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Sums of Nevanlinna functions and differential equations on star-shaped graphs.

Vyacheslav Pivovarchik[†], Harald Woracek

Abstract

Additive decompositions of a meromorphic function give rise to quotient representations of a particular form. We raise the question which quotient representations of a given function arise in this way. This question is answered by means of two characterizations via different terms. We pay particular attention to functions belonging to various subclasses of the Nevanlinna class of functions with nonnegative imaginary part throughout the upper half-plane. Our results lead to some direct and inverse spectral theorems for systems of strings or systems of Sturm-Liouville equations supported on a star-shaped graph.

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1 Introduction

Every function f which is meromorphic in the whole complex plane can, by the Weierstraß Factorization Theorem, be represented as a quotient of two entire functions, $f = P^{-1}Q$. Among all quotient representations, clearly, those where P and Q have no common zeros are of particular importance. Thinking of the theory of divisibility in the integral domain $H(\mathbb{C})$ of all entire functions, these are just the representations of an element f of its quotient field as a quotient $P^{-1}Q$ of two relatively prime elements of the ring. Such representations always exist and are unique up to units of the ring, i.e. up to zerofree entire functions.

Particular quotient representations of a meromorphic function f by not necessarily relatively prime entire functions can be obtained from additive decompositions of f: If $f_1, \dots, f_n$ are meromorphic functions with $\sum_{i=1}^n f_i = f$, and if each f_i is written as $f_i = P_i^{-1} Q_i$ with some entire functions P_i, Q_i, then

$$f = \sum_{i=1}^n \frac{Q_i}{P_i} = \frac{Q}{P} \quad \text{where} \quad P := \prod_{i=1}^n P_i,\ Q := \sum_{i=1}^n \Big(Q_i \prod_{\substack{j=1 \\ j\neq i}}^n P_j \Big) \tag{1.1}$$

Here the functions P and Q need not be relatively prime, even if we assume that for each $i \in \{1, \dots, n\}$ the two functions P_i and Q_i are.

We raise the question how to recognize from a pair (Q, P) of entire functions whether it arises in the way (1.1) from an additive decomposition of its quotient $f := P^{-1}Q$ into n summands. It turns out to be necessary to answer this question not for the full field $\mathcal{M}(\mathbb{C})$ of all functions meromorphic in the plane, but for certain subclasses of $\mathcal{K} \subseteq \mathcal{M}(\mathbb{C})$ in the sense that the function f and

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the summands f_i in the additive decomposition $f = \sum_{i=1}^n f_i$ are required to belong to $\mathcal{K}$. To be precise, we will have to deal with meromorphic functions which additionally belong to several specific subclasses of Nevanlinna functions. Among them the class $\mathcal{S}^{-1}$ of all Nevanlinna functions which are analytic in $\mathbb{C} \setminus [0,\infty)$ and take nonpositive values on $(-\infty, 0)$, and the class $\mathcal{N}^{\mathrm{ep}}$ of all Nevanlinna functions which are meromorphic in $\mathbb{C}\setminus[0,\infty)$ and have only finitely many poles in $(-\infty, 0)$.

1.1 Definition. Let $\mathcal{K} \subseteq \mathcal{M}(\mathbb{C})$ and let $P, Q \in H(\mathbb{C})$.

(*i*) The pair (Q, P) is called a 1-$\mathcal{K}$-pair, if $P^{-1}Q \in \mathcal{K}$ and P and Q have no common zeros.

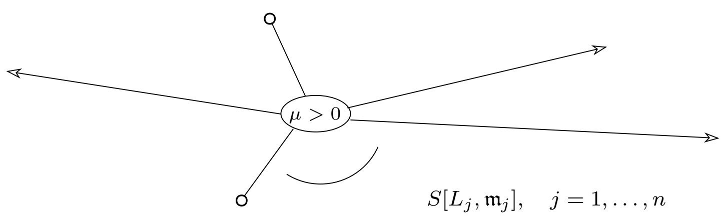
(*ii*) Let $n \in \mathbb{N}$, $n \geq 2$. The pair (Q, P) is called an n-$\mathcal{K}$-pair, if $P^{-1}Q \in \mathcal{K}$, there exist 1-$\mathcal{K}$-pairs $(Q_1, P_1), \dots, (Q_n, P_n)$ such that

$$P = \prod_{i=1}^n P_i, \quad Q = \sum_{i=1}^n \Big(Q_i \prod_{\substack{j=1\\ j\neq i}}^n P_j \Big), \tag{1.2}$$

and no representation of this kind is possible with less than n many 1-$\mathcal{K}$-pairs.

Under certain assumptions on the class $\mathcal{K}$, which will be made precise later, we will establish necessary and sufficient conditions for a pair (Q, P) to form an n-$\mathcal{K}$-pair. These conditions are given either in terms of the distribution and interrelation of the zeros of P and Q, cf. Theorem 4.2, or in terms of the distribution of zeros plus a reality condition on derivatives of the function $\phi(z) := P(z^2) - izQ(z^2)$ or $\psi(z) := Q(z^2) + izP(z^2)$, respectively, cf. Theorem 4.7.

The notion of n-$\mathcal{K}$-pairs is of some intrinsic interest, since it relates the additive and multiplicative structures of the ring $H(\mathbb{C})$. However, our motivation to introduce and investigate this notion arose from various concrete problems of mathematical physics. For example, consider a plane star-shaped graph composed of n strings, with (finite or infinite) respective lengths L_i and mass distributions $\mathfrak{m}_i(x)$, which are joined at one internal vertex:



(1.3)

The strings are stretched and the system is able to vibrate in the direction orthogonal to the equillibrium positions of the strings. The central vertex is assumed to be subject to viscouse friction with coefficient of damping $\mu > 0$.

Let us explain the phenomena which arise in this situation for the particular case of strings with finite length and total mass, where at the external vertices Dirichlet boundary conditions are imposed; imagine a vibrating spider web which is subject to damping in its center.

Denote by $s_i(z,s)$, $i=1,\dots,n$, the solution of the i-th string equation

$$y'(s)+\int_{[0,s]} zy(u)\,d\mathfrak{m}_i(u)=0,\ \ s\in(-\infty,L_i)$$

with boundary values $s_i(z,L_i)=0$, $\frac{\partial}{\partial s}s_i(z,s)|_{s=L_i}=1$, at the outer vertex. If the system were undamped, the associated operator model could be viewed as a quasilinear operator pencil z^2I-A with some selfadjoint operator A. The set of eigenfrequencies would equal the set of zeros of the function

$$\Phi_0(z):=\sum_{i=1}^n \frac{\partial}{\partial s}s_i(z^2,s)|_{s=0}\prod_{\substack{j=1\\ j\neq i}}^n s_j(z^2,0)$$

In our situation, due to the presence of damping, the associated operator model will consist of a quadratic operator pencil of the form $z^2R-iz\mu K+A$ with some selfadjoint operator A, a one-dimensional projection K, and a nonnegative operator R. The eigenfrequences of the system are equal to the zeros of the function

$$\Phi(z):=\Big(\sum_{i=1}^n \frac{\partial}{\partial s}s_i(z^2,s)|_{s=0}\prod_{\substack{j=1\\ j\neq i}}^n s_j(z^2,0)\Big)+iz\mu\prod_{i=1}^n s_i(z^2,0)$$

where $\mu>0$ denotes the coefficient of damping at the central vertex.

It is well-known, cf. [KK2], that the functions $s_i(z,s)$ and $\frac{\partial}{\partial s}s_i(z,s)$ are, for each fixed s, entire functions of z and do not have common zeros. Moreover, we have

$$\frac{\frac{\partial}{\partial s}s_i(z,s)|_{s=0}}{s_i(z,0)}\in\mathcal{S}^{-1}\,.$$

Defining entire functions P_i and Q_i as

$$P_i(z)=\mu s_i(z,0),\quad Q_i(z)=-\frac{\partial}{\partial s}s_i(z,s)|_{s=0}\,,$$

and letting P and Q be defined by (1.2), we see that the function $\Phi(z)$ is nothing else but

$$\Phi(z)=\frac{1}{\mu^{n-1}}\big[Q(z^2)+izP(z^2)\big]\,.$$

Thus the eigenfrequencies of the considered problem are described by the zeros of the function $Q(z^2)+izP(z^2)$ with an n-$\mathcal{K}$-pair (Q,P) where $\mathcal{K}=\mathcal{M}(\mathbb{C})\cap\mathcal{S}^{-1}$. Our investigations on n-$\mathcal{K}$-pairs will hence give rise to direct and inverse spectral theorems.

Let us point out one noteworthy consequence of the present description of the set of eigenfrequencies: Apparently, the eigenfrequencies of the above problem which are located in the open lower half-plane play a significantly different role than those lying on the real line, namely representing decaying states and not just stable states. From some general results we will obtain that, under some conditions, the number of real eigenfrequencies of the problem is bounded by the number of nonreal eigenfrequencies, where the term 'bounded' has to be interpreted by means of asymptotic density.

The content of the paper is arranged in six sections. After this introduction, in Section 2, we set up notation which will be used throughout the paper. Moreover, we recall some facts from the theory of strings. In Section 3, we introduce and investigate one property of a class $\mathcal{K}$ which is responsible for the validity of our characterizations of n-$\mathcal{K}$-pairs, cf. Definition 3.1. We show that this property is satisfied for subclasses of Nevanlinna functions which are defined by means of conditions on the data in the Herglotz-integral representation of a Nevanlinna function.

Section 4 is the core of this paper. We formulate and prove our main results Theorem 4.2, Theorem 4.7, and Theorem 4.15, where we give characterizations of n-$\mathcal{K}$-pairs (Q, P) in terms of the distribution of zeros of P, Q or of ϕ, ψ. These conditions can be significantly simplified for the particular case of the class $\mathcal{M}(\mathbb{C}) \cap \mathcal{S}^{-1}$, which is of importance in applications, cf. Corollary 4.6, Corollary 4.14, and Corollary 4.18. In Section 5, we discuss the relation between real and nonreal zeros of the function ψ. Boundedness of real zeros by nonreal zeros is expressed in terms of densites with respect to growth functions, cf. Theorem 5.4, Proposition 5.6. In the particular case of polynomials, a bound on the actual number of real zeros in terms of the number of nonreal zeros can be given, cf. Proposition 5.7. The relevant notation on zero-distribution and growth of entire functions will be recalled in the beginning of Section 5.

Finally, in Section 6, we turn to applications and discuss damped systems of differential equations on a star-shaped graph. The results of the previous sections give rise to some direct and inverse spectral results for the considered problems. First we deal with a system of string equations, and deduce a direct spectral theorem; we will employ Theorem 4.15. From the viewpoint of physical interpretation, unstable damped systems are not too meaningful. Hence, major interest lies in stable systems. For this case we will also obtain an inverse spectral theorem, i.e. make available a complete characterization of those point-sets which occur as spectra; the basic ingredients are Corollary 4.18 and classical inverse results on strings. Secondly, we investigate a system of Sturm-Liouville equations given on a star-shaped graph, and deduce a direct spectral theorem; we will again employ Theorem 4.15. In order to deal with inverse spectral problems for Sturm-Liouville equations, it is necessary to invoke considerations on the asymptotics of the spectrum in addition to Proposition 4.9. We will not touch upon these topics in the present paper; this will be subject of future work.

Let us note that the above mentioned direct spectral theorems include several statements made in earlier papers about the eigenfrequencies of a damped system of strings or Sturm-Liouville equations on a star-shaped graph as particular cases, for details see Remark 6.11, Remark 6.13.

2 Notation and preliminaries

In this section we collect some necessary notation and recall some results which will be needed throughout the paper.

A. Entire and meromorphic functions.

If $D \subseteq \mathbb{C}$ is an open set, we denote by $H(D)$ the set of all functions which are analytic on D, and by $\mathcal{M}(D)$ the set of all functions meromorphic in D.

2.1. Notation: Let $f \in \mathcal{M}(D)$.

(*i*) A function $\mathfrak{d}_f : D \to \mathbb{Z}$ is defined as follows: If

$$f(z) = \sum_{k=k_0}^{\infty} a_k (z-w)^k, \quad a_{k_0} \neq 0\,,$$

is the Laurent expansion of the function f at the point w, then $\mathfrak{d}_f(w) := k_0$.

(*ii*) Denote by $Z(f)$ and $\sigma(f)$ the set of all zeros of f and poles of f, respectively. In other words, $Z(f) = \{w \in D : \mathfrak{d}_f(w) > 0\}$ and $\sigma(f) = \{w \in D : \mathfrak{d}_f(w) < 0\}$.

(*iii*) A function $f^\# \in \mathcal{M}(D')$ with $D' := \{w \in \mathbb{C} : \overline{w} \in D\}$ is defined as $f^\#(z) := \overline{f(\overline{z})}$.

B. Classes of Nevanlinna functions.

The Nevanlinna class $\mathcal{N}$ is defined as the set of all functions $q \in H(\mathbb{C} \setminus \mathbb{R})$ with $q = q^\#$ which satisfy $\operatorname{Im} q(z) \geq 0$ for all points z in the open upper half plane $\mathbb{C}^+$. In the present paper also some subclasses of $\mathcal{N}$ will appear, namely:

(*i*) the class $\mathcal{N}^{\text{ep}}$ of essentially positive Nevanlinna functions, which is defined as the set of all functions $f \in \mathcal{N}$ which are analytic in $\mathbb{C} \setminus [0,\infty)$ with possible exception of finitely many poles.

(*ii*) the class $\mathcal{N}^{\text{ep}}_+$, which is defined as the set of all functions $f \in \mathcal{N}$ such that for some $\gamma \in \mathbb{R}$ we have $f \in H(\mathbb{C} \setminus [\gamma,\infty))$ and $f(z) > 0$ for $z \in (-\infty,\gamma)$.

(*iii*) the class $\mathcal{N}^{\text{ep}}_-$, which is defined as the set of all functions $f \in \mathcal{N}$ such that for some $\gamma \in \mathbb{R}$ we have $f \in H(\mathbb{C} \setminus [\gamma,\infty))$ and $f(z) \leq 0$ for $z \in (-\infty,\gamma)$.

(*iv*) the Stieltjes class $\mathcal{S}$, defined as the set of all functions $f \in \mathcal{N} \cap H(\mathbb{C} \setminus [0,\infty))$ such that $f(z) > 0$ for $z \in (-\infty,0)$;

(*v*) the class $\mathcal{S}^{-1}$, defined as the set of all functions $f \in \mathcal{N} \cap H(\mathbb{C} \setminus [0,\infty))$ such that $f(z) \leq 0$ for $z \in (-\infty,0)$;

Since in the present paper we mainly deal with function meromorphic in the whole complex plane, let us introduce the following notational convention: If $\mathcal{L}$ is any class of functions, then

$$\mathring{\mathcal{L}} := \mathcal{M}(\mathbb{C}) \cap \mathcal{L}\,,$$

e.g. $\mathring{\mathcal{S}}^{-1} = \mathcal{M}(\mathbb{C}) \cap \mathcal{S}^{-1}$.

Next we very briefly recall some properties of the above classes of Nevanlinna functions. More details can be found e.g. in [KK1] or [AD1], [AD2].

2.2. Integral representation for $\mathcal{N}$: If $q \in \mathcal{N}$, then there exist unique numbers $a, b \in \mathbb{R}$ with $b \geq 0$, and a unique positive Borel measure on $\mathbb{R}$ with

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty\,, \tag{2.1}$$

such that

$$q(z) = a + bz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\mu(t)\,. \tag{2.2}$$

Conversely, given $a, b \in \mathbb{R}$, $b \geq 0$, and a positive Borel measure μ with (2.1), then the right hand side of (2.2) defines a function which belongs to $\mathcal{N}$.

A function $q \in \mathcal{N}$ is meromorphic in the whole complex plane, i.e. belongs to $\mathring{\mathcal{N}}$, if and only if the corresponding measure μ is discrete. In this case $\sigma(q) = \operatorname{supp}\mu$, and the integral representation (2.2) writes as

$$q(z) = a + bz + \sum_{x_k \in \sigma(q)} \Big(\frac{1}{x_k - z} - \frac{x_k}{1 + x_k^2} \Big) r_k \tag{2.3}$$

where $r_k := -\operatorname{Res}(q, x_k)$. This series converges locally uniformly on $\mathbb{C} \setminus \sigma(q)$.

2.3. Integral representations for $\mathcal{S}$ and $\mathcal{S}^{-1}$:

(*i*) A function q belongs to the Stieltjes class if and only if it can be represented in the form (2.2) where the data (a, b, μ) has the following additional properties:

$$b = 0, \quad \operatorname{supp}\mu \subseteq [0, \infty), \int\limits_{[0,\infty)} \frac{d\mu(t)}{1 + |t|} < \infty, \quad a - \int\limits_{[0,\infty)} \frac{t}{1 + t^2}\, d\mu(t) \geq 0\,.$$

This means that we can write $q(z) = a' + \int_{[0,\infty)} \frac{d\mu(t)}{t-z}$ with some nonnegative real number a'.

(*ii*) A function q belongs to the class $\mathcal{S}^{-1}$ if and only if it can be represented in the form (2.2) where the data (a, b, μ) has the following additional properties:

$$\operatorname{supp}\mu \subseteq [0, \infty), \int\limits_{[0,\infty)} \frac{d\mu(t)}{t(1 + t^2)} < \infty, \quad a + \int\limits_{[0,\infty)} \frac{d\mu(t)}{t(1 + t^2)} \leq 0\,.$$

This means that we can write $q(z) = a' + bz + \int_{[0,\infty)} \big(\frac{1}{t-z} - \frac{1}{t}\big)\, d\mu(t)$ with some nonpositive real number a'.

2.4. Relations between the introduced subclasses of $\mathcal{N}$:

(*i*) We have

$$q(z) \in \mathcal{N} \iff -q(z)^{-1} \in \mathcal{N} \qquad q(z) \in \mathcal{N}^{\mathrm{ep}} \iff -q(z)^{-1} \in \mathcal{N}^{\mathrm{ep}}$$
$$q(z) \in \mathcal{N}^{\mathrm{ep}}_{+} \iff -q(z)^{-1} \in \mathcal{N}^{\mathrm{ep}}_{-} \qquad q(z) \in \mathcal{S} \iff -q(z)^{-1} \in \mathcal{S}^{-1}$$

(*ii*) We have

$$q(z) \in \mathcal{S} \iff \big(q(z) \in \mathcal{N} \text{ and } zq(z) \in \mathcal{N}\big) \iff zq(z^2) \in \mathcal{N}$$

(*iii*) We have

$$\mathcal{N}^{\mathrm{ep}} = \mathcal{N}^{\mathrm{ep}}_{+} \,\dot\cup\, \mathcal{N}^{\mathrm{ep}}_{-}, \quad \mathcal{S} \subseteq \mathcal{N}^{\mathrm{ep}}_{+}, \ \mathcal{S}^{-1} \subseteq \mathcal{N}^{\mathrm{ep}}_{-}$$

C. The Hermite-Biehler class.

The Hermite-Biehler class $\mathcal{H}B$ is defined as the set of all entire functions E which have no zeros in the open upper half plane $\mathbb{C}^+$, and satisfy

$$|E(\overline{z})| \leq |E(z)|,\ z \in \mathbb{C}^+ .$$

For an entire function E we set $A := \frac{1}{2}(E + E^\#)$ and $B := \frac{i}{2}(E - E^\#)$, so that $E = A - iB$. Then $E \in \mathcal{H}B$ if and only if A and B have no common nonreal zeros and the function $A^{-1}B$ belongs to the Nevanlinna class.

With a pair (A, B) of entire functions another entire function $\phi_{A,B}$ can be associated. This definition is motivated from the form of the characteristic functions appearing in applications, cf. §6, as well as from some results on symmetric and semibounded Hermite-Biehler functions, cf. [KWW2], [PW].

2.5 Definition. For $A, B \in H(\mathbb{C})$ define

$$\phi_{A,B}(z) := A(z^2) - izB(z^2) .$$

Note that the function $\phi_{A,B}$ satisfies the functional equation $\phi_{A,B}^\#(z) = \phi_{A,B}(-z)$.

2.6. $\phi_{A,B}$ as Hermite-Biehler function: Let $A, B \in H(\mathbb{C})$, $A = A^\#$, $B = B^\#$, be given. Then $\phi_{A,B} \in \mathcal{H}B$ if and only if A and B have no common zeros in $\mathbb{C} \setminus [0, \infty)$ and $A^{-1}B$ belongs to the Stieltjes class.

D. Strings.

In this, more elaborate, subsection we recall some facts about strings which will be used in this paper. We do not intend to go into the greatest possible generality, we content ourselves with what will be needed later on. Our standard reference concerning the theory of strings is [KK2], and most of the things we state below are extracted from this paper. Other referencse for the fundamental results on strings are [DK] or [Ka]; for the relationship with canonical systems see [KWW3].

A string $S[L, \mathfrak{m}]$ is a pair consisting of a number $L \in [0, \infty]$, and a non-negative (possibly infinite) Borel measure $\mathfrak{m}$ on $\mathbb{R} \cup \{+\infty\}$ with $\operatorname{supp} \mathfrak{m} \subseteq [0, L]$, $\mathfrak{m}([0, x]) < \infty$ for $x \in [0, L)$, and $\mathfrak{m}(\{L\}) = 0$. Denote by M the distribution function of $\mathfrak{m}$ which is normalized such that it is left-continuous and satisfies $M(0) = 0$, i.e. put

$$M(s) := \mathfrak{m}\big((-\infty, s)\big),\ s \in (-\infty, L] .$$

The equation of the string is the integral equation

$$y'(s) + \int_{[0,s]} zy(u)\, d\mathfrak{m}(u) = 0,\ s \in (-\infty, L) . \tag{2.4}$$

Thereby z is a complex parameter. Often this equation is also written in the form

$$\frac{d}{dM(s)}\Big(\frac{d}{ds} y(s)\Big) + zy(s) = 0 ,$$

understanding by $\frac{\partial}{\partial M}$ the Radon-Nikodym derivative. This equation arises when Fourier's method is applied to the partial differential equation

$$\frac{\partial}{\partial M(s)} \left(\frac{\partial v(s,t)}{\partial s} \right) - \frac{\partial^2}{\partial t^2} v(s,t) = 0 ,$$

which describes the vibrations of a string with nonhomogeneous mass distribution.

