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A SIXTH-ORDER NONLINEAR PARABOLIC EQUATION FOR QUANTUM SYSTEMS*

ANSGAR JÜNGEL[†] AND JOSIPA-PINA MILIŠIĆ[‡]

Abstract. The global-in-time existence of weak nonnegative solutions to a sixth-order nonlinear parabolic equation in one space dimension with periodic boundary conditions is proved. The equation arises from an approximation of the quantum drift-diffusion model for semiconductors and describes the evolution of the electron density in the semiconductor crystal. The existence result is based on two techniques. First, the equation is reformulated in terms of exponential and power variables, which allows for the proof of nonnegativity of solutions. The existence of solutions to an approximate equation is shown by fixed-point arguments. Second, a priori bounds uniformly in the approximation parameters are derived from the algorithmic entropy construction method which translates systematic integration by parts into polynomial decision problems. The a priori estimates are employed to show the exponential time decay of the solution to the constant steady state in the L^1 norm with an explicit decay rate. Furthermore, some numerical examples are presented.

Key words. Sixth-order diffusive equation, global existence of periodic solutions, algorithmic entropy construction, entropy dissipation, quantum semiconductors, long-time behavior of the solutions.

AMS subject classifications. 35K30, 35K55, 35Q40, 35B40.

1. Introduction. We prove the global-in-time existence of weak nonnegative solutions to the following initial-value problem on the one-dimensional torus \mathbb{T} with unit length:

$$n_t - \left[n \left(\frac{1}{n} (n(\log n)_{xx})_{xx} + \frac{1}{2} ((\log n)_{xx})^2 \right)_x \right]_x = 0, \quad x \in \mathbb{T}, \quad t > 0, \quad (1.1)$$

$$n(x, 0) = n_0(x), \quad x \in \mathbb{T}. \quad (1.2)$$

Equation (1.1) originates from a generalized quantum drift-diffusion model for semiconductors, derived from a Wigner-BGK model using a moment method [9]. It is obtained from the $O(\hbar^6)$ approximation of the quantum drift-diffusion model, where $\hbar = h/2\pi$ is the reduced Planck constant; see the appendix of [14]. The variable $n(x, t)$ represents the electron density in the semiconductor crystal and should be a nonnegative quantity. Expanding the derivatives, the above equation can be written as

$$n_t = (n_{xxxxx} - 3n^{-1}n_{xxxx}n_x - 5n^{-1}n_{xxx}n_{xx} + 8n^{-2}n_{xxx}n_x^2 - 18n^{-3}n_{xx}n_x^3 + 6n^{-4}n_x^5)_x,$$

which shows that (1.1) is of parabolic type.

Since about fifteen years, there is an increasing interest in the mathematical literature in higher-order nonlinear parabolic equations, due to their interesting mathematical structure and surprising properties. Probably the most famous example is

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the fourth-order thin-film equation

$$u_t + (u^\beta u_{xxx})_x = 0, \quad \beta > 0,$$

which describes the surface-tension-dominated motion of thin viscous films for the film height $u(x, t)$ and spreading droplets in the lubrication approximation [13] or, for $\beta = 1$, the thin neck of fluid in a Hele-Shaw cell [7]. This equation was first analyzed by Bernis and Friedman in [1]. An overview of some results can be found in [8]. Another interesting fourth-order equation arises in the context of spin systems [10]. Derrida et al. derived from a Toom model an equation which describes fluctuations of the interface between the regions of predominantly positive and negative particle spins, relative to the diagonal line:

$$n_t + (n(\log n)_{xx})_{xx} = 0, \quad (1.3)$$

where $n(x, t)$ is a suitable scaling limit of a random variable. Surprisingly, the same equation arises from the $O(\hbar^4)$ approximation of the quantum drift-diffusion model mentioned above. The so-called Derrida-Lebowitz-Speer-Spohn equation (1.3) was first analyzed in [2] for local positive smooth solutions and then in [17] for global nonnegative weak solutions. We refer to [16] for a review on known results for (1.3).

Sixth-order nonlinear parabolic equations appear in flow modeling, describing the spreading of thin viscous droplets under different driving forces:

$$u_t - (u^\beta u_{xxxxx})_x = 0, \quad \beta > 0.$$

The case $\beta = 3$, for instance, arises in a mathematical model of the oxidation of silicon in semiconductor devices [18] or for a moving boundary given by a beam of negligible mass on a surface of a thin film [23]. For a summary of mathematical results and more references we refer to [12]. Another sixth-order equation, with a higher-order p -Laplacian operator,

$$u_t - (u^\beta |u_{xxxxx}|^\gamma u_{xxxxx})_x = 0, \quad \beta, \gamma > 0,$$

is obtained for power-law fluids spreading on a horizontal substrate [19]. Notice that these sixth-order equations are of degenerate type, whereas (1.1) is uniformly parabolic. Other (linear and nonlinear) sixth-order parabolic equations appear in cellular detonation formation dynamics [21], higher-order phase field models [4], and surface diffusion processes in the presence of strong anisotropy and curvature-dependent energy [3].

Most of the above mentioned higher-order models are physically positivity preserving, i.e., the solutions are expected to be nonnegative if they do so initially. For such equations, there is generally no maximum principle which makes the mathematical analysis and in particular the proof of nonnegativity quite difficult. General mathematical theories for these equations do not exist, and several principle questions concerning global existence, uniqueness, and regularity of solutions, in full generality, remain open. The mathematical difficulties for (1.1) come from the logarithmic terms, which complicates the proof of local existence of nonnegative solutions, and from the strong nonlinearities, which make it difficult to derive a priori estimates for global existence results. To our knowledge, no rigorous results are known up to now for the sixth-order equation (1.1).

In order to overcome the above mentioned problems, we employ two techniques. The first technique is to write (1.1) in different variables, such as the exponential

variable $n = \exp y$ and the power variable $n = u^4$, and to solve (an approximation of) the resulting equation using fixed-point arguments. The advantage of both formulations is that for given (regular) y or u , the original function n becomes positive or nonnegative, respectively. More precisely, (1.1) is (for smooth positive solutions) equivalent to

$$n_t - (n(\log n)_{xxx})_{xxx} - 2(n((\log n)_{xx})^2)_{xx} = 0. \quad (1.4)$$

This formulation provides a symmetric structure in the sixth-order differential operator for the variable y , which is needed to solve the linearization in the fixed-point argument. Notice, however, that the fourth-order term may have a bad sign since the main part equals $-2(n_{xx}/n)n_{xxxx}$. This *exponential variable transformation* was also employed in [16, 17] for the fourth-order equation (1.3). It is used here for the first time for a sixth-order equation.

We will prove the existence of solutions to an *approximation* of (1.4), but we are not able to prove the existence of solutions in the formulation (1.4) since there are seemingly no a priori bounds for $y = \log n$. We need to work with another formulation which avoids quotients and logarithms, and thus, both formulations (1.1) and (1.4) are not appropriate. It turns out that the right variable is $u = \sqrt[4]{n}$, since (1.1) can be written (again for smooth positive solutions) as

$$\begin{aligned} n_t - 2\left(\sqrt{n}(\sqrt{n})_{xxx} - \frac{16}{3}\sqrt[4]{n}(\sqrt[4]{n})_x^3\right)_{xxx} + 12(\sqrt{n}(\sqrt[4]{n})_x^2)_{xxxx} \\ - 8\left((\sqrt{n})_{xx}^2 + \frac{16}{3}(\sqrt[4]{n})_x^4\right)_{xx} = 0. \end{aligned} \quad (1.5)$$

We have formulated the second- and third-order derivatives in terms of \sqrt{n} since the a priori estimate presented below provides a uniform H^3 bound for $\sqrt{n}(\cdot, t)$ and not for $\sqrt[4]{n}(\cdot, t)$. For $\sqrt[4]{n}(\cdot, t)$, only bounds for its first spatial derivative are available.

Our second technique is concerned with the derivation of a priori estimates needed in the existence analysis. From the physical application, we expect that the entropy functional

$$E(n) = \int_{\mathbb{T}} (n(\log n - 1) + 1) dx$$

is nonincreasing along the solutions to (1.1). The entropy dissipation $-dE/dt$, which is nonnegative, contains derivatives of the solution, thus providing gradient estimates. Such estimations can be obtained in principle by using $\log n$ as a test function in the weak formulation of (1.1) and by integrating by parts in a suitable way. For instance, we obtain from the formulation (1.4) for such a test function formally

$$\frac{dE}{dt} + \int_{\mathbb{T}} (n|(\log n)_{xxx}|^2 - 2n((\log n)_{xx})^3) dx = 0, \quad (1.6)$$

but it is not clear how to estimate the third-order term $((\log n)_{xx})^3$. The idea is to use systematic integration by parts, developed in [14] as the so-called *algorithmic entropy construction method*. In fact, it is shown in [14] that smooth positive solutions to (1.1) are satisfying the following estimate:

$$\frac{dE}{dt} + C \int_{\mathbb{T}} (|(\sqrt{n})_{xxx}|^2 + |(\sqrt[4]{n})_x|^6) dx \leq 0, \quad (1.7)$$

where $C > 0$ is some constant independent of n . In this paper, we make this estimate rigorous for weak nonnegative solutions and give an explicit value for the constant C . This so-called entropy–entropy dissipation inequality also allows us to prove the exponential decay of the solutions towards the steady state with an explicit rate involving C , valid for all $t > 0$.

