eta \cdot \nabla_x V_0 *_x \rho_0(t,x)\|_{C([0,T];L^2(\mathbb{R}^{2N}))} \tag{48}$$

to conclude that, due to the convergence of W_ϵ to f in $L^\infty(L^2) - w*$,

$$\lim_{\epsilon\to 0} |\langle \theta_\epsilon[V_0 *_x \rho_\epsilon]W_\epsilon, \phi\ \psi\rangle - \langle f, \nabla_\xi \phi \cdot (\nabla_x V_0 *_x \rho_0)\ \psi\rangle| = 0.$$

To prove (48), we majorate the expression by (taking for ex. $\text{supp}\hat{\phi} := K_x \times K_\eta$)

$$c\int_{-1/2}^{1/2} ds \|\nabla_x V_0 *_x \rho_\epsilon(t, x + \epsilon s\eta) - \nabla_x V_0 *_x \rho_0(t, x + \epsilon s\eta)\|_{C([0,T];L^2(K_x \times K_\eta))}$$

$$+c\int_{-1/2}^{1/2} ds \|\nabla_x V_0 *_x \rho_0(t, x + \epsilon s\eta) - \nabla_x V_0 *_x \rho_0(t, x)\|_{C([0,T];L^2(K_x \times K_\eta))} := L_\epsilon + N_\epsilon.$$

Concerning the first integral we immediately get

$$L_\epsilon \leq c\|\nabla_x V_0 *_x \rho_\epsilon(t, z) - \nabla_x V_0 *_x \rho_0(t, z)\|_{C([0,T];L^2(U_\phi))} \to 0 \quad as\ \epsilon \to 0,$$

from what above. For the second term N_ϵ, we observe that the sequence of measures $\rho_0(t, x+\epsilon y)$ (with y fixed) tends to $\rho_0(t, x)$ in $C([0, T]; \mathcal{M}-w*)$ as $\epsilon \to 0$. We show that $\nabla_x V_0 *_x \rho_0(t, x+\epsilon y) \to \nabla_x V_0 *_x \rho_0$ in $C([0, T]; L^2(K_x \times K_\eta))$: after the regularization h_n of F_A in ∇V_0 we proceed in analogy to (47). Since we work with 2 variables, the only different estimate is

$$\int_{-1/2}^{1/2} ds \sup_{t\in[0,T]} \| \|(h_n)_j *_x (\rho_0(t, x + \epsilon s\eta) - \rho_0(t, x))\|_{L^2(K_x)}\|_{L^2(K_\eta)} \leq$$

$$c\int_{-1/2}^{1/2} ds \Big(\int_{K_\eta} d\eta \big[\sup_{(t,x)\in[0,T]\times K_x} |(h_n)_j *_x (\rho_0(t, x + \epsilon s\eta) - \rho_0(t, x))|\big]^2\Big)^{1/2},$$

which tends to zero as $\epsilon \to 0$. Hence we get $N_\epsilon \to 0$ and this concludes the proof. □

5 Appendix

5.1 Compactness results

Lemma 5.1. *Let X be a separable Banach space, X' the dual space and $X' - w*$ the dual provided with the weak-star topology. If $\{f_n\}_{n\in\mathbb{N}} \subset C([0, T]; X' - w*)$ is a sequence of functions such that:*

1. *(uniform boundedness) $\exists c > 0$ such that* $\sup_{n\in\mathbb{N}} \|f_n(t)\|_{L^\infty([0,T];X')} < c$,
2. *(time equicontinuity) $\forall \phi \in X$, the sequence of functions $t \mapsto \langle f_n(t), \phi\rangle_{X'\times X}$ is uniformly continuous in $t \in [0, T]$, uniformly in $n \in \mathbb{N}$.*

Then $\{f_n\}_{n\in\mathbb{N}}$ is relatively compact in $C([0, T]; X' - w*)$.

Proof. (It is an adaptation of Lemma C.1 in [25]). Since X is separable, then the closed ball $B_R \subset X'$ is metrizable for the weak-star topology $\sigma(X', X)$, with distance

$$d(f, g) = \sum_{k\geq 1} \frac{1}{2^k} \frac{|\langle f - g, \phi_k\rangle_{X'\times X}|}{1 + |\langle f - g, \phi_k\rangle_{X'\times X}|},$$

where $\{\phi_k\}_{k\geq 1}$ is a dense set in X. By Banach-Alaoglu, B_R is sequentially compact for the weak-star topology. Hence, for each t fixed, the sequence $f_n(t)$ is relatively and sequentially compact in $X' - w*$. In order to apply the Ascoli-Arzelà theorem, it remains to show

$$\sup_{n\in\mathbb{N}} d(f_n(t), f_n(s)) \to 0 \quad as \quad |t-s| \to 0, \;\; t,\, s \in [0,T]\,.$$