To a string $S[L,\mathfrak{m}]$ an operator model is associated, namely one can consider the operator $T_{\max}$ which acts in the Hilbert space $L^2(\mathfrak{m})$ as

$$T_{\max}y := -\frac{d}{dM(s)}\Big(\frac{dy}{ds}\Big),$$

and whose domain is the set of all elements of $L^2(\mathfrak{m})$ such that this expression is well-defined and belongs to $L^2(\mathfrak{m})$. The adjoint of $T_{\max}$ will be denoted by $T_{\min}$. It is a symmetric operator in $L^2(\mathfrak{m})$, and has either defect index $(2,2)$ or $(1,1)$. If it has defect $(2,2)$ we speak of limit circle case, otherwise of limit point case.

2.7. Limit circle/point case: It has been shown in [KK2, (10.4)] that for a string $S[L,\mathfrak{m}]$ limit circle case prevails if and only if

$$\int_{[0,L]} u^2\, d\mathfrak{m}(u) < \infty .$$

We know from [KK2, §2] that for each value of parameter $z \in \mathbb{C}$ and each given initial value $(a,b) \in \mathbb{C}^2$, there exists a unique solution $y(z,s)$ of (2.4) with $y(z,0) = a$ and $y'(z,0-) = b$. Here, and throughout this paper, a prime will denote differentiation with respect to s. Moreover, note that for each solution y of (2.4) the function y' is continuous from the right but not necessarily from the left. Let us denote by $\varphi(z,s), \psi(z,s)$ the unique solutions of (2.4) with

$$\varphi(z,0) = 1,\ \varphi'(z,0-) = 0, \qquad \psi(z,0) = 0,\ \psi'(z,0-) = 1 . \tag{2.5}$$

The following facts have been shown in [KK2, §2].

2.8. Properties of φ,ψ as functions of z: Let $s \in [0,L)$ be fixed.

(*i*) The functions $\varphi(z,s), \varphi'(z,s), \psi(z,s), \psi'(z,s)$ are entire functions of z. They satisfy

$$\psi'(z,s)\varphi(z,s) - \varphi'(z,s)\psi(z,s) = 1 . \tag{2.6}$$

(*ii*) We have

$$\frac{\partial^n}{\partial z^n}\varphi(0,s) \begin{cases} = 1\,,\ n=0 \\ < 0\,,\ n \text{ odd} \\ > 0\,,\ n \text{ even}, n \geq 2 \end{cases} \qquad \frac{\partial^n}{\partial z^n}\varphi'(0,s) \begin{cases} = 0\,,\ n=0 \\ < 0\,,\ n \text{ odd} \\ > 0\,,\ n \text{ even}, n \geq 2 \end{cases}$$

$$\frac{\partial^n}{\partial z^n}\psi(0,s) \begin{cases} = s\,,\ n=0 \\ < 0\,,\ n \text{ odd} \\ > 0\,,\ n \text{ even}, n \geq 2 \end{cases} \qquad \frac{\partial^n}{\partial z^n}\psi'(0,s) \begin{cases} = 1\,,\ n=0 \\ < 0\,,\ n \text{ odd} \\ > 0\,,\ n \text{ even}, n \geq 2 \end{cases}$$

(*iii*) Each of the functions $\varphi(z,s), \varphi'(z,s), \psi(z,s), \psi'(z,s)$ takes real and positive values for $z \in (-\infty,0)$.

(*iv*) The functions $\frac{\varphi(z,s)}{\varphi'(z,s)}$ and $\frac{\psi(z,s)}{\psi'(z,s)}$ belong to the Stieltjes class, and satisfy

$$\lim_{z\to-\infty}\frac{\varphi(z,s)}{\varphi'(z,s)} = \lim_{z\to-\infty}\frac{\psi(z,s)}{\psi'(z,s)} = 0\,,$$

$$\lim_{z\nearrow 0}\frac{\varphi(z,s)}{\varphi'(z,s)}=+\infty,\quad \lim_{z\nearrow 0}\frac{\psi(z,s)}{\psi'(z,s)}=s\,.$$

The functions φ,ψ are related to the operator theory $T_{\min}$, since they are candidates for defect elements. Actually, the following statement holds true: The string $S[L,\mathfrak{m}]$ is in the limit circle case if and only if both $\varphi(z,s)$ and $\psi(z,s)$ belong to $L^2(\mathfrak{m})$. If $S[L,\mathfrak{m}]$ is in the limit point case, then for each $z\in\mathbb{C}\setminus\mathbb{R}$ there exists a unique number $Q(z)$ such that

$$Q(z)\varphi(z,s)-\psi(z,s)\in L^2(\mathfrak{m})\,. \tag{2.7}$$

Another classification of strings is the following: Put $l:=\sup(\operatorname{supp}\mathfrak{m})$, then the string $S[L,\mathfrak{m}]$ is called regular if $l<\infty$ and $M(l)<\infty$. Otherwise, if $l+m(l)=\infty$, it is called singular. Clearly, a regular string is in the limit circle case; for a singular string both, limit circle or limit point case, may occur.

We say that a string has discrete spectrum, if one (and hence all) selfadjoint extensions of $T_{\min}$ have discrete spectrum. Equivalently, one (and hence all) closed operators T with $T_{\min}\subseteq T\subseteq T_{\max}$ and $\rho(T)\neq\emptyset$ have compact resolvents. This property of a string has been characterized explicitly, cf. [KK2, 11.9°].

2.9. Strings with discrete spectrum: The string $S[L,\mathfrak{m}]$ has discrete spectrum if and only if it is of one of the following forms (again $l:=\sup(\operatorname{supp}\mathfrak{m})$):

1. $S[L,\mathfrak{m}]$ is regular;
2. $l=\infty$, $M(\infty)<\infty$, and $\lim_{s\to\infty}s(M(\infty)-M(s))=0$;
3. $l<\infty$, $M(l)=\infty$, and $\lim_{s\to l}M(s)(l-s)=0$.

Note that $S[L,\mathfrak{m}]$ always has discrete spectrum if limit circle case prevails.

Let $S[L,\mathfrak{m}]$ be a string, and let $\varphi(z,s),\psi(z,s)$ be the fundamental system of solutions defined by (2.5). Moreover, let $\gamma\in(-\infty,\infty]$. Then we consider the limit

$$q^{\gamma}_{L,\mathfrak{m}}(z):=\lim_{s\nearrow L}\frac{\psi'(z,s)\gamma+\psi(z,s)}{\varphi'(z,s)\gamma+\varphi(z,s)}\,. \tag{2.8}$$

For $\gamma=\infty$ the quotient on the right hand side is understood as $\frac{\psi'(z,s)}{\varphi'(z,s)}$. Of course, first of all, we have to investigate when this limit exists. Thereby we meet significantly different situations depending whether $S[L,\mathfrak{m}]$ is regular or singular and in the limit circle or limit point case.

2.10. $q^{\gamma}_{L,\mathfrak{m}}$ when regular: For each $\gamma\in(-\infty,\infty]$ the limit (2.8) exists locally uniformly for $z\in\mathbb{C}\setminus\mathbb{R}$. The function $q^{\gamma}_{L,\mathfrak{m}}$ belongs to $\mathcal{N}^{\text{ep}}_{+}\cap\mathcal{M}(\mathbb{C})$ and has at most one pole in $(-\infty,0)$. It belongs to the Stieltjes class if and only if $\gamma\in[0,\infty]$. Each function $\varphi(z,s),\varphi'(z,s),\psi(z,s),\psi'(z,s)$ has a continuous extensions to $s=L$. The respective limits when s tends to L exist locally uniformly for $z\in\mathbb{C}\setminus\mathbb{R}$. We can thus write

$$q^{\gamma}_{L,\mathfrak{m}}(z)=\frac{\psi'(z,L)\gamma+\psi(z,L)}{\varphi'(z,L)\gamma+\varphi(z,L)}\,.$$

The properties of $q^{\gamma}_{L,\mathfrak{m}}$ stated in 2.10 are only implicitly contained in [KK2]; we will provide a direct proof below, after Lemma 2.14.

2.11. $q^{\gamma}_{L,\mathfrak{m}}$ when singular/limit circle case: Assume that $\sup(\operatorname{supp}\mathfrak{m}) = L$. For each $\gamma \in [0,\infty]$ the limit (2.8) exists locally uniformly for $z \in \mathbb{C} \setminus [0,\infty)$ and does not depend on γ. The function obtained in this way, let us denote it by $\mathring{q}_{L,\mathfrak{m}}$, belongs to $\mathcal{S} \cap \mathcal{M}(\mathbb{C})$. Each function $\varphi'(z,s), \psi'(z,s)$ has a continuous extensions to $s = L$. We can thus write

$$\mathring{q}_{L,\mathfrak{m}}(z) = \frac{\psi'(z,L)}{\varphi'(z,L)}\,.$$

2.12. $q^{\gamma}_{L,\mathfrak{m}}$ when singular/limit point case: Assume that $\sup(\operatorname{supp}\mathfrak{m}) = L$. For each $\gamma \in (-\infty,\infty]$ the limit (2.8) exists locally uniformly for $z \in \mathbb{C} \setminus [0,\infty)$ and does not depend on γ. The function obtained in this way, let us denote it by $q_{L,\mathfrak{m}}$, is called the Titchmarsh-Weyl coefficient of the string $S[L,\mathfrak{m}]$. It belongs to the Stieltjes class $\mathcal{S}$. Moreover, it is meromorphic on $\mathbb{C}$ if and only if the string $S[L,\mathfrak{m}]$ has discrete spectrum. For each $z \in \mathbb{C} \setminus \mathbb{R}$ we have $q_{L,\mathfrak{m}}(z)\varphi(z,s) - \psi(z,s) \in L^2(\mathfrak{m})$.

It is a fundamental fact that an inverse theorem holds, cf. [KK2, Theorem 11.2].

2.13. Inverse Theorem for $\mathcal{S}$: Let $q \in \mathcal{S}$ be given. Then there exists a string $S[L,\mathfrak{m}]$ and a number $\gamma \in [0,\infty]$ such that $q = q^{\gamma}_{L,\mathfrak{m}}$.

Note here that if $S[L,\mathfrak{m}]$ is regular then γ is uniquely determined and belongs to $[0,\infty]$. If $S[L,\mathfrak{m}]$ is singular and in the limit circle (limit point) case, the choice of $\gamma \in [0,\infty]$ (or $\gamma \in (-\infty,\infty]$, respectively) is arbitrary.

An even more fundamental result states that the string $S[L,\mathfrak{m}]$ (together with the number γ if it is regular) will be uniquely determined by the given function $q \in \mathcal{S}$ if we require some additional normalization of $L,\mathfrak{m}$. We will not go into more details in this respect, since in our present work uniqueness aspects will not play a role; actually, in the situations considered later on, uniqueness will never hold, cf. Remark 6.7.

We will often work with a representation of the function $q^{\gamma}_{L,\mathfrak{m}}$ by means of a solution $s(z,s)$ satisfying a boundary condition at the right endpoint, rather than with its definition via the solutions φ,ψ. If $S[L,\mathfrak{m}]$ is given together with a number $\gamma \in (-\infty,\infty]$ in case $S[L,\mathfrak{m}]$ is regular, then for each $z \in \mathbb{C} \setminus \mathbb{R}$ there exists a (up to scalar multiples unique) nontrivial solution $s(z,s)$ of (2.4) which satisfies the boundary conditions

$$\gamma s'(z,L) + s(z,L) = 0 \quad \text{(regular)} \tag{2.9}$$

$$\lim_{s\nearrow L} s'(z,s) = 0 \quad \text{(singular/limit circle case)} \tag{2.10}$$

$$s(z,s) \in L^2(\mathfrak{m}) \quad \text{(singular/limit point case)}$$

If the spectrum of the string is discrete, such a function exists actually for all $z \in \mathbb{C}$. Note here that, if $S[L,\mathfrak{m}]$ is singular but in the limit circle case, due to (2.10) we consider only one specific selfadjoint extension of $T_{\min}$, although there exists a whole 1-parameter family of selfadjoint extensions. The only reason for this is that the exact formulation of the corresponding boundary conditions would require introduction of more notation. Related with this fact is also the, on first sight maybe surprising, behaviour of the limit $q^{\gamma}_{L,\mathfrak{m}}$ in 2.11.

2.14 Lemma. *We have $q^{\gamma}_{L,\mathfrak{m}}(z) = -\frac{s(z,0)}{s'(z,0)}$.*

Proof. Write $s(z,s) = A(z)\varphi(z,s) + B(z)\psi(z,s)$. Then it follows that $s'(z,s) = A(z)\varphi'(z,s) + B(z)\psi'(z,s)$. For $s = 0$ we obtain $A(z) = s(z,0)$, $B(z) = s'(z,0)$.

Consider first the case that $S[L,\mathfrak{m}]$ is regular. Setting $s = L$ and substituting in the boundary condition gives

$$0 = \gamma\big[A(z)\varphi'(z,L) + B(z)\psi'(z,L)\big] + \big[A(z)\varphi(z,L) + B(z)\psi(z,L)\big] =$$

$$= A(z)\big[\varphi'(z,L)\gamma + \varphi(z,L)\big] + B(z)\big[\psi'(z,L)\gamma + \psi(z,L)\big]\,,$$

and we obtain that

$$q^{\gamma}_{L,\mathfrak{m}}(z) = -\frac{A(z)}{B(z)} = -\frac{s(z,0)}{s'(z,0)}\,.$$

Next, let $S[L,\mathfrak{m}]$ be singular but in the limit circle case. Then the desired equality is obtained by letting s tend to L in the relation $s'(z,s) = s(z,0)\varphi'(z,s) + s'(z,0)\psi'(z,s)$.

Finally, assume that $S[L,\mathfrak{m}]$ is in the limit point case. Then $s(z,s)$ is linearly dependent with the function $q_{L,\mathfrak{m}}(z)\varphi(z,s) - \psi(z,s)$, cf. 2.11, (2.7). Again the assertion follows.

❑

Let us now come to the above promised proof of the portion of 2.10 which deals with the properties of $q^{\gamma}_{L,\mathfrak{m}}$: *The function* $q^{\gamma}_{L,\mathfrak{m}}$ *belongs to* $\mathcal{N}^{\mathrm{ep}}_{+} \cap \mathcal{M}(\mathbb{C})$ *and has at most one pole in* $(-\infty,0)$. *It belongs to the Stieltjes class if and only if* $\gamma \in [0,\infty]$.

Proof. Using the Lagrange identity [KK2, (1.20)] and the boundary condition (2.9), we obtain

$$(z-\overline{z})\int_0^L s(z,u)s(\overline{z},s)\,d\mathfrak{m}(u) = \int_0^L \big([zs(z,u)]s(\overline{z},s) - s(z,u)[\overline{z}s(\overline{z},s)]\big)\,d\mathfrak{m}(u) =$$

$$= \big[s(z,L)s'(\overline{z},L) - s'(z,L)s(\overline{z},L)\big] - \big[s(z,0)s'(\overline{z},0) - s'(z,0)s(\overline{z},0)\big] =$$

$$-s(z,0)s'(\overline{z},0) + s'(z,0)s(\overline{z},0) = |s'(z,0)|^2\Big(-\frac{s(z,0)}{s'(z,0)} + \frac{s(\overline{z},0)}{s'(\overline{z},0)}\Big)\,.$$

Thus $-\frac{s(z,0)}{s'(z,0)} \in \mathcal{N}$.

By (2.6) the functions $\psi'(z,L)\gamma + \psi(z,L)$ and $\varphi'(z,L)\gamma + \varphi(z,L)$ have no common zeros. Thus the poles and zeros of $q^{\gamma}_{L,\mathfrak{m}}$ coincides with the solutions of the respective equation

$$\frac{\varphi(z,L)}{\varphi'(z,L)} = -\gamma, \quad \frac{\psi(z,L)}{\psi'(z,L)} = -\gamma\,.$$

We are interested in possible solutions lying in $(-\infty,0)$. By 2.8, (iv), we see that the first equation possesses either no such solution or exactly one, depending whether $\gamma \in [0,\infty]$ or $\gamma \in (-\infty,0)$. Similarly, the second equation has no or exactly one solution in the interval $(-\infty,0)$, depending whether $\gamma \in (-\infty,-L] \cup [0,\infty]$ or $\gamma \in (-L,0)$.

❑

In the present work we will be particularly interested in strings which have a nonnegative spectrum.

2.15. Strings with nonnegative spectrum: Let a string $S[L,\mathfrak{m}]$ be given. Then we can consider the restriction V of $T_{\max}$ defined by the boundary condition

$$y'(0-) = 0\,.$$

The symmetric operator V^* has defect index $(1,1)$ or is selfadjoint, depending whether $S[L,\mathfrak{m}]$ is in the limit circle or limit point case.

(i) Assume that $S[L,\mathfrak{m}]$ is in the limit point case. Then the spectrum of V^* coincides with the set of poles of the Titchmarsh-Weyl coefficient $q_{L,\mathfrak{m}}$ and, hence, is contained in $[0,\infty)$.

If V^* has defect index $(1,1)$, we have to prescribe a second boundary condition in order to fix a selfadjoint extension of V^*, and by means of this may talk about the spectrum of the string.

(ii) Assume that $S[L,\mathfrak{m}]$ is regular. Then the set of all selfadjoint extensions of V^* is parameterized as $\{A_\gamma : \gamma\in(-\infty,\infty]\}$, where A_γ is the selfadjoint extension of V^* defined by the boundary condition $\gamma y'(L)+y(L)=0$. For $\gamma=\infty$, this condition is understood as $y'(L)=0$. The spectrum of A_γ coincides with the set of poles of the function $q^\gamma_{L,\mathfrak{m}}$. Hence, it is nonnegative if and only if $\gamma\in[0,\infty]$. Let us remark that the extension A_∞ is the Friedrichs extension of V^*.

(iii) Assume that $S[L,\mathfrak{m}]$ is singular but in the limit circle case. Then exactly one selfadjoint extension of V^*, namely its Friedrichs extension, is nonnegative. It is given by the boundary condition $\lim_{s\to L} y'(s)=0$, and its spectrum coincides with the set of poles of $\mathring{q}_{S,\mathfrak{m}}$.

E. n-$\mathcal{K}$-pairs.

Item (ii) in Definition 1.1 of an n-$\mathcal{K}$-pair with $n\geq 2$ can be reformulated as follows: Let $\mathcal{K}\subseteq\mathcal{M}(\mathbb{C})$ and $P,Q\in H(\mathbb{C})$. Then (Q,P) is an n-$\mathcal{K}$-pair if and only if there exist 1-$\mathcal{K}$-pairs (Q_j,P_j), $j=1,\dots,n$, such that

$$\frac{Q}{P}=\sum_{j=1}^n\frac{Q_j}{P_j},\quad P=\prod_{j=1}^n P_j\,, \tag{2.11}$$

and no such representation is possible with less than n many 1-$\mathcal{K}$-pairs.

Thus (Q,P) being an n-$\mathcal{K}$-pair implies that the function $P^{-1}Q$ can be decomposed into a sum of n functions, all of them belonging to $\mathcal{K}$. Conversely, however, not every additive decomposition of $P^{-1}Q$ will be suitable:

2.16 Lemma. *Let $\mathcal{K}\subseteq\mathcal{M}(\mathbb{C})$, $P,Q\in H(\mathbb{C})$, and $n\in\mathbb{N}$. Then (Q,P) is an n-$\mathcal{K}$-pair if and only if there exist $f_1,\dots,f_n\in\mathcal{K}$ such that*

$$\frac{Q}{P}=\sum_{j=1}^n f_j,\quad \mathfrak{d}_P=\sum_{j=1}^n\max\{-\mathfrak{d}_{f_j},0\}\,, \tag{2.12}$$

and no such decomposition is possible with less than n many elements of $\mathcal{K}$.

Proof. Assume first that (Q, P) is an n-$\mathcal{K}$-pair, and let (Q_j, P_j) be 1-$\mathcal{K}$-pairs which satisfy (2.11). Then $f_j := P_j^{-1}Q_j \in \mathcal{K}$ and $\mathfrak{d}_{f_j} = \mathfrak{d}_{Q_j} - \mathfrak{d}_{P_j}$. Since $\operatorname{supp}\mathfrak{d}_{Q_j} \cap \operatorname{supp}\mathfrak{d}_{P_j} = \emptyset$, we have

$$\max\{-\mathfrak{d}_{f_j}, 0\} = \mathfrak{d}_{P_j}\,, \tag{2.13}$$

and conclude that $\mathfrak{d}_P = \sum_{j=1}^n \max\{-\mathfrak{d}_{f_j}, 0\}$.

Conversely, let $f_1, \ldots, f_n \in \mathcal{K}$ be given according to (2.12). Choose $Q_j, P_j \in H(\mathbb{C})$ such that $\operatorname{supp}\mathfrak{d}_{Q_j} \cap \operatorname{supp}\mathfrak{d}_{P_j} = \emptyset$ and $f_j = P_j^{-1}Q_j$. Then (2.13) holds, and hence $\mathfrak{d}_P = \mathfrak{d}_{\prod_{j=1}^n P_j}$. Thus there exists a zerofree function $D \in H(\mathbb{C})$ with $P = D\prod_{j=1}^n P_j$. The pairs

$$(DQ_1, DP_1),\ (Q_2, P_2), \ldots, (Q_n, P_n)$$

are 1-$\mathcal{K}$-pairs and satisfy (2.11). We conclude that (Q, P) is an m-$\mathcal{K}$-pair with some $m \leq n$. If m were strictly less than n, we would obtain from the first paragraph of this proof a contradiction to the minimality requirement in the condition of the lemma.

❑

3 The pole-subset-property

In the study of n-$\mathcal{K}$-pairs the following property of a subclass $\mathcal{K}$ of $\mathcal{M}(\mathbb{C})$ plays a crucial role.

3.1 Definition. Let $\mathcal{K} \subseteq \mathcal{M}(\mathbb{C})$. We say that the class $\mathcal{K}$ has the pole-subset-property, if it satisfies:

(PSP) Whenever $f \in \mathcal{K}$ and a nonempty subset $T \subseteq \sigma(f)$ is given, then there exist $g, g_T \in \mathcal{K}$ such that

$$\sigma(g) = \sigma(f),\ \sigma(g_T) = T \quad \text{and} \quad f = g + g_T\,.$$

3.2 Example. A first example for a class with the pole-subset-property is the set $\mathcal{M}(\mathbb{C})$ itself. To see this, let $f \in \mathcal{M}(\mathbb{C})$ and $T \subseteq \sigma(f)$ be given. By the Mittag-Leffler Theorem there exists a function $g_T \in \mathcal{M}(\mathbb{C})$, all of whose poles are simple, such that $\sigma(g_T) = T$ and

$$\operatorname{Res}(g_T, w) \neq \operatorname{Res}(f, w),\ w \in T\,.$$

Put $g := f - g_T$. Then $g \in M(\mathbb{C})$, $\sigma(g) = \sigma(f)$ and $f = g + g_T$.

