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Dependence of the Weyl coefficient on singular interface conditions

Matthias Langer, Harald Woracek*

Abstract

We investigate the influence of interface conditions at a singularity of an indefinite canonical system on its Weyl coefficient. An explicit formula which parameterizes all possible Weyl coefficients of indefinite canonical systems with fixed Hamiltonian function is derived. This result is illustrated with two examples: the Bessel equation, which has a singular endpoint, and a Sturm–Liouville equation whose potential has an inner singularity, which arises from a continuation problem for a positive definite function.

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Keywords: indefinite canonical system, Weyl coefficient, singular potential

1 Introduction

A canonical system is a system of differential equations of the form

$$\frac{\partial}{\partial t}x(t,z) = zJH(t)x(t,z), \quad t \in [0,L)\,, \tag{1.1}$$

where $x = (x_1, x_2)^T$, $H(t)$ is a real and locally integrable 2×2-matrix valued function on $[0,L)$, $H(t) \geq 0$, which does not vanish on any set of positive measure, J denotes the symplectic matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and z is a complex parameter. The function H is called the Hamiltonian of the system (1.1). Canonical systems frequently arise in mathematical physics, for example in Hamiltonian mechanics or from the equation of a vibrating string, see, e.g. [2, 4, 13, 21]. Also, canonical systems can be viewed as natural generalizations of Sturm–Liouville equations. There are various approaches to an analysis of the equation (1.1), some of them employ operator theoretic methods, see, e.g. [3, 17, 19, 22, 23, 24, 35].

A canonical system is said to be in the limit point case at L, if $\int_0^L \operatorname{tr} H(t)\,dt = \infty$. A decisive role in the spectral analysis of canonical systems of this kind is

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played by the Weyl coefficient q_H associated with the Hamiltonian H. We will recall its construction later, cf. (2.5), at this stage let us only state its most important properties. It belongs to the class $\mathcal{N}_0$ of Nevanlinna functions, that is, q_H is analytic in $\mathbb{C} \setminus \mathbb{R}$, $q_H(\overline{z}) = \overline{q_H(z)}$ and $\operatorname{Im} q_H(z) \geq 0$ for $\operatorname{Im} z > 0$. The function q_H completely describes the spectrum of the problem (1.1) with boundary condition $x_1(0, z) = 0$, and the measure in its Herglotz integral representation can be used to construct a generalized Fourier transform. The Inverse Spectral Theorem due to L. de Branges states that the assignment $H \mapsto q_H$ yields a bijection of the set of all Hamiltonians (up to changes of scale) and the set $\mathcal{N}_0$, see [5]–[8] and also [37]. The proof of this deep result is contained in de Branges' theory of Hilbert spaces of entire functions, cf. [9]; many of its components can also be interpreted by means of the theory of symmetric and self-adjoint operators in a Hilbert space, in particular by means of M. G. Krein's theory of entire operators, cf. [18].

Recently, a generalization of the notion of a Hamiltonian and a canonical system to an indefinite (Pontryagin space) setting was given, cf. [28], [29]. Motivation to study an indefinite generalization of canonical systems can be drawn from various sources. For example, the class $\mathcal{N}_0$ has a generalization to an indefinite setting which has proved to be useful in various contexts and thus has been studied intensively, namely, the set $\mathcal{N}_{<\infty}$ of generalized Nevanlinna functions; we will recall its definition later, cf. (2.6). In view of de Branges' Inverse Spectral Theorem, it is natural to ask how the class of Hamiltonians has to be enlarged in order to have a bijective correspondence $H \mapsto q_H$ onto the set $\mathcal{N}_{<\infty}$ via a construction similar to the Weyl coefficient. On the other hand, in various contexts, differential equations of Sturm–Liouville type appear which have not too badly behaving singularities; for example the potential might be not locally integrable at a single point but satisfies only a weaker growth condition. It turned out that often constructions similar to the construction of the Titchmarsh–Weyl coefficient are possible and lead to generalized Nevanlinna functions, which again describe the spectrum of the given problem, see, e.g. [14, 15, 16, 31]. Hence it is a natural question how the most general singular differential expression looks like, such that building up a Weyl theory in the setting of $\mathcal{N}_{<\infty}$ is possible.

The answer is given by the notion of general Hamiltonians, whose definition will be provided later on in all detail, cf. Definition 2.1. For the moment let us content ourselves with the rough picture that a general Hamiltonian $\mathfrak{h}$ consists of a Hamiltonian function H which has finitely many inner singularities, i.e., is defined and locally integrable on a set of the form $[\sigma_0, \sigma_1) \cup (\sigma_1, \sigma_2) \cup \ldots \cup (\sigma_n, \sigma_{n+1})$, and of two collections of real parameters $\mathfrak{b}$, $\mathfrak{d}$. Thereby H models the potential which has singularities at $\sigma_1, \ldots, \sigma_n$, the parameters $\mathfrak{b}$ model a contribution of the singularities which is concentrated in these points, and $\mathfrak{d}$ models the part of the singularities which is in interaction with the local behaviour of H at the singularities. Intuitively we can think of a choice of $(\mathfrak{b}, \mathfrak{d})$ as a choice of singular interface conditions at $\sigma_1, \ldots, \sigma_n$.

Our present work addresses the following question: how does a different choice of the parameters $(\mathfrak{b}, \mathfrak{d})$, while keeping the Hamiltonian function H fixed, influence the spectral theory of the indefinite canonical system under consideration? More specifically, given a general Hamiltonian $\mathfrak{h} = (H, \mathfrak{b}, \mathfrak{d})$, we ask for an explicit description of the family of all Weyl coefficients of general Hamiltonians $\hat{\mathfrak{h}} = (H, \hat{\mathfrak{b}}, \hat{\mathfrak{d}})$ with the same Hamiltonian function than $\mathfrak{h}$ and arbitrary parameters $(\hat{\mathfrak{b}}, \hat{\mathfrak{d}})$. The answer is given in Theorem 5.4, which is the main result of this paper. In order to keep the technical effort of establishing explicit formulae bearable, we restrict ourselves to a certain special case, cf. Remark 2.3. For the case of a general Hamiltonian that arises from a Sturm–Liouville equation with a singularity at the left endpoint the formulae can be significantly simplified, cf. Corollary 5.5.

The question we raise and answer in the present paper seems natural from a theoretical point of view. However, our major motivation is found in the spectral theory of Sturm–Liouville problems with singular endpoints or inner singularities. We are going to explain this intriguing topic in detail for potentials with a singular endpoint. In the case of inner singularities similar phenomena occur and similar arguments can be applied.

Sturm–Liouville equations with singular endpoints and canonical systems.

Let us review the classical theory of Sturm–Liouville equations. Consider an equation of the form

$$-y''(t) + q(t)y(t) = \lambda y(t), \qquad t \in [0, \infty), \tag{1.2}$$

which is regular at 0 and in limit point case at infinity. Then the minimal operator is a symmetry with deficiency indices $(1, 1)$, i.e., for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there is, up to a constant, exactly one solution of (1.2) that is in $L^2(0, \infty)$. A realization of the Sturm–Liouville equation, i.e., a self-adjoint extension of the minimal operator, describes the behaviour of the equation and can be used to solve the eigenvalue problem. Direct and inverse spectral problems play an important role in the analysis of the equation.

With the potential $q(t)$ a scalar function can be associated: its Titchmarsh–Weyl coefficient. It is constructed as follows: let $\theta(t, \lambda)$ and $\phi(t, \lambda)$ be solutions of (1.2) that satisfy the initial conditions

$$\theta(0, \lambda) = 1, \;\; \theta'(0, \lambda) = 0, \qquad \phi(0, \lambda) = 0, \;\; \phi'(0, \lambda) = 1\,. \tag{1.3}$$

Such solutions exist for each $\lambda \in \mathbb{C}$ and are unique. Since the deficiency indices are $(1, 1)$, there exists a unique coefficient $m(\lambda)$ such that for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\theta(\,\cdot\,, \lambda) + m(\lambda)\phi(\,\cdot\,, \lambda) \in L^2(0, \infty)\,. \tag{1.4}$$

The function $m(\lambda)$ is called the Titchmarsh–Weyl coefficient of the equation (1.2) and is a Nevanlinna function. There is an intimate relation to the extension theory of the minimal operator, namely $m(\lambda)$ is, up to a constant, Krein's Q-function connected with the minimal operator and one particular self-adjoint extension.

The Titchmarsh–Weyl coefficient describes the spectrum of every self-adjoint realization of (1.2), hence solves the direct spectral problem. An inverse spectral problem is posed as follows: can the potential be recovered from the Titchmarsh–Weyl coefficient? The answer is yes. This deep result contains some classical inverse theorems, e.g. the recovery of the potential from two different spectra if they are discrete. As a Nevanlinna function, $m(\lambda)$ possesses a Herglotz integral representation. The measure involved in this representation can be used as spectral measure for a generalized Fourier transform. In particular, this shows that the spectral multiplicities of all self-adjoint realizations are 1.

Summarizing let us record: if the potential q is regular at 0, then

(*i*) The minimal operator has deficiency indices $(1,1)$. If the equation is considered only on a finite interval $(0,T)$, the corresponding minimal operator has compact resolvent.

(*ii*) For every $\lambda \in \mathbb{C}$ there exist solutions having the initial values (1.3). They depend analytically on $\lambda \in \mathbb{C}$.

(*iii*) There exists a Fourier transform into an L^2-space whose elements are scalar functions. In particular, the spectral multiplicity of any self-adjoint realization is 1.

(*iv*) The Titchmarsh–Weyl coefficient determines the potential uniquely.

If the potential q is singular at 0, i.e., not integrable at 0, but still in limit circle case, then the situation is very similar. Only, the fundamental system of solutions $\theta(\cdot,\lambda)$, $\phi(\cdot,\lambda)$ cannot be defined by initial conditions anymore; one has to use their asymptotic behaviour at 0 instead.

One way to approach these matters is to rewrite the Sturm–Liouville equation (1.2) as a canonical system (1.1). This is possible by making suitable transformation from y to the vector function x and setting $z^2 = \lambda$. Thereby the facts that Weyl's limit point case prevails at infinity and that Weyl's limit circle case prevails at 0 mean that $(x_0 \in (0,L))$

$$\int_{x_0}^{L} \operatorname{tr} H(t)\, dt = \infty \quad \text{and} \quad \int_0^{x_0} \operatorname{tr} H(t)\, dt < \infty\,,$$

respectively. The respective Weyl coefficients are related by $q_H(z) = -z/m(z^2)$. The theory of canonical systems is more general than the theory of Sturm–Liouville equations, i.e., there are many Hamiltonians which do not arise from rewriting a Sturm–Liouville equation. However, the above stated items (*i*)–(*iv*) are even valid for all canonical systems.

The situation changes drastically if the potential is so singular at 0 that also at this endpoint the equation is limit point. Then the minimal operator is self-adjoint; hence there is only one self-adjoint realization of (1.2). Concerning the above mentioned items related to the spectral theory of the equation, one can say: for real values of λ there need not exist any solution of the equation (1.2) belonging to L^2 at 0; if the equation is considered only on a finite interval $(0,T)$, the corresponding minimal operator might have continuous spectrum; a Fourier transform can be defined only into an L^2-space whose elements are 2-vector functions, and, actually, the spectral multiplicity of the (self-adjoint) minimal operator can be 2; in general, the potential cannot be recovered from a scalar function, but only from a 2×2-matrix Titchmarsh–Weyl function.

We see that for strong singularities of the potential in general a lot of the spectral theory breaks down. However, there are quite a few potentials known which, although being limit point at both endpoints, show a behaviour similar to the regular case. For example in [16] a class of strongly singular potentials was found for which there exists a family $\theta(\cdot,\lambda)$ of solutions which belong to

L^2 at 0 and which is defined and analytic on a neighbourhood of the real line. From this knowledge a scalar function $m(\lambda)$ is constructed quite similarly to the regular case. It is not anymore a Nevanlinna function but still gives rise to a scalar measure which can be used to define a generalized Fourier transform into a space of scalar functions (from which we obtain in particular that the spectral multiplicity of the self-adjoint operator is 1). For inverse problems for equations with certain types of singularities see [20], where the potential can be recovered from a scalar function.

When seeking for an explanation why some potentials—despite the fact that limit point case prevails—behave 'as if they were regular', the probably most convincing argument is to come up with an operator model which is naturally related to the potential and where the 'minimal operator' has deficiency indices $(1, 1)$. For some potentials this goal can be achieved by employing the theory of indefinite canonical systems. The fact that thereby one leaves the Hilbert space setting and deals with operator models in Pontryagin spaces (i.e. spaces with an indefinite inner product whose negative index is finite) is only a minor inconvenience.