For a fixed $h > 0$, we get

$$d(f_n(t), f_n(s)) \leq \sup_{1\leq k\leq h} \sup_{n\in\mathbb{N}} |\langle f_n(t) - f_n(s), \phi_k\rangle_{X'\times X}| + \frac{1}{2^h}.$$

The first term on the right hand side goes to zero as $t \to s$ due to the hypothesis

$$\sup_{n\in\mathbb{N}} |\langle f_n(t) - f_n(s), \phi_k\rangle_{X'\times X}| \to 0 \quad as \quad |t-s| \to 0.$$

One concludes noting that the second term is arbitrarily small as $h \to +\infty$. □

The previous Theorem is applied in section 4 to the pair $(X', X) = (\mathcal{A}', \mathcal{A})$ and to $(X', X) = (\mathcal{M}, C_0)$. In this last case we restate it explicitly.

Lemma 5.2. *Let $\mathcal{M}(\mathbb{R}^N)$ be the set of regular bounded signed measures on $\mathbb{R}^N$, $\mathcal{M} = (C_0(\mathbb{R}^N), \|\,\|_{L^\infty(\mathbb{R}^N)})'$, and let $\mathcal{M}(\mathbb{R}^N) - w*$ be the same space with the weak–star topology.*
If $\{\mu_n\}_{n\in\mathbb{N}} \subset C([0,T]; \mathcal{M}(\mathbb{R}^N) - w*)$ is a sequence such that:

1. *(uniform boundedness) $\exists c > 0$ such that $\sup_{n\in\mathbb{N}} |\mu_n(t)|(\mathbb{R}^N) < c\,, \;\; \forall\, t \in [0,T]$,*
2. *(time equicontinuity) $\forall \phi \in C_0(\mathbb{R}^N)$, the sequence of functions $t \mapsto \langle \mu_n(t), \phi\rangle$ is uniformly continuous in $t \in [0,T]$, uniformly in $n \in \mathbb{N}$.*

Then $\{\mu_n\}_{n\in\mathbb{N}}$ is relatively compact in $C([0,T]; \mathcal{M} - w*)$. That is, there exists $\mu \in C([0,T]; \mathcal{M} - w*)$ and a subsequence $\{n_j\}_{j\in\mathbb{N}}$ such that:

$$\forall \phi \in C_0(\mathbb{R}^N), \quad \langle \mu_{n_j}(t), \phi\rangle \to \langle \mu(t), \phi\rangle \;\; for\; j \to \infty$$

and $t \to \langle \mu(t), \phi\rangle$ is continuous in $[0,T]$.

5.2 Lieb-Thirring-type inequalities

In section 2 the following Lieb-Thirring-type inequalities (also known as generalized Sobolev inequalities, cf. [36]) are needed. For convenience we write them in the $\epsilon-$dependent form according to the classical scaling. A general statement is recovered by setting $\epsilon = 1$.

Lemma 5.3. *[26] Let $p \in [1,\infty]$, $R_\epsilon \in \mathcal{J}_p$, $R_\epsilon \geq 0$.*
Let ρ_ϵ, j_ϵ, $E^{kin}(R_\epsilon)$ be defined as in (11), (13) *and* (14). *Then:*

$$\|\rho_\epsilon\|_{L^q(\mathbb{R}^N)} \leq c_q \|R_\epsilon\|^{\alpha}_{\mathcal{J}_p} \, (E^{kin}(R_\epsilon))^{1-\alpha} \, \epsilon^{-2+2\alpha} \tag{49}$$

$$\|j_\epsilon\|_{L^r(\mathbb{R}^N)} \leq c_r \|R_\epsilon\|^{\beta}_{\mathcal{J}_p} \, (E^{kin}(R_\epsilon))^{1-\beta} \, \epsilon^{-2+2\beta+1} \tag{50}$$

$$q = \frac{(N+2)p-N}{Np-N+2}, \quad r = \frac{(N+2)p-N}{(N+1)p-N+1},$$

$$\alpha = \frac{2p}{(N+2)p-N}, \quad \beta = \frac{p}{(N+2)p-N},$$

Lemma 5.4. *[1] Let $p \in [1,\infty]$, $R_\epsilon \in \mathcal{J}_p$, $R_\epsilon \geq 0$.*
Let ρ_ϵ, j_ϵ, $E^{kin}(R_\epsilon)$ be defined as in (11), (13) *and* (14). *Then:*

