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ADAPTIVE BOUNDARY ELEMENT METHOD: SIMPLE ERROR ESTIMATORS AND CONVERGENCE

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Introduction. We consider Symm's integral equation in 2D with weakly singular integral operator

$$Vu(x) = -\frac{1}{2\pi}\int_\Gamma \log|x-y|\,u(y)\,ds_y. \tag{1}$$

Here, $\Gamma \subseteq \partial\Omega$ is an open piece of the boundary $\partial\Omega$ of a Lipschitz domain $\Omega \subset \mathbb{R}^2$. Provided $\operatorname{diam}(\Omega) < 1$, $\langle\!\langle u\,,\,v\rangle\!\rangle := \int_\Gamma Vu(x)v(x)\,ds_x$ defines an equivalent scalar product on $\mathcal{H} = \widetilde{H}^{-1/2}(\Gamma)$. For a given linear and continuous functional $\Phi \in \mathcal{H}^*$, the Lax-Milgram lemma thus proves the unique existence of (some unknown) $u \in \mathcal{H}$ with

$$\langle\!\langle u\,,\,v\rangle\!\rangle = \Phi(v) \quad \text{for all } v \in \mathcal{H}. \tag{2}$$

To approximate u by the lowest-order Galerkin scheme, let $\mathcal{T}_\ell$ be a triangulation of Γ and $X_\ell = \mathcal{P}^0(\mathcal{T}_\ell) := \big\{v_\ell : \Gamma \to \mathbb{R} \,:\, \forall T \in \mathcal{T}_\ell \quad v_\ell|_T \text{ is constant}\big\} \subset \mathcal{H}$. The (numerically computable) Galerkin solution $u_\ell \in X_\ell$ is the unique solution of

$$\langle\!\langle u_\ell\,,\,v_\ell\rangle\!\rangle = \Phi(v_\ell) \quad \text{for all } v_\ell \in X_\ell. \tag{3}$$

In a posteriori error analysis, one aims to provide a computable quantity η_ℓ which only depends on known and computed data, for instance, on u_ℓ and Φ such that

$$C_{\rm eff}^{-1}\eta_\ell \le |||u-u_\ell||| \le C_{\rm rel}\eta_\ell. \tag{4}$$

Here, $|||\cdot|||$ denotes the energy norm induced by $\langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle$. The lower and upper estimate are referred to as *efficiency* and *reliability* of η_ℓ, respectively, and local information of η_ℓ will be used to improve the mesh by local mesh-refinement.

***h*-*h*/2-based error estimators for Symm's integral equation.** The h-$h/2$-based strategy is one very basic and well-known technique for the a posteriori error estimation for Galerkin discretizations of energy minimization problems. Let $u_\ell \in X_\ell$ and $\widehat{u}_\ell \in \widehat{X}_\ell = \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$ be Galerkin solutions, where $\widehat{\mathcal{T}}_\ell$ is obtained by uniform refinement of $\mathcal{T}_\ell$. One then considers

$$\eta_\ell := |||\widehat{u}_\ell - u_\ell||| \tag{5}$$

to estimate the error $|||u-u_\ell|||$. By Galerkin orthogonality, η_ℓ is always efficient with known constant $C_{\rm eff} = 1$. Reliability of η_ℓ with $C_{\rm rel} = (1-\sigma^2)^{-1/2}$ follows from the saturation assumption

$$|||u-\widehat{u}_\ell||| \le \sigma\,|||u-u_\ell||| \quad \text{with some uniform constant} \quad \sigma \in (0,1). \tag{6}$$

Unlike to FEM, where (6) is proven for a sufficiently small mesh-size [11], the saturation assumption is open in the context of BEM but observed in practice [17].