In order to treat a given singular potential in this way, one first has to rewrite the equation (1.2) as an indefinite canonical system with some general Hamiltonian $\mathfrak{h} = (H, \mathfrak{b}, \mathfrak{d})$. Since our potential is defined and locally integrable on the open interval $(0, \infty)$, it is natural to use a general Hamiltonian which has just one singularity, namely at 0. It suggests itself to define the Hamiltonian function H on the interval $(0, \infty)$ from q by means of the same formulae as in the regular case. To the left of 0 we will just put a 'massless' interval in order to regard 0 as a singularity of $\mathfrak{h}$; this interval is described by a so-called indivisible interval (see (2.1)) in H. This choice is natural, since to the left of 0 there is no potential anyway.

The singular interface condition at 0 represented by the parameters $(\mathfrak{b}, \mathfrak{d})$ of $\mathfrak{h}$, which we did not choose yet, can be thought of as a singular boundary condition. The meaning of a choice of $\mathfrak{b}$ and $\mathfrak{d}$ is by no means clear. Actually, any choice has equal rights, gives rise to realizations of the equation (1.2), and can be used to deduce the desired direct and inverse spectral results. Sometimes a specific choice of $(\mathfrak{b}, \mathfrak{d})$ might be motivated from plausible physical conditions or from anticipating the outcome for the Titchmarsh–Weyl coefficient, e.g. by analogy to related regular equations. However, in general, the question arises how a change in the singular boundary condition $(\mathfrak{b}, \mathfrak{d})$, while sticking to the Hamiltonian function H naturally obtained from the potential, will affect the Titchmarsh–Weyl coefficient of $\mathfrak{h}$. This is the question we answer in Theorem 5.4.

Organization of the present paper.

We close this introductory section with a short description of the contents of this paper. In Section 2 we recall the definition of general Hamiltonians and maximal chains of matrices, and some results from earlier work which are needed in the present considerations. Maximal chains of matrices are the generalization to the indefinite setting of fundamental matrices of solutions of a canonical system. Between the singularities the rows of such a maximal chain of matrices satisfy the differential equation (1.1), at σ_0 it is the identity matrix and at $\sigma_1, \ldots, \sigma_n$ it is connected depending on $\mathfrak{b}$ and $\mathfrak{d}$. In Section 3 we deal with a transformation $\mathfrak{T}_m$ of matrices, which is the major technical tool for the proof of our main result

Theorem 5.4. The definition of $\mathfrak{T}_m$ might seem a bit ad hoc, but one should keep in mind that the same transformation was already successfully applied in the paper [27] in order to study the local structure of singularities in matrix chains. In that paper also a more intrinsic explanation of $\mathfrak{T}_m$ was provided.

Next, in Section 4, we introduce a perturbation of matrix chains depending on a parameter $\mathfrak{e} \in \mathbb{R} \times \mathbb{R} \times [0, \infty)$. It is shown that this perturbation is exactly a local version of changing the data $\mathfrak{b}, \mathfrak{d}$ in $\mathfrak{h} = (H, \mathfrak{b}, \mathfrak{d})$ translated into the language of matrix chains, cf. Proposition 4.6, Proposition 4.7. Section 5 is devoted to the statement and proof of Theorem 5.4. Thereby, a perturbation $q_{\mathfrak{h}}^{\mathfrak{e}}$ of the Weyl coefficient of a given general Hamiltonian $\mathfrak{h}$ is introduced, and the maximal chain whose Weyl coefficient equals $q_{\mathfrak{h}}^{\mathfrak{e}}$ is computed explicitly, see (5.9). This is obtained in the following way: first the transformation $\mathfrak{T}_m$ is applied to one matrix chain that is connected with the given Hamiltonian function H. This moves the singularity at σ_1 to the right so that the transformed matrix chain is now continuous at σ_1. Then the perturbation from Section 4 is applied and finally the inverse of the transformation $\mathfrak{T}_m$.

At the end of the paper, we illustrate the presently proposed method of approaching the spectral theory of singular Sturm–Liouville equations with two examples. First we investigate the Bessel equation. We chose this classical and well-studied equation since it beautifully shows the indefinite phenomena. Also, it is accessible to explicit computation and recently various attempts were made to obtain an intrinsic explanation for its comparatively nice behaviour known from classical studies, cf. [10, 14, 15, 31]. Secondly, we investigate a potential with an inner singularity, namely $q(t) = \frac{2}{(t-1)^2}$, $t \in [0, \infty)$. We chose this second example, since we find that the treatment of inner singularities within the framework of indefinite canonical systems is even more natural than for potentials with a singularity at the boundary. Moreover, this particular potential occurred previously in relation with a continuation problem for a positive definite function, and hence many of the necessary computations are readily available, cf. [32].

Finally, let us remark that the method instantiated in these two examples will apply to a wide class of potentials with singularities either at the boundary or in the interior. For example potentials involving a Dirac delta function and its derivatives. At the present stage it is unclear 'how strong' the singularity may be so that the proposed approach via indefinite canonical systems will work. To provide a thorough investigation of such situations, in particular to find explicit measures for the allowed strength of the singularity in the potential, will be subject of future work.

2 Indefinite canonical systems

In this section we provide the definitions of general Hamiltonians, maximal chains of matrices, and their Weyl coefficients.

Definition of general Hamiltonians.

First we have to introduce some preliminary notation. An interval (α, β) is called H-indivisible of type ϕ if

$$H(t) = h(t)\xi_\phi \xi_\phi^T, \quad t \in (\alpha, \beta), \tag{2.1}$$

where $\xi_\phi := (\cos\phi, \sin\phi)^T$ and $h(t)$ is some scalar function that is positive almost everywhere.

With any Hamiltonian H a number $\Delta(H) \in \mathbb{N} \cup \{0, \infty\}$ is associated, cf. [28, Definition 3.1], which in some sense measures the growth of H towards L. For example, $\Delta(H) = 0$ means that $\int_0^L \operatorname{tr} H(t)\, dt < \infty$; or if $\int_0^L \operatorname{tr} H(t)\, dt = \infty$ and the interval (L_1, L) is H-indivisible for some $L_1 < L$, then $\Delta(H) = 1$.

Assume that $\int_0^L \operatorname{tr} H(t)\, dt = \infty$. The Hamiltonian H is said to satisfy the condition (HS) if the resolvents of one and hence of all self-adjoint extensions of the minimal operator $T_{\min}(H)$ associated with H on $[0, L)$ are Hilbert–Schmidt operators. In this case, the growth of H towards L, as measured by $\Delta(H)$, is extremal in one direction ξ_ϕ in the sense that, for a unique angle $\phi \in [0, \pi)$, we have

$$\int_0^L \xi_\phi^T H(t) \xi_\phi \, dt < \infty \,, \tag{2.2}$$

cf. [30, Theorem 2.4]. This angle will be denoted by $\phi(H)$.

Let H be a function defined on an interval (L_-, L_+) which takes real and non-negative 2×2-matrices as values, is locally integrable on (L_-, L_+) and does not vanish on any set of positive measure. Fix $\alpha \in (L_-, L_+)$, and put $H_+(t) := H(\alpha + t)$, $t \in [0, L_+ - \alpha)$, and $H_-(t) := H(\alpha - t)$, $t \in [0, \alpha - L_-)$. Then $H_\pm$ are Hamiltonians. We say that H is in the limit point/circle case at L_+ or L_-, if H_+ or H_-, respectively, has this property. The conditions (HS_+) and (HS_-) and the numbers $\Delta_\pm(H)$ and $\phi_\pm(H)$ are defined similarly. These numbers do not depend on the choice of α. In the following we also call such a function H defined on an open interval (L_-, L_+) a Hamiltonian.

2.1 Definition. A *general Hamiltonian* $\mathfrak{h}$ is a collection of data of the following kind:

(*i*) $n \in \mathbb{N} \cup \{0\}$, $\sigma_0, \ldots, \sigma_{n+1} \in \mathbb{R} \cup \{\pm\infty\}$ with $\sigma_0 < \sigma_1 < \ldots < \sigma_{n+1}$,

(*ii*) Hamiltonians H_i, $i = 0, \ldots, n$, defined on the respective intervals (σ_i, σ_{i+1}),

(*iii*) numbers $\ddot{o}_1, \ldots, \ddot{o}_n \in \mathbb{N} \cup \{0\}$ and $b_{i,1}, \ldots, b_{i,\ddot{o}_i+1} \in \mathbb{R}$, $i = 1, \ldots, n$, with $b_{i,1} \neq 0$ in the case $\ddot{o}_i \geq 1$,

(*iv*) numbers $d_{i,0}, \ldots, d_{i,2\Delta_i - 1} \in \mathbb{R}$, $i = 1, \ldots, n$, where $\Delta_i := \max\{\Delta_+(H_{i-1}), \Delta_-(H_i)\}$,

(*v*) a finite subset E of $\{\sigma_0, \sigma_{n+1}\} \cup \bigcup_{i=0}^n (\sigma_i, \sigma_{i+1})$,

which is assumed to be subject to the following conditions:

(H1) H_0 is in the limit circle case at σ_0 and, if $n \geq 1$, in the limit point case at σ_1. H_i is in the limit point case at both endpoints σ_i and σ_{i+1}, $i = 1, \ldots, n-1$. If $n \geq 1$, then H_n is in the limit point case at σ_n.

(H2) For $i = 1, \ldots, n-1$ the interval (σ_i, σ_{i+1}) is not H_i-indivisible. If H_n is in the limit point case at σ_{n+1}, then also (σ_n, σ_{n+1}) is not H_n-indivisible.

(H3) We have $\Delta_i < \infty$, $i = 1, \dots, n$. Moreover, H_0 satisfies (HS$_+$), H_i satisfies (HS$_-$) and (HS$_+$) for $i = 1, \dots, n-1$, and H_n satisfies (HS$_-$).

(H4) We have $\phi_+(H_{i-1}) = \phi_-(H_i)$, $i = 1, \dots, n$.

(H5) Let $i \in \{1, \dots, n\}$. If for some $\epsilon > 0$ the interval $(\sigma_i - \epsilon, \sigma_i)$ is H_{i-1}-indivisible and the interval $(\sigma_i, \sigma_i + \epsilon)$ is H_i-indivisible, then $d_1 = 0$. If additionally $b_{i,1} = 0$, then also $d_0 < 0$.

(E1) $\sigma_0, \sigma_{n+1} \in E$, and $E \cap (\sigma_i, \sigma_{i+1}) \neq \emptyset$ for $i = 1, \dots, n-1$. If H_n is in the limit point case at σ_{n+1}, then also $E \cap (\sigma_n, \sigma_{n+1}) \neq \emptyset$. Let $i \in \{0, \dots, n\}$; if (α, σ_{i+1}) or (σ_i, α) is a maximal H_i-indivisible interval, then $\alpha \in E$.

(E2) No point of E is an inner point of an indivisible interval.

The number

$$\operatorname{ind}_- \mathfrak{h} := \sum_{i=1}^{n} \Big(\Delta_i + \Big[\frac{\ddot{o}_i}{2} \Big] \Big) + \big| \big\{ 1 \leq i \leq n : \ddot{o}_i \text{ odd}, b_{i,1} > 0 \big\} \big| \tag{2.3}$$

is called the *negative index* of the general Hamiltonian $\mathfrak{h}$. Moreover, $\mathfrak{h}$ is called *definite* if $\operatorname{ind}_- \mathfrak{h} = 0$, and *indefinite* otherwise. We say that $\mathfrak{h}$ is in the *limit point case* or *limit circle case* if H_n has the respective property at σ_{n+1}.

In order to shorten notation we shall write a Hamiltonian $\mathfrak{h}$ that is given by the data n, $\sigma_0, \dots, \sigma_{n+1}$, $H_0, \dots, H_n$, $\ddot{o}_1, \dots, \ddot{o}_n$, $b_{i,j}$, $d_{i,j}$, E as

$$\mathfrak{h} = (H, \mathfrak{b}, \mathfrak{d}),$$

where H represents the Hamiltonians H_i, including their number n and their domains of definition (σ_i, σ_{i+1}), $\mathfrak{b}$ represents the numbers $\ddot{o}_i$ and $b_{i,j}$, and $\mathfrak{d}$ represents the numbers $d_{i,j}$ and the subset E. However, we identify H also with a function defined on $\bigcup_{i=0}^{n} (\sigma_i, \sigma_{i+1})$ such that $H(t) = H_i(t)$ for $t \in (\sigma_i, \sigma_{i+1})$. This hopefully does not cause any confusion.