The entropy estimate (1.7) provides an H^3 bound for $\sqrt{n}(\cdot, t)$ and a $W^{1,6}$ estimate for $\sqrt[6]{n}(\cdot, t)$ (and $\sqrt[4]{n}(\cdot, t)$). Hence, we can interpret the formulation (1.5) in the sense of $H^{-3}(\mathbb{T})$. However, the entropy inequality is not sufficient to conclude the existence of solutions. The reason is that we are working with approximate solutions $u_\varepsilon = \sqrt[4]{n}$ to (1.5) and we need to show that $|u_{\varepsilon,x}|^4$ converges to $|u_x|^4$ in some sense. However, the above estimate shows only the *weak* convergence $u_{\varepsilon,x} \rightharpoonup u_x$ in L^6 (up to a subsequence) which does *not* imply that $|u_{\varepsilon,x}|^4 \rightharpoonup |u_x|^4$ in L^1 . We overcome this problem by exploiting an additional entropy estimate. In fact, we prove, in addition to (1.7), the estimate

$$\frac{dE}{dt} + C \int_{\mathbb{T}} |u_{\varepsilon,x} u_{\varepsilon,xx}|^2 dx \leq 0,$$

thus providing a uniform bound for $|u_{\varepsilon,x}(\cdot, t)|^2$ in H^1 which is sufficient to conclude that $|u_{\varepsilon,x}|^4 \rightharpoonup |u_x|^4$ holds in some sense.

Our main results are as follows.

THEOREM 1.1 (Existence of solutions). *Let $T > 0$ and let n_0 be a nonnegative measurable function on \mathbb{T} with finite entropy $E(n_0) < \infty$. Then there exists a nonnegative weak solution n to (1.2) and (1.5) satisfying*

$$\begin{aligned} n &\in W^{1,5/4}(0, T; H^{-3}(\mathbb{T})), \quad \sqrt{n} \in L^2(0, T; H^3(\mathbb{T})), \\ \sqrt[4]{n} &\in L^6(0, T; W^{1,6}(\mathbb{T})), \quad |(\sqrt[4]{n})_x|^2 \in L^2(0, T; H^1(\mathbb{T})). \end{aligned}$$

Equation (1.5) and the initial condition (1.2) are satisfied in the sense of $L^1(0, T; H^{-3}(\mathbb{T}))$.

THEOREM 1.2 (Exponential time decay). *Let n_0 be a nonnegative measurable function on \mathbb{T} with finite entropy $E(n_0) < \infty$ and unit mass $\int_{\mathbb{T}} n_0 dx = 1$. Let n be the weak solution to (1.5) constructed in Theorem 1.1. Then, for all $t > 0$,*

$$E(n(\cdot, t)) \leq \exp\left(-\frac{1}{2}(2\pi)^6 \kappa_2 t\right) E(n_0),$$

where $\kappa_2 = 2.73573\dots$ is the unique solution to the third-order equation

$$18\kappa^3 - 199\kappa^2 + 784\kappa - 1024 = 0. \tag{1.8}$$

Clearly, if the solution to (1.5) is positive and smooth, it solves the original formulation (1.1).

By the Csiszár-Kullback inequality (see, e.g., [5]), we obtain the following decay in the L^1 norm:

$$\|n(\cdot, t) - 1\|_{L^1(\mathbb{T})} \leq \sqrt{2E(n(\cdot, t))} \leq \exp\left(-\frac{1}{4}(2\pi)^6 \kappa_2 t\right) \sqrt{2E(n_0)}.$$

It is shown in [11] that the entropy along the solutions to the fourth-order equation (1.3) decay exponentially fast to the mean value with rate $2(2\pi)^4$. This rate seems to be optimal in view of the numerical results of [6]. The decay rate of the entropy of

the sixth-order equation is larger than the rate for the fourth-order equation which is a consequence of the stronger diffusion.

The paper is organized as follows. In the following section, we prove two bounds from below for the integral in (1.6) providing some a priori estimates. The existence result is proved in Section 3 and the long-time behavior of the solutions is shown in Section 4. In section 5, we present some numerical examples.

2. Some auxiliary inequalities. In this section we show that the integral in (1.6) can be bounded from below by some gradient terms.

LEMMA 2.1. *Let $n \in H^3(\mathbb{T})$ satisfy $n > 0$ a.e. Then*

$$\int_{\mathbb{T}} (n(\log n)_{xxx}^2 - 2n(\log n)_{xx}^3) dx \geq \kappa_1 \int_{\mathbb{T}} ((\sqrt{n})_{xxx}^2 + (\sqrt[4]{n})_x^2 (\sqrt[4]{n})_{xx}^2 + (\sqrt[6]{n})_x^6) dx,$$

where $\kappa_1 = 2.72070\dots$ equals the smallest root of the fourth-order polynomial

$$\frac{16379}{16384}\kappa^4 - \frac{762195}{2048}\kappa^3 + \frac{4153083}{1024}\kappa^2 - \frac{254741}{16}\kappa + 20736 = 0.$$

Here and in the following, we employ the notation $u_x^p = (u_x)^p$, $u_{xx}^p = (u_{xx})^p$ etc. for functions u and numbers p to avoid the brackets. Notice that the inequality of the above lemma can be deduced from the results of [14] for *some* (non-explicit) constant $\kappa_1 > 0$. Lemma 2.1 provides such a constant, which may be useful to compute the explicit rate of decay to equilibrium of the solutions.

Proof. The proof is based on the algorithmic technique of [14]. The main idea of this method is to reformulate integration by parts as a decision problem for polynomials. We wish to find $\kappa > 0$ such that the integral

$$J = \int_{\mathbb{T}} (n(\log n)_{xxx}^2 - 2n(\log n)_{xx}^3 - \kappa(\sqrt{n})_{xxx}^2 - \kappa(\sqrt[4]{n})_x^2 (\sqrt[4]{n})_{xx}^2 - \kappa(\sqrt[6]{n})_x^6) dx$$

is nonnegative. To this end, we set $n = u^4$ and expand the derivatives:

$$\begin{aligned} J = 16 \int_{\mathbb{T}} u^4 & \left[\left(12 - \frac{4\kappa}{729} \right) \left(\frac{u_x}{u} \right)^6 - 36 \left(\frac{u_x}{u} \right)^4 \frac{u_{xx}}{u} + 4 \left(\frac{u_x}{u} \right)^3 \frac{u_{xxx}}{u} \right. \\ & + \left(33 - \frac{37\kappa}{16} \right) \left(\frac{u_x}{u} \right)^2 \left(\frac{u_{xx}}{u} \right)^2 - \left(6 + \frac{3\kappa}{2} \right) \frac{u_x}{u} \frac{u_{xx}}{u} \frac{u_{xxx}}{u} - 8 \left(\frac{u_{xx}}{u} \right)^3 \\ & \left. + \left(1 - \frac{\kappa}{4} \right) \left(\frac{u_{xxx}}{u} \right)^2 \right] dx. \end{aligned}$$

In order to show the nonnegativity of the right-hand side, we employ the following integration-by-parts formulas:

$$\begin{aligned} I_1 &= \int_{\mathbb{T}} \left(u^4 \left(\frac{u_x}{u} \right)^5 \right)_x dx = \int_{\mathbb{T}} u^4 \left(- \left(\frac{u_x}{u} \right)^6 + 5 \left(\frac{u_x}{u} \right)^4 \frac{u_{xx}}{u} \right) dx, \\ I_2 &= \int_{\mathbb{T}} \left(u^4 \left(\frac{u_x}{u} \right)^3 \frac{u_{xx}}{u} \right)_x dx = \int_{\mathbb{T}} u^4 \left(3 \left(\frac{u_x}{u} \right)^2 \left(\frac{u_{xx}}{u} \right)^2 + \left(\frac{u_x}{u} \right)^3 \frac{u_{xxx}}{u} \right) dx, \\ I_3 &= \int_{\mathbb{T}} \left(u^4 \frac{u_x}{u} \left(\frac{u_{xx}}{u} \right)^2 \right)_x dx = \int_{\mathbb{T}} u^4 \left(\left(\frac{u_x}{u} \right)^2 \left(\frac{u_{xx}}{u} \right)^2 + \left(\frac{u_{xx}}{u} \right)^3 + 2 \frac{u_x}{u} \frac{u_{xx}}{u} \frac{u_{xxx}}{u} \right) dx. \end{aligned}$$

Clearly, in view of the periodic boundary conditions, $I_1 = I_2 = I_3 = 0$. There are more integration-by-parts formulas, but they involve derivatives of fourth and fifth order and are therefore not useful for the following analysis.