Next we give two construction methods for classes with (PSP). The proof of these statements is immediate from the definition.

3.3 Remark.

(*i*) Let $\mathcal{K}_i$, $i \in I$, be a family of subsets of $\mathcal{M}(\mathbb{C})$. If, for each $i \in I$, the class $\mathcal{K}_i$ satisfies (PSP), then also $\bigcup_{i\in I} \mathcal{K}_i$ does.

(*ii*) Assume that $\mathcal{K} \subseteq \mathcal{M}(\mathbb{C})$ has the property (PSP), and let $w_1, w_2 \in \mathbb{C}$, $u_1, u_2 \in \mathbb{C} \setminus \{0\}$. Then also

$$\mathcal{K}' := \big\{u_2 f(u_1 z + w_1) + w_2 : f \in \mathcal{K}\big\}$$

satisfies (PSP).

In the context of subclasses of $\mathcal{N}$ a source for the pole-subset-property is found in integral representations.

3.4 Example.

(*i*) Let $D \subseteq \mathbb{R}$, and define

$$\mathcal{N}_D := \big\{ f \in \mathcal{N} : \ f \text{ analytic on } \mathbb{C} \setminus D \big\} .$$

Then $\mathring{\mathcal{N}}_D$ has the pole-subset-property.

(*ii*) The set $\mathring{\mathcal{S}}$ of all meromorphic Stieltjes class functions has the pole-subset-property.

(*iii*) For $x_0, \alpha \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, $x_0 \in A$, denote by $\mathcal{N}_{A;x_0,\alpha}$ the set of all Nevanlinna functions f which are analytic in an open neighbourhood of A and satisfy $f(x_0) \leq \alpha$. Then $\mathring{\mathcal{N}}_{A;x_0,\alpha}$ has the pole-subset-property if and only if $\alpha \geq 0$.

Proof. Let $f \in \mathring{\mathcal{N}}$ and $T \subseteq \sigma(f)$ be given. Let a, b, and r_n be the data in the integral representation (2.3) of f, and define

$$g_T(z) := \sum_{x_n \in T} \Big(\frac{1}{x_n - z} - \frac{x_n}{1 + x_n^2} \Big) \frac{r_n}{2} . \tag{3.1}$$

Then g_T and $g := f - g_T$ are both Nevanlinna functions and, clearly, $\sigma(g_T) = T$, $\sigma(f - g_T) = \sigma(f)$ and $f = g + g_T$.

The assertion (*i*) is immediate: If $\sigma(f) \subseteq D$, clearly, also $\sigma(g_T), \sigma(g) \subseteq D$. Assume next that $f \in \mathring{\mathcal{S}}$. Put $\hat{a} := \sum_{x_n \in T} \frac{x_n}{1+x_n^2} \frac{r_n}{2}$, then $\hat{a} + g_T \in \mathring{\mathcal{S}}$, and

$$\big[f - (\hat{a} + g_T) \big](z) = (a - \hat{a}) + \sum_{x_n \in \sigma(f)} \Big(\frac{1}{x_n - z} - \frac{x_n}{1 + x_n^2} \Big) r_n'$$

with

$$r_n' = \begin{cases} r_n & , \ x_n \in \sigma(f) \setminus T \\ \frac{r_n}{2} & , \ x_n \in T \end{cases}$$

We have

$$a - \hat{a} \geq \sum_{x_n \in \sigma(f)} \frac{x_n}{1 + x_n^2} r_n - \sum_{x_n \in T} \frac{x_n}{1 + x_n^2} \frac{r_n}{2} = \sum_{x_n \in \sigma(f)} \frac{x_n}{1 + x_n^2} r_n' ,$$

and hence $f - (\hat{a} + g_T) \in \mathring{\mathcal{S}}$. This shows (*ii*).

We come to the proof of (*iii*). A function f belongs to the class $\mathcal{N}_{A;x_0,\alpha}$ if and only if the data (a, b, μ) in the integral representation (2.2) has the following additional properties:

$$A \cap \operatorname{supp} \mu = \emptyset, \ \ a + b x_0 + \int_{\mathbb{R}} \Big(\frac{1}{t - x_0} - \frac{t}{1 + t^2} \Big) \, d\mu(t) \leq \alpha .$$

Assume that $\alpha \geq 0$, and let $f \in \mathring{\mathcal{N}}_{A;x_0,\alpha}$ be given. Moreover, let g_T be as in (3.1) and put

$$\tilde{a} := - \sum_{x_n \in T} \Big(\frac{1}{x_n - x_0} - \frac{x_n}{1 + x_n^2} \Big) \frac{r_n}{2} .$$

Since $\alpha \geq 0$, we have $\tilde{a} + g_T \in \mathcal{N}_{A;x_0,\alpha}$. If we set $g := f - (\tilde{a} - g_T)$, we obtain

$$g(z) = (a - \tilde{a}) + \sum_{x_n \in \sigma(f)} \Big(\frac{1}{x_n - z} - \frac{x_n}{1 + x_n^2}\Big) r'_n$$

and

$$(a - \tilde{a}) + \sum_{x_n \in \sigma(f)} \Big(\frac{1}{x_n - x_0} - \frac{x_n}{1 + x_n^2}\Big) r'_n =$$
$$= \Big[a + \sum_{x_n \in \sigma(f)} \Big(\frac{1}{x_n - x_0} - \frac{x_n}{1 + x_n^2}\Big) r_n\Big] - \underbrace{\Big[\tilde{a} + \sum_{x_n \in T} \Big(\frac{1}{x_n - x_0} - \frac{x_n}{1 + x_n^2}\Big) \frac{r_n}{2}\Big]}_{=0} \leq \alpha \,.$$

Hence also $g \in \mathring{\mathcal{N}}_{A;x_0,\alpha}$.

Consider the case that $\alpha < 0$. Then, for each two functions $f_1, f_2 \in \mathcal{N}_{A;x_0,\alpha}$ we have $f_1(x_0) + f_2(x_0) \leq 2\alpha < \alpha$. Hence each function $f \in \mathcal{N}_{A;x_0,\alpha}$ with $f(x_0) = \alpha$ cannot be decomposed in the desired way.

❑

3.5 Example. Each of the classes $\mathring{\mathcal{N}}^{\mathrm{ep}}$, $\mathring{\mathcal{N}}_+^{\mathrm{ep}}$, $\mathring{\mathcal{N}}_-^{\mathrm{ep}}$, and $\mathring{\mathcal{S}}^{-1}$ has the pole-subset property.

Proof. These assertions follow, since we can write

$$\mathcal{N}^{\mathrm{ep}} = \bigcup_{\substack{M \subseteq (-\infty,0) \\ M \text{ finite}}} \mathcal{N}_{M \cup [0,\infty)}, \quad \mathcal{N}_+^{\mathrm{ep}} = \bigcup_{\gamma \in \mathbb{R}} \big\{ f(z + \gamma) : \, f \in \mathcal{S} \big\} \,, \tag{3.2}$$

$$\mathcal{N}_-^{\mathrm{ep}} = \bigcup_{x_0 \in \mathbb{R}} \mathcal{N}_{(-\infty,x_0];x_0,0}, \quad \mathcal{S}^{-1} = \mathcal{N}_{(-\infty,0];0,0} \,.$$

❑

3.6 Remark.

(*i*) Sometimes it is practical to have available the following consequence of (PSP): Let $f \in \mathcal{K}$ and $T_2, \ldots, T_n \subseteq \sigma(f)$ be nonempty. Then there exist $g_1, g_2, \ldots, g_n \in \mathcal{K}$ such that

$$\sigma(g_1) = \sigma(f), \ \sigma(g_j) = T_j, \ j = 2, \ldots, n, \quad \text{and} \quad f = \sum_{j=1}^{n} g_j \,.$$

This is seen by an obvious induction on n.

(*ii*) All the concrete classes of meromorphic functions we have considered so far actually possess the following property, stronger than (PSP):

(s-PSP) Whenever $f \in \mathcal{K}$ and nonempty subsets $T_1, T_2 \subseteq \sigma(f)$ with $T_1 \cup T_2 = \sigma(f)$ are given, then there exist $g_1, g_2 \in \mathcal{K}$ such that

$$\sigma(g_j) = T_j, \ j = 1, 2, \text{and} \quad f = g_1 + g_2 \,.$$

This follows by inspecting the proofs of the above examples. Actually, the definition (3.1) should be replaced by

$$\tilde{g}_1(z) := \sum_{x_n \in T_1 \setminus T_2} \Big(\frac{1}{x_n - z} - \frac{x_n}{1 + x_n^2}\Big) r_n + \sum_{x_n \in T_1 \cap T_2} \Big(\frac{1}{x_n - z} - \frac{x_n}{1 + x_n^2}\Big) \frac{r_n}{2},$$

$$\tilde{g}_2(z) := \sum_{x_n \in T_2 \setminus T_1} \Big(\frac{1}{x_n - z} - \frac{x_n}{1 + x_n^2}\Big) r_n + \sum_{x_n \in T_1 \cap T_2} \Big(\frac{1}{x_n - z} - \frac{x_n}{1 + x_n^2}\Big) \frac{r_n}{2},$$

and in some places the obvious adjustments should be made. We will not use this fact in the present paper, and therefore will not go into more detail. Let us only note that, similar as in item (i) of the present remark, we can inductively deduce the following property from (s-PSP): Let $f \in \mathcal{K}$, and nonempty subsets $T_1, \dots, T_n \subseteq \sigma(f)$ with $\bigcup_{j=1}^n T_j = \sigma(f)$ be given. Then there exist $g_1, \dots, g_n \in \mathcal{K}$ such that

$$\sigma(g_j) = T_j,\ j = 1, \dots, n, \quad \text{and} \quad f = \sum_{j=1}^n g_j .$$

4 Characterizations of n-$\mathcal{K}$-pairs

In this section we state and prove our main results on n-$\mathcal{K}$-pairs, which give characterizations in different terms. Besides the pole-subset-property, they also depend on the following properties of the class $\mathcal{K}$ under consideration:

($\mathbf{P}_1$) Each function $f \in \mathcal{K}$ has only simple poles.

($\mathbf{P}_2$) If $f_1, \dots, f_n \in \mathcal{K}$, then

$$\sigma\Big(\sum_{i=1}^n f_i\Big) = \bigcup_{i=1}^n \sigma(f_i) .$$

4.1 Remark.

(i) The condition (P_1) can equivalently be stated as follows: If $f \in \mathcal{K}$ and $f = P^{-1}Q$ with some functions $P, Q \in H(\mathbb{C})$, then $\mathfrak{d}_P(w) \leq \mathfrak{d}_Q(w) + 1$, $w \in \mathbb{C}$.

(ii) The class $\mathring{\mathcal{N}}$, and hence also each of its subclasses, satisfies (P_1) and (P_2).

4.1 Characterization in terms of zeros of P and Q

Let us first investigate how to recognize n-$\mathcal{K}$-pairs (Q, P) among all the possible quotient-representations of a function $f \in \mathcal{K}$ in terms of the zero sets of the functions P and Q.

4.2 Theorem. *Assume that $\mathcal{K} \subseteq \mathcal{M}(\mathbb{C})$ satisfies* (PSP), (P_1)*, and* (P_2)*. Let $P, Q \in H(\mathbb{C})$ be such that $P^{-1}Q \in \mathcal{K}$, and let $n \in \mathbb{N}$. Then (Q, P) is an n-$\mathcal{K}$-pair if and only if the following condition* (B) *holds:*

(B) *The functions P and Q satisfy*

(B$_1$) *If* $w \in \mathbb{C}$ *and* $P(w) = 0$*, then* $\mathfrak{d}_P(w) = \mathfrak{d}_Q(w) + 1$*,*

(B$_2$) $\max_{w\in\mathbb{C}} \mathfrak{d}_P(w) = n$*.*

Proof. To start with let us note that (B$_1$) is equivalent to the condition $\sigma(P^{-1}Q) = Z(P)$: The inclusion $\sigma(P^{-1}Q) \subseteq Z(P)$ always holds, and equality prevails if and only if for each zero w of P we have $\mathfrak{d}_P(w) > \mathfrak{d}_Q(w)$. However, by Remark 4.1, (i), this is equivalent to (B$_1$).
Step 1: Assume that (Q, P) is an n-$\mathcal{K}$-pair, and let Q_i, P_i, $i = 1, \dots, n$, be as in Definition 1.1. Clearly, we have

$$\sigma\Big(\frac{Q}{P}\Big) \subseteq Z(P) = \bigcup_{i=1}^{n} Z(P_i)\,.$$

Let $w \in Z(P_i)$ for some $i \in \{1, \dots, n\}$. Since P_i and Q_i have no common zeros, it follows that $w \in \sigma(P_i^{-1}Q_i)$. By (P$_2$) it follows that $w \in \sigma(P^{-1}Q)$. Thus $\sigma(P^{-1}Q) = Z(P)$, i.e. (B$_1$) holds.

Since, for each $i \in \{1, \dots, n\}$, the functions P_i and Q_i have no common zeros and $P_i^{-1}Q_i \in \mathcal{K}$, by (P$_1$) each function P_i can have only simple zeros. Thus $\max_{w\in\mathbb{C}} \mathfrak{d}_P(w) \leq n$.

Step 2: Assume that (Q, P) satisfies the conditions (B$_1$) and (B$_2$). We show that (Q, P) is an m-$\mathcal{K}$-pair with some $m \leq n$. Put

$$T_j := \big\{w \in \mathbb{C} : \mathfrak{d}_P(w) \geq j\big\},\ j = 1, \dots, n\,.$$

Note that $T_0 = Z(P) = \sigma(P^{-1}Q)$, and that none of T_j is empty. Remark 3.6, (i), furnishes us with functions $g_1, \dots, g_n \in \mathcal{K}$ such that $\sigma(g_j) = T_j$, $j = 1, \dots, n$, and $P^{-1}Q = \sum_{j=1}^n g_j$. We have

$$\max\big\{-\mathfrak{d}_{g_j}(w), 0\big\} = \begin{cases} 1 & ,\ w \in T_j \\ 0 & ,\ w \notin T_j \end{cases}$$

It follows that $\sum_{j=1}^n \max\{-\mathfrak{d}_{g_j}(w), 0\} = \mathfrak{d}_P$, and Lemma 2.16 yields that (Q, P) is an m-$\mathcal{K}$-pair with some $m \leq n$.

Step 3: The proof of the theorem is now easily completed. If (Q, P) is an n-$\mathcal{K}$-pair, then by Step 1 it satisfies (B$_1$) and (B$_2$) with some number $m \leq n$. However, if we had $m < n$, then Step 2 would yield a representation of (Q, P) with less than n many 1-$\mathcal{K}$-pairs, namely with at most m many. A contradiction to the minimality requirement in Definition 1.1. Conversely, if (Q, P) satisfies (B$_1$) and (B$_2$) with n, then by Step 2 it is an m-$\mathcal{K}$-pair with some $m \leq n$. However, by Step 1, this implies that $\max_{w\in\mathbb{C}} \mathfrak{d}_P(w) \leq m$, and we see that m must be equal to n.

❏

We will in §6 apply Theorem 4.2 with the classes $\mathring{\mathcal{N}}_-^{\mathrm{ep}}$ and $\mathring{\mathcal{S}}^{-1}$. In these cases, the conditions (B) can be stated in a more comprehensive way. The fact which makes this reformulation possible is that for each function $q \in \mathring{\mathcal{N}}^{\mathrm{ep}}$ the poles and zeros of q are contained in some semiaxis $[m, \infty)$ and interlace. We will first provide the necessary argument for the class $\mathring{\mathcal{N}}^{\mathrm{ep}}$, and then reduce to subclasses.

4.3 Proposition. *Let $P, Q \in H(\mathbb{C})$ be such that $P^{-1}Q \in \mathcal{N}^{ep}$ and let $n \in \mathbb{N}$. Then (Q, P) is an n-$\mathring{\mathcal{N}}^{\mathrm{ep}}$-pair if and only if the following conditions (i)–(v) hold:*

(*i*) *$Z(P) \cup Z(Q) \subseteq \mathbb{R}$ and*

$$m := \inf\big(Z(P) \cup Z(Q)\big) > -\infty\,. \tag{4.1}$$

Denote by (μ_k) and (ν_k) the (finite or infinite) sequences of zeros of P and Q, respectively, listed according to their multiplicities and arranged such that

$$m \leq \mu_1 \leq \mu_2 \leq \ldots \quad \textit{and} \quad m \leq \nu_1 \leq \nu_2 \leq \ldots\,.$$

(*ii*) *If $\nu_1 < \mu_1$, then $\nu_1 < \mu_1 \leq \nu_2 \leq \mu_2 \leq \nu_3 \leq \ldots$ and*

$$\forall\, k \in \mathbb{N} : \Big(\mu_k = \nu_{k+1} \iff \nu_{k+1} = \mu_{k+1}\Big)\,.$$

(*iii*) *If $\nu_1 = \mu_1$, then $\mu_1 = \nu_1 = \mu_2 \leq \nu_2 \leq \mu_3 \ldots$ and*

$$\forall\, k \geq 2 : \Big(\mu_k = \nu_k \iff \nu_k = \mu_{k+1}\Big)\,.$$

(*iv*) *If $\nu_1 > \mu_1$, then $\mu_1 < \nu_1 < \mu_2 \leq \nu_2 \leq \mu_3 \ldots$ and*

$$\forall\, k \geq 2 : \Big(\mu_k = \nu_k \iff \nu_k = \mu_{k+1}\Big)\,.$$

(*v*) $\max_{x\in\mathbb{R}} \mathfrak{d}_P(x) = n$.

Proof. Assume that (Q, P) is an n-$\mathring{\mathcal{N}}^{ep}$-pair, so that the conditions (B_1) and (B_2) hold. By (B_1) we have

$$\sigma\Big(\frac{Q}{P}\Big) = Z(P), \quad Z\Big(\frac{Q}{P}\Big) = Z(Q) \setminus Z(P)\,.$$

Since $P^{-1}Q \in \mathcal{N}^{\mathrm{ep}}$, this implies that $Z(P) \cup Z(Q) \subseteq \mathbb{R}$ and that (4.1) holds. Moreover, we see that (*v*) coincides with (B_2). Since the poles and zeros of $P^{-1}Q$ interlace, the conditions (*ii*)–(*iv*) are just another way to state (B_1), i.e. that, if a point x_0 belongs to $Z(P) \cap Z(Q)$, then there must be one more zero of P at x_0 than zeros of Q (thinking in terms of multiplicities).

Conversely, assume that (*i*)–(*v*) hold. Then, by (*i*), the condition (B_2) coincides with (*v*). Moreover, (*ii*)–(*iv*) yield (B_1). Thus (Q, P) is an n-$\mathring{\mathcal{N}}^{\mathrm{ep}}$-pair.

❑

We will pass to subclasses with help of two lemmata. The current aim, namely the below Corollary 4.6, could also be deduced directly from Theorem 4.2. However, in view of some later argumentations we prefer the approach via Lemma 4.4 and Lemma 4.5.

4.4 Lemma. *let $P, Q \in H(\mathbb{C})$, $P = P^{\#}$, $Q = Q^{\#}$ and let $n \in \mathbb{N}$. Moreover, let $\mathcal{K}$ be one of classes*

$$\mathring{\mathcal{N}}_{(-\infty,x_0];x_0,\alpha},\ \mathring{\mathcal{N}}^{\mathrm{ep}}_{-},\ \mathring{\mathcal{S}}^{-1}, \quad \mathring{\mathcal{S}},\ \mathring{\mathcal{N}}^{\mathrm{ep}}_{+}\,.$$

Then the following are equivalent:

(*i*) (Q,P) *is an* n-$\mathcal{K}$-*pair;*

(*ii*) (Q,P) *is an* n-$\mathring{\mathcal{N}}^{\rm ep}$-*pair and* $P^{-1}Q \in \mathcal{K}$.

Proof. First of all note that, since each of the stated classes $\mathcal{K}$ is contained in $\mathring{\mathcal{N}}^{\rm ep}$, an n-$\mathcal{K}$-pair (Q,P) certainly is an m-$\mathring{\mathcal{N}}^{\rm ep}$-pair with some $m \leq n$. Moreover, clearly, $P^{-1}Q \in \mathcal{K}$.

We will in the following show that each n-$\mathring{\mathcal{N}}^{\rm ep}$-pair (Q,P) with $P^{-1}Q \in \mathcal{K}$ is an m-$\mathcal{K}$-pair with some $m \leq n$. This will complete the proof of the asserted equivalence. Throughout the rest of this proof let an n-$\mathring{\mathcal{N}}^{\rm ep}$-pair (Q,P) be given.

Case $\mathcal{K} = \mathring{\mathcal{N}}_{(-\infty,x_0];x_0,\alpha}$*:* Assume that $P^{-1}Q \in \mathcal{N}_{(-\infty,x_0];x_0,\alpha}$. Let $f_1,\dots,f_n \in \mathcal{N}^{\rm ep}$ be chosen according to (2.12). By (P_2), this implies that $\sigma(f_j) \subseteq (x_0,\infty)$. Define

$$g_j(z) := f_j(z) - f_j(x_0),\ j=1,\dots,n-1, \quad g_n(z) := f_n(z) + \sum_{j=1}^{n-1} f_j(x_0)\,.$$

Then, clearly, $\sum_{j=1}^n g_j = \sum_{j=1}^n f_j = P^{-1}Q$. Moreover, $\sigma(g_j) = \sigma(f_j) \subseteq (x_0,\infty)$, $j=1,\dots,n$, $\max\{-\mathfrak{d}_{g_j},0\} = \max\{-\mathfrak{d}_{f_j},0\}$, and

$$g_j(x_0) = 0,\ j=1,\dots,n-1, \quad g_n(x_0) = \frac{Q}{P}(x_0) \leq \alpha\,.$$

Thus $g_j \in \mathring{\mathcal{N}}_{(-\infty,x_0];x_0,\alpha}$, $j=1,\dots,n$, and we conclude that (Q,P) is an m-$\mathring{\mathcal{N}}_{(-\infty,x_0];x_0,\alpha}$-pair with some $m \leq n$.