2.2 Remark. Intuitively, this notion can be understood as follows: its purpose is to model an indefinite canonical system. So we deal with the differential equation $f' = zJHf$ given on an interval (σ_0, σ_{n+1}) which involves some kind of singularities which are located at the points σ_i, $i = 1, \dots, n$. Condition (H1) says that the differential equation is regular at σ_0, so that the initial value problem at σ_0 is well posed, but that $\sigma_1, \dots, \sigma_n$ actually are singularities. Moreover, and this is the condition (H2), two adjacent singularities σ_i and σ_{i+1} must be separated by more than just a single indivisible interval. The meaning of (H3) is that the growth of H_i towards a singularity is not too fast. Moreover, (H4) is an interface condition at σ_i.

The numbers $\ddot{o}_i \in \mathbb{N} \cup \{0\}$ and $b_{i,1}, \dots, b_{i,\ddot{o}_i+1}$ model the part of the singularity σ_i which is concentrated at σ_i, whereas the numbers $d_{i,0}, \dots, d_{i,2\Delta_i - 1}$ model the part of this singularity which is in interaction with the local behaviour around σ_i. The elements of E in the vicinity of σ_i determine quantitatively what local here means, more precisely, the points in E split the set $\bigcup_{i=0}^{n} (\sigma_i, \sigma_{i+1})$ into pieces that contain only one singularity. The freedom of this interaction is, by the first part of (H5), restricted if to both sides of σ_i indivisible intervals adjoin. The possibility that on both sides of σ_i indivisible intervals adjoin and at the same time $b_{i,1} = 0$, can occur by the second part of (H5) only in the case of 'indivisible intervals of negative length', the simplest possible kind of singularity.

2.3 Remark. We will subsequently confine our interest to general Hamiltonians with negative index 1. Let us explicitly state which data is needed to obtain an object of this kind. In order to have $\operatorname{ind}_- \mathfrak{h} = 1$, the general Hamiltonian $\mathfrak{h}$ has to consists of: two Hamiltonians H_0 and H_1 defined on intervals (σ_0, σ_1) and (σ_1, σ_2), respectively, which are subject to the conditions of Definition 2.1 and satisfy $\Delta = 1$; a number $\ddot{o} \in \{0, 1\}$; a number $b_1 \in \mathbb{R}$ which, if $\ddot{o} = 1$, is negative; in case $\ddot{o} = 1$ another number $b_2 \in \mathbb{R}$; real numbers d_0, d_1; a finite subset E, which can be chosen of the form $\{s_0, s_1\}$ with $s_0 = \sigma_0$, $s_1 \in (\sigma_1, \sigma_2)$.

Weyl theory for indefinite canonical systems.

Let us recall the construction of the Weyl coefficient of a canonical system: let $\upsilon(t,z) = (\upsilon(t,z)_{ij})_{i,j=1}^2$ be the 2×2-matrix solution of

$$\begin{gathered}\frac{\partial}{\partial t}\upsilon(t,z)J = z\upsilon(t,z)H(t),\ t \in [0, L)\,,\\ \upsilon(0,z) = I.\end{gathered} \tag{2.4}$$

Note that the rows of υ are solutions of (1.1). Then, for each fixed $z \in \mathbb{C} \setminus \mathbb{R}$ and $t \in [0, L)$, the function $q_{z,t}(\tau) := \upsilon(t,z) \star \tau$, $\tau \in \mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$, maps the closed upper half plane onto a disk; here we denote by $\mathbb{C}^+$ the open upper half plane, and for a 2×2-matrix function $M = (m_{ij})_{i,j=1}^2$ and a scalar function α we denote by $M \star \alpha$ the expression

$$M \star \alpha := \frac{m_{11}\alpha + m_{12}}{m_{21}\alpha + m_{22}}\,.$$

If t increases, the disks $q_{z,t}(\mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\})$ form a nested sequence. In the limit $t \nearrow L$ we thus obtain a limit disk. It degenerates to a single point if and only if

$$\int_0^L \operatorname{tr} H(t)\, dt = +\infty\,.$$

In this case, one says that for the Hamiltonian H Weyl's limit point case prevails (otherwise, one says that H is in the limit circle case), and defines the Weyl coefficient of H as

$$q_H(z) := \lim_{t \nearrow L} q_{z,t}(\tau), \quad \tau \in \mathbb{R} \cup \{\infty\}\,. \tag{2.5}$$

This limit does not depend on $\tau \in \mathbb{R} \cup \{\infty\}$ and exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$.

In order to build up a Weyl theory for indefinite canonical systems, one has to have available an analogue of the fundamental solution $\upsilon(t,z)$ in (2.4). This is achieved by the notion of maximal chains of matrices. Their definition also requires some preliminary notation.

Let W be a 2×2-matrix valued function

$$W = (w_{ij})_{i,j=1}^2 : \mathbb{C} \to \mathbb{C}^{2\times 2}$$

such that the entries w_{ij} are entire functions, $w_{ij}(\overline{z}) = \overline{w_{ij}(z)}$, $\det W \equiv 1$, and $W(0) = I$. If $\kappa \in \mathbb{N} \cup \{0\}$, we write $W \in \mathcal{M}_\kappa$ if the 2×2-matrix valued kernel

$$H_W(w, z) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}$$

has κ negative squares on $\mathbb{C}$. We put

$$\mathcal{M}_{\leq\kappa} := \bigcup_{0 \leq \nu \leq \kappa} \mathcal{M}_\nu, \quad \mathcal{M}_{<\infty} := \bigcup_{\nu \in \mathbb{N} \cup \{0\}} \mathcal{M}_\nu\,,$$

and write $\operatorname{ind}_- W = \kappa$ to express that a matrix function W belongs to $\mathcal{M}_\kappa$.

Matrices of the class $\mathcal{M}_{<\infty}$ which are linear polynomials play a special role. Recall that a linear polynomial matrix W belongs to $\mathcal{M}_{<\infty}$ if and only if

$$W(z) = W_{(l,\phi)}(z) := \begin{pmatrix} 1 - lz \sin\phi \cos\phi & lz \cos^2\phi \\ -lz \sin^2\phi & 1 + lz \sin\phi \cos\phi \end{pmatrix}$$

for some $l \in \mathbb{R}$ and $\phi \in [0, \pi)$. In this case the number of negative squares of the kernel H_W is equal to 0 or 1, depending whether $l \geq 0$ or $l < 0$. Matrices of the form $W_{(l,\phi)}$ are related to indivisible intervals, actually we have

$$\frac{\partial}{\partial t} W_{(t,\phi)}(z)J = zW_{(t,\phi)}(z)\xi_\phi\xi_\phi^T, \quad t \in [0, l]\,.$$

For a matrix function W we denote by $\mathfrak{t}(W)$ the trace functional $\mathfrak{t}(W) := \operatorname{tr}(W'(0)J)$.

2.4 Definition. A mapping $\omega : \mathcal{I} \to \mathcal{M}_{<\infty}$ is called a *maximal chain of matrices* if the following axioms are satisfied:

(W1) Its domain $\mathcal{I}$ is of the form $(\sigma_0, \sigma_1) \cup \ldots \cup (\sigma_n, \sigma_{n+1})$, where $\sigma_0 < \sigma_1 < \ldots < \sigma_n < \sigma_{n+1} \leq \infty$.

(W2) The function ω is not constant on any interval contained in $\mathcal{I}$.

(W3) For all $s, t \in \mathcal{I}$, $s \leq t$, we have $\omega(s)^{-1}\omega(t) \in \mathcal{M}_{<\infty}$ and

$$\operatorname{ind}_- \omega(t) = \operatorname{ind}_- \omega(s) + \operatorname{ind}_- \omega(s)^{-1}\omega(t)\,.$$

(W4) If $t \in \mathcal{I}$ and for some $W \in \mathcal{M}_{<\infty}$, $W \neq I$, we have $W^{-1}\omega(t) \in \mathcal{M}_{<\infty}$ and $\operatorname{ind}_- \omega(t) = \operatorname{ind}_- W + \operatorname{ind}_- W^{-1}\omega(t)$, then there exists a number $s \in \mathcal{I}$ such that $W = \omega(s)$.

(W5) We have $\lim_{t \nearrow \sigma_{n+1}} \mathfrak{t}(\omega(t)) = +\infty$. If $n \geq 1$, there exist numbers $s, t \in (\sigma_n, \sigma_{n+1})$, such that $\omega(s)^{-1}\omega(t)$ is not a linear polynomial.

The set of all maximal chains will be denoted by $\mathfrak{M}_{<\infty}$. The matrices $\omega_{st} := \omega(s)^{-1}\omega(t)$ are called *transfer matrices*.

It was proved in [27, Lemma 3.5] that the function $\operatorname{ind}_- \omega(t)$ is constant on each connected component of $\mathcal{I}$ and takes different values on different components. Moreover, by (W3), it is non-decreasing. In particular, it is bounded and attains its maximum on $\mathcal{I}_\infty$. This allows us to define

$\operatorname{ind}_- \omega := \max_{t\in\mathcal{I}} \operatorname{ind}_- \omega(t)$. The set of all maximal chains ω with $\operatorname{ind}_- \omega = \kappa$ will be denoted by $\mathfrak{M}_\kappa$. It was also proved in [27, Lemma 3.5] that for any chain $\omega \in \mathfrak{M}_{<\infty}$ we have $\lim_{t\searrow\sigma_0} \omega(t) = I$. Hence, we can always extend a maximal chain ω continuously to $\mathcal{I} \cup \{\sigma_0\}$ by putting $\omega(\sigma_0) := I$.

Due to the condition $\lim_{t\nearrow\sigma_{n+1}} \mathfrak{t}(\omega(t)) = +\infty$ in (W5), for any maximal chain of matrices the limit

$$q_\omega := \lim_{t\nearrow\sigma_{n+1}} \omega(t) \star \tau$$

exists locally uniformly on $\mathbb{C}\setminus\mathbb{R}$ for $\tau \in \mathbb{R}\cup\{\infty\}$ and does not depend on τ. The function q_ω is a generalized Nevanlinna function, actually $\operatorname{ind}_- q_\omega = \operatorname{ind}_- \omega$, cf. [26, Lemma 8.2, Lemma 8.5]. Recall here that a function q belongs to the class $\mathcal{N}_\kappa$, $\kappa \in \mathbb{N}_0$, if it is meromorphic in $\mathbb{C}\setminus\mathbb{R}$, $q(\overline{z}) = \overline{q(z)}$ for every z in the domain of q and the kernel

$$K_q(w,z) = \frac{q(z) - \overline{q(w)}}{z - \overline{w}} \tag{2.6}$$

has κ negative squares. We also write $\operatorname{ind}_- q = \kappa$ if $q \in \mathcal{N}_\kappa$. The set of generalized Nevanlinna functions is then defined by $\mathcal{N}_{<\infty} := \bigcup_{\kappa=0}^{\infty} \mathcal{N}_\kappa$.

With a general Hamiltonian $\mathfrak{h}$ there can be associated a maximal chain $\omega_\mathfrak{h}$, cf. [29]. On the intervals (σ_i, σ_{i+1}) it is a solution of the differential equation in (2.4) and the initial condition at σ_0 is $\omega_\mathfrak{h}(\sigma_0) = I$. The jump over the singularities $\sigma_1, \ldots, \sigma_n$ is determined by the data $\mathfrak{b}, \mathfrak{d}$; however, this relation is highly implicit; note that by (H1) the limits $\lim_{t\searrow\sigma_i} \omega_\mathfrak{h}(t)$ and $\lim_{t\nearrow\sigma_i} \omega_\mathfrak{h}(t)$ do not exist. Moreover, one has $\operatorname{ind}_- \omega_\mathfrak{h} = \operatorname{ind}_- \mathfrak{h}$.

The *Weyl coefficient of* $\mathfrak{h}$ is defined as the function $q_\mathfrak{h} := q_{\omega_\mathfrak{h}}$. The indefinite analogue of de Branges' inverse spectral theorem states that the assignment $\mathfrak{h} \mapsto q_\mathfrak{h}$ yields a bijection of the set of all general Hamiltonians (up to changes of scale) and the set $\mathcal{N}_{<\infty}$ of all generalized Nevanlinna functions, cf. [29].

Some more preliminaries on chains of matrices.

Chains which can be obtained from each other by a change of variable will share their important properties. This idea is formalized by the notion of reparameterization.