$$\|\rho_\epsilon\|_{L^q(\mathbb{R}^N)} \leq c_{q,p} \|R_\epsilon\|^{\alpha}_{\mathcal{J}_p} \, (E^{kin}(R_\epsilon))^{1-\alpha} \, \epsilon^{-2+2\alpha}$$

$$\|j_\epsilon\|_{L^r(\mathbb{R}^N)} \leq c_{r,p} \|R_\epsilon\|^{\beta}_{\mathcal{J}_p} \, (E^{kin}(R_\epsilon))^{1-\beta} \, \epsilon^{-2+2\beta+1}$$

with

$$\tfrac{(N+2)p-N}{Np-N+2} \leq q \begin{cases} \leq \infty & N=1 \\ < \infty & N=2 \\ \leq \frac{N}{N-2} & N \geq 3 \end{cases}, \quad \tfrac{(N+2)p-N}{(N+1)p-N+1} \leq r \begin{cases} \leq 2 & N=1 \\ < 2 & N=2 \\ \leq \frac{N}{N-1} & N \geq 3 \end{cases}$$

$\alpha = \frac{N-q(N-2)}{2q}$, $\quad \beta = \frac{N-r(N-1)}{2r}$
and $\|\epsilon \nabla \rho_\epsilon\|_{L^r(\mathbb{R}^N)}$ satisfies the same estimates as $\|j_\epsilon\|_{L^r(\mathbb{R}^N)}$.

Remark 5.5. Note the following relations, which are widely used to get estimates uniformly in ϵ, when $R_\epsilon \in \mathcal{J}_p$ and $R_\epsilon \geq 0$. The important fact is that the constants involved do not depend explicitly on ϵ, but only on the trace, the kinetic energy of R_ϵ and the L^2-norm of the Wigner function. We remember also that the term $\epsilon \nabla \rho_\epsilon$ satisfies the same estimates as j_ϵ.
For $p = 1$:

$$\|\rho_\epsilon\|_{L^1(\mathbb{R}^N)} = \|R_\epsilon\|_{\mathcal{J}_1},$$

$$\|j_\epsilon\|_{L^1(\mathbb{R}^N)} \leq \|R_\epsilon\|^{1/2}_{\mathcal{J}_1} \, (E^{kin}(R_\epsilon))^{1/2}. \tag{51}$$

For $p = 2$ and relation (26):

$$\|\rho_\epsilon\|_{L^{(N+4)/(N+2)}(\mathbb{R}^N)} \leq c(N) \, \|W_\epsilon\|^{4/(N+4)}_{L^2(\mathbb{R}^{2N})} \, (E^{kin}(R_\epsilon))^{N/(N+4)}, \tag{52}$$

$$\|j_\epsilon\|_{L^{(N+4)/(N+3)}(\mathbb{R}^N)} \leq c(N) \, \|W_\epsilon\|^{2/(N+4)}_{L^2(\mathbb{R}^{2N})} \, (E^{kin}(R_\epsilon))^{(N+2)/(N+4)}.$$

By interpolation between the two cases above, we get:

$$\|\rho_\epsilon\|_{L^q(\mathbb{R}^N)} \leq c(N, \mathrm{Tr}(R_\epsilon), \|W_\epsilon\|_{L^2}) \, (E^{kin}(R_\epsilon))^{(1-\theta)\, N/(N+4)}, \tag{53}$$

$$\|j_\epsilon\|_{L^r(\mathbb{R}^N)} \leq c(N, \mathrm{Tr}(R_\epsilon), \|W_\epsilon\|_{L^2}) \, (E^{kin}(R_\epsilon))^{(1-\mu)\, (N+2)/(N+4)}. \tag{54}$$

with $\theta = (N+4-q(N+2))/2$, $\mu = N+4-r(N+3)$.
We present a last consequence of the above inequalities, useful for the a-priori estimates of Lemma 2.4.
By interpolation between $\|\rho_\epsilon\|_{L^1}$ and $\|\rho_\epsilon\|_{L^q}$ of Lemma 5.4,

$$\|\rho_\epsilon\|_{L^m(\mathbb{R}^3)} \quad \leq \quad c(N, \mathrm{Tr}(R_\epsilon))\ (E^{kin}(R_\epsilon))^{(1-\theta)(1-\alpha)}\ \epsilon^{-2(1-\theta)(1-\alpha)},$$

with $\theta = (m-q)/1-q$. Therefore, for $N=3,\ m=2,\ q=3$, we have

$$\|\rho_\epsilon\|_{L^2(\mathbb{R}^3)} \leq c(\mathrm{Tr}(R_\epsilon))\ (E^{kin}(R_\epsilon))^{1/2}\epsilon^{-1}. \tag{55}$$

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