Case $\mathcal{K} = \mathring{\mathcal{S}}^{-1}, \mathring{\mathcal{N}}_-^{\rm ep}$*:* Since $\mathcal{S}^{-1} = \mathcal{N}_{(-\infty,0];0,0}$ the case $\mathcal{K} = \mathring{\mathcal{S}}^{-1}$ has already been covered. Next, we have $\mathcal{N}_-^{\rm ep} = \bigcup_{\gamma\in\mathbb{R}} \mathcal{N}_{(-\infty,\gamma];\gamma,0}$. Hence, if $P^{-1}Q \in \mathcal{N}_-^{\rm ep}$, then this quotient belongs to $\mathcal{N}_{(-\infty,\gamma];\gamma,0}$ for some $\gamma \in \mathbb{R}$. By the already settled case, (Q,P) is an m-$\mathring{\mathcal{N}}_{(-\infty,\gamma];\gamma,0}$-pair with some $m \leq n$, and hence also an m'-$\mathring{\mathcal{N}}_-^{\rm ep}$-pair with some $m' \leq n$.

Case $\mathcal{K} = \mathring{\mathcal{S}}$*:* Assume that $P^{-1}Q \in \mathcal{S}$, and put $a := \lim_{x\to-\infty} P(x)^{-1}Q(x)$. Let again $f_1,\dots,f_n \in \mathring{\mathcal{N}}^{\rm ep}$ be chosen according to (2.12). By (P_2) it follows that $\sigma(f_j) \subseteq [0,\infty)$. Since each of the functions $f_j(x)$ is nondecreasing on $(-\infty,0)$, we have

$$\sum_{j=1}^n \lim_{x\to-\infty} f_j(x) = \lim_{x\to-\infty} P(x)^{-1}Q(x) = a \geq 0\,.$$

Thus $a_j := \lim_{x\to-\infty} f_j > -\infty$. Define functions

$$g_j := \begin{cases} f_1 - a_1 + a & ,\ j=1 \\ f_j - a_j & ,\ j=2,\dots,n \end{cases}$$

Then $g_j \in \mathcal{S}$ and $\max\{-\mathfrak{d}_{g_j},0\} = \max\{-\mathfrak{d}_{f_j},0\} = \mathfrak{d}_P$, and we conclude that (Q,P) is an m-$\mathring{\mathcal{S}}$-pair with some $m \leq n$.

Case $\mathcal{K} = \mathring{\mathcal{N}}_+^{\rm ep}$*:* This case is now deduced similar as the case of $\mathring{\mathcal{N}}_-^{\rm ep}$ above using (3.2).

❑

4.5 Lemma. *Let $P, Q \in H(\mathbb{C})$ be such that $P^{-1}Q \in \mathcal{N}^{ep}$, and assume that (Q, P) satisfies (B_1). Then $P^{-1}Q$ belongs to*

(*i*) $\mathcal{N}_{+}^{\text{ep}}$*, if and only if* $\min Z(P) \leq \min Z(Q)$.

(*ii*) $\mathcal{S}$*, if and only if* $0 \leq \min Z(P) \leq \min Z(Q)$.

(*iii*) $\mathcal{N}_{-}^{\text{ep}}$*, if and only if* $\min Z(P) > \min Z(Q)$.

(*iv*) $\mathcal{S}^{-1}$*, if and only if* $\min Z(P) > \min Z(Q) \geq 0$.

Proof. We have $\sigma(P^{-1}Q) = Z(P)$ and $Z(P^{-1}Q) = Z(Q) \setminus Z(P)$. Moreover, $P^{-1}(x)Q(x)$ is nondecreasing on each interval between two poles. Thus, for example, $P^{-1}(x)Q(x) \leq 0$ for all $x \in (-\infty, 0)$ if and only if the first of its zeros lies left of the first of its poles and is nonnegative. This shows (*iv*); the other cases are treated similarly.

❑

4.6 Corollary. *Let $P, Q \in H(\mathbb{C})$ be such that $P^{-1}Q \in \mathcal{N}^{ep}$ and let $n \in \mathbb{N}$. Then (Q, P) is an*

(*i*) *n-$\mathring{\mathcal{N}}_{+}^{\text{ep}}$-pair if and only if the conditions (i) and (iii)–(v) of Proposition 4.3, and* $\min Z(P) \leq \min Z(Q)$ *hold.*

(*ii*) *n-$\mathring{\mathcal{S}}$-pair if and only if* $0 \leq \min Z(P) \leq \min Z(Q)$*, and the conditions (iii)–(v) of Proposition 4.3 hold.*

(*iii*) *n-$\mathring{\mathcal{N}}_{-}^{\text{ep}}$-pair if and only if the conditions (i), (ii), and (v) of Proposition 4.3, and* $\min Z(P) > \min Z(Q)$ *hold.*

(*iv*) *n-$\mathring{\mathcal{S}}^{-1}$-pair if and only if* $0 \leq \min Z(P) < \min Z(Q)$*, and the conditions (ii) and (v) of Proposition 4.3 hold.*

Proof. Let us prove the assertion (*i*); the other items are deduced in the same manner. Assume first that (Q, P) is an n-$\mathring{\mathcal{N}}_{+}^{\text{ep}}$-pair. Then (*i*)–(*v*) of Proposition 4.3 hold. Since, by Theorem 4.2, (Q, P) satisfies (B_1), Lemma 4.5 shows $\min Z(P) \leq \min Z(Q)$.

Conversely, assume that the conditions stated in the present item (*i*) hold. Then the condition (*ii*) of Proposition 4.3 is trivially satisfied, and we condlude that (Q, P) is an n-$\mathring{\mathcal{N}}^{\text{ep}}$-pair. Thus it satisfies (B_1), and we obtain from Lemma 4.5 that $P^{-1}Q \in \mathcal{N}_{+}^{\text{ep}}$. According to Lemma 4.4, it follows that (Q, P) actually is an n-$\mathring{\mathcal{N}}_{+}^{\text{ep}}$-pair.

❑

4.2 Characterization in terms of $\phi_{P,Q}$ or $\phi_{Q,-P}$

Our second task is to translate the property of (Q, P) being an n-$\mathcal{K}$-pair into properties of the function $\phi_{P,Q}$ or $\phi_{Q,-P}$, respectively, cf. Definition 2.5. This is, actually, more involved than dealing with the zero sets of P and Q. We will in the sequel denote by $[x]$ the integer part of a real number x.

4.7 Theorem. *Assume that* $\mathcal{K} \subseteq \mathcal{M}(\mathbb{C})$ *satisfies* (PSP), (P_1), *and* (P_2). *Let* $P, Q \in H(\mathbb{C})$ *be such that* $P^{-1}Q \in \mathcal{K}$ *and let* $n \in \mathbb{N}$. *Assume, moreover, that* $P = P^{\#}$, $Q = Q^{\#}$, *and that the function* P *has at least one zero in* $\mathbb{C}$. *Then* (Q, P) *is an* n-$\mathcal{K}$*-pair if and only if one (and hence both) of the following (equivalent) conditions* (C) *and* (D) *holds:*

(C) *Put* $\phi(z) := \phi_{P,Q}(z) = P(z^2) - izQ(z^2)$ *and* $\delta_w := \min\{\mathfrak{d}_\phi(w), \mathfrak{d}_\phi(\overline{w})\}$, $w \in \mathbb{C}$. *Then*

- (C_1) *If* $w \in \mathbb{C}$ *and* $\delta_w > 0$, *then* $\phi^{(\delta_w)}(w) + \overline{\phi^{(\delta_w)}(\overline{w})} = 0$.
- (C_2) $\max\left(\{\delta_w : w \in \mathbb{C} \setminus \{0\}\} \cup \{[\frac{1}{2}\delta_0]\}\right) = n - 1$.

(D) *Put* $\psi(z) := \phi_{Q,-P}(z) = Q(z^2) + izP(z^2)$ *and* $\epsilon_w := \min\{\mathfrak{d}_\psi(w), \mathfrak{d}_\psi(\overline{w})\}$, $w \in \mathbb{C}$. *Then*

- ($D_{1,1}$) *If* $w \in \mathbb{C} \setminus \{0\}$ *and* $\epsilon_w > 0$, *then* $\psi^{(\epsilon_w)}(w) - \overline{\psi^{(\epsilon_w)}(\overline{w})} = 0$.
- ($D_{1,2}$) *If* $\epsilon_0 > 1$, *then* ϵ_0 *is even and* $\psi^{(\epsilon_0+1)}(0) = 0$.
- (D_2) $\max\left(\{\epsilon_w : w \in \mathbb{C} \setminus \{0\}\} \cup \{[\frac{1}{2}\epsilon_0]\}\right) = n - 1$.

4.8 Remark. Let us make the following facts explicit:

(*i*) If the function P in the statement of Theorem 4.7 has no zeros, then trivially (Q, P) is a 1-$\mathcal{K}$-pair.

(*ii*) We have $\delta_w > 0$ or $\epsilon_w > 0$, respectively, if and only if either w is a real zero of ϕ or ψ, or $(w, \overline{w})$ is a conjugate pair of nonreal zeros of ϕ or ψ. If w is a real zero of ϕ or ψ, then $\delta_w = \mathfrak{d}_\phi(w)$ or $\epsilon_w = \mathfrak{d}_\psi(w)$, respectively.

(*iii*) If (C_1) or ($D_{1,1}$), respectively, hold, then for each pair $(w, \overline{w})$ of nonreal zeros of ϕ or ψ we have $\delta_w = \mathfrak{d}_\phi(w) = \mathfrak{d}_\phi(\overline{w})$ or $\epsilon_w = \mathfrak{d}_\psi(w) = \mathfrak{d}_\psi(\overline{w})$.

(*iv*) If $w \in \mathbb{R}$, then we have $\overline{\phi^{(m)}(\overline{w})} = \overline{\phi^{(m)}(w)}$ and $\overline{\psi^{(m)}(\overline{w})} = \overline{\psi^{(m)}(w)}$, respectively. Thus, for real points w, the requirement in (C_1) or ($D_{1,1}$) says nothing else but $\operatorname{Re}\phi^{(\mathfrak{d}_\phi(w))}(w) = 0$ or $\operatorname{Im}\psi^{(\mathfrak{d}_\psi(w))}(w) = 0$, respectively.

(*v*) The functions ϕ and ψ satisfies the functional equations $\phi^{\#}(z) = \phi(-z)$ and $\psi^{\#}(z) = \psi(-z)$. Hence also $(\phi^{\#})^{(m)}(z) = (-1)^m\phi^{(m)}(-z)$ and $(\psi^{\#})^{(m)}(z) = (-1)^m\psi^{(m)}(-z)$. Thus the requirement in (C_1) or ($D_{1,1}$), respectively, could also be written as

$$\phi^{(\delta_w)}(w) + (-1)^{\delta_w}\phi^{(\delta_w)}(-w) = 0, \quad \psi^{(\delta_w)}(w) - (-1)^{\delta_w}\psi^{(\delta_w)}(-w) = 0\,. \tag{4.2}$$

(*vi*) Using the symmetry of ϕ, it is easy to see that the condition (C_1) implies $\mathfrak{d}_{\phi_{P,Q}}(0) \in \{0\} \cup (2\mathbb{N} - 1)$.

Proof (of 'n*-*$\mathcal{K}$*-pair* $\iff$ (C)*').* Since $P = P^{\#}$ and $Q = Q^{\#}$, we have $\phi^{\#}(z) = P(z^2) + izQ(z^2)$. Thus $2P(z^2) = \phi(z) + \phi^{\#}(z)$ and $-2izQ(z^2) = \phi(z) - \phi^{\#}(z)$. With account of $\delta_\omega > 0$ this shows that

$$\delta_w = \min\left\{\mathfrak{d}_\phi(w), \mathfrak{d}_\phi(\overline{w})\right\} = \min\left\{\mathfrak{d}_{P\circ X^2}(w), \mathfrak{d}_{X\cdot(Q\circ X^2)}(w)\right\}.$$

Since, for every entire function f,

$$\mathfrak{d}_{f\circ X^2}(w)=\begin{cases}\mathfrak{d}_f(w^2)\,, & w\neq 0\\ 2\mathfrak{d}_f(0)\,, & w=0\end{cases}\,,\quad \mathfrak{d}_{X\cdot f}(w)=\begin{cases}\mathfrak{d}_f(w) & , w\neq 0\\ \mathfrak{d}_f(0)+1\,, & w=0\end{cases} \tag{4.3}$$

it follows that

$$\delta_w=\begin{cases}\min\big\{\mathfrak{d}_P(w^2),\mathfrak{d}_Q(w^2)\big\} & , \quad w\neq 0\\ \min\big\{2\mathfrak{d}_P(0),2\mathfrak{d}_Q(0)+1\big\}\,, & \quad w=0\end{cases} \tag{4.4}$$

In particular, we see that $\delta_w>0$ implies $P(w^2)=0$. Moreover, we compute

$$2\frac{d^m(P\circ X^2)}{dX^m}(z)=\phi^{(m)}(z)+\big(\phi^{\#}\big)^{(m)}(z)=\phi^{(m)}(z)+\big(\phi^{(m)}\big)^{\#}(z) \tag{4.5}$$

Step 1, $(B_1)\Rightarrow(C_1)$*:* Let $w\in\mathbb{C}$ with $\delta_w>0$ be given. Then $P(w^2)=0$ and hence, by (B_1), $\mathfrak{d}_P(w^2)=\mathfrak{d}_Q(w^2)+1$. We obtain from (4.4) and (4.3) that

$$\delta_w=\left\{\begin{array}{ll}\mathfrak{d}_P(w^2)-1\,, & w\neq 0\\ 2\mathfrak{d}_P(0)-1\,, & w=0\end{array}\right\}=\mathfrak{d}_{P\circ X^2}(w)-1 \tag{4.6}$$

Hence, by (4.5),

$$\phi^{(\delta_w)}(w)+\big(\phi^{(\delta_w)}\big)^{\#}(w)=2\frac{d^{\delta_w}(P\circ X^2)}{dX^{\delta_w}}(w)=0\,.$$

Step 2, $(C_1)\Rightarrow(B_1)$*:* Let $w\in\mathbb{C}$ with $P(w)=0$ be given. In order to establish (B_1) it is enough to show that $\mathfrak{d}_P(w)>\mathfrak{d}_Q(w)$, since by (P_1) always $\mathfrak{d}_P(w)\leq\mathfrak{d}_Q(w)+1$.

Assume on the contrary that $\mathfrak{d}_P(w)\leq\mathfrak{d}_Q(w)$. In particular, this yields that $Q(w)=0$. Let v be a square root of w, then $\phi(v)=\phi(\overline{v})=0$, i.e. $\delta_v>0$. Since $2P(z^2)=\phi(z)+\phi^{\#}(z)$, the relation

$$\frac{d^m(P\circ X^2)}{dX^m}(v)=0$$

holds for each $m<\delta_v$. As it is seen from (4.5), the validity of (C_1) implies that it also holds for $m=\delta_v$. This shows that $\mathfrak{d}_{P\circ X^2}(v)>\delta_v$. Using (4.3), (4.4), and our indirect hypothesis $\mathfrak{d}_P(w)\leq\mathfrak{d}_Q(w)$, we derive the following contradiction (note here that $v=0$ if and only $w=0$):

$$\mathfrak{d}_P(w)=\left\{\begin{array}{ll}\mathfrak{d}_{P\circ X^2}(v) & , \ w\neq 0\\ \frac12\mathfrak{d}_{P\circ X^2}(0)\,, & w=0\end{array}\right\}>\left\{\begin{array}{ll}\delta_v & , \ w\neq 0\\ \frac12\delta_0\,, & w=0\end{array}\right\}=$$

$$=\left\{\begin{array}{ll}\min\{\mathfrak{d}_P(w),\mathfrak{d}_Q(w)\} & , \ w\neq 0\\ \frac12\min\{2\mathfrak{d}_P(0),2\mathfrak{d}_Q(0)+1\}\,, & w=0\end{array}\right\}=\mathfrak{d}_P(w)$$

Step 3: We show that (B_1) implies

$$\begin{aligned}\max\Big(\big\{\delta_w:\,w\in\mathbb{C}\setminus\{0\}\big\}\cup\big\{\big[\tfrac{1}{2}\delta_0\big]\big\}\Big)=&\\ =\max\big\{\mathfrak{d}_P(u):\,u\in\mathbb{C}\big\}-1&\end{aligned} \tag{4.7}$$

Both sides of (4.7) are nonnegative numbers; recall here that by assumption the function P has at least one zero. Hence, in order to establish (4.7), it is enough to look at the nonzero elements on either side of (4.7).

If $w \in \mathbb{C} \setminus \{0\}$ and $\delta_w > 0$, then $\delta_w = \mathfrak{d}_P(w^2) - 1$, cf. (4.6). Thus δ_w is less or equal to the maximum on the right side of (4.7). If $\delta_0 > 1$, then, again by (4.6), we have $[\frac{1}{2}\delta_0] = [\mathfrak{d}_P(0) - \frac{1}{2}] = \mathfrak{d}_P(0) - 1$. Together it follows that the inequality '$\leq$' in (4.7) holds.

For the converse, let first $u \in \mathbb{C} \setminus \{0\}$, $\mathfrak{d}_P(u) > 1$. Then $\mathfrak{d}_Q(u) = \mathfrak{d}_P(u) - 1 > 0$, and hence $\delta_w > 0$ where w denotes a square root of u. Thus $\mathfrak{d}_P(u) - 1 = \delta_w$. If $\mathfrak{d}_P(0) > 1$, then $\mathfrak{d}_Q(0) > 0$ and it follows that $\delta_0 > 1$, actually $\delta_0 \geq 3$. Thus $[\frac{1}{2}\delta_0] = \mathfrak{d}_P(0) - 1$. Together the inequality '$\geq$' in (4.7) follows.

Step 4, finish of proof: If (Q, P) is an n-$\mathcal{K}$-pair, then by Theorem 4.2 the conditions (B_1) and (B_2) hold. By Step 1 (C_1) follows, and by Step 3 (4.7) holds. However, (4.7) together with (B_2) gives (C_2).

Conversely, assume that (C_1) and (C_2) hold. Step 2 yields (B_1), and Step 3 gives in turn (4.7). Now (4.7) in conjunction with (C_2) implies (B_2). By Theorem 4.2, (Q, P) is an n-$\mathcal{K}$-pair.

❑

The case of the function ψ is treated in a much similar way.

Proof (of 'n-$\mathcal{K}$-pair $\iff$ (D)'). We have $\psi^\#(z) = Q(z^2) - izP(z^2)$, and thus $2Q(z^2) = \psi(z) + \psi^\#(z)$ and $2izP(z^2) = \psi(z) - \psi^\#(z)$. This implies that

$$\epsilon_w = \begin{cases} \min\left\{\mathfrak{d}_Q(w^2), \mathfrak{d}_P(w^2)\right\} & , \quad w \neq 0 \\ \min\left\{2\mathfrak{d}_Q(0), 2\mathfrak{d}_P(0) + 1\right\} & , \quad w = 0 \end{cases} \tag{4.8}$$

In particular, we see that, for $w \in \mathbb{C} \setminus \{0\}$, $\epsilon_w > 0$ implies that $P(w^2) = 0$. Similarly, $\epsilon_0 > 1$ implies $P(0) = 0$. Moreover, we compute

$$\begin{aligned} &2iz\frac{d^m(P \circ X^2)}{dX^m}(z) + 2im\frac{d^{m-1}(P \circ X^2)}{dX^{m-1}}(z) = \\ &= \psi^{(m)}(z) - \left(\psi^\#\right)^{(m)}(z) = \psi^{(m)}(z) - \left(\psi^{(m)}\right)^\#(z)\,. \end{aligned} \tag{4.9}$$

Step 1, (B_1)$\Rightarrow$($D_{1,1}$),($D_{1,2}$)*:* Let $w \in \mathbb{C} \setminus \{0\}$ with $\epsilon_w > 0$ be given. Then $P(w^2) = 0$, and hence $\mathfrak{d}_P(w^2) = \mathfrak{d}_Q(w^2) + 1$. It follows from (4.8) that $\epsilon_w = \mathfrak{d}_Q(w^2)$, in particular $\epsilon_w < \mathfrak{d}_P(w^2) = \mathfrak{d}_{P \circ X^2}(w)$. We conclude from (4.9) that

$$\psi^{(\epsilon_w)}(w) - \overline{\psi^{(\epsilon_w)}(\overline{w})} = 0\,.$$

Assume next that $\epsilon_0 > 1$. Again it follows that $P(0) = 0$ and thus that $\mathfrak{d}_P(0) = \mathfrak{d}_Q(0) + 1$. This implies, by (4.8), that $\epsilon_0 = 2\mathfrak{d}_Q(0)$. In particular, ϵ_0 is even. Since the function $Q \circ X^2$ is even, we have $(Q \circ X^2)^{(m)}(0) = 0$ whenever m is odd. Since $\epsilon_0 = 2\mathfrak{d}_Q(0) < 2\mathfrak{d}_P(0)$, we have $(P \circ X^2)^{(m)}(0) = 0$ for $m \leq \epsilon_0$. Together we obtain that

$$\psi^{(\epsilon_0+1)}(0) = (Q \circ X^2)^{(\epsilon_0+1)}(0) + i(\epsilon_0 + 1)(P \circ X^2)^{(\epsilon_0)}(0) = 0\,.$$

Step 2, $(D_{1,1}),(D_{1,2})\Rightarrow(B_1)$*:* Let $w \in \mathbb{C}$ with $P(w) = 0$ be given. In order to establish (B_1) it suffices to show $\mathfrak{d}_P(w) > \mathfrak{d}_Q(w)$.

Assume on the contrary that $\mathfrak{d}_P(w) \leq \mathfrak{d}_Q(w)$. Then also $Q(w) = 0$ and, if v denotes a square root of w, therefore $\epsilon_v > 0$. In case $w = 0$, actually $\epsilon_0 > 1$. We will next show that

$$\frac{d^m(P \circ X^2)}{dX^m}(v) = 0,\ m \leq \epsilon_v\,. \tag{4.10}$$

If $v \neq 0$, this follows by an inductive argument starting from $P(w) = 0$, proceeding step by step with the help of (4.9) and, in the last step $m = \epsilon_v$, using $(D_{1,1})$. Consider the case that $v = 0$. Then the right hand side of (4.9), evaluated at $z = 0$, trivially vanishes whenever $m < \epsilon_0$. If $m = \epsilon_0$, it vanishes since, by $(D_{1,2})$, ϵ_0 is an even number, $Q \circ X^2$ and $P \circ X^2$ are even function and $Q \circ X^2$ takes real values on the real line. Finally, for $m = \epsilon_0 + 1$, it vanishes by $(D_{1,2})$. Again an inductive argument using (4.9), this time up to $m = \epsilon_0 + 1$, will apply and give (4.10).