2.5 Definition. Let $\mathcal{J}_1, \mathcal{J}_2$ be subsets of $\mathbb{R}$ and let $\omega_i : \mathcal{J}_i \to \mathcal{M}_{<\infty}$, $i = 1, 2$. Then we say that ω_2 is a *reparameterization* of ω_1 if there exists an increasing and bijective map $\alpha : \mathcal{J}_2 \to \mathcal{J}_1$ such that $\omega_2 = \omega_1 \circ \alpha$. In this case we write $\omega_2 \leftrightsquigarrow \omega_1$.

The relation $\leftrightsquigarrow$ yields an equivalence relation on $\mathfrak{M}_{<\infty}$. Clearly, each of the subsets $\mathfrak{M}_\kappa$, $\kappa \in \mathbb{N} \cup \{0\}$, is saturated with respect to $\leftrightsquigarrow$.

Intervals where the chain is of a particularly simple form often play an exceptional role. If $s_1, s_2 \in [0, L) \setminus \{\sigma_1, \ldots, \sigma_n\}$, $s_1 < s_2$, the interval (s_1, s_2) is called *indivisible of length* l *and type* ϕ if $\omega(s_1)^{-1}\omega(s_2) = W_{(l,\phi)}$. If $l > 0$, then (s_1, s_2) is contained in the domain of ω, and

$$(\omega(s_1)^{-1}\omega(t))_{t\in[s_1,s_2]} \leftrightsquigarrow (W_{(t,\phi)})_{t\in[0,l]} \, .$$

Note that $(W_{(t,\phi)})_{t\in[0,l]}$ satisfies the differential equation (2.4) with $H(t) = \xi_\phi \xi_\phi^T$ for $t \in (0, l)$, i.e., the interval $(0, l)$ is H-indivisible of type ϕ. If, on the other hand, $l<0$, then there exists exactly one point σ_i which is contained in (s_1,s_2), and

$$(\omega(s_1)^{-1}\omega(t))_{t\in[s_1,s_2]\setminus\{\sigma_i\}} \leftrightsquigarrow (W_{(-\frac{1}{t}+\frac{l}{2},\phi)})_{t\in[\frac{2}{l},-\frac{2}{l}]\setminus\{0\}}\,.$$

An interval (s_1,σ_i) or (σ_i,s_2) which has the property that for all t in this interval the matrix $\omega(s_1)^{-1}\omega(t)$ or $\omega(t)^{-1}\omega(s_2)$, respectively, is a linear polynomial is called *indivisible of infinite length*.

We will also need the notion of finite maximal chains, which are bounded analogues of a maximal chain.

2.6 Definition. A mapping $\omega:\mathcal{I}\to\mathcal{M}_{<\infty}$ is called a *finite maximal chain of matrices* if

(W1$_f$) the set $\mathcal{I}$ is of the form $[0,L]\setminus\{\sigma_1,\dots,\sigma_n\}$, where $0<\sigma_1<\dots<\sigma_n<L<\infty$.

and it satisfies the axioms (W2), (W3) and (W4). The set of all finite maximal chains will be denoted by $\mathfrak{M}^f_{<\infty}$.

The same reasoning which led to the proof of [27, Lemma 3.5], shows that $\omega(0)=I$ for any finite maximal chain ω.

A finite maximal chain can always be extended to a maximal chain in various ways, cf. [27, Lemma 3.7]. In fact, such extensions are obtained by appending another chain. A formalization of this procedure gives rise to the following notion of linking chains.

2.7 Definition. Let $\mathcal{J}_1,\mathcal{J}_2\subseteq\mathbb{R}$ and let $\omega_i:\mathcal{J}_i\to\mathcal{M}_{<\infty}$, $i=1,2$. Assume that $\sup\mathcal{J}_1\in\mathcal{J}_1$ and $\inf\mathcal{J}_2\in\mathcal{J}_2$, $\omega_2(\inf\mathcal{J}_2)=I$. Then we define a map ω as follows: choose increasing bijections φ_1 of $[\inf\mathcal{J}_1,\sup\mathcal{J}_1]$ onto $[0,1]$, φ_2 of $[\inf\mathcal{J}_2,\sup\mathcal{J}_2]$ onto $[1,2]$, and let $\omega_1\uplus\omega_2:\varphi_1(\mathcal{J}_1)\cup\varphi_2(\mathcal{J}_2)\to\mathcal{M}_{<\infty}$ be defined as

$$(\omega_1\uplus\omega_2)(t):=\begin{cases}\omega_1(\varphi_1^{-1}(t)) & \text{for } t\in\varphi_1(\mathcal{J}_1),\\ \omega_1(\sup\mathcal{J}_1)\cdot\omega_2(\varphi_2^{-1}(t)) & \text{for } t\in\varphi_2(\mathcal{J}_2).\end{cases}$$

Note that these definitions agree for $t=1$. We say that the function $\omega_1\uplus\omega_2$ is obtained by *linking* ω_1 *and* ω_2.

It is easy to see that the operation $\uplus$ is associative up to reparameterization, i.e., $\omega_1\uplus(\omega_2\uplus\omega_3)\leftrightsquigarrow(\omega_1\uplus\omega_2)\uplus\omega_3$. Moreover, if $\omega_1\leftrightsquigarrow\omega_1'$ and $\omega_2\leftrightsquigarrow\omega_2'$, then also $\omega_1\uplus\omega_2\leftrightsquigarrow\omega_1'\uplus\omega_2'$.

In our context the following fact, which follows from the discussion concerning linking of chains at the end of [26, §7], is of interest.

2.8 Remark. Let $\omega_1\in\mathfrak{M}^f_\kappa$, $\omega_2\in\mathfrak{M}_0$ and assume that neither of the following hold:

(*i*) ω_1 ends with an indivisible interval of infinite length and ω_2 is just an indivisible interval of the same type and infinite length.

(*ii*) ω_1 ends with an indivisible interval of negative length l_1 and ω_2 starts with an indivisible interval of the same type and length $l_2\geq -l_1$.

Then $\omega_1\uplus\omega_2\in\mathfrak{M}_{\kappa_1}$.

Sometimes also the following notations are practical.

2.9 Definition. Let $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathbb{R}$ and let $\omega_i : \mathcal{J}_i \to \mathcal{M}_{<\infty}$, $i = 1, 2$. Assume that $\sup \mathcal{J}_1 \notin \mathcal{J}_1$ and $\inf \mathcal{J}_2 \notin \mathcal{J}_2$. Then we define a map $\omega_1 \dot{\uplus} \omega_2$ by the following procedure: again choose increasing bijections $\varphi_1 : [\inf \mathcal{J}_1, \sup \mathcal{J}_1] \to [0, 1]$ and $\varphi_2 : [\inf \mathcal{J}_2, \sup \mathcal{J}_2] \to [1, 2]$. Define $\omega_1 \dot{\uplus} \omega_2 : \varphi_1(\mathcal{J}_1) \cup \varphi_2(\mathcal{J}_2) \to \mathcal{M}_{<\infty}$ as

$$\omega_1 \dot{\uplus} \omega_2(t) := \begin{cases} \omega_1(\varphi_1^{-1}(t)), & t \in \varphi_1(\mathcal{J}_1), \\ \omega_2(\varphi_2^{-1}(t)), & t \in \varphi_2(\mathcal{J}_2). \end{cases}$$

In the same way like $\uplus$, the operation $\dot{\uplus}$ is associative and compatible with reparameterizations.

2.10 Definition. Let $\mathcal{J} \subseteq \mathbb{R}$ and let $\omega : \mathcal{J} \to \mathcal{M}_{<\infty}$. Let $\hat{\mathcal{J}}$ be the set of all points $t \in \overline{\mathcal{J}}$ such that the limit $\lim_{s \to t, s \in \mathcal{J}} \omega(s)$ exists. Then we can define a function $\mathrm{C}\omega : \hat{\mathcal{J}} \to \mathcal{M}_{<\infty}$ by

$$\mathrm{C}\omega(t) := \begin{cases} \omega(t), & t \in \mathcal{J}, \\ \lim_{s \to t, s \in \mathcal{J}} \omega(s), & t \in \hat{\mathcal{J}} \setminus \mathcal{J}. \end{cases}$$

We speak of completion of the given function ω.

Sometimes it is useful to apply the following transformation

$$\hat{\omega}(t) := N_\alpha \omega(t) N_\alpha^*,$$

where

$$N_\alpha := \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{2.7}$$

and $\alpha \in [0, \pi)$. The corresponding transformation for the Hamiltonian is

$$\hat{H}(t) = N_\alpha H(t) N_\alpha^*, \tag{2.8}$$

which changes the direction: $\phi(\hat{H}) = \phi(H) - \alpha$. For two general Hamiltonians of the form $\mathfrak{h} = (H, \mathfrak{b}, \mathfrak{d})$, $\hat{\mathfrak{h}} = (\hat{H}, \mathfrak{b}, \mathfrak{d})$ with $\hat{H} = N_\alpha H N_\alpha^*$, the Weyl coefficients are related as follows: $q_{\hat{\mathfrak{h}}} = N_\alpha \star q_{\mathfrak{h}}$.

3 The transformation $\mathfrak{T}_m$

We will employ the transformation $\mathcal{T}_m$ of matrices, cf. [27, §4]. Let us recall the definition; later we extend the transformation to chains of matrices (which will be denoted by $\mathfrak{T}_m$).

3.1 Definition. Let $W = (W_{ij})_{i,j=1}^2$ be an entire matrix function with $W(0) = I$, and let $m \in \mathbb{R} \setminus \{0\}$. Put

$$\alpha(W, m) := 1 - m W_{21}'(0)$$

and

$$\beta(W, m) := m \frac{W_{21}''(0)}{2} + m W_{21}'(0) W_{11}'(0) - 2 W_{11}'(0)\,.$$

We say $W \in \operatorname{dom} \mathcal{T}_m$ if $\alpha(W, m) \neq 0$, and in this case define

$$\mathcal{T}_m(W) := \begin{pmatrix} 1 & -\frac{m}{z} \\ 0 & 1 \end{pmatrix} W(z) \begin{pmatrix} \frac{1}{\alpha(W,m)} & m\big(\frac{\beta(W,m)}{\alpha(W,m)} + \frac{1}{z}\big) \\ 0 & \alpha(W, m) \end{pmatrix}.$$

It was proved in [27] that $\mathcal{T}_m(W)$ is entire and takes the value I at $z = 0$. Moreover, if $W \in \mathcal{M}_\kappa$ then $\mathcal{T}_m(W) \in \mathcal{M}_{\kappa'}$ with

$$\kappa' = \kappa + \begin{cases} 0 & \text{if } \alpha(W,m) > 0, \\ 1 & \text{if } \alpha(W,m) < 0,\ m < 0, \\ -1 & \text{if } \alpha(W,m) < 0,\ m > 0. \end{cases} \tag{3.1}$$

For later reference let us state the following facts, which were shown in [27].

3.2 Remark.

(*i*) The transformations $\mathcal{T}_m$ and $\mathcal{T}_{-m}$ are inverses of each other: if $W \in \operatorname{dom}\mathcal{T}_m$, then $\mathcal{T}_m(W) \in \operatorname{dom}\mathcal{T}_{-m}$ and

$$\mathcal{T}_{-m}\big(\mathcal{T}_m(W)\big) = W\,.$$

This is also reflected in the formulae

$$\alpha(\mathcal{T}_m(W),-m) = \frac{1}{\alpha(W,m)}, \quad \frac{\beta(\mathcal{T}_m(W),-m)}{\alpha(\mathcal{T}_m(W),-m)} = \frac{\beta(W,m)}{\alpha(W,m)}\,. \tag{3.2}$$

(*ii*) The transformation $\mathcal{T}_m$ preserves indivisible intervals; i.e., if $W_1, W_2 \in \operatorname{dom}\mathcal{T}_m$ satisfy $W_1^{-1}W_2 = W_{(l,\phi)}$, then $\mathcal{T}_m(W_1)^{-1}\mathcal{T}_m(W_2) = W_{(\tilde{l},\tilde{\phi})}$ with some appropriately chosen numbers $\tilde{l}, \tilde{\phi}$.

(*iii*) The value $\mathfrak{t}(\mathcal{T}_m(W))$ is explicitly given as

$$\begin{aligned} \mathfrak{t}(\mathcal{T}_m(W)) = m\frac{\beta(W,m)}{\alpha(W,m)}\Big(W_{11}'(0) - m\frac{W_{21}''(0)}{2}\Big) + m\frac{W_{11}''(0)}{2} - m^2\frac{W_{21}'''(0)}{6} \\ + \alpha(W,m)W_{12}'(0) - \alpha(W,m)m\frac{W_{22}''(0)}{2} - \frac{W_{21}'(0)}{\alpha(W,m)}\,. \end{aligned} \tag{3.3}$$

In the present context the following observation will be of importance.