The idea of systematic integration by parts is to formulate the integrands of the above integrals as polynomials by translating the derivatives to polynomial variables:

$$\xi_1 = \frac{u_x}{u}, \quad \xi_2 = \frac{u_{xx}}{u}, \quad \xi_3 = \frac{u_{xxx}}{u}.$$

Then the integrand of J , up to the factor u^4 , translates to

$$\begin{aligned} S_0(\xi) &= \left(12 - \frac{4\kappa}{729}\right)\xi_1^6 - 36\xi_1^4\xi_2 + 4\xi_1^3\xi_3 + \left(33 - \frac{37\kappa}{16}\right)\xi_1^2\xi_2^2 - \left(6 + \frac{3\kappa}{2}\right)\xi_1\xi_2\xi_3 \\ &\quad - 8\xi_2^3 + \left(1 - \frac{\kappa}{4}\right)\xi_3^2, \end{aligned}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$, and the integrands of I_j become

$$T_1(\xi) = -\xi_1^6 + 5\xi_1^4\xi_2, \quad T_2(\xi) = 3\xi_1^2\xi_2^2 + \xi_1^3\xi_3, \quad T_3(\xi) = \xi_1^2\xi_2^2 + \xi_2^3 + 2\xi_1\xi_2\xi_3.$$

These functions are called *shift polynomials*. Hence, the integral J is nonnegative if there exist constants $c_1, c_2, c_3 \in \mathbb{R}$ such that for all $\xi \in \mathbb{R}^3$,

$$S(\xi) = (S_0 + c_1T_1 + c_2T_2 + c_3T_3)(\xi) \geq 0.$$

The polynomial S is indefinite in ξ_2 since the largest exponent is three. Therefore, we choose $c_3 = 8$ such that the coefficient of ξ_2^3 in S vanishes. Then

$$S(\xi) = a_1\xi_1^6 + a_2\xi_1^4\xi_2 + a_3\xi_1^3\xi_3 + a_4\xi_1^2\xi_2^2 + a_5\xi_1\xi_2\xi_3 + a_6\xi_3^2,$$

where

$$\begin{aligned} a_1 &= 12 - \frac{4\kappa}{729} - c_1, & a_2 &= -36 + 5c_1, & a_3 &= 4 + c_2, \\ a_4 &= 41 - \frac{37\kappa}{16} + 3c_2, & a_5 &= 10 - \frac{3\kappa}{2}, & a_6 &= 1 - \frac{\kappa}{4}. \end{aligned}$$

The sign of the polynomial S can be determined under the following conditions.

LEMMA 2.2 (Lemma 12 of [14]). *Let the real polynomial*

$$S(\xi) = a_1\xi_1^6 + a_2\xi_1^4\xi_2 + a_3\xi_1^3\xi_3 + a_4\xi_1^2\xi_2^2 + a_5\xi_1\xi_2\xi_3 + a_6\xi_3^2$$

for $\xi = (\xi_1, \xi_2, \xi_3)$ be given and let $a_6 > 0$. Then the statement $S(\xi) \geq 0$ for all $\xi \in \mathbb{R}^3$ is equivalent to

$$\begin{aligned} &\text{either (i) } 4a_4a_6 - a_5^2 = 2a_2a_6 - a_3a_5 = 0 \quad \text{and} \quad 4a_1a_6 - a_3^2 \geq 0, \\ &\text{or (ii) } 4a_4a_6 - a_5^2 > 0 \quad \text{and} \quad 4a_1a_4a_6 - a_1a_5^2 - a_2^2a_6 - a_3^2a_4 + a_2a_3a_5 \geq 0. \end{aligned}$$

We remark that the condition $a_6 > 0$ is equivalent to $\kappa < 4$. The equations $4a_4a_6 - a_5^2 = 0$ and $2a_2a_6 - a_3a_5 = 0$ are a linear system in the variables c_1 and c_2 ,

$$\begin{aligned} 3(4 - \kappa)c_2 + \frac{\kappa^2}{16} - \frac{81}{4}\kappa + 64 &= 0, \\ \left(10 - \frac{5}{2}\kappa\right)c_1 - \left(10 - \frac{3}{2}\kappa\right)c_2 + 24\kappa - 112 &= 0. \end{aligned}$$

Inserting the unique solution (c_1, c_2) of this system into the inequality of (i) yields the following condition on κ :

$$\frac{16379}{16384}\kappa^4 - \frac{762195}{2048}\kappa^3 + \frac{4153083}{1024}\kappa^2 - \frac{254741}{16}\kappa + 20736 \geq 0.$$

This inequality is satisfied if and only if κ is smaller than the smaller root $\kappa_1^* = 2.72070\dots$ or larger than the larger root $\kappa_2^* = 361.16783\dots$. In view of the condition $0 < \kappa < 4$, we conclude that all $\kappa \leq \kappa_1^*$ are admissible. We notice that condition (ii) does not provide a better result. \square

The constant κ_1 in Lemma 2.1 can be slightly improved if we take into account the dissipation $\int_{\mathbb{T}}(\sqrt{n})^2_{xxx} dx$ only. This result will be used for the computation of the decay rate of Theorem 1.2.

LEMMA 2.3. *Let $n \in H^3(\mathbb{T})$ satisfy $n > 0$ a.e. Then*

$$\int_{\mathbb{T}} (n(\log n)^2_{xxx} - 2n(\log n)^3_{xx}) dx \geq \kappa_2 \int_{\mathbb{T}} (\sqrt{n})^2_{xxx} dx, \quad (2.1)$$

where κ_2 is defined in (1.8).

Proof. We wish to find $\kappa > 0$ such that

$$\begin{aligned} J &= \int_{\mathbb{T}} (n(\log n)^2_{xxx} - 2n(\log n)^3_{xx} - \kappa(\sqrt{n})^2_{xxx}) dx \\ &= 16 \int_{\mathbb{T}} u^4 \left[12 \left(\frac{u_x}{u} \right)^6 - 36 \left(\frac{u_x}{u} \right)^4 \frac{u_{xx}}{u} + 4 \left(\frac{u_x}{u} \right)^3 \frac{u_{xxx}}{u} \right. \\ &\quad \left. + \left(33 - \frac{9}{4}\kappa \right) \left(\frac{u_x}{u} \right)^2 \left(\frac{u_{xx}}{u} \right)^2 - \left(6 + \frac{3}{2}\kappa \right) \frac{u_x}{u} \frac{u_{xx}}{u} \frac{u_{xxx}}{u} - 8 \left(\frac{u_{xx}}{u} \right)^3 \right. \\ &\quad \left. + \left(1 - \frac{\kappa}{4} \right) \left(\frac{u_{xxx}}{u} \right)^2 \right] dx \geq 0. \end{aligned}$$

With the notations of the previous proof, we associate to the integrand (up to the factor u^4) the polynomial

$$\begin{aligned} S_0(\xi) &= 12\xi_1^6 - 36\xi_1^4\xi_2 + 4\xi_1^3\xi_3 + \left(33 - \frac{9}{4}\kappa \right) \xi_1^2\xi_2^2 - \left(6 + \frac{3}{2}\kappa \right) \xi_1\xi_2\xi_3 - 8\xi_2^3 \\ &\quad + \left(1 - \frac{\kappa}{4} \right) \xi_3^2. \end{aligned}$$

Employing the same shift polynomials T_i as in the previous proof, the integral J is nonnegative if there are constants $c_1, c_2, c_3 \in \mathbb{R}$ such that for all $\xi \in \mathbb{R}^3$,

$$S(\xi) = (S_0 + c_1T_1 + c_2T_2 + c_3T_3)(\xi) \geq 0.$$

We choose as in the previous proof $c_3 = 8$ in order to eliminate the term ξ_2^3 . This gives the polynomial

$$S(\xi) = a_1\xi_1^6 + a_2\xi_1^4\xi_2 + a_3\xi_1^3\xi_3 + a_4\xi_1^2\xi_2^2 + a_5\xi_1\xi_2\xi_3 + a_6\xi_3^2,$$

where

$$\begin{aligned} a_1 &= 12 - c_1, & a_2 &= -36 + 5c_1, & a_3 &= 4 + c_2, \\ a_4 &= 41 - \frac{9}{4}\kappa + 3c_2, & a_5 &= 10 - \frac{3}{2}\kappa, & a_6 &= 1 - \frac{\kappa}{4}. \end{aligned}$$

The two equations of condition (i) in Lemma 2.2 are linear equations for c_1 and c_2 ,

$$\begin{aligned} 3(4 - \kappa)c_2 - 20\kappa + 64 &= 0, \\ \left(10 - \frac{5}{2}\kappa\right)c_1 + \left(\frac{3}{2}\kappa - 10\right)c_2 + 24\kappa - 112 &= 0. \end{aligned}$$