We see from (4.10) that $\mathfrak{d}_{P\circ X^2}(v) > \epsilon_v$. Together with (4.8) and $\mathfrak{d}_P(w) \leq \mathfrak{d}_Q(w)$, we derive a contradiction:

$$\mathfrak{d}_P(w) = \left\{\begin{array}{ll} \mathfrak{d}_{P\circ X^2}(v) & ,\ \ w \neq 0 \\ \frac{1}{2}\mathfrak{d}_{P\circ X^2}(0)\,, & w = 0 \end{array}\right\} > \left\{\begin{array}{ll} \epsilon_v & ,\ \ w \neq 0 \\ \frac{1}{2}\epsilon_0\,, & w = 0 \end{array}\right\} =$$

$$= \left\{\begin{array}{ll} \min\{\mathfrak{d}_Q(w), \mathfrak{d}_P(w)\} & ,\ \ w \neq 0 \\ \frac{1}{2}\min\{2\mathfrak{d}_Q(0), 2\mathfrak{d}_P(0)+1\}\,, & w = 0 \end{array}\right\} \geq \mathfrak{d}_P(w)$$

Step 3: We show that (B_1) implies

$$\begin{aligned} \max\Big(\big\{\epsilon_w :\ w \in \mathbb{C}\setminus\{0\}\big\} \cup \big\{\big[\frac{1}{2}\epsilon_0\big]\big\}\Big) &= \\ = \max\big\{\mathfrak{d}_P(u) :\ u \in \mathbb{C}\big\} - 1 \end{aligned} \tag{4.11}$$

Since we assume that P has at least one zero, the right hand side of this relation, let us denote it by M_r, is nonnegative. Thus, if $\epsilon_w = 0$ or $\epsilon_0 \leq 1$, certainly $\epsilon_w \leq M_r$ or $[\frac{1}{2}\epsilon_0] \leq M_r$, respectively. Consider a point $w \in \mathbb{C}\setminus\{0\}$ with $\epsilon_w > 0$. Then, as we saw in Step 1, $\epsilon_w < \mathfrak{d}_P(w^2)$. Thus $\epsilon_w \leq \mathfrak{d}_P(w^2) - 1 \leq M_r$. Finally, assume that $\epsilon_0 > 1$. Then $\epsilon_0 = 2\mathfrak{d}_Q(0) = 2\mathfrak{d}_P(0) - 2$, and thus $[\frac{1}{2}\epsilon_0] = \mathfrak{d}_P(0) - 1 \leq M_r$. We see that the inequality '$\leq$' in (4.11) holds.

For the converse note first that the maximum on the left side of (4.11) is trivially nonnegative. Let $u \in \mathbb{C}$ be given such that $\mathfrak{d}_P(u) > 1$, and denote by w a square root of u. Then $Q(0) = 0$, and hence $\epsilon_w > 0$ in case $u \neq 0$, and $\epsilon_0 > 1$ in case $u = 0$, respectively. It follows from (4.8) and (B_1) that $\epsilon_w = \mathfrak{d}_P(u) - 1$ or $\epsilon_0 = 2\mathfrak{d}_P(0) - 2$, respectively, and hence that $\mathfrak{d}_P(u) - 1$ is less or equal to the maximum on the left side of (4.11). Thus, in (4.11), also '$\geq$' holds.

Step 4: The proof is now finished in exactly the same way as in the last step of the proof of equivalence with (C).

❑

Next we will deduce characterizations for (Q, P) being an n-$\mathcal{K}$-pair for some concrete classes $\mathcal{K}$. These results are not just reformulations of the conditions

(D) of Theorem 4.7; in contrast to Theorem 4.2 and Theorem 4.7, in the below Proposition 4.9, Corollary 4.13, and Corollary 4.14, we do not require the a priori knowledge that $P^{-1}Q \in \mathcal{K}$.

In view of our later needs, we confine our attention to the function ψ, and the classes $\mathring{\mathcal{N}}^{\rm ep}$, $\mathring{\mathcal{N}}^{\rm ep}_{-}$, and $\mathring{\mathcal{S}}^{-1}$. However, a similar investigation could be undertaken for the function ϕ instead of ψ and/or for the classes $\mathring{\mathcal{N}}^{\rm ep}_{+}$, $\mathring{\mathcal{S}}$ instead of $\mathring{\mathcal{N}}^{\rm ep}$, $\mathring{\mathcal{S}}^{-1}$, or $\mathring{\mathcal{N}}^{\rm ep}_{-}$. The key result is the following characterization for $\mathcal{K} = \mathring{\mathcal{N}}^{\rm ep}$; the other characterizations will be deduced with the help of Lemma 4.4 and Lemma 4.5.

4.9 Proposition. *Let $P, Q \in H(\mathbb{C})$, $P = P^{\#}$, $Q = Q^{\#}$, let $n \in \mathbb{N}$, and put $\psi := \phi_{Q,-P}$. Then (Q, P) is an n-$\mathring{\mathcal{N}}^{\rm ep}$-pair if and only if the following conditions (i)–$(viii)$ are satisfied:*

- (i) *The function ψ has only finitely many conjugate pairs of nonreal zeros. Each such pair $(w, \overline{w})$ is located on the imaginary axis, and w and $\overline{w}$ have the same multiplicity.*

Let $(i\lambda_1, -i\lambda_1), \ldots, (i\lambda_p, -i\lambda_p)$, $0 < \lambda_1 < \ldots < \lambda_p$, be all the conjugate pairs of nonreal zeros of ψ, put $\epsilon_j := \mathfrak{d}_\psi(i\lambda_j) = \mathfrak{d}_\psi(-i\lambda_j)$, and

$$\Lambda(z) := \prod_{j=1}^{p} (z - i\lambda_j)^{\epsilon_j} (z + i\lambda_j)^{\epsilon_j} = \prod_{j=1}^{p} (z^2 + \lambda_j^2)^{\epsilon_j} .$$

- (ii) *The function $\Lambda^{-1}\psi$ has only finitely many zeros in $\mathbb{C}^+$. These are all simple and located on the imaginary axis.*

Let $iy_1, \ldots, iy_\kappa$, $0 < y_1 < \ldots < y_\kappa$, be all the zeros of $\Lambda^{-1}\psi$ in $\mathbb{C}^+$, and put

$$Y(z) := \prod_{j=1}^{\kappa} \Big(1 - \frac{z}{iy_j}\Big) .$$

- (iii) *The function $(Y\Lambda)^{-1}\psi$ belongs to the class $\mathcal{H}B$.*
- (iv) *We have $\psi^{(\epsilon_j)}(i\lambda_j) = (-1)^{\epsilon_j} \psi^{(\epsilon_j)}(-i\lambda_j)$, $j = 1, \ldots, p$.*
- (v) *If $w \in \mathbb{R} \setminus \{0\}$ is a zero of ψ with multiplicity α, then $\operatorname{Im} \psi^{(\alpha)}(w) = 0$.*
- (vi) *$\mathfrak{d}_\psi(0) \in \{0, 1\} \cup 2\mathbb{N}$. If $\mathfrak{d}_\psi(0) > 1$, then $\psi^{(\mathfrak{d}_\psi(0)+1)}(0) = 0$.*
- (vii) *For each $k \in \{2, \ldots, \kappa\}$, the number $\sum_{w \in [-iy_{k-1}, -iy_k]} (\mathfrak{d}_\psi(w) - \mathfrak{d}_\Lambda(w))$ is odd. The number $\sum_{w \in (0, -iy_1]} (\mathfrak{d}_\psi(w) - \mathfrak{d}_\Lambda(w))$ is odd if $\mathfrak{d}_\psi(0) = 1$ and even otherwise.*
- $(viii)$ $\max\big(\{\epsilon_j : j = 1, \ldots, p\} \cup \{\mathfrak{d}_\psi(x) : x \in \mathbb{R} \setminus \{0\}\} \cup \{[\frac{1}{2}\mathfrak{d}_\psi(0)]\}\big) = n - 1.$

Proof. Let us first settle the case that P is zerofree. Then the function $\psi := \phi_{Q,-P}$ has no conjugate pairs of nonreal zeros and no real zeros. Hence the conditions (i), (iv), (v) are void, the condition (vi) is satisfied, and the value of the maximum in $(viii)$ is 0. By [PW, Theorem 3.1], the conditions (ii), (iii), and (vii) together are equivalent to $P^{-1}Q \in \mathcal{N}^{\rm ep}$ (recall here also [KWW2, Remark 4.2, (iii)]). In turn, this is equivalent to (Q, P) being a 1-$\mathring{\mathcal{N}}^{\rm ep}$-pair. We see that the equivalence asserted in the present proposition holds.

For the rest of this proof we assume that P has at least one zero. Assume that the conditions (i)–$(viii)$ are satisfied. Our first task is to show that $P^{-1}Q \in \mathcal{N}^{\mathrm{ep}}$. To this end we will employ [PW, Theorem 3.1]. Put $E := \Lambda^{-1}\psi$. Then, clearly, $E^{\#}(z) = E(-z)$. Condition (iii) together with, e.g., [PW, Remark 2.3, (ii)] implies that the overall hypothesis of this theorem is satisfied.

Write $E(z) = A(z^2) - izB(z^2)$ with $A, B \in H(\mathbb{C})$, $A = A^{\#}$, $B = B^{\#}$. In view of (vi), the conditions (ii) and (vii) are exactly what is needed to apply [PW, Theorem 3.1]. We obtain $A(z) - iB(z) \in \mathcal{H}B$ and that $\inf Z(A) > -\infty$. This, however, shows that $A^{-1}B \in \mathcal{N}^{\mathrm{ep}}$.

Next we compute

$$Q(z^2) + izP(z^2) = \psi(z) = \Lambda(z)E(z) = \Lambda(z)A(z^2) - iz\Lambda(z)B(z^2) =$$

$$= \Big(\prod_{j=1}^{p} (z^2 + \lambda_j^2)^{\epsilon_j} \cdot A(z^2) \Big) - iz \Big(\prod_{j=1}^{p} (z^2 + \lambda_j^2)^{\epsilon_j} \cdot B(z^2) \Big) .$$

Hence

$$Q(z) = \prod_{j=1}^{p} (z + \lambda_j^2)^{\epsilon_j} \cdot A(z), \quad P(z) = - \prod_{j=1}^{p} (z + \lambda_j^2)^{\epsilon_j} \cdot B(z) , \tag{4.12}$$

and it follows that $-Q^{-1}P \in \mathcal{N}^{\mathrm{ep}}$. However, a function f belongs to $\mathcal{N}^{\mathrm{ep}}$ if and only if $-f^{-1}$ does, and this gives $P^{-1}Q \in \mathcal{N}^{\mathrm{ep}}$.

In the next step we show that (i)–$(viii)$ impliy that (D) holds, and thus that (Q, P) is an n-$\mathring{\mathcal{N}}^{\mathrm{ep}}$-pair: The conditions (iv) and (v) are exactly ($\mathrm{D}_{1,1}$), and (vi) gives ($\mathrm{D}_{1,2}$). Since $\epsilon_w > 0$ if and only if either $(w, \overline{w})$ is a conjugate pair of nonreal zeros of ψ or if w is a real zero of ψ, the condition $(viii)$ is the same as (D_2).

Conversely, assume that (Q, P) is an n-$\mathring{\mathcal{N}}^{\mathrm{ep}}$-pair. Then $P^{-1}Q \in \mathcal{N}^{\mathrm{ep}}$, and hence has no poles off the real axis and only finitely many poles in $(-\infty, 0)$. By (B_1) this implies that P has no zeros in $\mathbb{C} \setminus \mathbb{R}$ and only finitely many zeros in $(-\infty, 0)$. We conclude from (4.8) that (i) holds. Let A, B be defined by (4.12), and put $E(z) := A(z^2) - izB(z^2)$. Then, by [PW, Proposition 2.10] the overall hypothesis of [PW, Theorem 3.1] is satisfied, and this theorem itself yields (ii) and (vii). From [PW, Remark 2.3, (ii)] we now obtain (iii). The condition ($\mathrm{D}_{1,1}$) gives (iv) and (v), from ($\mathrm{D}_{1,2}$) we obtain (vi), and finally (D_2) is just $(viii)$.

❑

4.10 Remark. Sometimes it is practical to note that in the situation of Proposition 4.9 the set of zeros of the function ψ can be split into two disjoint parts: Assume that P and Q satisfy the overall hypothesis of Proposition 4.9, and that (Q, P) is an n-$\mathring{\mathcal{N}}^{\mathrm{ep}}$-pair for some $n \in \mathbb{N}$. Put

$$M_1 := \big\{ w \in Z(\psi) \setminus \{0\} : \epsilon_w = 0 \big\} \cup \begin{cases} \{0\} & , \ \epsilon_0 = 1 \\ \emptyset & , \ \text{otherwise} \end{cases}$$

$$M_2 := \big\{ w \in Z(\psi) \setminus \{0\} : \epsilon_w > 0 \big\} \cup \begin{cases} \{0\} & , \ \epsilon_0 > 1 \\ \emptyset & , \ \text{otherwise} \end{cases}$$

and let

$$d_1(w) := \begin{cases} \mathfrak{d}_\psi(w) & , \ w \in M_1 \\ 0 & , \ w \notin M_1 \end{cases}, \qquad d_2(w) := \begin{cases} \mathfrak{d}_\psi(w) & , \ w \in M_2 \\ 0 & , \ w \notin M_2 \end{cases}$$

Then, by (*iv*) and (*vi*) of Proposition 4.9, see also Remark 4.8, (*iii*), the sets M_1 and M_2 are disjoint. Moreover, clearly, $d_1 + d_2 = \mathfrak{d}_\psi$.

In the set M_1 there are collected all nonreal zeros of ψ which are not part of a conjugate pair of zeros (and 0 if it is a simple zero). As it is seen by the description of their distribution in Proposition 4.9, compare with [PW, Theorem 3.1], these are those zeros responsible for $P^{-1}Q$ belonging to $\mathcal{N}^{\text{ep}}$.

In the set M_2 there are collected all conjugate pairs of nonreal zeros, and all real zeros (with exception of 0 if it is a simple zero). These are those zeros which are responsible that (Q, P) is only an n-$\mathring{\mathcal{N}}^{\text{ep}}$-pair and not a 1-$\mathring{\mathcal{N}}^{\text{ep}}$-pair.

The numbers p, κ, ϵ_j in Proposition 4.9, and the number of zeros of P in $(-\infty, 0)$, are related in several ways. For example it is apparent that

$$\sum_{j=1}^{p} \epsilon_j = N - \hat{N},$$

where N and $\hat{N}$ denote the number of zeros of P located in $(-\infty, 0)$ counted with or without, respectively, their multiplicities.

The following relation is less obvious. It can be deduced with help of some elementary properties of indefinite Hermite-Biehler functions. For the definition of this term and a collection of some properties of such functions, the reader is referred to [PW, §2].

4.11 Corollary. *Assume that $P, Q \in H(\mathbb{C})$, $P = P^\#$, $Q = Q^\#$, form an n-$\mathring{\mathcal{N}}^{\text{ep}}$-pair, and let the numbers p and κ be as in Proposition 4.9. Then $p \leq \kappa$.*

Proof. Denote $E := \Lambda^{-1}\psi$, and let us reformulate condition (*iv*) of Proposition 4.9 in terms of E. To this end, note that $\psi = \Lambda E$, and Λ has a zero of multiplicity ϵ_j at $\pm i\lambda_j$. Hence

$$\psi^{(\epsilon_j)}(\pm i\lambda_j) = \Lambda^{(\epsilon_j)}(\pm i\lambda_j) E(\pm i\lambda_j), \ j = 1, \ldots, p.$$

However,

$$\Lambda^{(\epsilon_j)}(i\lambda_j) = \epsilon_j!(2i\lambda_j)^{\epsilon_j} \prod_{\substack{i=1 \\ i \neq j}}^{p} \left(\lambda_i^2 - \lambda_j^2\right), \quad \Lambda^{(\epsilon_j)}(-i\lambda_j) = \epsilon_j!(-2i\lambda_j)^{\epsilon_j} \prod_{\substack{i=1 \\ i \neq j}}^{p} \left(\lambda_i^2 - \lambda_j^2\right),$$

and hence (*iv*) is equivalent to $E(i\lambda_j) = E(-i\lambda_j)$, $j = 1, \ldots, p$.

By [PW, Remark 2.3, (*ii*)], the function E belongs to the indefinite Hermite-Biehler class with negative index κ. This implies that the total multiplicity of all points in $\mathbb{C}^+$ where the function $E^{-1}E^\#$ attains a value of modulus 1 does not exceed κ, cf. e.g. [KL].

❑

4.12 Remark. In the situation of Corollary 4.11, also the number κ can be estimated by the number of zeros of P: Let again $\hat{N}$ denote the number of zeros of P in $(-\infty,0)$ counted without multiplicites. Then $\kappa \leq \ell(\hat{N})$ where ℓ is some function which grows only linearly. This estimate follows from [KWW1, Corollary 4.4]. The bound obtained in this way is very rough (and not worth to be given explicitly). It is more interesting, especially in view of our later applications, that in the situation of n-$\mathring{\mathcal{N}}_-^{\mathrm{ep}}$-pairs an exact formula can be given, cf. the below Corollary 4.13.

From Proposition 4.9 we can also deduce characterizations of n-$\mathring{\mathcal{N}}_-^{\mathrm{ep}}$-pairs and n-$\mathring{\mathcal{S}}^{-1}$-pairs. In the case of $\mathring{\mathcal{N}}_-^{\mathrm{ep}}$ not much changes; matters are equally complicated. The conditions for (Q,P) being an n-$\mathring{\mathcal{S}}^{-1}$-pair, however, are significantly simpler.

4.13 Corollary. *Let $P,Q \in H(\mathbb{C})$, $P = P^\#$, $Q = Q^\#$, let $n \in \mathbb{N}$, and put $\psi := \phi_{Q,-P}$. Then (Q,P) is an n-$\mathring{\mathcal{N}}_-^{\mathrm{ep}}$-pair if and only if it satisfies the conditions (i)–$(viii)$ of Proposition 4.9 and*

(ix) The function

$$p_e : \begin{cases} \mathbb{R} & \to \quad \mathbb{R} \\ t & \mapsto \quad (\Lambda^{-1}\psi)(it) + (\Lambda^{-1}\psi)(-it) \end{cases}$$

has a zero in (y_κ,∞).

In this case, p_e has exactly one zero outside of (y_κ,∞). Moreover, we have

$$\kappa = \begin{cases} \hat{N} & , \ \lim_{x \nearrow 0} \frac{Q(x)}{P(x)} \in (-\infty,0] \\ \hat{N}+1 & , \ \textit{otherwise} \end{cases}$$

Proof. In view of Lemma 4.4, we only need to investigate whether $P^{-1}Q \in \mathcal{N}_-^{\mathrm{ep}}$ or, equivalently, whether $q := -Q^{-1}P \in \mathcal{N}_+^{\mathrm{ep}}$. The points $y_1,\ldots,y_\kappa$ are exactly the solutions of the equation $q(-t^2) = -\frac{1}{t}$. The zeros of p_e are exactly the squares roots of the poles of q in $(-\infty,0)$. The present assertion follows from [PW, Lemma 3.2, (3.2)].

In order to obtain the formula for κ, note that $-Q^{-1}P \in \mathcal{N}_+^{\mathrm{ep}}$ allows to apply [KWW1, Example 4.5] rather than [KWW1, Corollary 4.4].

❑

4.14 Corollary. *Let $P,Q \in H(\mathbb{C})$, $P = P^\#$, $Q = Q^\#$, let $n \in \mathbb{N}$, and put $\psi := \phi_{Q,-P}$. Then (Q,P) is an n-$\mathring{\mathcal{S}}^{-1}$-pair if and only if the following conditions (i)–(iv) are satisfied:*

(i) $\psi \in \mathcal{HB}$.

(ii) If $w \in \mathbb{R}\setminus\{0\}$ is a zero of ψ with multiplicity α, then $\operatorname{Im}\psi^{(\alpha)}(w) = 0$.

(iii) $\mathfrak{d}_\psi(0) \in \{0,1\}$.

(iv) $\max\{\mathfrak{d}_\psi(x) : x \in \mathbb{R}\setminus\{0\}\} = n-1$.

Proof. Let us first settle the case that P has no zeros. In this case the condition (ii) is void, (iii) is satisfied, and the value of the maximum in (iv) is equal to 0. By 2.6, the condition (i) is equivalent to $P^{-1}Q \in \mathcal{S}^{-1}$ which in turn is equivalent to (Q,P) being a 1-$\mathring{\mathcal{S}}^{-1}$-pair. This proves the required equivalence.

For the rest of this proof assume that P has at least one zero. Assume that (Q,P) is an n-$\mathring{\mathcal{S}}^{-1}$-pair. Then $-Q^{-1}P \in \mathcal{S}$. Moreover, $Z(P) \subseteq (0,\infty)$ and $Z(Q) \subseteq [0,\infty)$. In particular, the functions P and Q cannot have common zeros in $(-\infty,0)$. It follows from 2.6 that $\psi \in \mathcal{HB}$. Since $P(0) \neq 0$, we also obtain $\mathfrak{d}_\psi(0) \in \{0,1\}$. The presently required conditions (ii) and (iv) are just (v) and $(viii)$ of Proposition 4.9, and are therefore fullfilled. Note here that $\psi \in \mathcal{HB}$ implies that ψ has no zeros in $\mathbb{C}^+$, and thus that $p = \kappa = 0$ and $\Lambda = Y = 1$ in Proposition 4.9.

Conversely, assume that (i)–(iv) are satisfied. Due to (i) we have $-Q^{-1}P \in \mathcal{S}$, i.e. $P^{-1}Q \in \mathcal{S}^{-1}$. As we already noted, $p = \kappa = 0$ and $\Lambda = Y = 1$ in Proposition 4.9. Thus also the condition (iii) of Proposition 4.9 is satisfied and (i), (ii), (iv) and (vii) are void. The conditions (v), (vi) and $(viii)$ of Proposition 4.9 hold by the present (ii), (iii) and (iv). Combining Proposition 4.9 with Lemma 4.4, we conclude that (Q,P) is an n-$\mathring{\mathcal{S}}^{-1}$-pair.