3.3 Lemma. *Let $\omega(t), \omega(s), \hat{\omega}(t), \hat{\omega}(s) \in \operatorname{dom}\mathcal{T}_m$. Then we have*

$$\mathcal{T}_m(\omega(t))^{-1}\mathcal{T}_m(\omega(s)) = \mathcal{T}_m(\hat{\omega}(t))^{-1}\mathcal{T}_m(\hat{\omega}(s)) \tag{3.4}$$

if and only if

$$\hat{\omega}(t)^{-1}\hat{\omega}(s) = A(t)^{-1}\cdot\omega(t)^{-1}\omega(s)\cdot A(s), \tag{3.5}$$

with

$$A(t) := \begin{pmatrix} \frac{\alpha(\hat{\omega}(t),m)}{\alpha(\omega(t),m)} & -\frac{m(\beta(\hat{\omega}(t),m)-\beta(\omega(t),m))}{\alpha(\omega(t),m)\alpha(\hat{\omega}(t),m)} + \frac{m}{z}\big(\frac{1}{\alpha(\hat{\omega}(t),m)} - \frac{1}{\alpha(\omega(t),m)}\big) \\ 0 & \frac{\alpha(\omega(t),m)}{\alpha(\hat{\omega}(t),m)} \end{pmatrix}$$

Proof. From the definition of $\mathcal{T}_m$ we see that

$$\mathcal{T}_m(\omega(t))^{-1}\mathcal{T}_m(\omega(s)) =$$

$$= \begin{pmatrix} \alpha(\omega(t),m) & -m\big(\frac{\beta(\omega(t),m)}{\alpha(\omega(t),m)} + \frac{1}{z}\big) \\ 0 & \frac{1}{\alpha(\omega(t),m)} \end{pmatrix} \omega_{ts} \begin{pmatrix} \frac{1}{\alpha(\omega(s),m)} & m\big(\frac{\beta(\omega(s),m)}{\alpha(\omega(s),m)} + \frac{1}{z}\big) \\ 0 & \alpha(\omega(s),m) \end{pmatrix}$$

From this, and the same relation with ω replaced by $\hat\omega$, it follows that (3.4) is equivalent to

$$\hat\omega_{ts}=\begin{pmatrix}\alpha(\hat\omega(t),m) & -m\big(\frac{\beta(\hat\omega(t),m)}{\hat\alpha(\omega(t),m)}+\frac{1}{z}\big)\\ 0 & \frac{1}{\alpha(\hat\omega(t),m)}\end{pmatrix}^{-1}\begin{pmatrix}\alpha(\omega(t),m) & -m\big(\frac{\beta(\omega(t),m)}{\alpha(\omega(t),m)}+\frac{1}{z}\big)\\ 0 & \frac{1}{\alpha(\omega(t),m)}\end{pmatrix}\times$$

$$\times\,\omega_{ts}\begin{pmatrix}\frac{1}{\alpha(\omega(s),m)} & m\big(\frac{\beta(\omega(s),m)}{\alpha(\omega(s),m)}+\frac{1}{z}\big)\\ 0 & \alpha(\omega(s),m)\end{pmatrix}\begin{pmatrix}\frac{1}{\alpha(\hat\omega(s),m)} & m\big(\frac{\beta(\hat\omega(s),m)}{\alpha(\hat\omega(s),m)}+\frac{1}{z}\big)\\ 0 & \alpha(\hat\omega(s),m)\end{pmatrix}^{-1}$$

This is, however, equivalent to the asserted form of $\hat\omega_{ts}$.

❑

We will employ an additivity property of the functions $\alpha(W,m)$ and $\beta(W,m)$.

3.4 Lemma. *Let W,V be entire, $W(0)=V(0)=I$, $\det W=1$, and let $m\in\mathbb{R}$. Then*

$$\alpha(WV,m)=\alpha(W,m)-mV_{21}'(0)\,,$$
$$\beta(WV,m)=\beta(W,m)+\beta(V,m)+2mW_{21}'(0)V_{11}'(0)\,.$$

Proof. We have

$$(WV)'(0)=W'(0)+V'(0),\ (WV)''(0)=W''(0)+2W'(0)V'(0)+V''(0)\,.$$

From this the first asserted relation is immediate. For the second relation we compute

$$\begin{aligned}\beta(WV,m)=&\frac{m}{2}\big(W_{21}''(0)+2\big[W_{21}'(0)V_{11}'(0)+W_{22}'(0)V_{21}'(0)\big]+V_{21}''(0)\big)\\ &+m(W_{21}'(0)+V_{21}'(0))(W_{11}'(0)+V_{11}'(0))-2(W_{11}'(0)+V_{11}'(0))\\ =&\,\beta(W,m)+\beta(V,m)+2mW_{21}'(0)V_{11}'(0)+mW_{22}'(0)V_{21}'(0)\\ &+mV_{21}'(0)W_{11}'(0)\,.\end{aligned}$$

Since we assumed that $\det W=1$, we have $W_{22}'(0)=-W_{11}'(0)$, and this gives the desired equality.

❑

3.5 Corollary. *Let ω be a chain of matrices and let (s_-,s_+) be an indivisible interval of type* 0*. Then the functions $\alpha(\omega(t),m)$ and $\beta(\omega(t),m)$ are constant on (s_-,s_+).*

Proof. For a matrix $W_{(l,0)}$ we clearly have ${W_{(l,0)}}_{21}'(0)={W_{(l,0)}}_{11}'(0)=0$ and $\beta(W_{(l,0)},m)=0$. Let $t\in(s_-,s_+)$ be given, then $\omega(t)=\omega(s_-)W_{(l(t),0)}$, and hence

$$\alpha(\omega(t),m)=\alpha(\omega(s_-),m)-m{W_{(l,0)}}_{21}'(0)=\alpha(\omega(s_-),m)$$

and

$$\beta(\omega(t),m)=\beta(\omega(s_-),m)+\beta(W_{(l,0)},m)+2m\omega_{t,21}'(0){W_{(l,0)}}_{11}'(0)=\beta(\omega(s_-),m)\,.$$

❑

The transformation $\mathcal{T}_m$ can be applied to chains of matrices, cf. [27, §4, §6]. In fact, it can be used to locally decrease or increase the negative index of a chain depending whether $m > 0$ or $m < 0$. In particular it allows to locally remove or produce singularities. We shall, for the convenience of the reader, explicitly discuss the situation which occurs in the present context. Let us first describe what happens when singularities are produced.

Let $\omega \in \mathfrak{M}_0^f$, $\omega : [0, L] \to \mathcal{M}_0$, and assume that $m < 0$ is such that $\alpha(\omega(L), m) < 0$. The function $\omega(t)'_{21}(0)$ depends continuously on t, cf. [27, Lemma 3.5], and is locally non-increasing. Hence also $\alpha(\omega(t), m)$ is continuous and, since $m < 0$, locally non-increasing. Moreover, $\alpha(\omega(0), m) = 1$, and hence there exist points $\sigma_-, \sigma \in (0, L)$ such that

$$\alpha(\omega(t), m) \begin{cases} > 0 & \text{for } t \in [0, \sigma_-), \\ = 0 & \text{for } t \in [\sigma_-, \sigma], \\ < 0 & \text{for } t \in (\sigma, L]. \end{cases} \tag{3.6}$$

The transfer matrix $\omega_{\sigma_-\sigma}$ belongs to $\mathcal{M}_0$ and has the property that

$$\omega'_{\sigma_-\sigma,21}(0) = -\frac{1}{m}\Big(\alpha(\omega(\sigma), m) - \alpha(\omega(\sigma_-), m)\Big) = 0\,.$$

Since $\frac{\omega_{\sigma_-\sigma,11}}{\omega_{\sigma_-\sigma,21}} \in \mathcal{N}_0$ and $\omega_{\sigma_-\sigma,11}(0) = 1$, $\omega_{\sigma_-\sigma,21}(0) = 0$, this implies that $\omega_{\sigma_-\sigma,21}$ vanishes identically and hence

$$\omega_{\sigma_-\sigma} = W_{(l,0)}$$

where $l := \mathfrak{t}(\omega(\sigma)) - \mathfrak{t}(\omega(\sigma_-))$. This shows that we can write

$$\omega \rightsquigarrow \omega|_{[0,\sigma_-]} \uplus \big(W_{(t,0)}\big)_{t\in[0,l]} \uplus \big(\omega_{\sigma t}\big)_{t\in[\sigma,L]}\,.$$

¿From the results of [27] we now obtain the following.

3.6 Corollary. *Let $\omega \in \mathfrak{M}_0^f$, $\omega : [0, L] \to \mathcal{M}_0$, be given. Let $m < 0$ be such that $\alpha(\omega(L), m) < 0$, and let σ_- and σ be defined according to* (3.6). *Then the chain*

$$\mathfrak{T}_m(\omega) := \mathcal{T}_m \circ \omega|_{[0,\sigma_-)} \dot{\uplus}\, \mathcal{T}_m \circ \omega|_{(\sigma,L]}$$

belongs to $\mathfrak{M}_1^f$. Its singularity has the property that $(\mathfrak{T}_m(\omega(t)))'_{21}(0)$ is unbounded when t approaches the singularity.

Proof. By the definition of σ_- and σ we have $\omega(t) \in \operatorname{dom} \mathcal{T}_m$ for all $t \in [0, \sigma_-) \cup (\sigma, L]$. By (3.1) we have $\mathcal{T}_m(\omega(t)) \in \mathcal{M}_{\kappa(t)}$ where

$$\kappa(t) = \begin{cases} 0 & \text{for } t \in [0, \sigma_-), \\ 1 & \text{for } t \in (\sigma, L]. \end{cases}$$

Note that, clearly, $\mathcal{T}_m(\omega(0)) = I$.

Let $\varpi \in \mathfrak{M}_1^f$ be the finite maximal chain going downwards from $\mathcal{T}_m(\omega(L))$. By [27, Lemma 4.5], we have

$$\operatorname{ind}_-\big(\mathcal{T}_m(\omega(t))^{-1}\mathcal{T}_m(\omega(s))\big) = \operatorname{ind}_- \mathcal{T}_m(\omega(s)) - \operatorname{ind}_- \mathcal{T}_m(\omega(t))\,;$$

hence, by (W4), each matrix $\mathcal{T}_m(\omega(t))$, $t \in [0,\sigma_-)\cup(\sigma,L]$, occurs in ϖ. However, by (3.3) the function $\mathfrak{t}(\mathcal{T}_m(\omega(t)))$ depends continuously on $t \in [0,\sigma_-) \cup (\sigma, L]$, and satisfies

$$\lim_{t\nearrow\sigma_-} \mathfrak{t}(\mathcal{T}_m(\omega(t))) = +\infty,\ \lim_{t\searrow\sigma} \mathfrak{t}(\mathcal{T}_m(\omega(t))) = -\infty\,.$$

Hence, by [25, Theorem 13.1], every matrix $\varpi(s)$ is equal to a matrix $\mathcal{T}_m(\omega(t))$, i.e., we have $\varpi = \mathfrak{T}_m(\omega)$.

Since $(\mathcal{T}_m(\omega(t)))'_{21}(0) = \frac{1}{\alpha(\omega(t),m)}\omega(t)'_{21}(0)$, we see that $(\mathcal{T}_m(\omega(t)))'_{21}(0)$ is unbounded when t approaches σ_- from the left or σ from the right.

❑

Now we shall describe how singularities can be removed. Let $\varpi \in \mathfrak{M}_1^f$, $\varpi : [0,\sigma) \cup (\sigma, L] \to \mathcal{M}_{<\infty}$, and assume that $\varpi(t)'_{21}(0)$ is unbounded when t tends to σ. Moreover, let $m > 0$ be such that $\alpha(\varpi(L), m) < 0$. Since $m > 0$, the function $\alpha(\varpi(t), m)$ is locally non-decreasing. Moreover, $\alpha(\varpi(0), m) = 1$ and $\alpha(\varpi(L), m) < 0$; thus

$$\alpha(\varpi(t), m) \begin{cases} > 0 & \text{for } t \in [0, \sigma), \\ < 0 & \text{for } t \in (\sigma, L]. \end{cases}$$

It follows that $\varpi(t) \in \operatorname{dom} \mathcal{T}_m$ for all $t \in \operatorname{dom} \varpi$ and $\mathcal{T}_m(\varpi(t)) \in \mathcal{M}_0$, cf. (3.1). By [27, Lemma 4.5] each matrix $\mathcal{T}_m(\varpi(t))$ belongs to the finite maximal chain going downwards from $\mathcal{T}_m(\varpi(L))$. Moreover, the chain $(\mathcal{T}_m(\varpi(L)))_{t\in\operatorname{dom}\varpi}$ is almost maximal as the following corollary shows.