Inserting the unique solution (c_1, c_2) into the inequality in condition (i) yields the following inequality for κ :

$$18\kappa^3 - 199\kappa^2 + 784\kappa - 1024 \leq 0.$$

All $\kappa \leq \kappa_2$, where κ_2 is the unique solution to (1.8), satisfy this inequality. We notice that condition (ii) gives the stronger restriction $\kappa < \kappa_2$. \square

3. Existence of solutions. This section is devoted to the proof of Theorem 1.1. The proof is divided in several steps.

3.1. Existence of a time-discrete problem. Let $T > 0$ and $\tau > 0$ be given. We wish to find, for given w , a solution $u \in H^3(\mathbb{T})$ to the semidiscrete equation

$$\begin{aligned} \frac{1}{\tau}(u^4 - w^4) &= 2\left(u^2(u^2)_{xxx} - \frac{16}{3}uu_x^3\right)_{xxx} - 12(u^2u_x^2)_{xxxx} \\ &\quad + 8\left((u^2)_{xx}^2 + \frac{16}{3}u_x^4\right)_{xx}. \end{aligned} \quad (3.1)$$

LEMMA 3.1. *Let w be a nonnegative measurable function satisfying $E(w^4) < \infty$ and $\|w^4\|_{L^1(\mathbb{T})} = 1$. Then there exists a solution $u \in W^{1,6}(\mathbb{T})$ to (3.1) satisfying $u^2 \in H^3(\mathbb{T})$ and $u_x^2 \in H^1(\mathbb{T})$. Furthermore, u^4 has unit mass and the physical entropy is dissipated in the sense*

$$E(u^4) + \tau C \int_{\mathbb{T}} \left((u^2)_{xxx}^2 + (u_x^2)_x^2 + u_x^6 \right) dx \leq E(w^4), \quad (3.2)$$

where $C > 0$ is a constant independent of τ .

Proof. In the proof we work simultaneously in the variables u and $y = \log(u^4)$.

Step 1: solution of a regularized problem. We define first a regularized problem. The solution to (3.1) is then obtained as the limit of the solutions to the regularized problem. We reformulate (3.1) by introducing the logarithmic variable $y = \log(u^4)$ as

$$\frac{1}{\tau}(u^4 - w^4) = (u^4 y_{xxx})_{xxx} + 32(u^2 u_{xx}^2 - 2u u_x^2 u_{xx} + u_x^4)_{xx} + \varepsilon(y_{xxxxxx} - y), \quad (3.3)$$

where $\varepsilon > 0$ is a regularization parameter. The first term on the right-hand side comes from the formulation (1.4). The second term equals

$$32(u^2 u_{xx}^2 - 2u u_x^2 u_{xx} + u_x^4)_{xx} = 2(u^4 (\log(u^4))_{xx}^2)_{xx}.$$

The third term guarantees coercivity of the right-hand side with respect to y .

In order to solve (3.3), we employ the Leray-Schauder fixed-point theorem (see, e.g., [24, Theorem B.5]). Let $\sigma \in [0, 1]$ and $v \in W^{2,4}(\mathbb{T})$. We define for $y, z \in H^3(\mathbb{T})$,

$$\begin{aligned} a(y, z) &= \int_{\mathbb{T}} (\sigma v^4 y_{xxx} z_{xxx} + \varepsilon y_{xxx} z_{xxx} + \varepsilon y z) dx, \\ f(z) &= -\frac{\sigma}{\tau} \int_{\mathbb{T}} (v^4 - w^4) z dx + 32\sigma \int_{\mathbb{T}} (v^2 v_{xx}^2 - 2v v_x^2 v_{xx} + v_x^4) z_{xx} dx. \end{aligned}$$

The bilinear form a is continuous and coercive since

$$a(y, y) \geq \varepsilon \int_{\mathbb{T}} (y_{xxx}^2 + y^2) dx \geq C\varepsilon \|y\|_{H^3(\mathbb{T})}^2,$$

where $C > 0$ is some constant. Moreover, f is continuous on $H^3(\mathbb{T})$ since $w^4 \in L^1(\mathbb{T}) \hookrightarrow H^{-1}(\mathbb{T})$ and $v \in W^{2,4}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ implies that $v^2 v_{xx}^2 - 2v v_x^2 v_{xx} + v_x^4 \in L^2(\mathbb{T})$. Therefore, the Lax-Milgram lemma provides the existence of a unique solution $y \in H^3(\mathbb{T})$ to the linear problem

$$a(y, z) = f(z) \quad \text{for all } z \in H^3(\mathbb{T}).$$

This defines the fixed-point operator $S : W^{2,4}(\mathbb{T}) \times [0, 1] \rightarrow W^{2,4}(\mathbb{T})$, $S(v, \sigma) = u = e^{y/4}$. Notice that in one space dimension, $H^3(\mathbb{T})$ embeds continuously into $W^{2,4}(\mathbb{T})$ and thus, S is well-defined. It holds $S(v, 0) = 1$. By standard results for elliptic equations, S is continuous and compact since the embedding $H^3(\mathbb{T}) \hookrightarrow W^{2,4}(\mathbb{T})$ is compact. It remains to prove a uniform bound for all fixed points of $S(\cdot, \sigma)$.

Let $u \in H^3(\mathbb{T})$ be a fixed point of $S(\cdot, \sigma)$ for some $\sigma \in [0, 1]$. Then, by construction, $u = e^{y/4} > 0$ is a solution to (3.3). The convexity of $\phi(s) = s(\log s - 1) + 1$ implies that

$$\begin{aligned} \frac{\sigma}{\tau} (E(u^4) - E(w^4)) &= \frac{\sigma}{\tau} \int_{\mathbb{T}} (\phi(u^4) - \phi(w^4)) dx \\ &\leq \frac{\sigma}{\tau} \int_{\mathbb{T}} \phi'(u^4)(u^4 - w^4) dx = \frac{\sigma}{\tau} \int_{\mathbb{T}} \log(u^4)(u^4 - w^4) dx \\ &= \sigma \int_{\mathbb{T}} (-u^4 y_{xxx}^2 + 32(u^2 u_{xx}^2 - 2u u_x^2 u_{xx} + u_x^4) y_{xx}) dx - \varepsilon \int_{\mathbb{T}} (y_{xxx}^2 + y^2) dx \\ &= -\sigma \int_{\mathbb{T}} (u^4 (\log(u^4))_{xxx}^2 - 2u^4 (\log(u^4))_{xx}^3) dx - \varepsilon \int_{\mathbb{T}} (y_{xxx}^2 + y^2) dx. \end{aligned}$$

Lemma 2.1 shows that

$$\int_{\mathbb{T}} (u^4 (\log u)_{xxx}^2 - 2u^4 (\log(u^4))_{xx}^3) dx \geq \kappa_1 \int_{\mathbb{T}} ((u^2)_{xxx}^2 + u_x^2 u_{xx}^2 + (u^{2/3})_x^6) dx,$$

giving the estimate

$$\begin{aligned} E(u^4) + \tau \kappa_1 \int_{\mathbb{T}} ((u^2)_{xxx}^2 + u_x^2 u_{xx}^2 + (u^{2/3})_x^6) dx \\ + \tau \varepsilon \int_{\mathbb{T}} (y_{xxx}^2 + y^2) dx \leq E(w^4). \end{aligned} \quad (3.4)$$

Thus, y and y_{xxx} are bounded in $L^2(\mathbb{T})$ (with a bound which depends on ε). By the Gagliardo-Nirenberg inequality, we infer a bound for y in $H^3(\mathbb{T})$. This shows that $u = e^{y/4}$ is bounded in $H^3(\mathbb{T})$ and hence, also in $W^{2,4}(\mathbb{T})$. This proves the desired bound for all fixed points of $S(\cdot, \sigma)$.