❑

4.3 Characterization in terms of zeros of $\phi_{Q,-P}$

It is of major interest to formulate the conditions of Proposition 4.9, Corollary 4.13, and Corollary 4.14 as purely as possible in terms of the zero-distribution of ψ. Again we restrict ourselves to what will be needed in applications, namely to the function ψ and the classes $\mathring{\mathcal{N}}^{\text{ep}}$, $\mathring{\mathcal{N}}^{\text{ep}}_{-}$, $\mathring{\mathcal{S}}^{-1}$. Similar results can be shown for ϕ and for the classes $\mathring{\mathcal{N}}^{\text{ep}}_{+}$, $\mathring{\mathcal{S}}$.

4.15 Theorem. *Let $d : \mathbb{C} \to \mathbb{N}_0$ have discrete support. Then there exists an n-$\mathring{\mathcal{N}}^{\text{ep}}$-pair (Q,P) such that $d = \mathfrak{d}_{\phi_{Q,-P}}$, if and only if d satisfies the following conditions:*

- (i) *There exist only finitely many nonreal conjugate pairs $(w,\overline{w})$ such that $\min\{d(w), d(\overline{w})\} > 0$. Each such pair $(w,\overline{w})$ is located on the imaginary axis, and satifies $d(w) = d(\overline{w})$.*

Let $(i\lambda_1, -i\lambda_1), \ldots, (i\lambda_p, -i\lambda_p)$, $0 < \lambda_1 < \ldots < \lambda_p$, be all the nonreal conjugate pairs as in (i), and denote by d_Λ the function

$$d_\Lambda(w) := \begin{cases} d(w) & , \ w = \pm i\lambda_j, \ j = 1,\ldots,p \\ 0 & , \ \textit{otherwise} \end{cases}$$

- (ii) *There exist only finitely many points w in the open upper half plane, such that $d(w) - d_\Lambda(w) > 0$. Each such point w lies on the imaginary axis and satisifes $d(w) = 1$.*

Denote the points in (ii) as $iy_1, \ldots, iy_\kappa$, $0 < y_1 < \ldots < y_\kappa$, and let Y be the function

$$Y(z) := \prod_{j=1}^{\kappa} \Big(1 - \frac{z}{iy_j}\Big) \,.$$

(*iii*) *We have* $d(-\overline{w}) = d(w)$, $w \in \mathbb{C}$, *and* $d(0) \in \{0,1\} \cup 2\mathbb{N}$.

(*iv*) *For each* $k \in \{2,\dots,\kappa\}$, *the number* $\sum_{w\in[-iy_{k-1},-iy_k]}(d(w) - d_\Lambda(w))$ *is odd. The number* $\sum_{w\in(0,-iy_1]}(d(w) - d_\Lambda(w))$ *is odd if* $d(0) = 1$ *and even otherwise.*

(*v*) $\max\big(\{d(i\lambda_j) : j = 1,\dots,p\} \cup \{d(x) : x \in \mathbb{R}\setminus\{0\}\} \cup \{[\frac{1}{2}d(0)]\}\big) = n - 1.$

(*vi*) *We have* $\sum_{\operatorname{Im} w<0} d(w) \operatorname{Im}\frac{1}{w} < \infty$.

Choose a function $c : \mathbb{C} \to \mathbb{N}_0$ *such that* $c(w) = c(-\overline{w})$, $w \in \mathbb{C}$, $\operatorname{supp} c = (\operatorname{supp} d) \cap \{w \in \mathbb{C} : \operatorname{Im} w < 0, \operatorname{Re} w \neq 0\}$, *and* $\sum_{w\in\mathbb{C}\setminus\{0\}} d(w)\big|\frac{r}{w}\big|^{c(w)+1} < \infty$, $r > 0$. *By* (*vi*) *an entire function* E *is well-defined by*

$$E(z) := \prod_{\operatorname{Im} w<0}\Big(1 - \frac{z}{w}\Big)^{d(w)} \exp\Big(d(w)\sum_{k=1}^{c(w)}\frac{z^k}{k}\operatorname{Re}\frac{1}{w^k}\Big). \tag{4.13}$$

(*vii*) *There exists a number* $a \geq 0$ *such that*

(*a*) $\arg Y(w) + \arg E(w) \equiv aw \mod \pi$, $w \in (\operatorname{supp} d) \cap \mathbb{R}$;

(*b*) *If* $d(0) > 1$, *then* $a = \sum_{j=1}^{\kappa}\frac{1}{y_j} - \sum_{\operatorname{Im} w<0}\operatorname{Im}\frac{1}{w}$;

(*c*) $e^{-2a\lambda_j}\frac{E(-i\lambda_j)}{E(i\lambda_j)} = (-1)^\delta\frac{Y(i\lambda_j)}{Y(-i\lambda_j)}$, $j = 1,\dots,p$.

Proof. We need to reformulate the the conditions of Proposition 4.9.

Step 1: Assume that $P, Q \in H(\mathbb{C})$, $P = P^\#$, $Q = Q^\#$, and put $\psi(z) := \phi_{Q,-P}$, $d := \mathfrak{d}_\psi$. The following correspondences are obvious:

condition of Proposition 4.9		present condition
(*i*)	$\Longleftrightarrow$	(*i*)
$\mathfrak{d}_\Lambda$	$=$	d_Λ
(*ii*)	$\Longleftrightarrow$	(*ii*)
Y	$=$	Y
(*vi*), first part	$\Longleftrightarrow$	(*iii*), second part
(*vii*)	$\Longleftrightarrow$	(*iv*)
(*viii*)	$\Longleftrightarrow$	(*v*)
(*iii*)	$\overset{[L]}{\Longrightarrow}$	(*vi*)
$\psi^\#(z) = \psi(-z)$	$\Longrightarrow$	(*iii*), first part

Step 2: The relation between (*iv*)–(*vi*) of Proposition 4.9 and the present condition (*vii*) is not so straightforward. Assume that ψ is as in Step 1, and satisfies (*i*)–(*iii*) of Proposition 4.9. By Krein's Factorization Theorem and symmetry of ψ, cf. [L, Lehrsatz VII.6], [PW, §2], there exists a function c with the stated properties, a number $a \geq 0$, and an entire function D with $D(z) = D^\#(z) = D(-z)$, $\operatorname{supp}\mathfrak{d}_D \subseteq \mathbb{R}\setminus\{0\}$, such that ($\delta := d(0)$)

$$\psi(z) = \Lambda(z)Y(z)z^\delta D(z)e^{-iaw}E(z),$$

where E is defined by (4.13).

Let $w \in \mathbb{R} \setminus \{0\}$ be a zero of ψ with multiplicity α. Then

$$\psi^{(\alpha)}(w) = \Lambda(w)Y(w)w^\delta D^{(\alpha)}(w)e^{-iaw}E(w)\,,$$

and hence

$$\arg \psi^{(\alpha)}(w) \equiv \arg Y(w) - aw + \arg E(w) \mod \pi\,. \tag{4.14}$$

Next, we compute

$$\psi^{(\delta+1)}(0) = \binom{\delta+1}{1}\delta!\big[\Lambda Y D e^{-iaz}E\big]'(0) = (\delta+1)!\big[\underbrace{\Lambda'(0)}_{=0}D(0)+$$

$$+\Lambda(0)Y'(0)D(0) + \Lambda(0)\underbrace{D'(0)}_{=0}+\Lambda(0)D(0)(-ia) + \Lambda(0)D(0)E'(0)\,.$$

We have $Y'(0) = -\sum_{j=1}^{\kappa}\frac{1}{iy_j}$ and $E'(0) = -\sum_{\operatorname{Im} w<0}\frac{1}{w}$. By symmetry, actually, $\sum_{\operatorname{Im} w<0}\frac{1}{w} = i\sum_{\operatorname{Im} w<0}\operatorname{Im}\frac{1}{w}$. It follows that

$$\psi^{(\delta+1)}(0) = i\Lambda(0)D(0)\Big(\sum_{j=1}^{\kappa}\frac{1}{y_j} - a - \sum_{\operatorname{Im} w<0}\operatorname{Im}\frac{1}{w}\Big)\,. \tag{4.15}$$

Finally, note that

$$\Lambda^{(\epsilon_j)}(i\lambda_j) = (-1)^{\epsilon_j}\Lambda^{(\epsilon_j)}(-i\lambda_j)\,,$$

and hence

$$\begin{aligned}\psi^{(\epsilon_j)}(i\lambda_j) - (-1)^{\epsilon_j}\psi^{(\epsilon_j)}(-i\lambda_j) &= \Lambda^{(\epsilon_j)}(i\lambda_j)Y(i\lambda_j)(i\lambda_j)^\delta D(i\lambda_j)e^{a\lambda_j}E(i\lambda_j)-\\ &-(-1)^{\epsilon_j}\Lambda^{(\epsilon_j)}(-i\lambda_j)Y(-i\lambda_j)(-i\lambda_j)^\delta D(-i\lambda_j)e^{-a\lambda_j}E(-i\lambda_j) =\\ &= (-1)^{\epsilon_j}\Lambda^{(\epsilon_j)}(-i\lambda_j)(i\lambda_j)^\delta D(i\lambda_j)\cdot\\ &\cdot\big[Y(i\lambda_j)e^{a\lambda_j}E(i\lambda_j) - Y(-i\lambda_j)(-1)^\delta e^{-a\lambda_j}E(-i\lambda_j)\big]\,.\end{aligned} \tag{4.16}$$

We see that the following correspondences hold:

condition of Proposition 4.9		present condition
(iv)	$\overset{(4.16)}{\Longleftrightarrow}$	*(vii)/(c)*
(v)	$\overset{(4.14)}{\Longleftrightarrow}$	*(vii)/(a)*
(vi), second part	$\overset{(4.15)}{\Longleftrightarrow}$	*(vii)/(b)*

Step 3, completion of proof: Assume that there exists an n-$\mathring{\mathcal{N}}^{\mathrm{ep}}$-pair (Q,P) such that $d = \mathfrak{d}_\psi$ with $\psi := \phi_{Q,-P}$. The function ψ satisfies all the conditions in Proposition 4.9, and thus, by the above correspondences, all conditions of the present theorem.

Conversely, assume that d satisfies the present conditions. Choose a function D with $D(z) = D^{\#}(z) = D(-z)$ and

$$\mathfrak{d}_D(w) = \begin{cases} d(w) & ,\ w \in \mathbb{R}\setminus\{0\}\\ 0 & ,\ \text{otherwise}\end{cases}$$

and define

$$\psi(z) := \Lambda(z)Y(z)z^{d(0)}D(z)e^{-iaz}E(z)\,.$$

By the symmetry of d, we have $\psi^{\#}(z) = \psi(-z)$. Thus we can write $\psi = \phi_{Q,-P}$ with some functions $P, Q \in H(\mathbb{C})$, $P = P^{\#}$, $Q = Q^{\#}$. Moreover, ψ satisfies the conditions (i)–(iii) of Proposition 4.9. By the above correspondences, ψ satisifes all the conditions of Proposition 4.9. We conclude that (Q, P) is an n-$\mathring{\mathcal{N}}^{\text{ep}}$-pair. Clearly, $d = \mathfrak{d}_\psi$.

❑

4.16 Remark.

(i) The conditions in Theorem 4.15 do not depend on the choice of the function c.

(ii) The condition (vii) in Theorem 4.15 is of course very implicit and hard to verify. The appearence of a condition of this type can already be observed in the particular case of finite spectra, i.e. polynomial functions Φ, where all computations can be carried out explicitly, for some cases see [BP]. It seems that this is an intrinsic complication and cannot be removed.

(iii) Assume that d satisfies the conditions of Theorem 4.15. Then $\kappa \geq p$. If $d(0) > 1$, then $\kappa > 0$. If $p > 0$ or $d(0) > 1$, then the number a is unique.

4.17 Corollary. *Let notation be as in Theorem 4.15. Then there exists an n-$\mathring{\mathcal{N}}^{\text{ep}}_{-}$-pair (Q, P) such that $d = \mathfrak{d}_{\phi_{Q,-P}}$, if and only if d satisfies the conditions (i)–(vii) of Theorem 4.15 and, additionally,*

$(vii)/(d)$ *There exists a number $\lambda \in (y_\kappa, \infty)$ such that $e^{-2a\lambda}\frac{E(-i\lambda)}{E(i\lambda)} = (-1)^{\delta+1}\frac{Y(i\lambda)}{Y(-i\lambda)}$.*

Proof. Let notation be as in Step 2 of the proof of Theorem 4.15. Then we compute

$$(\Lambda^{-1}\psi)(it) + (\Lambda^{-1}\psi)(-it) =$$
$$= Y(it)(it)^\delta D(it)e^{at}E(it) + Y(-it)(-it)^\delta D(-it)e^{-at}E(-it) =$$
$$= (it)^\delta D(it) \cdot \left[Y(it)e^{at}E(it) + Y(-it)(-1)^\delta e^{-at}E(-it)\right].$$

Thus the condition (ix) of Corollary 4.13 is equivalent to the present condition $(vii)/(d)$. The proof of the present assertion is completed by the same arguments used in Step 3 of the proof of Theorem 4.15.

❑

4.18 Corollary. *Let notation be as in Theorem 4.15. Then there exists an n-$\mathring{\mathcal{S}}^{-1}$-pair (Q, P) such that $d = \mathfrak{d}_{\phi_{Q,-P}}$, if and only if d satisfies the following conditions:*

(i) *We have* $\operatorname{supp} d \subseteq \mathbb{C}_- \cup \mathbb{R}$.

(ii) *We have $d(-\overline{w}) = d(w)$, $w \in \mathbb{C}$, and $d(0) \in \{0, 1\}$.*

(iii) $\max\{d(x) : x \in \mathbb{R} \setminus \{0\}\} \leq n - 1$.

(*iv*) *We have* $\sum_{\operatorname{Im} w<0} d(w) \operatorname{Im} \frac{1}{w} < \infty$.

(*v*) *There exists a number* $a \geq 0$ *such that* $\arg E(w) \equiv aw \mod \pi$, $w \in (\operatorname{supp} d) \cap \mathbb{R}$.

Proof. This assertion follows by combining the correspondences established in Step 1 and Step 2 of the proof of Theorem 4.15 with Corollary 4.14, and using $\psi(z) = z^{d(0)} D(z) e^{-iaz} E(z)$.

❑

4.19 Corollary. *Let* (Q, P) *be an* n-$\mathring{\mathcal{N}}^{\text{ep}}$*-pair, and assume that* $\operatorname{supp} \mathfrak{d}_{\phi_{Q,-P}} \subseteq \mathbb{C}^- \cup \mathbb{R}$. *Then* (Q, P) *is an* n-$\mathring{\mathcal{S}}^{-1}$*-pair.*

Proof. The function $\mathfrak{d}_{\phi_{Q,-P}}$ satisifes the conditions of Theorem 4.15. By the present assumption, we have $p = \kappa = 0$. By Remark 4.16, (*iii*), thus also $\mathfrak{d}_{\phi_{Q,-P}}(0) \leq 1$. Moreover, $Y = 1$. Alltogether, we see that $\mathfrak{d}_{\phi_{Q,-P}}$ satisfies the conditions of Corollary 4.18. Hence, there exists an n-$\mathring{\mathcal{S}}^{-1}$-pair$(\tilde{Q}, \tilde{P})$ such that $\mathfrak{d}_{\phi_{Q,-P}} = \mathfrak{d}_{\phi_{\tilde{Q},-\tilde{P}}}$.

Let G be the zerofree entire function which satisfies $\phi_{\tilde{Q},-\tilde{P}} = G\phi_{Q,-P}$, and define H by the relation $H(z^2) = \frac{1}{2}(G(z) + G(-z))$. Then $\tilde{Q} = HQ$ and $\tilde{P} = HP$, and hence $\tilde{P}^{-1}\tilde{Q} = P^{-1}Q$. In particular, $P^{-1}Q \in \mathcal{S}^{-1}$, and it follow from Lemma 4.4 that (Q, P) is an n-$\mathring{\mathcal{S}}^{-1}$-pair.

❑

4.20 Example. Although each n-$\mathring{\mathcal{N}}^{\text{ep}}$-pair (Q, P) with $\operatorname{supp} \mathfrak{d}_{\phi_{Q,-P}} \subseteq \mathbb{C}^- \cup \mathbb{R}$ is an n-$\mathring{\mathcal{S}}^{-1}$-pair, not every aditive decomposition of $P^{-1}Q$ into summands of $\mathring{\mathcal{N}}^{\text{ep}}$ is suitable to show this. Let us consider the following example: Let

$$Q(z) := -2z^2 + 4z - 1, \quad P(z) := (1 - z)(2 - z)\,,$$

and

$$Q_1(z) := 1, P_1(z) := (1 - z), \quad Q_2(z) := 2z - 3, P_2(z) := 2 - z$$
$$\tilde{Q}_1(z) := z, \tilde{P}_1(z) := (1 - z), \quad \tilde{Q}_2(z) := z - 1, \tilde{P}_2(z) := 2 - z$$

Then a short computation shows that

$$\frac{Q(z)}{P(z)} = \frac{Q_1(z)}{P_1(z)} + \frac{Q_2(z)}{P_2(z)} = \frac{\tilde{Q}_1(z)}{\tilde{P}_1(z)} + \frac{\tilde{Q}_2(z)}{\tilde{P}_2(z)}\,.$$

Moreover, $\tilde{P}_1^{-1}\tilde{Q}_1, \tilde{P}_2^{-1}\tilde{Q}_2 \in \mathcal{S}^{-1}$, but $P_1^{-1}Q_1 \in \mathcal{N}_-^{\text{ep}} \setminus \mathcal{S}^{-1}$.

5 Relation between real and nonreal zeros of ψ

Theorem 4.7 has an interesting consequence on the distribution of the zeros of the function $\psi := \phi_{Q,-P}$ for an n-$\mathcal{K}$-pair (Q, P). As we have already remarked in the introduction, the physical interpretation in applications suggests to consider the real and nonreal zeros of ψ separately. It is a noteworthy fact that, roughly speaking, the number of real zeros is bounded by the number of nonreal zeros. This fact is explained by making use of the relation between the growth of an entire function and the distribution of its zeros. Similar results hold for the function $\phi := \phi_{P,Q}$, however, in view of our applications we will again restrict ourselves to the consideration of ψ.

5.1 Some notation concerning growth and distribution of zeros of an entire function

In this preliminary subsection we recall some notation and results on growth and zero-distribution of entire functions with respect to a general growth function. Terms and notions related to the usual order and type of an entire function, like e.g. convergence exponent, genus, etc., will be freely used throughout the text. Standard references for all these items are e.g. [L], [LG], or [Ru].

First we define how to measure the growth of an entire function. A function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is called a growth function, if it satisfies the following axioms:

(gf1) The limit $\rho(\lambda) := \lim_{r\to\infty} \frac{\log\lambda(r)}{\log r}$ exists and is a finite nonnegative number;

(gf2) For all sufficiently large values of r, the function λ is differentiable and $\lim_{r\to\infty} r\frac{\lambda'(r)}{\lambda(r)} = \rho(\lambda)$;

(gf3) $\log r = o(\lambda(r))$.

The conditions (gf1) and (gf2) ensure that we have available Valiron's theory of proximate orders, as well as the theory of value distribution of meromorphic functions. The condition (gf3), that λ grows sufficiently rapidly, is imposed to exclude trivial cases and is no essential restriction.

Classical examples of growth functions are functions of the form

$$\lambda(r) = r^\alpha (\log r)^\beta$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$.

If F is an entire function and λ is a growth function, the λ-type of F is defined as the number

$$\sigma_F^\lambda := \limsup_{|z|\to\infty} \frac{\log^+ |F(z)|}{\lambda(|z|)} \in [0,\infty] .$$

Next we define how to measure the density of a sequence of complex numbers with respect to a growth function λ. If $(a_m)_{m\in\mathbb{N}}$ is a sequence of complex numbers, denote by $n_r(a_m)$ the number of all terms a_m with modulus at most r. The upper λ-density $\Delta^\lambda(a_m)$ of the sequence (a_m) is then defined as

$$\Delta^\lambda(a_m) := \limsup_{r\to\infty} \frac{n_r(a_m)}{\lambda(r)} \in [0,\infty] .$$

Also a certain measure for the regularity of the distribution of a sequence plays a role. Define

$$\delta^\lambda(a_m) := \frac{1}{\rho(\lambda)} \limsup_{r\to\infty} \frac{r^{\rho(\lambda)}}{\lambda(r)} \Big| \sum_{|a_m|<r} \frac{1}{a_m^{\rho(\lambda)}} \Big| ,$$

and put

$$\gamma^\lambda(a_m) := \max\left\{\Delta^\lambda(a_m), \delta^\lambda(a_m)\right\} .$$

For an entire function F denote by (a_m^F) the sequence of zeros of F listed according to their multiplicities. The number of zeros of a function F will always be limited by its growth; the following result can be found e.g. in [L], cf. Lemma 4 and the proof of Theorem 15.

5.1. *Zero-distribution governed by growth:* *Let λ be a growth function. Then there exists a positive number $c(\lambda)$ such that*

$$\Delta^\lambda(a_m^F) \leq c(\lambda)\sigma_F^\lambda,\ F \in H(\mathbb{C})\,,$$

and, in case $\rho(\lambda) \in \mathbb{N}$, that

$$\delta^\lambda(a_m^F) \leq c(\lambda)\sigma_F^\lambda,\ F \in H(\mathbb{C})\,.$$

As for a converse, the matters are more complicated. The first obstacle is the presence of an exponential factor in the Hadamard factorization of F. This can be easily overcome by considering the canonical product only. However, another obstacle appears when $\rho(\lambda)$ is an integer, and this is of intrinsic nature. In case $\rho(\lambda) \in \mathbb{N}$, the growth of a canonical product depends not only on the density of its zeros, but also on the regularity of their distribution. A proof of the following statements (i) and (ii) can be found e.g. in [L], cf. the proof of Theorem 17 and Theorem 18.