3.7 Corollary. *Consider a chain* $\varpi \in \mathfrak{M}_1^f$, $\operatorname{dom}\varpi = [0,\sigma) \cup (\sigma, L]$, *with* $\lim_{t\to\sigma} |\varpi(t)'_{21}(0)| = \infty$. *Moreover, let* $m > 0$ *and assume that* $\alpha(\varpi(L), m) < 0$. *Then the chain*

$$\mathfrak{T}_m(\varpi) := \mathrm{C}\big(\mathcal{T}_m \circ \varpi|_{[0,\sigma)}\big) \uplus \big(W_{(t,0)}\big)_{t\in[0,e]} \uplus \mathrm{C}\big(\mathcal{T}_m \circ \varpi|_{(\sigma,L]}\big)\,,$$

where $e := \lim_{t\searrow\sigma} \mathfrak{t}(\mathcal{T}_m \circ \varpi(t)) - \lim_{t\nearrow\sigma} \mathfrak{t}(\mathcal{T}_m \circ \varpi(t))$, *belongs to* $\mathfrak{M}_0^f$.

Proof. Denote by ω the finite maximal chain going downwards from $\mathcal{T}_m(\varpi(L))$, and let $\iota : \operatorname{dom}\varpi \to \operatorname{dom}\omega$ be such that $\mathcal{T}_m(\varpi) = \omega \circ \iota$. By (3.2) we have

$$\lim_{t\nearrow\sigma} \alpha(\omega \circ \iota(t), -m) = \lim_{t\searrow\sigma} \alpha(\omega \circ \iota(t), -m) = 0\,.$$

By (3.3) the function $\mathfrak{t}(\mathcal{T}_m(\varpi(t)))$ is continuous on $[0,\sigma) \cup (\sigma, L]$. Clearly, $\lim_{t\searrow 0} \mathcal{T}_m(\varpi(t)) = I$ and $\lim_{t\nearrow L} \mathcal{T}_m(\varpi(t)) = \mathcal{T}_m(\varpi(L))$.

If we put $\sigma_- := \lim_{t\nearrow\sigma} \iota(t)$, $\sigma_+ := \lim_{t\searrow\sigma} \iota(t)$, then

$$\omega(\sigma_-) = \lim_{t\nearrow\sigma} \mathcal{T}_m(\varpi(t)),\ \omega(\sigma_+) = \lim_{t\searrow\sigma} \mathcal{T}_m(\varpi(t))\,.$$

Moreover, $\omega'_{\sigma_-\sigma_+,21}(0) = 0$, and hence $\omega_{\sigma_-\sigma_+} = W_{(e,0)}$ for some appropriate number $e \geq 0$. Altogether, we obtain that the chain $\mathfrak{T}_m(\varpi)$ as defined in the statement of the corollary is equal to ω.

❑

3.8 Remark. The transforms $\mathfrak{T}_m$ and $\mathfrak{T}_{-m}$ are inverses of each other in the following sense: let $\omega \in \mathcal{M}_0^f$, $\operatorname{dom}\omega = [0, L]$, and $m < 0$ with $\alpha(\omega(L), m) < 0$ be given. Then the construction of Corollary 3.7 can be applied to the chain $\mathfrak{T}_m(\omega)$ and the number $-m$, and we have $\mathfrak{T}_{-m}(\mathfrak{T}_m(\omega)) = \omega$. Conversely, let $\varpi \in \mathfrak{M}_1^f$ and $m > 0$ be given such that the hypothesis of Corollary 3.7 are satisfied. Then Corollary 3.6 can be applied to the chain $\mathfrak{T}_m(\varpi)$ and the number $-m$, and we have $\mathfrak{T}_{-m}(\mathfrak{T}_m(\varpi)) = \varpi$.

4 A perturbation of chains

Throughout this section let $\omega \in \mathfrak{M}_0^f$, $\omega : [0, L] \to \mathcal{M}_0$, and $m < 0$ be fixed, and assume that $\alpha(\omega(L), m) < 0$. Let σ_- and σ be defined as in (3.6) and put $l := \mathfrak{t}(\omega(\sigma)) - \mathfrak{t}(\omega(\sigma_-))$ so that

$$\omega \leftrightsquigarrow \omega|_{[0,\sigma_-]} \uplus \big(W_{(t,0)}\big)_{t\in[0,l]} \uplus \big(\omega_{\sigma t}\big)_{t\in[\sigma,L]} .$$

Let $\mathfrak{e} := (e_1, e_2, e_3) \in \mathbb{R} \times \mathbb{R} \times [0, \infty)$. We define a perturbed chain $\omega_{\mathfrak{e}}$, see Definition 4.3 below. Before we can do so, however, we need the following supplement to [27, Lemma 4.2] and one corollary of these results.

4.1 Lemma. *Let $M \in \mathcal{M}_\kappa$ be given.*

(*i*) *Let $\chi, \lambda \in \mathbb{R} \setminus \{0\}$, $\upsilon, \nu \in \mathbb{R}$. Then the matrix*

$$\tilde{M} := \begin{pmatrix} \chi & -\upsilon \\ 0 & \frac{1}{\chi} \end{pmatrix} M \begin{pmatrix} \frac{1}{\lambda} & \nu \\ 0 & \lambda \end{pmatrix} \tag{4.1}$$

is entire and satisfies $\tilde{M}(0) = I$ if and only if $\chi = \lambda$ and $\upsilon = \nu$. In this case $\tilde{M} \in \mathcal{M}_\kappa$.

(*ii*) *Assume that $M'_{21}(0) = 0$, and let $\chi, \lambda, u, \mu \in \mathbb{R} \setminus \{0\}$, $\upsilon, \nu \in \mathbb{R}$ be given. Then the matrix*

$$\tilde{M}(z) := \begin{pmatrix} \chi & -\upsilon - \frac{u}{z} \\ 0 & \frac{1}{\chi} \end{pmatrix} M \begin{pmatrix} \frac{1}{\lambda} & \nu + \frac{\mu}{z} \\ 0 & \lambda \end{pmatrix}$$

is entire and satisfies $\tilde{M}(0) = I$, if and only if $\lambda = \chi$, $u = \mu$ and

$$\upsilon = \nu + \mu\Big(2M'_{11}(0) - \frac{\mu}{\lambda}\frac{M''_{21}(0)}{2}\Big) .$$

In this case $\tilde{M} \in \mathcal{M}_\kappa$.

Proof. Necessity in (*i*) is clear since we must have $\tilde{M}(0) = I$. Sufficiency follows since the factors in (4.1) are iJ-unitary.

The assertion (*ii*) follows by inspecting the explicit form of $\tilde{M}$, cf. the set of formulae at the beginning of the proof of [27, Lemma 4.2, in particular equation (IV)], and by repeating the arguments for counting the negative index of $\tilde{M}$. ❑

4.2 Corollary. *Let W be an entire matrix functions with $W(0) = I$ and let $e_1, e_2, \varepsilon_1, \varepsilon_2 \in \mathbb{R}$, $\lambda \in \mathbb{R} \setminus \{0\}$. Then the matrix function*

$$\hat{W} = \begin{pmatrix} 1 & -e_1 - \frac{e_2}{z} \\ 0 & 1 \end{pmatrix} W \begin{pmatrix} \frac{1}{\lambda} & \varepsilon_1 + \frac{\varepsilon_2}{z} \\ 0 & \lambda \end{pmatrix}$$

is entire and takes the value I at $z = 0$ if and only if

$$\lambda = 1 - e_2 W_{21}'(0), \qquad \varepsilon_2 = e_2,$$
$$\varepsilon_1 = \frac{e_1 - 2e_2 W_{11}'(0) + e_2^2 W_{21}'(0) W_{11}'(0) + \frac{e_2^2}{2} W_{21}''(0)}{1 - e_2 W_{21}'(0)}. \tag{4.2}$$

Proof. Solve the equations 1.,2. and 3. of [27, Lemma 4.2] and Lemma 4.1, respectively, for λ, ν, μ.

❑

Now we are ready to define the perturbed chain $\omega_{\mathfrak{e}}$.

4.3 Definition. Choose $S \in (\sigma, L]$ such that $1 - e_2 \omega_{\sigma t,21}'(0) > 0$, $t \in [\sigma, S]$. Such a choice is possible by the continuity of $\omega_{\sigma t,21}'(0)$ and the fact that $\omega_{\sigma\sigma,21}'(0) = 0$. Define

$$\hat{W}_{\sigma t} := \begin{pmatrix} 1 & -e_1 - \frac{e_2}{z} \\ 0 & 1 \end{pmatrix} \omega_{\sigma t} \begin{pmatrix} \frac{1}{\lambda(t)} & \varepsilon_1(t) + \frac{\varepsilon_2(t)}{z} \\ 0 & \lambda(t) \end{pmatrix} , \quad t \in [\sigma, S] ,$$

where $\lambda, \varepsilon_1, \varepsilon_2$ are given by

$$\lambda(t) := 1 - e_2 \omega_{\sigma t,21}'(0), \qquad \varepsilon_2(t) := e_2 ,$$
$$\varepsilon_1(t) := \frac{e_1 - 2e_2 \omega_{\sigma t,11}'(0) + e_2^2 \omega_{\sigma t,21}'(0) \omega_{\sigma t,11}'(0) + \frac{e_2^2}{2} \omega_{\sigma t,21}''(0)}{1 - e_2 \omega_{\sigma t,21}'(0)}. \tag{4.3}$$

With this notation put

$$\omega_{\mathfrak{e}} := \omega|_{[0,\sigma_-]} \uplus \big(W_{(t,0)}\big)_{t \in [0,e_3]} \uplus \big(\hat{W}_{\sigma t}\big)_{t \in [\sigma,S]} .$$

We will always assume that $\omega_{\mathfrak{e}}$ is parameterized such that $\omega_{\mathfrak{e}}(t) = \omega(\sigma_-) W_{(e_3,0)} \hat{W}_{\sigma t}$ for $t \in [\sigma, S]$.

4.4 Lemma. *We have $\omega_{\mathfrak{e}} \in \mathfrak{M}_0^f$.*

Proof. In view of Remark 2.8 it is enough to show that $(\hat{W}_{\sigma t})_{t \in [\sigma,S]} \in \mathfrak{M}_0^f$. Thereby, clearly, $\hat{W}_{\sigma\sigma} = I$. Next note that, since $\lambda(t) > 0$ for $t \in [\sigma, S]$, by [27, Lemma 4.2] and Lemma 4.1 all matrices

$$\hat{W}_{\sigma t}(z)^{-1} \hat{W}_{\sigma s}(z) = \begin{pmatrix} \lambda(t) & -\varepsilon_1(t) - \frac{\varepsilon_2(t)}{z} \\ 0 & \frac{1}{\lambda(t)} \end{pmatrix} \omega_{ts} \begin{pmatrix} \frac{1}{\lambda(s)} & \varepsilon_1(s) + \frac{\varepsilon_2(s)}{z} \\ 0 & \lambda(s) \end{pmatrix}$$

where $\sigma \leq t \leq s \leq S$, belong to $\mathcal{M}_0$. Moreover, cf. the explicit formulae for $\hat{W}_{\sigma t}$ given in the proof of [27, Lemma 4.2],

$$\hat{W}_{\sigma t,21}'(0) = \frac{1}{\lambda(t)} \omega_{\sigma t,21}'(0) , \tag{4.4}$$

$$\hat W'_{\sigma t,12}(0) = \varepsilon_1(t)\omega'_{\sigma t,11}(0) + \varepsilon_2(t)\frac{\omega''_{\sigma t,11}(0)}{2} + \lambda(t)\omega'_{\sigma t,12}(0) - e_1\lambda(t)\omega'_{\sigma t,22}(0)$$
$$- e_2\lambda(t)\frac{\omega''_{\sigma t,22}(0)}{2} - e_1\varepsilon_1(t)\omega'_{\sigma t,21}(0) - (e_1\varepsilon_2(t) + e_2\varepsilon_1(t))\frac{\omega''_{\sigma t,21}(0)}{2}$$
$$- e_2\varepsilon_2(t)\frac{\omega'''_{\sigma t,21}(0)}{6} .$$

Hence, since $\omega_{\sigma t}$ depends continuously on t with respect to locally uniform convergence, also $\mathfrak{t}(\hat W_{\sigma t}) = \hat W'_{\sigma t,12}(0) - \hat W'_{\sigma t,21}(0)$ depends continuously on $t \in [\sigma, S]$. It follows that $(\hat W_{\sigma t})_{t\in[\sigma,S]} \in \mathfrak{M}_0^f$.