The Leray-Schauder theorem then provides a solution u to $S(u, 1) = u$, which we denote by u_ε . By construction, we can write $u_\varepsilon = \exp(y_\varepsilon/4) > 0$ for some $y_\varepsilon \in H^3(\mathbb{T})$. This fixed point is a periodic solution to the following equation:

$$\begin{aligned} \frac{1}{\tau} (u_\varepsilon^4 - w^4) &= (u_\varepsilon^4 (\log u_\varepsilon^4)_{xxx})_{xxx} + 32(u_\varepsilon^2 u_{\varepsilon,xx}^2 - 2u_\varepsilon u_{\varepsilon,x}^2 u_{\varepsilon,xx} + u_{\varepsilon,x}^4)_{xx} \\ &\quad + \varepsilon (y_{\varepsilon,xxxxxx} - y_\varepsilon), \end{aligned}$$

which can be written equivalently as

$$\begin{aligned} \frac{1}{\tau}(u_\varepsilon^4 - w^4) &= 2\left(u_\varepsilon^2(u_\varepsilon^2)_{xxx} - \frac{16}{3}u_\varepsilon u_{\varepsilon,x}^3\right)_{xxx} - 12(u_\varepsilon^2 u_{\varepsilon,x}^2)_{xxxx} \\ &\quad + 8\left((u_\varepsilon^2)_{xx}^2 + \frac{16}{3}u_{\varepsilon,x}^4\right)_{xx} + \varepsilon(y_{\varepsilon,xxxxxx} - y_\varepsilon). \end{aligned} \quad (3.5)$$

Step 2: limit $\varepsilon \rightarrow 0$. We have to show that, as $\varepsilon \rightarrow 0$, u_ε converges to some function u , which solves (3.1). The estimate (3.4) shows that the sequence $(u_\varepsilon^{2/3})$ is bounded in $W^{1,6}(\mathbb{T})$ and that (u_ε^2) is bounded in $H^3(\mathbb{T})$. Since $W^{1,6}(\mathbb{T})$ embeds continuously into $L^\infty(\mathbb{T})$, the first bound implies that also (u_ε) is bounded in $W^{1,6}(\mathbb{T})$. Thus, for a subsequence which is not relabeled, as $\varepsilon \rightarrow 0$,

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } W^{1,6}(\mathbb{T}), \quad u_\varepsilon^2 \rightharpoonup g \quad \text{weakly in } H^3(\mathbb{T}).$$

Furthermore, since $W^{1,6}(\mathbb{T})$ embeds compactly into $L^\infty(\mathbb{T})$, again for a subsequence, $u_\varepsilon \rightarrow u$ strongly in $L^\infty(\mathbb{T})$. Hence, $g = u^2$. Moreover, $u_\varepsilon^2 \rightarrow u^2$ strongly in $W^{2,\infty}(\mathbb{T})$. These results show that

$$u_\varepsilon^2(u_\varepsilon^2)_{xxx} \rightharpoonup u^2(u^2)_{xxx} \quad \text{weakly in } L^2(\mathbb{T}), \quad (3.6)$$

$$(u_\varepsilon^2)_{xx}^2 \rightarrow (u^2)_{xx}^2 \quad \text{strongly in } L^\infty(\mathbb{T}). \quad (3.7)$$

Estimate (3.4) shows that $u_{\varepsilon,x}^2$ is bounded in $H^1(\mathbb{T})$. Therefore, for a subsequence, $u_{\varepsilon,x}^2 \rightharpoonup z$ weakly in $H^1(\mathbb{T})$ and strongly in $L^\infty(\mathbb{T})$ for some function $z \in H^1(\mathbb{T})$. The limit function z can be identified with u_x^2 using the Browder-Minty idea. Indeed, for any smooth v , we can pass to the limit $\varepsilon \rightarrow 0$ in

$$0 \leq \int_{\mathbb{T}} (u_{\varepsilon,x}^2 - v_x^2)(u_{\varepsilon,x} - v_x) dx,$$

leading to

$$0 \leq \int_{\mathbb{T}} (z - v_x^2)(u_x - v_x) dx.$$

Choosing $v_x = u_x + \delta z$ for some $\delta \in \mathbb{R}$ and $z \in L^2(\mathbb{T})$ and letting $\delta \rightarrow 0$ for $\delta > 0$ and then for $\delta < 0$ gives

$$0 = \int_{\mathbb{T}} (z - u_x^2)z dx \quad \text{for all } z.$$

Thus, $z = u_x^2$ and

$$u_{\varepsilon,x}^2 \rightarrow u_x^2 \quad \text{and} \quad u_{\varepsilon,x}^4 \rightarrow u_x^4 \quad \text{strongly in } L^\infty(\mathbb{T}).$$

This convergence result implies immediately that

$$u_{\varepsilon,x}^3 \rightarrow u_x^3 \quad \text{strongly in } L^\infty(\mathbb{T}), \quad (3.8)$$

$$u_\varepsilon u_{\varepsilon,x}^3 \rightarrow u u_x^3 \quad \text{strongly in } L^\infty(\mathbb{T}). \quad (3.9)$$

The uniform bound of $(u_{\varepsilon,x}^2)$ in $H^1(\mathbb{T})$ implies that, for a subsequence, $(u_{\varepsilon,x}^2)_x \rightharpoonup (u_x^2)_x$ weakly in $L^2(\mathbb{T})$. Therefore, since u_ε^2 and $(u_\varepsilon^2)_x$ converge strongly in $L^\infty(\mathbb{T})$,

$$(u_\varepsilon^2 u_{\varepsilon,x}^2)_x = u_\varepsilon^2 (u_{\varepsilon,x}^2)_x + (u_\varepsilon^2)_x u_{\varepsilon,x}^2 \rightharpoonup (u^2 u_x^2)_x \quad \text{weakly in } L^2(\mathbb{T}). \quad (3.10)$$

The entropy estimate (3.4) shows that $\sqrt{\varepsilon}\|y_\varepsilon\|_{H^3(\mathbb{T})}$ is uniformly bounded. Hence, for a subsequence,

$$\varepsilon y_\varepsilon \rightarrow 0 \quad \text{strongly in } H^3(\mathbb{T}). \quad (3.11)$$

The convergence results (3.6), (3.7), and (3.9)-(3.11) allow us to perform the limit $\varepsilon \rightarrow 0$ in (3.5), which holds in the sense of $H^{-3}(\mathbb{T})$. Therefore, u is a nonnegative weak solution to (3.1).

Step 3: end of the proof. We need to verify that u^4 has unit mass and that (3.2) holds. Conservation of mass follows from the weak formulation of (3.1) by using $z = 1$ as a test function. It follows from (3.4) that u_ε satisfies, for some constant $C > 0$,

$$E(u_\varepsilon^4) + \tau C \int_{\mathbb{T}} ((u_\varepsilon^2)_{xxx}^2 + u_{\varepsilon,x}^6 + (u_{\varepsilon,x}^2)_x^2) dx \leq E(w^4).$$

Since $(u_\varepsilon^2)_{xxx}$, $u_{\varepsilon,x}^3$, and $(u_{\varepsilon,x}^2)_x$ converge weakly in $L^2(\mathbb{T})$, we conclude that

$$\begin{aligned} E(u^4) + \tau C \int_{\mathbb{T}} ((u^2)_{xxx}^2 + u_x^6 + (u_x^2)_x^2) dx \\ \leq E(u^4) + \tau C \liminf_{\varepsilon \rightarrow 0} \int_{\mathbb{T}} ((u_\varepsilon^2)_{xxx}^2 + u_{\varepsilon,x}^6 + (u_{\varepsilon,x}^2)_x^2) dx \leq E(w^4). \end{aligned}$$

This finishes the proof. \square

REMARK 3.2 (Regularity). Lemma 3.1 shows that the square root of the particle density satisfies the regularity $\sqrt{n} = u^2 \in H^3(\mathbb{T})$. Actually, we can show a slightly improved result. We claim that $n = u^4 \in H^4(\mathbb{T})$. In order to see this, we observe that (3.1) can be rewritten, in the sense of distributions, as

$$\frac{1}{\tau}(u^4 - w^4) = (u^4)_{xxxxx} - 4[3(u^2)_x(u^2)_{xxx} - 4(u^2)_{xx}u_x^2 + (u^2)_{xx}^2 + 4(u^2)_x(u_x^2)_x]_{xx}.$$

Since the terms inside of the bracket $[\dots]$ are $L^2(\mathbb{T})$ functions, we infer that $(u^4)_{xxxxx} \in H^{-2}(\mathbb{T})$ and therefore $u^4 \in H^4(\mathbb{T})$.

It is known that whenever a function u^2 has bounded second-order derivatives, then u has bounded first-order derivatives [20] and

$$\|u_x\|_{L^\infty(\mathbb{T})} \leq c\|(u^2)_{xx}\|_{L^\infty(\mathbb{T})}$$

for some constant $c > 0$. Now, since $u^2 \in H^3(\mathbb{T}) \hookrightarrow W^{2,\infty}(\mathbb{T})$, we obtain by the above result that $\sqrt[4]{n} = u \in W^{1,\infty}(\mathbb{T})$.

