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A NON-CONFORMING FINITE ELEMENT METHOD FOR CONVEX VARIATIONAL PROBLEMS

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Introduction. For a domain $\Omega \subset \mathbb{R}^n$, a strictly convex density $W : \mathbb{R}^{m\times n} \to \mathbb{R}\cup\{+\infty\}$ with $W(F) \geq c(|F|^p - 1)$ for some $p \in (1,\infty)$, and $f \in \mathrm{L}^{p'}(\Omega)^m$, we define the energy functional

$$\mathcal{J}(v) = \int_\Omega \Big(W(\nabla v) - f \cdot v \Big)\,\mathrm{dx}. \tag{1}$$

Furthermore, for given $g \in \mathrm{W}^{1,p}(\Omega)^m$ and $\Gamma_i \subset \partial\Omega$, $i = 1,\dots,m$, we define the admissible set

$$\mathcal{A} = \big\{ v \in \mathrm{W}^{1,p}(\Omega)^m : v_i|_{\Gamma_i} = g_i|_{\Gamma_i}, i = 1,\dots,m \big\}. \tag{2}$$

Throughout, we assume that the surface measure of the sets Γ_i is non-zero, for all $i = 1,\dots,m$. In that case, the direct method of the calculus of variations [3] establishes the existence of a minimizer of $\mathcal{J}$ in $\mathcal{A}$,

$$u = \operatorname{argmin} \mathcal{J}(\mathcal{A}), \tag{3}$$

which is even unique due to strict convexity. Since u, in general, is not accessible analytically, one aims at the numerical solution of (3).

Foss, Hrusa, and Mizel [4] showed that (3), even under the fairly strong conditions posed above, may exhibit the *Lavrentiev gap phenomenon*,

$$\inf \mathcal{J}(\mathcal{A} \cap \mathrm{W}^{1,\infty}(\Omega)) > \inf \mathcal{J}(\mathcal{A}), \tag{4}$$

which is the focus of the present paper.

Let $\mathcal{T}_h$ be a shape-regular simplicial triangulation of Ω with mesh-size h, and let $\mathrm{P}_1(\mathcal{T}_h)$ be the space of continuous, piecewise affine finite element functions. Then, the P_1-finite element discretization of (3) reads

$$u_h = \operatorname{argmin} \mathcal{J}\big(\mathcal{A} \cap \mathrm{P}_1(\mathcal{T}_h)^m\big). \tag{5}$$

In the presence of a Lavrentiev gap, however, the inclusion $\mathrm{P}_1(\mathcal{T}_h) \subset W^{1,\infty}$ immediately implies that $\mathcal{J}(u_h) \not\to \mathcal{J}(u)$, i.e., the P_1-FEM cannot converge to the "correct" limit. The purpose of this talk is to show that, by contrast, the non-conforming Crouzeix-Raviart finite element method can be employed for a successful discretization of (3).

Crouzeix-Raviart FEM for (3). Let $\mathcal{E}_h$ be the set of faces of elements and, for $E \in \mathcal{E}_h$, let m_E be the face midpoint. Let $\mathrm{CR}(\mathcal{T}_h)$ be the Crouzeix-Raviart finite element space, which consists of all piecewise affine functions, which are continuous in all face midpoints m_E. We assume that the mesh $\mathcal{T}_h$ respects the sets Γ_i, for all $i = 1,\dots,m$. We then discretize the admissible set $\mathcal{A}$ by

$$\mathcal{A}_h = \big\{ \Pi_h v : v \in \mathcal{A} \big\} = \big\{ v_h \in \mathrm{CR}(\mathcal{T}_h)^m : v_{h,i}(m_E) = \Pi_h g_i(m_E), \forall E \subset \Gamma_i \big\}, \tag{6}$$

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where $\Pi_h : \mathrm{W}^{1,p}(\Omega)^m \to \mathrm{CR}(\mathcal{T}_h)^m$ denotes the Crouzeix-Raviart interpolant

$$\Pi_h v(m_E) = |E|^{-1} \int_E v \,\mathrm{d}s \qquad \forall E \in \mathcal{E}_h. \tag{7}$$

Extending the gradient operator to the space $\mathrm{CR}(\mathcal{T}_h)$, where it denotes the piecewise gradient, we also extend the function $\mathcal{J}$ to $\mathcal{A}_h$ in an obvious way. We thus can compute the *discrete minimizers*

$$u_h = \operatorname{argmin} \mathcal{J}(\mathcal{A}_h). \tag{8}$$

As for the continuous formulation (3), the direct method of the calculus of variations proves that the Crouzeix-Raviart FEM (8) has a unique solution.