❑

4.5 Lemma. *We have*

$$\alpha(\omega_{\mathfrak{e}}(t), m) = \frac{\alpha(\omega(t), m)}{\lambda(t)},\ \ t \in (\sigma, S] .$$

In particular $\alpha(\omega_{\mathfrak{e}}(t), m) < 0$ *for* $t \in (\sigma, S]$.

Proof. Since to the left of the indivisible interval whose right endpoint is σ the chains $\omega_{\mathfrak{e}}$ and ω coincide and by Corollary 3.5 the number $\alpha(\omega_{\mathfrak{e}}(\sigma), m)$ is constant on this interval, we have

$$\alpha(\omega_{\mathfrak{e}}(\sigma), m) = \alpha(\omega(\sigma), m) .$$

Since $\alpha(\omega(\sigma), m) = 0$, we obtain from Lemma 3.4 that

$$\begin{aligned} \alpha(\omega(t), m) &= \alpha(\omega(\sigma), m) - m\omega'_{\sigma t,21}(0) = -m\omega'_{\sigma t,21}(0) , \\ \alpha(\omega_{\mathfrak{e}}(t), m) &= \alpha(\omega_{\mathfrak{e}}(\sigma), m) - m\hat W'_{\sigma t,21}(0) = -m\hat W'_{\sigma t,21}(0). \end{aligned} \tag{4.5}$$

Using (4.4) we conclude that

$$\alpha(\omega_{\mathfrak{e}}(t), m) = -m\frac{\omega'_{\sigma t,21}(0)}{\lambda(t)} = \frac{\alpha(\omega(t), m)}{\lambda(t)} .$$

❑

By our assumptions and the previous lemma we may apply Corollary 3.6 to ω as well as to $\omega_{\mathfrak{e}}$ and, in this way, obtain two chains $\mathfrak{T}_m(\omega)$ and $\mathfrak{T}_m(\omega_{\mathfrak{e}})$ belonging to $\mathfrak{M}_1^f$. We assume that $\mathfrak{T}_m(\omega)$ and $\mathfrak{T}_m(\omega_{\mathfrak{e}})$ are parameterized such that $\mathfrak{T}_m(\omega)(t) = \mathcal{T}_m(\omega(t))$, $t \in (\sigma, S]$, and $\mathfrak{T}_m(\omega_{\mathfrak{e}})(t) = \mathcal{T}_m(\omega_{\mathfrak{e}}(t))$, $t \in (\sigma, S]$.

4.6 Proposition. *The chains* $\mathfrak{T}_m(\omega)$ *and* $\mathfrak{T}_m(\omega_{\mathfrak{e}})$ *coincide to the left of the singularity* σ. *We have*

$$\mathfrak{T}_m(\omega_{\mathfrak{e}})_{ts} = \mathfrak{T}_m(\omega)_{ts}, \quad \sigma < t \le s \le S .$$

Proof. We shall employ Lemma 3.3. To this end we have to compute $\frac{\alpha(\omega_{\mathfrak{e}}(t),m)}{\alpha(\omega(t),m)}$ and $\beta(\omega_{\mathfrak{e}}(t), m) - \beta(\omega(t), m)$.

We have already seen in Lemma 4.5 that

$$\frac{\alpha(\omega_{\mathfrak{e}}(t),m)}{\alpha(\omega(t),m)}=\frac{1}{\lambda(t)}\,. \tag{4.6}$$

From this and (4.5) it also follows that

$$\begin{aligned}\frac{1}{\alpha(\omega_{\mathfrak{e}}(t),m)}-\frac{1}{\alpha(\omega(t),m)}&=\frac{\lambda(t)-1}{\alpha(\omega(t),m)}=\\&=\frac{\big(1-e_2\omega'_{\sigma t,21}(0)\big)-1}{-m\omega'_{\sigma t,21}(0)}=\frac{e_2}{m}\,.\end{aligned} \tag{4.7}$$

Again, since to the left of the indivisible interval whose right endpoint is σ the chains $\omega_{\mathfrak{e}}$ and ω coincide and $\beta(\omega_{\mathfrak{e}}(\sigma),m)$ is constant on this indivisible interval, cf. Corollary 3.5, we have

$$\beta(\omega_{\mathfrak{e}}(\sigma),m)=\beta(\omega(\sigma),m)\,.$$

By Lemma 3.4 we have for $t>\sigma$

$$\beta(\omega_{\mathfrak{e}}(t),m)=\beta(\omega_{\mathfrak{e}}(\sigma),m)+\beta(\hat{W}_{\sigma t},m)+2m\omega_{\mathfrak{e}}(\sigma)'_{21}(0)\hat{W}'_{\sigma t,11}(0)\,,$$

$$\beta(\omega(t),m)=\beta(\omega(\sigma),m)+\beta(\omega_{\sigma t,21},m)+2m\omega(\sigma)'_{21}(0)\omega'_{\sigma t,11}(0)\,.$$

Since $m\omega(\sigma)'_{21}(0)=1-\alpha(\omega(\sigma),m)=1$ and also $m\omega_{\mathfrak{e}}(\sigma)'_{21}(0)=1$,

$$\beta(\omega_{\mathfrak{e}}(t),m)-\beta(\omega(t),m)=\beta(\hat{W}_{\sigma t},m)-\beta(\omega_{\sigma t,21},m)+2(\hat{W}'_{\sigma t,11}(0)-\omega'_{\sigma t,11}(0))\,.$$

From the definition of $\hat{W}_{\sigma t}$ and (4.4) we find that

$$\hat{W}''_{\sigma t,21}(0)=\frac{\omega''_{\sigma t,21}(0)}{\lambda(t)}\,,$$

$$\hat{W}'_{\sigma t,11}(0)=\frac{\omega'_{\sigma t,11}(0)}{\lambda(t)}-e_1\frac{\omega'_{\sigma t,21}(0)}{\lambda(t)}-e_2\frac{\omega''_{\sigma t,21}(0)}{2\lambda(t)}\,.$$

Hence we can compute further

$$\begin{aligned}&\beta(\omega_{\mathfrak{e}}(t),m)-\beta(\omega(t),m)\\&=\Big(\frac{m}{2}\hat{W}''_{\sigma t,21}(0)+m\hat{W}'_{\sigma t,21}(0)\hat{W}'_{\sigma t,11}(0)-2\hat{W}'_{\sigma t,11}(0)\Big)\\&\quad-\Big(\frac{m}{2}\omega''_{\sigma t,21}(0)+m\omega'_{\sigma t,21}(0)\omega'_{\sigma t,11}(0)-2\omega'_{\sigma t,11}(0)\Big)\\&\quad+2\big(\hat{W}'_{\sigma t,11}(0)-\omega'_{\sigma t,11}(0)\big)\\&=\frac{m}{2}\frac{\omega''_{\sigma t,21}(0)}{\lambda(t)}+m\frac{\omega'_{\sigma t,21}(0)}{\lambda(t)}\Big(\frac{\omega'_{\sigma t,11}(0)}{\lambda(t)}-e_1\frac{\omega'_{\sigma t,21}(0)}{\lambda(t)}-e_2\frac{\omega''_{\sigma t,21}(0)}{2\lambda(t)}\Big)\\&\quad-\Big(\frac{m}{2}\omega''_{\sigma t,21}(0)+m\omega'_{\sigma t,21}(0)\omega'_{\sigma t,11}(0)\Big)\\&=\frac{m}{2}\omega''_{\sigma t,21}(0)\Big(\frac{1}{\lambda(t)}-\frac{e_2}{\lambda(t)^2}\omega'_{\sigma t,21}(0)-1\Big)-m\omega'_{\sigma t,21}(0)^2\frac{e_1}{\lambda(t)^2}\\&\quad+m\omega'_{\sigma t,11}(0)\omega'_{\sigma t,21}(0)\Big(\frac{1}{\lambda(t)^2}-1\Big)\,.\end{aligned}$$

It follows that

$$
\begin{aligned}
&\lambda(t)^2\big(\beta(\omega_{\mathfrak{e}}(t),m)-\beta(\omega(t),m)\big)\\
&=\frac{m}{2}\omega''_{\sigma t,21}(0)\big(\lambda(t)-e_2\omega'_{\sigma t,21}(0)-\lambda(t)^2\big)-m\omega'_{\sigma t,21}(0)^2 e_1\\
&\quad+m\omega'_{\sigma t,11}(0)\omega'_{\sigma t,21}(0)\big(1-\lambda(t)^2\big)\\
&=\frac{m}{2}\omega''_{\sigma t,21}(0)\big(-e_2^2\omega'_{\sigma t,21}(0)^2\big)-m\omega'_{\sigma t,21}(0)^2 e_1\\
&\quad+m\omega'_{\sigma t,11}(0)\omega'_{\sigma t,21}(0)\big(2e_2\omega'_{\sigma t,21}(0)-e_2^2\omega'_{\sigma t,21}(0)^2\big)\\
&=-m\omega'_{\sigma t,21}(0)^2\Big(e_2^2\frac{\omega''_{\sigma t,21}(0)}{2}+e_1-2e_2\omega'_{\sigma t,11}(0)+e_2^2\omega'_{\sigma t,11}(0)\omega'_{\sigma t,21}(0)\Big)\\
&=-\frac{1}{m}\alpha(\omega(t),m)^2\,\lambda(t)\varepsilon_1(t)\,.
\end{aligned}
$$

Using this computation and (4.6), we conclude that

$$
\begin{aligned}
-\frac{m(\beta(\omega_{\mathfrak{e}}(t),m)-\beta(\omega(t),m))}{\alpha(\omega_{\mathfrak{e}}(t),m)\alpha(\omega(t),m)}&=\frac{-m\lambda(t)^2(\beta(\omega_{\mathfrak{e}}(t),m)-\beta(\omega(t),m))}{\lambda(t)^2\alpha(\omega_{\mathfrak{e}}(t),m)\alpha(\omega(t),m)}=\\
&=\frac{\alpha(\omega(t),m)^2\lambda(t)\varepsilon_1(t)}{\lambda(t)^2\alpha(\omega_{\mathfrak{e}}(t),m)\alpha(\omega(t),m)}=\varepsilon_1(t)\,.
\end{aligned}
\tag{4.8}
$$

Since, for $\sigma<t\le s\le S$, we have

$$
\omega_{\mathfrak{e}}(t)^{-1}\omega_{\mathfrak{e}}(s)=\hat{W}_{\sigma t}^{-1}\hat{W}_{\sigma s}=
$$

$$
=\begin{pmatrix}\frac{1}{\lambda(t)} & \varepsilon_1(t)+\frac{\varepsilon_2(t)}{z}\\ 0 & \lambda(t)\end{pmatrix}^{-1}\omega_{ts}\begin{pmatrix}\frac{1}{\lambda(s)} & \varepsilon_1(s)+\frac{\varepsilon_2(s)}{z}\\ 0 & \lambda(s)\end{pmatrix},
$$

we conclude from (4.6), (4.7) and (4.7) that the hypothesis (3.5) of Lemma 3.3 is satisfied.

❑

Next we show that Proposition 4.6 has a converse, i.e., that every chain that has the same transfer matrices as the given chain ω is of the form $\omega_{\mathfrak{e}}$.

4.7 Proposition. *Let $\hat{\omega}\in\mathfrak{M}_0^f$, $\hat{\omega}:[0,\hat{L}]\to\mathcal{M}_0$, be given and assume that $\alpha(\hat{\omega}(\hat{L}),m)<0$. Let $\hat{\sigma}_-,\hat{\sigma}$ be defined as in (3.6). Assume that there exist continuous and strictly increasing embeddings*

$$
\iota_+:[\hat{\sigma},\hat{L}]\to[\sigma,L],\ \ \iota_-:[0,\hat{\sigma}_-]\to[0,\sigma_-]\,,
$$

with $\iota_+(\hat{\sigma})=\sigma$ and ι_- bijective, such that

$$
\hat{\omega}|_{[0,\hat{\sigma}_-]}=\omega\circ\iota_-
$$

and

$$
\mathfrak{T}_m(\hat{\omega})_{ts}=\mathfrak{T}_m(\omega)_{\iota_+(t)\iota_+(s)},\ \ \hat{\sigma}<t\le s\le\hat{L}\,.
$$

Then there exists a triple $\mathfrak{e}\in\mathbb{R}\times\mathbb{R}\times[0,\infty)$ such that $\omega_{\mathfrak{e}}\circ\iota_+=\hat{\omega}|_{[\hat{\sigma},\hat{L}]}$.