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Since the energy norm $|||\cdot|||$ is nonlocal, the error estimator η_ℓ does not provide information for a local mesh-refinement. Using a local inverse estimate from [18] and a local approximation result from [6], one may prove estimator equivalence [17]

$$C_{\rm apx}^{-1}\eta_\ell \le \mu_\ell := \|h_\ell^{1/2}(\widehat{u}_\ell - u_\ell)\|_{L^2(\Gamma)} \le C_{\rm inv}\eta_\ell, \tag{7}$$

where $h_\ell \in L^\infty(\Gamma)$ denotes the local mesh-width $h_\ell|_T = \mathrm{diam}(T)$ for $T \in \mathcal{T}_\ell$. The local contributions $\mu_\ell(T) := \mathrm{diam}(T)^{1/2}\|\widehat{u}_\ell - u_\ell\|_{L^2(T)}$ of μ_ℓ are then used for the marking strategy in an adaptive mesh-refining algorithm.

Convergence of adaptive Galerkin BEM. Based on the error estimators η_ℓ and μ_ℓ from the previous section and based on a fixed parameter $\theta \in (0,1)$, the usual adaptive algorithm reads as follows: Until η_ℓ is sufficiently small, do:

(i) Refine $\mathcal{T}_\ell$ uniformly to obtain $\widehat{\mathcal{T}}_\ell$.
(ii) Compute discrete solutions u_ℓ and $\widehat{u}_\ell$.
(iii) Find minimal set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that

$$\theta \sum_{T\in\mathcal{T}_\ell} \mu_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell} \mu_\ell(T)^2. \tag{8}$$

(iv) Refine at least marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$.
(v) Increase counter $\ell \mapsto \ell+1$ and iterate.

Convergence of this type of algorithms has first been proven in [10], where also the marking criterion (8) is introduced. The latter work considered the residual error estimator for a P1-FEM discretization of the Poisson problem, and it is assumed that data oscillations on the initial mesh are sufficiently small. In [21], the resolution of the data oscillations is included into the adaptive algorithm. The convergence analysis is based on reliability and the so-called *discrete local efficiency* of the residual error estimator, which relies on an *interior node property* for the local refinement. The main idea of the convergence proof then is to show that the error is contractive up to the data oscillations. In [9], this has been weakened in the sense that it is proven that a weighted sum of error and error estimator yields a contraction property without requiring (discrete local) efficiency.

Only recently, analogous results for adaptive BEM could be derived, and a first convergence result reads as follows [16]: Provided that μ_ℓ is reliable and that marked elements are halved, there are constants $\kappa, \gamma \in (0,1)$ such that

$$\Delta_\ell^2 := |||u - u_\ell|||^2 + |||u - \widehat{u}_\ell|||^2 + \gamma\,\mu_\ell^2 \quad \text{satisfies} \quad \Delta_{\ell+1} \le \kappa\,\Delta_\ell. \tag{9}$$

In particular, this implies convergence $u_\ell \to u$ as $\ell \to \infty$. The proof of (9) requires that the local contributions of μ_ℓ used for marking, have an h-weighting factor. Therefore, the analysis might carry over to adaptive algorithms steered by h-weighted residual error estimators [2, 4, 5] or averaging error estimators [6, 7, 8], whereas the two-level error estimators [12, 13, 19, 20, 22] and the Faermann error estimator [3, 14, 15] seem to need further arguments.

Concluding Remarks. The convergence proof of [16] for adaptive Galerkin BEM also applies to hypersingular integral equations and mixed formulations in 2D and 3D. For 3D, however, our proof — as well as the available a posteriori error analysis from [4, 5, 6, 7, 8, 13, 15, 17, 19, 22] — is restricted to the case of isotropic mesh-refinement, whereas anisotropic mesh-refinement is needed to resolve edge singularities efficiently.

Despite of convergence, even the question of optimal convergence rates of the adaptive FEM based on residual error estimators is well-understood. Whereas prior works [1, 23] used an additional coarsening step to prove optimality, recent works [9, 24] prove optimality for the standard algorithm steered by the residual error estimator. The latter analysis relies on a *discrete local reliability* of the error estimator, which remains open for adaptive Galerkin BEM. This will be a major topic for future research.

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