5.2. *Growth governed by zero-distribution:* *Let (a_m) be a sequence of nonzero complex numbers whose convergence exponent is finite, let p denote the genus of this sequence, and put*

$$F(z) := \prod \Big(1 - \frac{z}{a_m}\Big) \exp\Big(\sum_{l=0}^{p} \frac{z^l}{l}\frac{1}{a_m^l}\Big)\,.$$

(i) *Let λ be a growth function with $\rho(\lambda) \notin \mathbb{N}$. Then*

$$\sigma_F^\lambda \leq C(\lambda)\Delta^\lambda(a_m)\,.$$

(ii) *Let λ be a growth function with $\rho(\lambda) \in \mathbb{N}$. Then*

$$\sigma_F^\lambda \leq C(\lambda)\gamma^\lambda(a_m)\,.$$

Thereby $C(\lambda)$ is a positive number which depends only on λ (but not on F).

5.2 Comparison of real and nonreal zeros

Let ψ be an entire function of finite order. We will throughout this section keep the following notation:

5.3. Notational conventions: Put $\epsilon_w := \min\{\mathfrak{d}_\psi(w), \mathfrak{d}_\psi(\overline{w})\}$, $w \in \mathbb{C}$. Let (x_m) denote the sequence of real nonzero zeros of ψ listed according to their multiplicites, and let (λ_m) denote the sequence of all nonreal points w with $\epsilon_w > 0$ where each λ_m is listed exactly ϵ_{λ_m} times. Let p_1 denote the genus of the sequence of the points (λ_m), let p_2 be the genus of (x_m), and define

$$\Lambda(z) := \prod \Big(1 - \frac{z}{\lambda_m}\Big)\Big(1 - \frac{z}{\overline{\lambda_m}}\Big) \exp\Big(2\sum_{l=1}^{p_1} \frac{z^l}{l} \operatorname{Re} \frac{1}{\lambda_m^l}\Big)$$

$$X(z) := \prod \Big(1 - \frac{z}{x_m}\Big) \exp\Big(\sum_{l=1}^{p_2} \frac{z^l}{l}\frac{1}{x_m^l}\Big)$$

Then the function $(\Lambda X)^{-1}\psi$ has no real zeros (with possible exception of a zero at the origin) and no conjugate pairs of nonreal zeros. Let (a_m) denote the sequence of zeros of $(\Lambda X)^{-1}\psi$ where each zero is listed repeatedly according to its multiplicity, denote by p_3 the genus of this sequence, and put

$$A(z) := \prod_{a_m \neq 0} \Big(1 - \frac{z}{a_m}\Big) \exp\Big(\sum_{l=1}^{p_3} \frac{z^l}{l} \frac{1}{a_m^l}\Big)$$

By the Hadamard Factorization Theorem we can write ψ in the form

$$\psi(z) = z^m e^{d_1(z)} \Lambda(z) X(z) \cdot e^{id_2(z)} A(z)\,,$$

where $m := \mathfrak{d}_\psi(0)$, and where d_1 and d_2 are polynomials with real coefficients whose degree does not exceed the order of ψ.

We will now show in a somewhat more general formulation that in the presence of (D) the number of real zeros of ψ is limited by the number of its nonreal zeros. Actually, we give a rather general formulation of such a type of result.

5.4 Theorem. *Let ψ be an entire function of finite order ρ, and let λ be a growth function with $\rho(\lambda) = \rho$. Let notation like ϵ_w, x_m, d_2 etc., be as in 5.3. Assume that ψ satisfies* $(\mathrm{D}_{1,1})$ *and* (D_2) *(with some number n). Then the following hold:*

(a) *If $\rho \notin \mathbb{N}$, then*
$$\Delta^\lambda(x_m) + \Delta^\lambda(\lambda_m) \leq c(\lambda, n) \Delta^\lambda(a_m)\,.$$

(b) *If $\rho \in \mathbb{N}$ and either* $\deg d_2 < \rho$ *or $r^\rho = o(\lambda(r))$, then*
$$\Delta^\lambda(x_m) + \Delta^\lambda(\lambda_m) \leq c(\lambda, n) \gamma^\lambda(a_m)\,.$$

(c) *If $\rho \in \mathbb{N}$ and $r^\rho = O(\lambda(r))$, then*
$$\gamma^\lambda(a_m) < \infty \ \Rightarrow\ \Delta^\lambda(x_m) + \Delta^\lambda(\lambda_m) < \infty\,.$$

Proof. First let us introduce one more notation. For a sequence (w_m) of complex numbers, denote $\hat{n}_r(\lambda_m) := \#\{w_m : |w_m| \leq r\}$, i.e. the number of terms of the sequence (w_m) whose modulus does not exceed r where each point is counted only once (and not according to the number of its appearances in the sequence). Note that, trivially, $\hat{n}_r(w_m) \leq n_r(w_m)$.
Step 1: Define an entire function Ψ as

$$\Psi := e^{id_2(z)} A(z) - e^{-id_2(z)} A^\#(z)\,,$$

and denote by (b_m) the sequence of zeros of Ψ listed according to their multiplicities. Observe that we have

$$\psi(z) - \psi^\#(z) = z^m e^{d_1(z)} \Lambda(z) X(z) \cdot \Psi(z)\,.$$

It follows from $(\mathrm{D}_{1,1})$ that $Z(\Lambda \cdot X) \subseteq Z(\Psi)$. Since the sequences (x_m) and (λ_m) are disjoint this implies that

$$\hat{n}_r(x_m) + \hat{n}_r(\lambda_m) \leq \hat{n}_r(b_m)\,.$$

By (D_2) we have $\sup\{\mathfrak{d}_{\Lambda\cdot X}(w) : w \in \mathbb{C}\} \leq n-1$, and hence $n_r(x_m) \leq (n-1)\hat{n}_r(x_m)$ and $n_r(\lambda_m) \leq (n-1)\hat{n}_r(\lambda_m)$. It follows that

$$n_r(x_m) + n_r(\lambda_m) \leq (n-1)\big(\hat{n}_r(x_m) + \hat{n}_r(\lambda_m)\big)\,,$$

and we obtain the estimate

$$\Delta^\lambda(x_m) + \Delta^\lambda(\lambda_m) \leq (n-1)\Delta^\lambda(b_m)\,.$$

Step 2: It remains to estimate $\Delta^\lambda(b_m)$ in terms of the sequence (a_m). To this end observe that by 5.1 we have $\Delta^\lambda(b_m) \leq c(\lambda)\sigma^\lambda_\Psi$, that by the definition of Ψ we have $\sigma^\lambda_\Psi \leq \sigma^\lambda_{e^{id_2}A}$, and that by 5.2 we have

$$\sigma^\lambda_A \leq \begin{cases} C(\lambda)\Delta^\lambda(a_m) & ,\ \rho \notin \mathbb{N} \\ C(\lambda)\gamma^\lambda(a_m) & ,\ \rho \in \mathbb{N} \end{cases}$$

In order to fill the gap in our series of estimates, notice that either of the hypothesis in (a) or (b) implies $\sigma^\lambda_{e^{id_2}A} = \sigma^\lambda_A$, and that the hypothesis in (c) implies that

$$\sigma^\lambda_A < \infty \ \Rightarrow\ \sigma^\lambda_{e^{id_2}A} < \infty$$

This completes the proof of the theorem.

❑

For the case $\rho \in \mathbb{N}$ it is desireable to obtain some information on $\delta^\lambda(x_m)$ and $\delta^\lambda(\lambda_m)$. In general this will not be possible. However, under some additional hypothesis, which are satisfied e.g. in the context of n-$\mathring{\mathcal{N}}^{\mathrm{ep}}$-pairs, at least something can be said.

5.5 Remark. Assume that ψ is an entire function which satisfies the functional equation $\psi^\#(z) = \psi(-z)$, and let ρ be any odd integer. Then

$$\sum_{|x_m|<r} \frac{1}{x_m^\rho} = \sum_{|\lambda_m|<r} \frac{1}{\lambda_m^\rho} = 0\,.$$

This follows since, by the symmetry of ψ, we always have $\epsilon_w = \epsilon_{-w}$.

5.6 Proposition. *Let notation and hypotheses by as in Theorem 5.4 and its proof, and put*

$$L := \limsup_{r\to\infty} \frac{r^\rho}{\lambda(r)} \Big| \sum_{\substack{|b_m|<r \\ b_m \notin \mathbb{R}}} \frac{1}{b_m^\rho} \Big|\,.$$

Then the following hold:

(b') *If either* $\deg d_2 < \rho$ *or* $r^\rho = o(\lambda(r))$, *then*

$$\delta^\lambda(x_m) \leq \tilde{c}(\lambda, n)\big(\gamma^\lambda(a_m) + L\big)\,.$$

(c') *If* $r^\rho = O(\lambda(r))$, *then*

$$\big(\gamma^\lambda(a_m) < \infty \text{ and } L < \infty\big) \ \Rightarrow\ \delta^\lambda(x_m) < \infty\,.$$

Proof. Denote by $(\hat{x}_m)$ the sequence which contains the same points as (x_m), but where each point is listed only once. Since ρ is even, we have $x_m^\rho > 0$ and $b_m^\rho > 0$ whenever $b_m \in \mathbb{R}$. Since $Z(X) \subseteq Z(\Psi)$, i.e. each point $\hat{x}_m$ appears among the points b_m, this implies

$$0 \leq \sum_{|x_m|<r} \frac{1}{x_m^\rho} \leq (n-1) \sum_{|\hat{x}_m|<r} \frac{1}{\hat{x}_m^\rho} \leq (n-1) \sum_{\substack{|b_m|<r \\ b_m \in \mathbb{R}}} \frac{1}{b_m^\rho}$$

Moreover, we have

$$\Big| \sum_{\substack{|b_m|<r \\ b_m \in \mathbb{R}}} \frac{1}{b_m^\rho} \Big| \leq \Big| \sum_{|b_m|<r} \frac{1}{b_m^\rho} \Big| + \Big| \sum_{\substack{|b_m|<r \\ b_m \notin \mathbb{R}}} \frac{1}{b_m^\rho} \Big|$$

and we obtain that $\delta^\lambda(x_m) \leq \delta^\lambda(b_m) + L$.

The proof of the present assertions is completed with the same argumentation as in Step 2 of the proof of Theorem 5.4

❑

We close this section with a remark on the case of polynomial functions ψ. Let ψ be a polynomial, and let notation like ϵ_w, x_m etc., be as in 5.3. Then we denote by N_X the number of nonzero real zeros of ψ, by N_Λ twice the number of nonreal conjugate pairs of zeros of ψ, and by N_A the number of all remaing nonzero zeros of ψ. Thereby each of these numbers are understood including multiplicities. In other words,

$$N_X = \sum_{w \in \mathbb{R}} \epsilon_w = \deg X,\ N_\Lambda = \sum_{w \notin \mathbb{R}} \epsilon_w = \deg \Lambda,\ N_A = \deg A\,.$$

5.7 Proposition. *Let ψ be a polynomial, and assume that ψ satisfies* $(\mathrm{D}_{1,1})$ *and* (D_2) *(with some number n). Then*

$$N_X + N_\Lambda \leq (n-1)N_A\,.$$

Proof. Let us denote, moreover, by $\hat{N}_X$ and $\hat{N}_\Lambda$ the respective numbers of zeros where each point is counted only once. Then, since $Z(\Lambda \cdot X) \subseteq Z(\Psi)$, we have

$$\hat{N}_X + \hat{N}_\Lambda \leq \deg \Psi \leq \deg A = N_A\,.$$

However, the multiplicity of a zero of Λ or X cannot exceed $n-1$. Thus $N_x \leq (n-1)\hat{N}_X$ and $N_\Lambda \leq (n-1)\hat{N}_\Lambda$.

❑

Let us show by an example that the assumptions in Theorem 5.4, (b), are actually needed to obtain an estimate of the asserted form. More precisely, in the situation of Theorem 5.4, (c), there need not prevail an estimate of the form as in (b).

5.8 Example. Denote

$$P_1(z) = P_2(z) := 2\frac{\sin\sqrt{z}}{\sqrt{z}}, \quad Q_1(z) = Q_2(z) := \cos\sqrt{z}\,,$$

and let P, Q be as in (1.1). Then $P(z) := 4(\frac{\sin\sqrt{z}}{\sqrt{z}})^2$, $Q(z) := 4\frac{\sin\sqrt{z}}{\sqrt{z}}\cos\sqrt{z}$, and hence

$$\phi_{Q,-P}(z) = 4\frac{\sin z}{z}e^{iz}\,.$$

Since $P_j^{-1}Q_j \in \mathcal{S}$, the function $\phi_{Q,-P}$ satisfies the conditions ($D_{1,1}$) and (D_2). Nevertheless, its zeros are exactly the points $k\pi$, $k \in \mathbb{Z}\setminus\{0\}$.

6 Systems of differential equations supported on a star-shaped graph

In this section we consider various systems of differential equations given on the plane star-shaped graph (1.3). We assume that at the central vertex damping is present, and impose interface conditions at the central vertex which ensure continuity of solutions. Moreover, if necessary, at the outer vertices boundary conditions will be fixed.

6.1 Direct problem for a system of strings

Consider a plane star-shaped graph (1.3) composed of n strings $S[L_j, \mathfrak{m}_j]$, $j = 1, \dots, n$, which are tied together at the central vertex. We assume that each string satisfies the condition of 2.9, so that its spectrum is discrete. The strings are stretched and the system is able to vibrate in the direction orthogonal to the equilibrium position of the strings. We suppose that the central vertex is subject to viscous friction. Denote by $v_j(s,t)$ the transversal displacement at the time t of the point lying on the j-th edge of the graph at a distance s from the central vertex, and let $\mu > 0$ be the coefficient of damping at the central vertex. Then this vibrating system is described by the following equations:

$$\frac{\partial}{\partial M_j(s)}\Big(\frac{\partial v_j(s,t)}{\partial s}\Big) - \frac{\partial^2}{\partial t^2}v_j(s,t) = 0, \quad j = 1,\dots,n,\ s \in (0, L_j)\,. \tag{6.1}$$

$$v_1(0,t) = v_2(0,t) = \ldots = v_n(0,t)\,, \tag{6.2}$$

$$\sum_{j=1}^{n}\frac{\partial}{\partial s}v_j(s,t)\Big|_{s=-0} - \mu\frac{\partial}{\partial t}v_1(0,t) = 0\,. \tag{6.3}$$

Here the condition (6.2) is due to our assumption that the strings are tied at their common boundary point, and the condition (6.3) describes the damping which is present at the central vertex and is known as the Kirchhoff condition.

Substituting $v_j(s,t) = e^{-i\lambda t}u_j(\lambda, s)$, leads to the following problem:

$$\frac{\partial^2}{\partial M_j(s)\partial s}u_j(\lambda,s) + \lambda^2 u_j(\lambda,s) = 0, \quad j = 1,\dots,n,\ s \in (0, L_j)\,, \tag{6.4}$$

$$u_1(\lambda,0) = u_2(\lambda,0) = \ldots = u_n(\lambda,0)\,, \tag{6.5}$$

$$\sum_{j=1}^{n}\frac{\partial}{\partial s}u_j(\lambda,s)\Big|_{s=-0} + i\mu\lambda u_1(\lambda,0) = 0\,. \tag{6.6}$$

The eigenvalues of this problem are just the eigenfrequencies of the damped vibrating system under consideration. In order to give meaning to the notion of eigenvalues, we have to impose boundary conditions at the outer vertices: If $S[L_j, \mathfrak{m}_j]$ is regular, let $\gamma \in (-\infty, \infty]$ and require that

$$\gamma_j \frac{\partial u_j(\lambda, s)}{\partial s}\Big|_{s=L_j} + u_j(\lambda, L_j) = 0 \quad \text{(regular case)}\,. \tag{6.7}$$

If $S[L_j, \mathfrak{m}_j]$ is singular but in limit circle case, require that

$$\lim_{x \to L_j} \frac{\partial u_j(\lambda, s)}{\partial s}\Big|_{s=x} = 0 \quad \text{(singular/limit circle)}\,. \tag{6.8}$$

If $S[L_j, \mathfrak{m}_j]$ is in limit point case, there is no additional requirement necessary.

6.1 Remark. Since the proceedure undertaken in the sequel will repeat itself in several instances, it is worth to provide a comprehensive outline:

Step 1: Assume that we are given some data. In the present situation this is

$$\begin{array}{ll} \rightsquigarrow & n \in \mathbb{N}; \\ \rightsquigarrow & \text{strings } S[L_j, \mathfrak{m}_j],\ j = 1, \dots, n, \text{ with discrete spectrum;} \\ \rightsquigarrow & \text{numbers } \gamma_j \in (-\infty, \infty] \text{ for those values of } j \in \{1, \dots, n\} \\ & \text{for which } S[L_j, \mathfrak{m}_j] \text{ is regular;} \\ \rightsquigarrow & \mu > 0. \end{array} \tag{6.9}$$

Then we can state the problem (6.4)–(6.8), and the spectrum of this problem is discrete and consists of eigenvalues of finite multiplicity.

Step 2: A direct and an inverse spectral problem suggests itself: Characterize the spectra (including multiplicities) of problems of the form introduced in Step 1.

Step 3: From data given according to Step 1, we construct an entire function which describes the spectrum of the problem stated in Step 1 as its zeroset (including multiplicities).

Step 4: We invoke our results of the previous sections to solve spectral problems as posed in Step 2.

Let us proceed to the construction of the function Φ. This is done in exactly the same way as in [P3, §3]. Assume that data $\langle n, S[L_j, \mathfrak{m}_j], \gamma_j, \mu \rangle$ is given according to (6.9). Since we assume that $S[L_j, \mathfrak{m}_j]$ has discrete spectrum, there exists for each $\lambda \in \mathbb{C}$ a nontrivial solution $S_j(\lambda, s) \in L^2(\mathfrak{m}_j)$ of (6.4) which satisfies (6.7), (6.8), and this solution is unique up to scalar multiples. For each fixed $s \in [0, L_j)$ the function $S_j(\lambda, s)$ is an entire functions of λ. Moreover, $S_j(\lambda, s)$ and $\frac{\partial S_j(\lambda,s)}{\partial s}$ are even functions of λ. Note that, with the notation of §2.D, we have $S_j(\lambda, s) = s_j(\lambda^2, s)$.

The system (6.4)–(6.8) has a nontrivial solution if and only if the linear system of equation

$$\begin{pmatrix} S_1(\lambda,0) & -S_2(\lambda,0) & 0 & \dots & 0 \\ S_1(\lambda,0) & 0 & -S_3(\lambda,0) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_1(\lambda,0) & 0 & 0 & \dots & -S_n(\lambda,0) \\ \frac{\partial S_1(\lambda,s)}{\partial s}|_{s=0} + i\mu\lambda S_1(\lambda,0) & \frac{\partial S_2(\lambda,s)}{\partial s}|_{s=0} & \frac{\partial S_3(\lambda,s)}{\partial s}|_{s=0} & \dots & \frac{\partial S_n(\lambda,s)}{\partial s}|_{s=0} \end{pmatrix} \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = 0$$

has a nontrivial solution $(C_1, \dots, C_n)$. The determinant of the matrix of this linear system, however, computes as

$$\Phi(\lambda) := \Big(\sum_{j=1}^n \frac{\partial}{\partial s} S_j(\lambda, s)\Big|_{s=-0} \prod_{\substack{i=1\\ i\neq j}}^n S_i(\lambda, 0)\Big) + i\lambda\mu \prod_{j=1}^n S_j(\lambda, 0) \,. \tag{6.10}$$

Thus the spectrum of the problem (6.4)–(6.8) equals the set of zeros of Φ, and this equality includes multiplicities.
Let us introduce functions

$$P_j(z) =: \mu S_j(\sqrt{z}, 0) = \mu s_j(z, 0), \quad Q_j(z) =: \frac{\partial}{\partial s} S_j(\sqrt{z}, s)\Big|_{s=-0} = s_j'(z, 0-)\,,$$

and let $P(z)$ and $Q(z)$ be defined as in (1.1), i.e.

$$P := \prod_{j=1}^n P_j, \quad Q := \sum_{j=1}^n \Big(Q_j \prod_{\substack{l=1\\ l\neq j}}^n P_l\Big)\,. \tag{6.11}$$

Then

$$\Phi(\lambda) = \frac{1}{\mu^{n-1}} \phi_{Q,-P}(\lambda)\,,$$

in particular, $\mathfrak{d}_\Phi = \mathfrak{d}_{\phi_{Q,-P}}$.

A direct spectral theorem is obtained as corollary of the results given in §4 and §5.

6.2 Theorem. *Let data $\langle n, S[L_j, \mathfrak{m}_j], \gamma_j, \mu\rangle$ be given according to (6.9), and let $d : \mathbb{C} \to \mathbb{N}_0$ be the function which assigns to each point $w \in \mathbb{C}$ its multiplicity as an eigenvalue of the problem (6.4)–(6.8) (understanding $d(w) = 0$ to mean that w does not belong to the spectrum). Choose a function $c : \mathbb{C} \to \mathbb{N}_0$ such that $c(w) = c(-\overline{w})$, $w \in \mathbb{C}$, $\operatorname{supp} c \subseteq (\operatorname{supp} d) \cap \{w \in \mathbb{C} : \operatorname{Im} w < 0, \operatorname{Re} w \neq 0\}$, and $\sum_{w\in\mathbb{C}\setminus\{0\}} d(w)\big|\frac{r}{w}\big|^{c(w)+1} < \infty$, $r > 0$. Let E be the entire function defined as*

$$E(z) := \prod_{\operatorname{Im} w<0} \Big(1 - \frac{z}{w}\Big)^{d(w)} \exp\Big(d(w) \sum_{k=1}^{c(w)} \frac{z^k}{k} \operatorname{Re} \frac{1}{w^k}\Big)\,.$$

Then the following hold:

(i) There exist only finitely many nonreal conjugate pairs $(w, \overline{w})$ such that $\min\{d(w), d(\overline{w})\} > 0$. Each such pair $(w, \overline{w})$ is located on the imaginary axis, and satifies $d(w) = d(\overline{w})$.

Let $(i\lambda_1, -i\lambda_1), \dots, (i\lambda_p, -i\lambda_p)$, $0 < \lambda_1 < \dots < \lambda_p$, be all the nonreal conjugate pairs as in (i), and denote by d_Λ the function

$$d_\Lambda(w) := \begin{cases} d(w) & , \ w = \pm i\lambda_j, \ j = 1, \dots, p \\ 0 & , \ \textit{otherwise} \end{cases}$$

(ii) There exist only finitely many points w in the upper half plane, such that $d(w) - d_\Lambda(w) > 0$. Each such point w lies on the imaginary axis and satisifes $d(w) = 1$.