It is crucial to observe that the Crouzeix-Raviart interpolant (7) satisfies

$$\mathcal{J}(\Pi_h v) \le \mathcal{J}(v) + C\, h\, \|f\|_{\mathrm{L}^{p'}(\Omega)} \|\nabla v\|_{\mathrm{L}^p(\Omega)} \quad \forall v \in \mathrm{W}^{1,p}(\Omega)^m. \tag{9}$$

Based on this estimate, the first main result of our presentation is the following theorem from [6], which proves *a priori* convergence of the Crouzeix-Raviart FEM even in the presence of the Lavrentiev gap phenomenon (4).

Theorem 1. *With u and u_h the solutions of* (3) *and* (8)*, respectively, there holds*

$$\begin{cases} u_h \to u & \text{strongly in } \mathrm{L}^p, \\ \nabla u_h \to \nabla u & \text{strongly in } \mathrm{L}^p, \\ \mathcal{J}(u_h) \to \mathcal{J}(u) & \end{cases} \tag{10}$$

as $h \searrow 0$. □

Adaptive Crouzeix-Raviart FEM for (3). The Lavrentiev gap phenomenon (4) is closely linked to the occurrence of singularities in the solution to the variational problem (3). It therefore seems to be natural to consider adaptive mesh refinement techniques.

In the following we replace the global mesh-size h by a local mesh-size function h_ℓ, where ℓ denotes the refinement level. We change the notation from $\mathcal{T}_h$ to $\mathcal{T}_\ell$, u_h to u_ℓ, and so forth. Let $[u_\ell]$ denote the jump of u_ℓ across internal faces, and denote $[u_\ell]_i = u_{\ell,i} - g_i$ on Γ_i, and $[u_\ell]_i = 0$ on $\partial\Omega \setminus \Gamma_i$, respectively.

By careful examination of the proof of Theorem 1, one finds that the condition $h \searrow 0$ may be relaxed to

$$\|h_\ell f\|_{\mathrm{L}^{p'}(\Omega)} + \|h_\ell^{1/p}[u_\ell]\|_{\mathrm{L}^p(\cup\mathcal{E}_h)} \xrightarrow{\ell\to\infty} 0. \tag{11}$$

More precisely, (11) implies convergence of the Crouzeix-Raviart approximations u_ℓ to the solution u of (3) in the sense of Theorem 1 as $\ell \to \infty$. This motivates the definition of the convergence indicator

$$\eta_\ell = \sum_{E\in\mathcal{E}_\ell} \eta_\ell(E) = \sum_{E\in\mathcal{E}_\ell} \Big(\|h_\ell f\|^{p'}_{\mathrm{L}^{p'}(\omega_E)} + \|h_\ell^{1/p}[u_\ell]\|^p_{\mathrm{L}^p(E)} \Big), \tag{12}$$

where ω_E denotes the patch of a face $E \in \mathcal{E}_h$. This indicator is then used to steer the following adaptive algorithm:

0. INPUT: initial mesh $\mathcal{T}_0$, marking parameter $\theta \in (0, 1]$
1. COMPUTE: $u_\ell = \operatorname{argmin} \mathcal{J}(\mathcal{A}_\ell)$.
2. ESTIMATE: compute convergence indicators $\eta_\ell(E)$, for all $E \in \mathcal{E}_\ell$.

3. MARK: construct $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell$ so that

$$\sum_{E \in \mathcal{M}_\ell} \eta_\ell(E) \geq \theta \eta_\ell$$

4. REFINE: refine all faces $E \in \mathcal{M}_\ell$, so that the local mesh-size is reduced by a fixed ratio, to obtain a new mesh $\mathcal{T}_{\ell+1}$. Continue at 1.

Our second main result [7] states that this algorithm is convergent. Our argument is based on a localized version of (9) and thus is restricted to non-conforming finite elements. Contrary to preceding works on adaptive Crouzeix-Raviart FEM [2], our proof is neither based on the explicit use of the Euler-Lagrange equations nor on any kind of (generalized) Galerkin orthogonality. In particular, we note that the indicator η_ℓ from (12) does not provide a reliable upper bound for the error $u - u_\ell$, in general. This makes it even more surprising that one can prove convergence of the adaptive scheme.

Theorem 2. *The sequence of indicators computed in the above algorithm satisfies* $\eta_\ell \to 0$ *as* $\ell \to \infty$ *and thus guarantees* (11). *In particular, there holds*

$$\left\{ \begin{array}{rcll} u_\ell & \to & u & \text{strongly in } \mathrm{L}^p, \\ \nabla u_\ell & \to & \nabla u & \text{strongly in } \mathrm{L}^p, \\ \mathcal{J}(u_\ell) & \to & \mathcal{J}(u), & \end{array} \right. \tag{13}$$

as $\ell \to \infty$. □

The preceding theorem allows for a numerical verification of Lavrentiev gaps as follows: besides the adaptive Crouzeix-Raviart solution (11), we compute a conforming $\mathrm{P}_1(\mathcal{T}_h)$ approximation (5) of (3). Comparing the difference of the energies, one may see whether a Lavrentiev gap occurs or not.

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