3.2. A priori estimates. Let $T > 0$ and $\tau > 0$ be given such that $T/\tau \in \mathbb{N}$. We define the step function $u^{(\tau)} : [0, T] \rightarrow L^2(\mathbb{T})$ recursively as follows. Let $u_0 = \sqrt[4]{n_0}$ and for given $k \in \mathbb{N}$, let $u_k \in H^3(\mathbb{T})$ be a nonnegative solution to (3.1) with $w = u_{k-1}$. Now define $u^{(\tau)}(t) = u_k$ for $(k-1)\tau < t \leq k\tau$. Then $u^{(\tau)}$ satisfies the equation

$$\begin{aligned} \frac{1}{\tau}((u^{(\tau)})^4 - (\sigma_\tau u^{(\tau)})^4) &= 2\left((u^{(\tau)})^2((u^{(\tau)})^2)_{xxx} - \frac{16}{3}u^{(\tau)}(u^{(\tau)})_x^3\right)_{xxx} \\ &\quad - 12((u^{(\tau)})^2(u^{(\tau)})_x^2)_{xxxx} + 8\left((u^{(\tau)})^2\right)_{xx}^2 + \frac{16}{3}(u^{(\tau)})_x^4\bigg|_{xx}, \quad (3.12) \end{aligned}$$

where σ_τ denotes the shift operator $(\sigma_\tau u^{(\tau)})(t) = u^{(\tau)}(t - \tau)$ for $\tau \leq t < T$. By (3.2) and taking into account the elementary inequality $x^4(\log x^4 - 1) \geq x - e$ for all $x \geq 0$,

we obtain the following a priori estimates:

$$\|u^{(\tau)}\|_{L^\infty(0,T;L^4(\mathbb{T}))} + \|(u^{(\tau)})^2\|_{L^2(0,T;H^3(\mathbb{T}))} \leq C, \quad (3.13)$$

$$\|(u^{(\tau)})^2_x\|_{L^2(0,T;H^1(\mathbb{T}))} + \|u^{(\tau)}\|_{L^6(0,T;W^{1,6}(\mathbb{T}))} \leq C, \quad (3.14)$$

where $C > 0$ denotes here and in the following a generic constant not depending on τ . For the limit $\tau \rightarrow 0$, we need more estimates.

LEMMA 3.3. *The function $u^{(\tau)}$ satisfies*

$$\tau^{-1}\|(u^{(\tau)})^4 - (\sigma_\tau u^{(\tau)})^4\|_{L^{5/4}(0,T;H^{-3}(\mathbb{T}))} \leq C. \quad (3.15)$$

Proof. We have to show that the right-hand side of (3.12) is uniformly bounded in $L^{5/4}(0,T;H^{-3}(\mathbb{T}))$. To simplify the notation, we write $u = u^{(\tau)}$. First, we remark that by the Gagliardo-Nirenberg inequality, it holds for $\theta = 1/6$,

$$\begin{aligned} \|u\|_{L^{24}(0,T;L^\infty(\mathbb{T}))}^{24} &\leq C \int_0^T \|u(\cdot, t)^2\|_{H^3(\mathbb{T})}^{12\theta} \|u(\cdot, t)^2\|_{L^2(\mathbb{T})}^{12(1-\theta)} dt \\ &\leq C \|u\|_{L^\infty(0,T;L^4(\mathbb{T}))}^{24(1-\theta)} \int_0^T \|u^2(\cdot, t)\|_{H^3(\mathbb{T})}^{12\theta} dt \leq C. \end{aligned} \quad (3.16)$$

Now we estimate the right-hand side of (3.12) term by term. By the Hölder inequality with respect to t , applied for $p = 8/3$ and $p' = 8/5$, we obtain

$$\begin{aligned} \|u^2(u^2)_{xxx}\|_{L^{5/4}(0,T;L^2(\mathbb{T}))}^{5/4} &\leq \int_0^T \|u(\cdot, t)\|_{L^\infty(\mathbb{T})}^{5/2} \|u(\cdot, t)^2\|_{H^3(\mathbb{T})}^{5/4} dt \\ &\leq \|u\|_{L^{20/3}(0,T;L^\infty(\mathbb{T}))}^{5/2} \|u^2\|_{L^2(0,T;H^3(\mathbb{T}))}^{5/4} \leq C, \end{aligned} \quad (3.17)$$

taking into account the estimates (3.13) and (3.16). Applying the Hölder inequality with $p = 24/9$ and $p' = 24/15$ yields

$$\begin{aligned} \|uu_x^3\|_{L^{5/4}(0,T;L^2(\mathbb{T}))}^{5/4} &\leq \int_0^T \|u(\cdot, t)\|_{L^\infty(\mathbb{T})}^{5/4} \|u(\cdot, t)\|_{W^{1,6}(\mathbb{T})}^{15/4} dt \\ &\leq \|u\|_{L^{10/3}(0,T;L^\infty(\mathbb{T}))}^{5/4} \|u\|_{L^6(0,T;W^{1,6}(\mathbb{T}))}^{15/4} \leq C, \end{aligned} \quad (3.18)$$

in view of (3.14) and (3.16). The estimates (3.13), (3.14), and (3.16) show that

$$\begin{aligned} \|(u^2 u_x^2)_x\|_{L^{5/4}(0,T;L^2(\mathbb{T}))} &= \|uu_x(u^2)_{xx}\|_{L^{5/4}(0,T;L^2(\mathbb{T}))} \\ &\leq \|u\|_{L^{15/2}(0,T;L^\infty(\mathbb{T}))} \|u_x\|_{L^6(0,T;L^6(\mathbb{T}))} \|(u^2)_{xx}\|_{L^2(0,T;L^2(\mathbb{T}))} \leq C. \end{aligned} \quad (3.19)$$

Next, we apply the Gagliardo-Nirenberg inequality with $\theta = 7/10$ and then the Hölder inequality with $p = 8/7$ and $p' = 8$:

$$\begin{aligned} \|(u^2)_{xx}^2\|_{L^{5/4}(0,T;L^2(\mathbb{T}))}^{5/4} &= \|(u^2)_{xx}\|_{L^{5/2}(0,T;L^4(\mathbb{T}))}^{5/2} \\ &\leq C \int_0^T \|u(\cdot, t)^2\|_{H^3(\mathbb{T})}^{5\theta/2} \|u(\cdot, t)^2\|_{L^\infty(\mathbb{T})}^{5(1-\theta)/2} dt \\ &= C \int_0^T \|u(\cdot, t)^2\|_{H^3(\mathbb{T})}^{7/4} \|u(\cdot, t)\|_{L^\infty(\mathbb{T})}^{3/2} dt \\ &\leq C \|u^2\|_{L^2(0,T;H^3(\mathbb{T}))}^{7/4} \|u\|_{L^{12}(0,T;L^\infty(\mathbb{T}))}^{3/2} \leq C, \end{aligned} \quad (3.20)$$

by (3.13) and (3.16). Finally, we conclude from (3.14) that

$$\|u_x^4\|_{L^{5/4}(0,T;L^1(\mathbb{T}))} = \|u_x\|_{L^5(0,T;L^4(\mathbb{T}))}^4 \leq \|u\|_{L^5(0,T;W^{1,4}(\mathbb{T}))}^4 \leq C. \quad (3.21)$$

Since $L^1(\mathbb{T})$ embeds continuously into $H^{-1}(\mathbb{T})$, the above estimates show that

$$\begin{aligned} \tau^{-1}\|u - \sigma_\tau u\|_{L^{5/4}(0,T;H^{-3}(\mathbb{T}))} &\leq 2\left\|u^2(u^2)_{xxx} - \frac{16}{3}uu_x^3\right\|_{L^{5/4}(0,T;L^2(\mathbb{T}))} \\ &\quad + 12\left\|(u^2u_x^2)_x\right\|_{L^{5/4}(0,T;L^2(\mathbb{T}))} + 8\left\|(u^2)_{xx}^2 + \frac{16}{3}u_x^4\right\|_{L^{5/4}(0,T;H^{-1}(\mathbb{T}))} \end{aligned}$$

is uniformly bounded. \square

LEMMA 3.4. *The function $u^{(\tau)}$ satisfies*

$$\|(u^{(\tau)})^4\|_{L^{5/4}(0,T;H^3(\mathbb{T}))} \leq C.$$

Proof. We set again $u = u^{(\tau)}$ and observe that

$$(u^4)_{xxx} = 24uu_x^3 + 12u^2(u_x^2)_x + 2u^2(u^2)_{xxx}.$$

The first and last terms are uniformly bounded in $L^{5/4}(0,T;L^2(\mathbb{T}))$, taking into account (3.17) and (3.18). Furthermore, the Hölder inequality with $p = 8/3$ and $p' = 8/5$ and the bounds (3.14) and (3.16) imply that

$$\begin{aligned} \|u^2(u_x^2)_x\|_{L^{5/4}(0,T;L^2(\mathbb{T}))} &\leq \int_0^T \|u(\cdot, t)\|_{L^\infty(\mathbb{T})}^{5/2} \|u_x(\cdot, t)\|_{H^1(\mathbb{T})}^{5/4} dt \\ &\leq \|u\|_{L^{20/3}(0,T;L^\infty(\mathbb{T}))}^{5/2} \|u_x\|_{L^2(0,T;H^1(\mathbb{T}))}^{5/4} \leq C. \end{aligned}$$

Hence, $(u^4)_{xxx}$ is uniformly bounded in $L^{5/4}(0,T;L^2(\mathbb{T}))$. Furthermore, u^4 is uniformly bounded in $L^{5/4}(0,T;L^2(\mathbb{T}))$. This gives the desired estimate. \square