Denote the points in (ii) as $iy_1,\dots,iy_\kappa$, $0<y_1<\dots<y_\kappa$, *and let* Y *be the function*

$$Y(z):=\prod_{j=1}^{\kappa}\Big(1-\frac{z}{iy_j}\Big).$$

(*iii*) *We have* $d(-\overline{w})=d(w)$, $w\in\mathbb{C}$, *and* $d(0)\in\{0,1\}\cup 2\mathbb{N}$.

(*iv*) *For each* $k\in\{2,\dots,\kappa\}$, *the number* $\sum_{w\in[-iy_{k-1},-iy_k]}(d(w)-d_\Lambda(w))$ *is odd. The number* $\sum_{w\in(0,-iy_1]}(d(w)-d_\Lambda(w))$ *is odd if* $d(0)=1$ *and even otherwise.*

(*v*) $\max\big(\{d(i\lambda_j):\, j=1,\dots,p\}\cup\{d(x):\, x\in\mathbb{R}\setminus\{0\}\}\cup\{[\frac{1}{2}d(0)]\}\big)\le n-1$.

(*vi*) *We have* $\sum_{\operatorname{Im} w<0} d(w)\operatorname{Im}\frac{1}{w}<\infty$.

(*vii*) *There exists a number* $a\ge 0$ *such that*

- (*a*) $\arg Y(w)+\arg E(w)\equiv aw \mod \pi$, $w\in(\operatorname{supp} d)\cap\mathbb{R}$;
- (*b*) *If* $d(0)>1$, *then* $a=\sum_{j=1}^{\kappa}\frac{1}{y_j}-\sum_{\operatorname{Im} w<0}\operatorname{Im}\frac{1}{w}$;
- (*c*) $e^{-2a\lambda_j}\frac{E(-i\lambda_j)}{E(i\lambda_j)}=(-1)^\delta\frac{Y(i\lambda_j)}{Y(-i\lambda_j)}$, $j=1,\dots,p$.
- (*d*) *There exists a number* $\lambda\in(y_\kappa,\infty)$ *such that* $e^{-2a\lambda}\frac{E(-i\lambda)}{E(i\lambda)}=-\frac{Y(i\lambda)}{Y(-i\lambda)}$.

(*viii*) *We have* $p\le\kappa$ *and* $\kappa+\sum_{j=1}^{p}\epsilon_j\le n$.

(*ix*) *Denote by* (a_m) *and* (x_m) *(finite or infinite) sequences such that* $\{a_m\}=(\operatorname{supp} d)\cap\{w\in\mathbb{C}:\operatorname{Im} w<0\}$ *and* $\{x_m\}=(\operatorname{supp} d)\cap\mathbb{R}$, *where each point* w *is listed exactly* $d(w)$ *times. Let* ρ *and* ρ' *be the convergence exponents of* (a_m) *and* (x_m), *respectively, and let* $\Delta:=\Delta^{r^\rho}(a_m)$, $\Delta':=\Delta^{r^\rho}(x_m)$, $\gamma:=\gamma^{r^\rho}(a_m)$. *Then*

- (*a*) *We have* $\rho'\le\rho$;
- (*b*) *If* $\rho'<\rho$ *or* $\rho'=\rho\notin 2\mathbb{N}$, *then* $\Delta'\le c(\rho,n)\Delta$ *with some constant* $c(\rho,n)$ *which depends only on* ρ *and* n;
- (*c*) *If* $\rho'=\rho\in 2\mathbb{N}$, *then* $\gamma<\infty$ *implies that* $\Delta'<\infty$.

Proof. By Lemma 2.14 and 2.10–2.12, we have $P_j^{-1}Q_j\in\mathcal{N}_-^{\mathrm{ep}}$. Moreover, $s(z,0)$ and $s'(z,0-)$ have no common zeros, i.e. (Q_j,P_j) forms a 1-$\mathring{\mathcal{N}}_-^{\mathrm{ep}}$-pair. Thus (Q,P) is an m-$\mathring{\mathcal{N}}_-^{\mathrm{ep}}$-pair with some $m\le n$. Moreover, as we have noted above, $d=\mathfrak{d}_{\phi_{Q,-P}}$. The assertions (*i*)–(*vii*) are just what has been shown in Theorem 4.15 and Corollary 4.17.

The assertion (*viii*) follows with an easy counting argument: Each function $-Q_j^{-1}P_j$ belongs to $\mathcal{N}_+^{\mathrm{ep}}$ and has at most one pole in $(-\infty,0)$. Thus each $P_j^{-1}Q_j$ has at most one pole in $(-\infty,0)$, and we conclude that $P^{-1}Q$ has at most n poles in $(-\infty,0)$. It follows from Corollary 4.13 that $\kappa+\sum_{j=1}^{p}\epsilon_j\le n+1$. If equality holds, then $P^{-1}Q$ has exactly n poles (including multiplicities) and $\lim_{x\nearrow 0}P^{-1}(x)Q(x)\in(0,+\infty]$. This, however, would imply that at least one of the functions $P_j^{-1}Q_j\in\mathcal{N}_-^{\mathrm{ep}}$ must in the same time have a pole in $(-\infty,0)$ and satisfy $\lim_{x\nearrow 0}P_j^{-1}(x)Q_j(x)\in(0,+\infty]$. We obtain that $Q_j^{-1}P_j$ has two poles in $(-\infty,0)$, a contradiction.

Finally, the assertion (ix) can be deduced with some standard arguments from Theorem 5.4, if we keep in mind that $\phi_{Q,-P}$ has only finitely many zeros in $\mathbb{C}^+$.

❑

6.3 Remark. The inverse spectral problem remains unsolved. Thereby the obstacle is not that we deal with a whole graph of strings, but the lack of a description of the totality of all functions arising as $q^{\gamma}_{L,\mathfrak{m}}$, $\gamma \in (-\infty, \infty]$, for strings $S[L, \mathfrak{m}]$, and an inverse theorem analogous to 2.13 for this class of functions. To answer these questions, however, seems to be a hard task.

6.4 Remark. This theorem contains some earlier results as particular cases. For example some parts of [KN, Theorem 3.1] can be obtained.

From the viewpoint of the physical interpretation of (6.1)–(6.3) as a damped system of strings, eigenfrequencies whose eigenfunctions have exponentially increasing amplitudes, or edges with infinite length or infinite total mass, do not make much sense. Hence, main interest lies in damped systems of regular strings for which the spectrum of (6.4)–(6.8) lies in the closed lower half-plane. The question which systems of strings have spectrum contained in $\mathbb{C}^- \cup \mathbb{R}$ can be answered in a most satisfactory way, see the below Corollary 6.8.

The question which spectra correspond to a system of regular strings remains unsolved; in the below Corollary 6.5 we just give a necessary condition which is certainly far from being sufficient. However, it already shows that for this question the asymptotic behaviour of the spectrum will play a role. To find a complete answer is probably quite difficult; recall that already the necessary and sufficient conditions given in [KK2, 11.11°] for a single string to be regular are unpleasantly implicit.

6.5 Corollary. *Assume that a collection $\langle n, S[L_j, \mathfrak{m}_j], \gamma_j, \mu\rangle$ of regular strings is given, and let $(a_m)_{m\in\mathbb{N}}$ be the sequence which contains each point of the spectrum of the problem* (6.4)–(6.8) *as often as its multiplicity prescribes. Let $\Delta := \Delta^r(a_m)$ be the density of the sequence $(a_m)_{m\in\mathbb{N}}$ with respect to the growth function $\lambda(r) := r$, cf. §5.1. Then*

$$\Delta \leq e \cdot \sum_{j=1}^{n} \sqrt{2L_j M(L_j)}\,.$$

Proof. If $S[L, \mathfrak{m}]$ is a regular string, then the functions $\varphi(z, L), \psi(z, L)$ are entire functions of order $\frac{1}{2}$ and their $r^{\frac{1}{2}}$-type does not exceed $\sqrt{2LM(L)}$, cf. [KK2, (2.27)], [BW, Proposition 2.3]. Hence the function Φ defined in (6.10) is of finite exponential type, and its type does not exceed $\sum_{j=1}^{n} \sqrt{2L_j M(L_j)}$. The asserted estimate follows from [L, Hilfssatz I.4].

❑

6.2 Direct and inverse problem for a system of strings with nonnegative spectrum

Let us turn to a closer investigation of strings with a nonnegative spectrum. In this case, also the inverse problem can be solved. Assume that the following data is given:

$$\begin{array}{ll} \rightsquigarrow & n \in \mathbb{N}; \\ \rightsquigarrow & \text{strings } S[L_j, \mathfrak{m}_j],\ j = 1, \ldots, n, \text{ with discrete spectrum;} \\ \rightsquigarrow & \text{numbers } \gamma_j \in [0, \infty] \text{ for those values of } j \in \{1, \ldots, n\} \\ & \text{for which } S[L_j, \mathfrak{m}_j] \text{ is regular;} \\ \rightsquigarrow & \mu > 0. \end{array} \tag{6.12}$$

Then we can consider the problem (6.4)–(6.8), and proceed along the lines indicated in Remark 6.1.

Since the now considered situation (6.12) is a subcase of the previous situation (6.9), we know that again the function Φ defined by (6.10) will describe the spectrum of the problem. Of course, we also know that the spectrum of (6.4)–(6.8) will satisfy all conditions stated in Theorem 6.2.

6.6 Theorem. *Let $d : \mathbb{C} \to \mathbb{N}_0$ be a function with discrete support. In order that there exists data $\langle n, S[L_j, \mathfrak{m}_j], \gamma_j, \mu\rangle$ as in* (6.12)*, such that d describes the spectrum of the problem* (6.4)–(6.8) *buildt with this data, it is necessary and sufficient that the following conditions* (i)–(v) *hold:*

(i) $\operatorname{supp} d \subseteq \mathbb{C}^- \cup \mathbb{R}$*;*

(ii) *We have $d(-\overline{w}) = d(w)$, $w \in \mathbb{C}$, and $d(0) \in \{0, 1\}$;*

(iii) $\max\{d(x) : x \in \mathbb{R} \setminus \{0\}\} < \infty$*;*

(iv) *We have $\sum_{\operatorname{Im} w < 0} d(w) \operatorname{Im} \frac{1}{w} < \infty$;*

Let functions $c(w)$ and $E(z)$ be as in Theorem 6.2.

(v) *There exists a number $a \geq 0$ such that $\arg E(w) \equiv aw \mod \pi$, $w \in (\operatorname{supp} d) \cap \mathbb{R}$.*

Proof. Assume first that $\langle n, S[L_j, \mathfrak{m}_j], \gamma_j, \mu\rangle$ is given according to (6.12), and let notation be as in §6.1. Since $\gamma_j \in [0, \infty]$, we know that $P_j^{-1} Q_j \in \mathcal{S}^{-1}$. Hence, we can apply Corollary 4.18, and conclude that the present conditions (i)–(v) hold.

Conversely, assume that (i)–(v) hold. Then, again by Corollary 4.18, there exists $n \in \mathbb{N}$ and an n-$\mathring{\mathcal{S}}^{-1}$-pair (Q, P), such that $d = \mathfrak{d}_{\phi_{Q,-P}}$. Choose 1-$\mathring{\mathcal{S}}^{-1}$-pairs (Q_j, P_j), $j = 1, \ldots, n$, such that

$$\frac{Q}{P} = \sum_{j=1}^{n'} \frac{Q_j}{P_j}, \quad P = \prod_{j=1}^{n'} P_j$$

holds. By 2.13, there exist strings $S[L_j, \mathfrak{m}_j]$ and numbers $\gamma_j \in [0, \infty]$ whenever $S[L_j, \mathfrak{m}_j]$ is regular, such that

$$-\frac{P_j(z)}{Q_j(z)} = -\frac{s_j(z, 0)}{s_j'(z, 0-)}.$$

Choose a number $\mu > 0$, set

$$\tilde{P}_j(z) := \mu s_j(z, 0), \quad \tilde{Q}_j(z) := s_j'(z, 0-),$$

and let $\tilde{P}$ and $\tilde{Q}$ be defined accordingly. Since the two functions P_j and Q_j, as well as the two functions $\tilde{P}_j$ and $\tilde{Q}_j$, have no common zeros, we can find a zerofree function D_j such that

$$P_j(z) = D_j(z)\tilde{P}_j(z), \quad Q_j(z) = D_j(z)\tilde{Q}_j(z)\,.$$

Then

$$\phi_{Q,-P}(z) = \prod_{j=1}^{n} D_j(z^2) \cdot \phi_{\tilde{Q},-\tilde{P}}(z)\,,$$

in particular, $\mathfrak{d}_{\phi_{Q,-P}} = \mathfrak{d}_{\phi_{\tilde{Q},-\tilde{P}}}$. We see that the spectrum induced by the data $\langle n, S[L_j, \mathfrak{m}_j], \gamma_j, \mu\rangle$ is exactly described by d.

❑

6.7 Remark. Assume that d satisfies the conditions of Theorem 6.6. The data $\langle n, S[L_j, \mathfrak{m}_j], \gamma_j, \mu\rangle$ representing d as the spectrum of (6.4)–(6.8), is by no means unique. For example already the choice of the number $\mu > 0$ was arbitrary. More interesting is the following notice, which is seen from proof of Theorem 6.6: *Each representation*

$$\frac{Q}{P} = \sum_{j=1}^{n'} \frac{Q_j}{P_j}, \quad P = \prod_{j=1}^{n'} P_j \tag{6.13}$$

where $n' \in \mathbb{N}$ and where (Q_j, P_j) are 1-$\mathring{\mathcal{S}}^{-1}$-pairs, yields a system of strings $\langle n, S[L_j, \mathfrak{m}_j], \gamma_j, \mu\rangle$ whose spectrum realizes d.

However, there exists a large variety of essentially different representations (6.13).

6.8 Corollary. *Let data $\langle n, S[L_j, \mathfrak{m}_j], \gamma_j, \mu\rangle$ be given according to* (6.9). *Then the spectrum of the problem* (6.4)–(6.8) *is contained in $\mathbb{C}^- \cup \mathbb{R}$ if and only if there exists data $\langle n', S[L'_j, \mathfrak{m}'_j], \gamma'_j, \mu'\rangle$ as in* (6.12), *i.e. with all numbers γ'_j nonnegative, which gives rise to the same spectrum (including multiplicities) as $\langle n, S[L_j, \mathfrak{m}_j], \gamma_j, \mu\rangle$.*

Proof. Assume first that $\gamma_j \in [0, \infty]$ for all j fof which $S[L_j, \mathfrak{m}_j]$ is regular. Then, by Theorem 6.6, the spectrum of the problem (6.4)–(6.8) lies in the closed lower half-plane.

Conversely, let d be the function which describes the spectrum including its multiplicites, and assume that $\operatorname{supp} d \subseteq \mathbb{C}^- \cup \mathbb{R}$. Then, using the notation of Theorem 6.2, we have $Y = 1$. Moreover, by Remark 4.16, (*iii*), we must have $d(0) \in \{0, 1\}$. Now Theorem 6.6 ensures the existence of the reqired data $\langle n', S[L'_j, \mathfrak{m}'_j], \gamma'_j, \mu'\rangle$.

❑

6.9 Remark. Let us note explicitly that it can happen that the spectrum induced by data $\langle n, S[L_j, \mathfrak{m}_j], \gamma_j, \mu\rangle$ lies in the closed lower half-plane although some of the numbers γ_j are negative. An example can be constructed corresponding to Example 4.20.

Using the characterization of n-$\mathring{\mathcal{S}}^{-1}$-pairs in terms of the zeros of P and Q, cf. Corollary 4.6, we obtain the following statement.

6.10 Corollary. *Let* $\langle n, S[L_j, \mathfrak{m}_j], \gamma_j, \mu\rangle$ *be given according to* (6.12), *and let* P *and* Q *be the functions defined in* (6.11). *Then* P *and* Q *have only real and nonnegative zeros. Denote the (finite or infinite) sequences of zeros of* P *and* Q *by* (μ_k) *and* (ν_k), *respectively, where each zero is listed according to their multiplicities. Assume that these sequences are arranged such that*

$$\mu_1 \leq \mu_2 \leq \dots \quad \textit{and} \quad \nu_1 \leq \nu_2 \leq \dots .$$

Then $\nu_1 < \mu_1 \leq \nu_2 \leq \mu_2 \leq \nu_3 \leq \dots$ *and*

$$\forall\, k \in \mathbb{N} : \Big(\mu_k = \nu_{k+1} \iff \nu_{k+1} = \mu_{k+1}\Big) .$$

Moreover, eaxh fixed point $x_0 \in \mathbb{R}$ *can occur at most* n *times in the sequence* (μ_k). ❑

6.11 Remark. Corollary 6.10 contains several earlier results as particular cases:

(*i*) [P1, Theorem 1] for $n = 2$ and $\gamma_j = 0$, $j = 1, 2$;

(*ii*) [HM, Theorem 5.1] for $\gamma_j \in R$ and $n = 2$;

(*iii*) [P2, Lemma 1.15] for $n = 3$ and $\gamma_j = \infty$, $j = 1, 2, 3$;

(*iv*) [P3, Theorem 3.17] for $n \in \mathbb{N}$ and $\gamma_j = 0$.

6.3 Direct problem for a system of Sturm-Liouville equations

Assume that we are given the data

- ⇝ $n \in \mathbb{N}$;
- ⇝ real and square-integrable potentials q_j, $j = 1, \dots, n$, which are defined on respective intervals $[0, a_j]$, $a_j \in (0, \infty)$;
- ⇝ numbers $\gamma_j \in (-\infty, \infty]$, $j = 1, \dots, n$;
- ⇝ $\alpha > 0$ and $\beta \in \mathbb{R}$.

(6.14)

Then we can state the problem:

$$y_j'' + \lambda^2 y_j - q(x) y_j = 0, \quad j = 1, \dots, n,\ x \in [0, a_j]\,, \tag{6.15}$$

$$y_1(\lambda, 0) = y_2(\lambda, 0) = \dots = y_n(\lambda, 0)\,, \tag{6.16}$$

$$\sum_{j=1}^{n} y_j'(\lambda, 0) + (i\alpha\lambda + \beta) y_1(\lambda, 0) = 0\,, \tag{6.17}$$

$$\gamma_j y_j'(\lambda, a_j) + y_j(\lambda, a_j) = 0, \quad j = 1, \dots, n\,. \tag{6.18}$$

The condition (6.18) is in the case $\gamma_j = \infty$ again understood as $y_j'(\lambda, a_j) = 0$. The spectrum of this problem is discrete, see e.g. [M, §1.3].

Such equations occurs from various physical problems. In general, non-real poles of the resolvent in $\mathbb{C} \setminus \mathbb{R}$ are called resonances. Physically, while real eigenvalues represent real energy levels and states in which the particles are permanently localized, unless disturbed, resonances correspond to quasi-stationary (metastable) states that only exist for a finite time, proportional to the inverse of the negative imaginary part of the resonance, and have energy proportional to the real part of the resonances, cf. [B]. In the setting of quantum mechanics it makes sense to allow eigenvalues in the upper half plane. These correspond to resonances on the so called unphysical sheet.

We employ the same method as in §6.1 to obtain a description of the spectrum of a problem of the form (6.15)–(6.18).

Again the computation of [P3, §3] works, and shows that the spectrum of the problem (6.15)–(6.18) (including multiplicities) is equal to the set of zeros of a certain entire function, namely the function Φ defined as follows: Let s_j, $j = 1, \ldots, n$, be nontrivial solutions of (6.15) which satisfy the boundary condition (6.18), and put

$$\Phi(\lambda) := \Big(\sum_{j=1}^{n} \frac{\partial}{\partial s} s_j(\lambda, s)\Big|_{s=-0} \prod_{\substack{i=1,\ldots,n \\ i\neq j}} s_i(\lambda, 0)\Big) + (\beta + i\lambda\alpha) \prod_{j=1}^{n} s_j(\lambda, 0) .$$

Introduce functions

$$P_j(z) =: \alpha \frac{1}{\alpha} s_j(\sqrt{z}, 0), \quad Q_j(z) =: \frac{\partial}{\partial s} s_j(\sqrt{z}, s)\Big|_{s=-0} + \frac{\beta}{n} s_j(\sqrt{z}, 0) ,$$

and set

$$P := \prod_{j=1}^{n} P_j, \quad Q := \sum_{j=1}^{n} \Big(Q_j \prod_{\substack{l=1 \\ l\neq j}}^{n} P_l \Big) .$$

Then

$$\Phi(\lambda) = \frac{1}{\alpha^{n-1}} \phi_{Q,-P}(\lambda) ,$$

and we see that $\mathfrak{d}_\Phi = \mathfrak{d}_{\phi_{Q,-P}}$.

The function $q_j := s_j(\sqrt{z}, 0)^{-1} \frac{\partial}{\partial s} s_j(\sqrt{z}, s)|_{s=0}$ belongs to the Nevanlinna class $\mathcal{N}$, cf. e.g. [L, VII.4]. By the known asymptotics of the functions s_j and s_j', cf. [M, Lemma 1.3.2], the function q_j actually belongs to $\mathcal{N}_-^{\text{ep}}$ and satisfies $\lim_{z\to-\infty} q_j(z) = -\infty$. Thus also

$$\frac{Q_j}{P_j} = \frac{1}{\alpha} q_j + \frac{\beta}{n\alpha} \in \mathcal{N}_-^{\text{ep}} .$$

We see that (Q, P) is an n-$\mathring{\mathcal{N}}_-^{\text{ep}}$-pair and, therefore, obtain a direct spectral theorem similar to Theorem 6.2.

6.12 Theorem. *Let data $\langle n, q_j, \gamma_j, \alpha, \beta\rangle$ be given according to* (6.14)*, and let $d : \mathbb{C} \to \mathbb{N}_0$ be the function which assigns to each point $w \in \mathbb{C}$ its multiplicity as an eigenvalue of the problem* (6.15)–(6.18) *understanding $d(w) = 0$ to mean that w does not belong to the spectrum). Then exactly the same assertion as written in Theorem 6.2 hold.* ❑

6.13 Remark. Again we obtain several earlier results as corollaries:

(*i*) For $n = 1$, $\beta = 0$, $\alpha = 1$ the problem (6.15)–(6.18) is nothing else but the Regge problem, cf. [Re]. Theorem 6.12 now implies [S, Theorem 6], see also [Ko].

(*ii*) For $n = 1$, and arbitrary $\beta \in R, \alpha \in (0, 1) \cup (1, \infty)$, Theorem 6.12 implies [PvM, Theorem 3.1].

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