3.3. The limit $\tau \rightarrow 0$. The a priori estimates of the previous subsection are sufficient to pass to the limit $\tau \rightarrow 0$. In fact, by Lemmas 3.3 and 3.4, we can apply the Aubin lemma (Theorem 5 in [22]) to obtain the existence of a subsequence of $((u^{(\tau)})^4)$, which is not relabeled, such that $(u^{(\tau)})^4 \rightarrow n$ strongly in $L^{5/4}(0,T;W^{2,p}(\mathbb{T}))$ for all $p \leq \infty$ as $\tau \rightarrow 0$ for some limit function n . Here, we have employed the compact embedding of $H^3(\mathbb{T})$ into $W^{2,p}(\mathbb{T})$ for all $p \leq \infty$. In particular, $(u^{(\tau)})$ converges pointwise a.e. Since $(u^{(\tau)})^4$ is nonnegative, also the limit n is nonnegative, and the function $u = \sqrt[4]{n}$ is well defined. Moreover, it holds

$$\tau^{-1}((u^{(\tau)})^4 - (\sigma_\tau u^{(\tau)})^4) \rightharpoonup (u^4)_t \quad \text{weakly in } L^{5/4}(0,T;H^{-3}(\mathbb{T})), \quad (3.22)$$

$$(u^{(\tau)})^2 \rightharpoonup u^2 \quad \text{weakly in } L^2(0,T;H^3(\mathbb{T})), \quad (3.23)$$

The strong convergence of $(u^{(\tau)})^4$ to u^4 in $L^{5/4}(0,T;L^\infty(\Omega))$ and the inequality $|a - b|^4 \leq |a^4 - b^4|$ for all $a, b > 0$ imply that

$$\begin{aligned} \|u^{(\tau)} - u\|_{L^5(0,T;L^\infty(\mathbb{T}))}^5 &\leq \int_0^T \|(u^{(\tau)})^4 - u^4\|_{L^\infty(\mathbb{T})}^{5/4} dt \\ &= \|(u^{(\tau)})^4 - u^4\|_{L^{5/4}(0,T;L^\infty(\mathbb{T}))}^{5/4} \rightarrow 0, \end{aligned}$$

which means that

$$u^{(\tau)} \rightarrow u \quad \text{strongly in } L^5(0, T; L^\infty(\mathbb{T})). \quad (3.24)$$

This result allows us to prove the strong convergence of $(u^{(\tau)})^2$ to u^2 in $L^2(0, T; H^2(\mathbb{T}))$. Indeed, the Gagliardo-Nirenberg inequality with $\theta = 3/5$ and then the Hölder inequality give

$$\begin{aligned} \|(u^{(\tau)})^2 - u^2\|_{L^2(0, T; H^2(\mathbb{T}))}^2 &\leq C \int_0^T \|(u^{(\tau)})^2 - u^2\|_{H^3(\mathbb{T})}^{2\theta} \|(u^{(\tau)})^2 - u^2\|_{L^\infty(\mathbb{T})}^{2(1-\theta)} dt \\ &\leq C \|(u^{(\tau)})^2 - u^2\|_{L^2(0, T; H^3(\mathbb{T}))}^{2\theta} \|(u^{(\tau)})^2 - u^2\|_{L^2(0, T; L^\infty(\mathbb{T}))}^{2(1-\theta)}. \end{aligned}$$

By (3.13), first factor is bounded, whereas the second factor converges to zero, by the above result. Thus,

$$(u^{(\tau)})^2 \rightarrow u^2 \quad \text{strongly in } L^2(0, T; H^2(\mathbb{T})). \quad (3.25)$$

As a consequence, we infer that

$$\begin{aligned} (u^{(\tau)})^2 (u_x^{(\tau)})^2 &= \frac{1}{4} [(u^{(\tau)})^2]_x^2 \rightarrow \frac{1}{4} (u^2)_x^2 = u^2 u_x^2 \quad \text{strongly in } L^1(0, T; L^1(\mathbb{T})), \\ [(u^{(\tau)})^2]_{xx}^2 &\rightarrow (u^2)_{xx}^2 \quad \text{strongly in } L^1(0, T; L^1(\mathbb{T})). \end{aligned}$$

Taking into account the uniform bounds (3.19) and (3.20), it follows that

$$\begin{aligned} (u^{(\tau)})^2 (u_x^{(\tau)})^2 &\rightharpoonup u^2 u_x^2 \quad \text{weakly in } L^{5/4}(0, T; L^2(\mathbb{T})), \\ [(u^{(\tau)})^2]_{xx}^2 &\rightharpoonup (u^2)_{xx}^2 \quad \text{weakly in } L^{5/4}(0, T; L^2(\mathbb{T})). \end{aligned} \quad (3.26)$$

In view of the estimate (3.19), we obtain further

$$((u^{(\tau)})^2 (u_x^{(\tau)})^2)_x \rightharpoonup (u^2 u_x^2)_x \quad \text{weakly in } L^{5/4}(0, T; L^2(\mathbb{T})). \quad (3.27)$$

The strong convergence of $u^{(\tau)}$ (see (3.24)) and the weak convergence of the sequence $[(u^{(\tau)})^2]_{xxx}$ (see (3.23)) imply that $(u^{(\tau)})^2 [(u^{(\tau)})^2]_{xxx}$ converges weakly to $u^2 (u^2)_{xxx}$ in $L^{10/7}(0, T; L^2(\mathbb{T}))$. In particular,

$$(u^{(\tau)})^2 [(u^{(\tau)})^2]_{xxx} \rightharpoonup u^2 (u^2)_{xxx} \quad \text{weakly in } L^{5/4}(0, T; L^2(\mathbb{T})). \quad (3.28)$$

Since $u_x^{(\tau)}$ converges to u_x weakly in $L^6(0, T; L^6(\mathbb{T}))$ and $[(u^{(\tau)})^2]_x$ converges to $(u^2)_x$ strongly in $L^2(0, T; L^\infty(\mathbb{T}))$, the following convergence holds:

$$u^{(\tau)} (u_x^{(\tau)})^2 = \frac{1}{2} u_x^{(\tau)} [(u^{(\tau)})^2]_x \rightharpoonup \frac{1}{2} u_x (u^2)_x = u u_x^2 \quad \text{weakly in } L^{3/2}(0, T; L^6(\mathbb{T})).$$

This result and the weak convergence of $u_x^{(\tau)} [(u^{(\tau)})^2]_{xx}$ to $u_x (u^2)_{xx}$ in $L^{3/2}(0, T; L^{3/2}(\mathbb{T}))$ show that

$$(u^{(\tau)})_x^3 = u_x^{(\tau)} [(u^{(\tau)})^2]_{xx} - [u^{(\tau)} (u^{(\tau)})_x^2]_x$$

converges weakly to $u_x (u^2)_{xx} - (u u_x^2)_x = u_x^3$ in $L^{3/2}(0, T; H^{-1}(\mathbb{T}))$. In fact, this convergence holds even in $L^2(0, T; L^2(\mathbb{T}))$ since $(u_x^{(\tau)})$ is bounded in $L^6(0, T; L^6(\mathbb{T}))$:

$$(u^{(\tau)})_x^3 \rightharpoonup u_x^3 \quad \text{weakly in } L^2(0, T; L^2(\mathbb{T})). \quad (3.29)$$

As a consequence, we obtain from (3.24):

$$u^{(\tau)}(u^{(\tau)})_x^3 \rightharpoonup uu_x^3 \quad \text{weakly in } L^{5/4}(0, T; L^2(\mathbb{T})). \quad (3.30)$$

It remains to prove that $(u^{(\tau)})_x^4$ converges to u_x^4 . For this, we first show that $[(u^{(\tau)})_x^3]_x$ is uniformly bounded:

$$\begin{aligned} \|[(u^{(\tau)})_x^3]_x\|_{L^{3/2}(0, T; L^{3/2}(\mathbb{T}))} &= \frac{3}{2} \|u_x^{(\tau)} [(u^{(\tau)})_x^2]_x\|_{L^{3/2}(0, T; L^{3/2}(\mathbb{T}))} \\ &\leq \frac{3}{2} \|u_x^{(\tau)}\|_{L^6(0, T; L^6(\mathbb{T}))} \|[(u^{(\tau)})_x^2]_x\|_{L^2(0, T; L^2(\mathbb{T}))} \leq C. \end{aligned}$$

Now, let $\phi \in L^{15/2}(0, T; H^1(\mathbb{T}))$ be a test function. Then, by integrating by parts,

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}} ((u^{(\tau)})_x^4 - u_x^4) \phi dx dt \\ &= \int_0^T \int_{\mathbb{T}} (u^{(\tau)})_x^3 (u^{(\tau)} - u_x) \phi dx dt + \int_0^T \int_{\mathbb{T}} u_x ((u^{(\tau)})_x^3 - u_x^3) \phi dx dt \\ &= - \int_0^T \int_{\mathbb{T}} [(u^{(\tau)})_x^3]_x (u^{(\tau)} - u) \phi dx dt - \int_0^T \int_{\mathbb{T}} (u^{(\tau)})_x^3 (u^{(\tau)} - u) \phi_x dx dt \\ &\quad + \int_0^T \int_{\mathbb{T}} ((u^{(\tau)})_x^3 - u_x^3) u_x \phi dx dt. \end{aligned}$$

Since $[(u^{(\tau)})_x^3]_x$ is bounded in $L^{3/2}(0, T; L^{3/2}(\mathbb{T}))$ and $u^{(\tau)}$ converges to u strongly in $L^5(0, T; L^\infty(\mathbb{T}))$, the first summand converges to zero. The same conclusion holds for the second summand. By (3.29) and since $u_x \phi \in L^2(0, T; L^6(\mathbb{T}))$, the third summand also converges to zero. This proves that

$$(u^{(\tau)})_x^4 \rightharpoonup u_x^4 \quad \text{weakly in } L^{3/2}(0, T; H^{-1}(\mathbb{T})). \quad (3.31)$$

In view of the convergence results (3.22), (3.26)-(3.28), (3.30), and (3.31), we can perform the limit $\tau \rightarrow 0$ in (3.12) to conclude that u solves

$$(u^4)_t = 2 \left(u^2 (u^2)_{xxx} - \frac{16}{3} uu_x^3 \right)_{xxx} - 12 (u^2 u_x^2)_{xxxx} + 8 \left((u^2)_{xx}^2 + \frac{16}{3} u_x^4 \right)_{xx}$$

in the sense of $L^{5/4}(0, T; H^{-3}(\mathbb{T}))$. This proves Theorem 1.1.

4. Exponential time decay of the solutions. We prove Theorem 1.2. Let $k \in \mathbb{N}$, $\tau > 0$, and $t \in ((k-1)\tau, k\tau]$. Furthermore, let u_k be the periodic solution to (3.1) constructed in Lemma 3.1. Then, by the proof of Lemma 3.1, using the inequality of Lemma 2.3 instead of Lemma 2.1,

$$E(u_k^4) + \tau \kappa_2 \int_{\mathbb{T}} (u^2)_{xxx}^2 dx \leq E(u_{k-1}^4), \quad k \in \mathbb{N},$$

where κ_2 is given in (1.8). We employ the generalized logarithmic Sobolev inequality with optimal constant (see Corollary 4.3 in [11]):

$$\int_{\mathbb{T}} v^2 \log \left(\frac{v^2}{\|v^2\|_{L^1(\mathbb{T})}} \right) dx \leq \frac{2}{(2\pi)^6} \int_{\mathbb{T}} v_{xxx}^2 dx, \quad v \in H^3(\mathbb{T}),$$

applied to $v = u_k^2$. Then, since u_0^4 is assumed to have unit mass, $\|u_k^4\|_{L^1(\mathbb{T})} = \|u_0^4\|_{L^1(\mathbb{T})} = 1$,

$$E(u_k^4) \leq \frac{2}{(2\pi)^6} \int_{\mathbb{T}} (u_k^2)_{xxx}^2 dx,$$

and the above entropy inequality becomes

$$E(u_k^4) + \frac{1}{2} \tau \kappa_2 (2\pi)^6 E(u_k^4) \leq E(u_{k-1}^4), \quad k \in \mathbb{N}.$$

This recursive inequality can be solved:

$$E(u_k^4) \leq E(u_0^4) \left(1 + \frac{1}{2} \tau \kappa_2 (2\pi)^6\right)^{-k}, \quad k \in \mathbb{N}.$$

For $t \in ((k-1)\tau, k\tau]$, we can write this inequality as

$$E((u^{(\tau)})^4) \leq E(u_0^4) \left(1 + \frac{1}{2} \tau \kappa_2 (2\pi)^6\right)^{-t/\tau}.$$

Since $u^{(\tau)}(x, t)$ converges a.e. to $u(x, t)$ as $\tau \rightarrow 0$ and $(1 + \frac{1}{2} \tau \kappa_2 (2\pi)^6)^{-t/\tau}$ converges to $\exp(-\frac{1}{2} \tau \kappa_2 (2\pi)^6 t)$, we can perform the limit $\tau \rightarrow 0$ in the above inequality which implies the assertion.

5. Numerical examples. The numerical examples of this section illustrate the long-time behavior of the solution to (1.1). They also underline the preservation of nonnegativity and the nonmonotonic behavior of the solutions. For the numerical experiments we choose, as in [2, 17], the initial datum

$$n_0(x) = \varepsilon + \cos^{2m}(\pi x), \quad x \in (0, 1), \quad (5.1)$$

where $\varepsilon > 0$ is a small parameter. We take $\varepsilon = 10^{-4}$ and $m = 1$ or $m = 8$. For the computations, we discretize (1.1) in time by the implicit Euler scheme and in space by central finite differences. The discrete nonlinear system is solved by the Newton method; no damping was necessary. The initial guess is chosen as the solution of the previous time level. The time step is $\Delta t = 10^{-10}$, and we used 100 grid points.

Figures 5.1 and 5.2 show the evolution of the initial datum with $m = 1$ and $m = 8$, respectively. The solution for $m = 1$ increases rapidly until it reaches the constant steady state $n_{\text{inf}} = 1$. On the other hand, the solution for $m = 8$ starts with one higher-order minimum, bifurcates into two local minima and reduces to one extremum again. In particular, the evolution is not monotone. This behavior is similar to that of the fourth-order equation (1.3) [2, 17].

The difference $n(\cdot, t) - n_{\text{inf}}$ in the L^1 norm versus time t is presented in Figures 5.3 and 5.4. Notice that we use a logarithmic scale for the ordinate. In both cases, we see that the decay of the difference is approximately exponential confirming the analytical result of Theorem 1.2.

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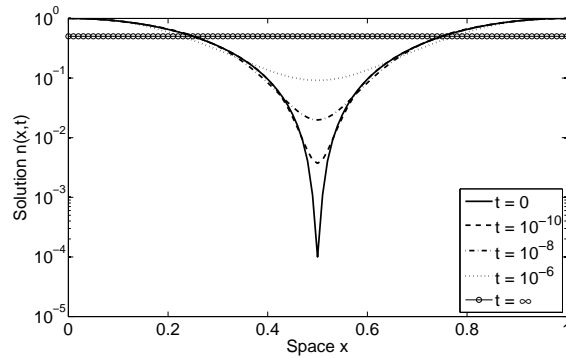


FIG. 5.1. Time evolution of the solution $n(x,t)$ with initial condition (5.1) with $m = 1$.

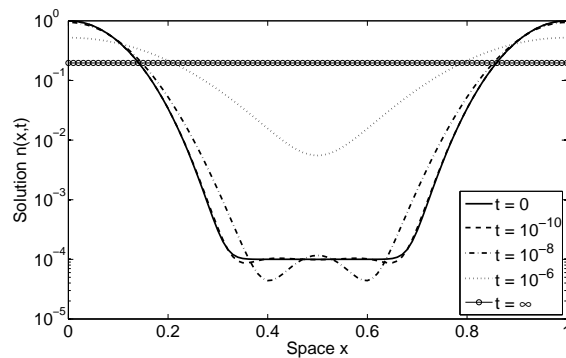


FIG. 5.2. Time evolution of the solution $n(x,t)$ with initial condition (5.1) with $m = 8$.

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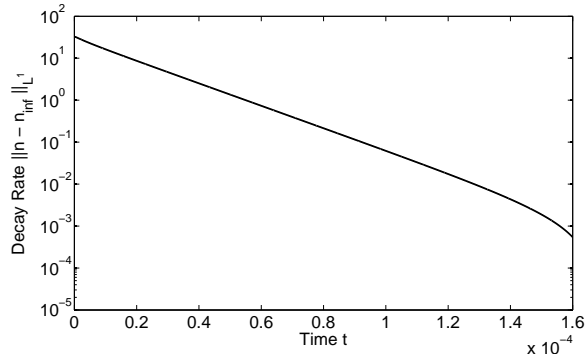


FIG. 5.3. Decay of the difference $\|n(\cdot, t) - 1\|_{L^1(0,1)}$ using the initial condition (5.1) with $m = 1$.

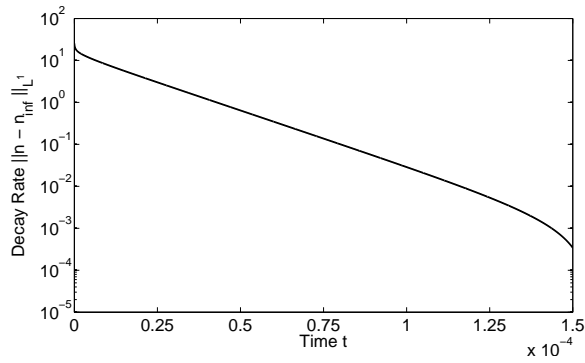


FIG. 5.4. Decay of the difference $\|n(\cdot, t) - 1\|_{L^1(0,1)}$ using the initial condition (5.1) with $m = 8$.

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