Proof. Without loss of generality let us assume that $\hat{\sigma} = \sigma$ and ι_+ is the inclusion map $[\hat{\sigma}, \hat{L}] \subseteq [\sigma, L]$. By Lemma 3.3 we have

$$\hat{\omega}_{ts} = A(t)^{-1}\omega_{ts}A(s),\ \sigma < t \leq s \leq \hat{L}\,.$$

We have a closer look at the entries of $A(t)$. First note that, by our assumption that $\mathfrak{T}_m(\hat{\omega})$ and $\mathfrak{T}_m(\omega)$ have the same transfer matrices,

$$\alpha(\mathcal{T}_m(\hat{\omega}(s)), -m) - \alpha(\mathcal{T}_m(\omega(s)), -m) = \alpha(\mathcal{T}_m(\hat{\omega}(t)), -m) - \alpha(\mathcal{T}_m(\omega(t)), -m)$$

for $\sigma < t \leq s \leq \hat{L}$. Since, by (4.21) of [27], for any matrix $W \in \operatorname{dom} \mathcal{T}_m$

$$\alpha(\mathcal{T}_m(W), -m) = \frac{1}{\alpha(W, m)}\,,$$

we conclude that the number

$$e_2(\hat{\omega}, \omega) := m\Big(\frac{1}{\alpha(\hat{\omega}(t), m)} - \frac{1}{\alpha(\omega(t), m)}\Big)$$

does not depend on $t \in (\sigma, \hat{L}]$. Since $\lim_{t\searrow\sigma} \alpha(\omega(t), m) = 0$, this also implies that

$$\lim_{t\searrow\sigma} \frac{\alpha(\omega(t), m)}{\alpha(\hat{\omega}(t), m)} = 1\,.$$

For arbitrary $t \in (\sigma, \hat{L}]$ we can write $A(t) = \omega_{t\hat{L}} A(\hat{L}) \hat{\omega}_{\hat{L}t}$. Hence the limit $\lim_{t\searrow\sigma} A(t)$ exists; in particular, also the limit

$$e_1(\hat{\omega}, \omega) := -m \lim_{t\searrow\sigma} \frac{\beta(\hat{\omega}(t), m) - \beta(\omega(t), m)}{\alpha(\hat{\omega}(t), m)\alpha(\omega(t), m)}$$

exists. Let $s \in (\sigma, \hat{L}]$ be fixed; then for arbitrary $t \in (\sigma, \hat{L}]$ we have $\hat{\omega}_{ts} = A(t)^{-1}\omega_{ts}A(s)$. If in this relation we let t tend to σ, we obtain

$$\hat{\omega}_{\sigma s} = \begin{pmatrix} 1 & -e_1(\hat{\omega}, \omega) - \frac{e_2(\hat{\omega}, \omega)}{z} \\ 0 & 1 \end{pmatrix} \omega_{\sigma s} A(s)$$

By Corollary 4.2 we must have

$$A(s) = \begin{pmatrix} \frac{1}{\lambda(s)} & \varepsilon_1(s) + \frac{\varepsilon_2(s)}{z} \\ 0 & \lambda(s) \end{pmatrix}$$

where $\lambda, \varepsilon_1, \varepsilon_2$ are defined by (4.3) with $e_1 = e_1(\hat{\omega}, \omega)$ and $e_2 = e_2(\hat{\omega}, \omega)$. This just says that we have

$$\hat{\omega}_{\sigma s} = \hat{W}_{\sigma s},\ \sigma \leq s \leq \hat{L}\,.$$

Put $e_3(\hat{\omega}) := \mathfrak{t}(\hat{\omega}(\sigma)) - \mathfrak{t}(\hat{\omega}(\hat{\sigma}_-))$ and $\mathfrak{e} := (e_1(\hat{\omega}, \omega), e_2(\hat{\omega}, \omega), e_3(\hat{\omega}))$; then

$$\begin{aligned} \hat{\omega} &= \hat{\omega}|_{[0,\hat{\sigma}_-]} \uplus W_{(e_3(\hat{\omega}),0)} \uplus \big(\hat{\omega}_{\sigma t}\big)_{t\in[\sigma,\hat{L}]} \\ &= \omega|_{[0,\sigma_-]} \uplus W_{(e_3(\hat{\omega}),0)} \uplus \big(\hat{W}_{\sigma t}\big)_{t\in[\sigma,\hat{L}]} = \omega_{\mathfrak{e}}|_{[0,\hat{L}]}\,. \end{aligned}$$

❑

5 Main theorem

Let $\mathfrak{h} = (H, \mathfrak{b}, \mathfrak{d})$ be an indefinite Hamiltonian in limit point case with negative index 1. Since $\operatorname{ind}_- \mathfrak{h} = 1$, $\mathfrak{h}$ can have only one singularity, i.e., $H = (H_0, H_1)$, where H_0 is defined on $[\sigma_0, \sigma_1)$ and H_1 on (σ_1, σ_2). Moreover, by Remark 2.3, $\Delta = 1$, and hence $\mathfrak{d} = (d_0, d_1)$. Also, $\ddot{o} \in \{0, 1\}$, and $b_1 < 0$ in the case $\ddot{o} = 1$. Moreover, we assume that

$$\int_{\sigma_0}^{\sigma_1} (1,0) H_0(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt < \infty \,, \tag{5.1}$$

which is no essential restriction because the Hamiltonian can always be transformed using (2.8) such that (5.1) holds, cf. (2.2). Denote the Weyl coefficient of $\mathfrak{h}$ by $q_{\mathfrak{h}}$, so that $q_{\mathfrak{h}} \in \mathcal{N}_1$.

Let $\upsilon \in \mathfrak{M}_1$ be the unique maximal chain of matrices whose Weyl coefficient is $q_{\mathfrak{h}}$. Without loss of generality we assume that υ is parameterized similarly as H, i.e., that $\operatorname{dom} \upsilon = [\sigma_0, \sigma_1) \cup (\sigma_1, \sigma_2)$, and that $\upsilon(t)$ is a solution of the differential equation

$$\frac{\partial}{\partial t} \upsilon(t) J = z \upsilon(t) H(t), \quad t \in [\sigma_0, \sigma_1) \cup (\sigma_1, \sigma_2) \,,$$
$$\upsilon(\sigma_0) = I \,.$$

Note that the function $\upsilon(t)$ can be computed explicitly from H, by solving the canonical differential equation, only on the interval $[\sigma_0, \sigma_1)$. Moreover, note that $\upsilon(t)$ is also a function of z, and we identify $\upsilon(t)(z)$ and $\upsilon(t, z)$ as it appears, e.g. in (5.8).

Due to our condition (5.1), we have $\lim_{t \to \sigma_1} |\upsilon(t)'_{21}(0)| = \infty$. Hence there exists $L > \sigma_1$, such that $\alpha(\upsilon(L), 1) < 0$. Define $\omega := \mathfrak{T}_1(\upsilon|_{[\sigma_0, L] \setminus \{\sigma_1\}})$. Then $\omega \in \mathfrak{M}_0^f$ and is, if appropriately parameterized, explicitly given by

$$\omega(t) := \begin{cases} \mathcal{T}_1(\upsilon(t+l)), & t \in [\sigma_0 - l, \sigma_-), \\ \left[\lim_{s \nearrow \sigma_1} \mathcal{T}_1(\upsilon(s)) \right] \cdot \begin{pmatrix} 1 & (t - \sigma_-) z \\ 0 & 1 \end{pmatrix}, & t \in [\sigma_-, \sigma_1], \\ \mathcal{T}_1(\upsilon(t)), & t > \sigma_1, \end{cases}$$

where l and σ_- are defined by the relation

$$l = \sigma_1 - \sigma_- = \lim_{s \searrow \sigma_1} \mathfrak{t}\big(\mathcal{T}_1(\upsilon(s))\big) - \lim_{s \nearrow \sigma_1} \mathfrak{t}\big(\mathcal{T}_1(\upsilon(s))\big), \tag{5.2}$$

cf. Corollary 3.7. Note here that it follows from (5.1) that the limits on the right-hand side of this relation exist. Actually, the limits

$$\lim_{s \searrow \sigma_1} \mathcal{T}_1(\upsilon(s)), \ \lim_{s \nearrow \sigma_1} \mathcal{T}_1(\upsilon(s))$$

exist locally uniformly on $\mathbb{C}$ and belong to $\mathcal{M}_0$. Recall from [27, Theorem 4.4] that ω can be continued to a maximal chain $\hat{\omega}$ whose Weyl coefficient is equal to $q_{\mathfrak{h}}(z) - \frac{1}{z}$.

To simplify notation, let us denote

$$\begin{aligned} \alpha(t) &:= 1 - \upsilon(t,z)'_{21}(0) \,, \\ \beta(t) &:= \frac{\upsilon(t,z)''_{21}(0)}{2} + \upsilon(t,z)'_{21}(0) \upsilon(t,z)'_{11}(0) - 2\upsilon(t,z)'_{11}(0) \,, \end{aligned} \tag{5.3}$$

instead of $\alpha(\upsilon(t),1)$ and $\beta(\upsilon(t),1)$. Here primes denote differentiation with respect to the variable z and an evaluation after this denotes evaluation of z.

5.1 Proposition. *The limit*

$$M := \lim_{t\nearrow\sigma_1} \upsilon(t,z)\begin{pmatrix} \frac{1}{\alpha(t)} & \frac{\beta(t)}{\alpha(t)} + \frac{1}{z} \\ 0 & \alpha(t) \end{pmatrix}, \tag{5.4}$$

exists locally uniformly on $\mathbb{C}\setminus\{0\}$.

The function $\tau := M^{-1} \star q_{\mathfrak{h}}$ *belongs to* $\mathcal{N}_0 \cup \mathcal{N}_1$, *and* $\lim_{y\to+\infty} \frac{1}{iy}\tau(iy) = l$.

Proof. We compute

$$\upsilon(t,z)\begin{pmatrix} \frac{1}{\alpha(\upsilon(t),1)} & \frac{\beta(\upsilon(t),1)}{\alpha(\upsilon(t),1)} + \frac{1}{z} \\ 0 & \alpha(\upsilon(t),1) \end{pmatrix} = \tag{5.5}$$

$$= \begin{pmatrix} 1 & \frac{1}{z} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\frac{1}{z} \\ 0 & 1 \end{pmatrix} \upsilon(t,z)\begin{pmatrix} \frac{1}{\alpha(\upsilon(t),1)} & \frac{\beta(\upsilon(t),1)}{\alpha(\upsilon(t),1)} + \frac{1}{z} \\ 0 & \alpha(\upsilon(t),1) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{z} \\ 0 & 1 \end{pmatrix} \mathcal{T}_1(\upsilon(t,z)).$$

Thus the limit (5.4) exists locally uniformly on $\mathbb{C}\setminus\{0\}$.

Let $t\in[\sigma_0,\sigma_1)$. The matrix $\mathcal{T}_1(\upsilon(t))$ belongs to ω and thus also to $\hat{\omega}$. Hence there exists $\tau_t \in \mathcal{N}_0\cup\mathcal{N}_1$ such that $\mathcal{T}_1(\upsilon(t)) \star \tau_t = q_{\mathfrak{h}} - \frac{1}{z}$. We compute

$$\Big[\upsilon(t)\begin{pmatrix} \frac{1}{\alpha(\upsilon(t),1)} & \frac{\beta(\upsilon(t),1)}{\alpha(\upsilon(t),1)} + \frac{1}{z} \\ 0 & \alpha(\upsilon(t),1) \end{pmatrix}\Big]^{-1} \star q_{\mathfrak{h}} = \Big[\begin{pmatrix} 1 & \frac{1}{z} \\ 0 & 1 \end{pmatrix}\mathcal{T}_1(\upsilon(t))\Big]^{-1} \star q_{\mathfrak{h}} =$$

$$= \Big[\mathcal{T}_1(\upsilon(t))^{-1}\begin{pmatrix} 1 & -\frac{1}{z} \\ 0 & 1 \end{pmatrix}\Big] \star q_{\mathfrak{h}} = \mathcal{T}_1(\upsilon(t))^{-1} \star (q_{\mathfrak{h}} - \frac{1}{z}) = \tau_t.$$

The limit $t\nearrow\sigma_1$ on the left hand side of this relation exists and is equal to τ. Since $\mathcal{N}_0\cup\mathcal{N}_1$ is closed, we obtain $\tau\in\mathcal{N}_0\cup\mathcal{N}_1$.

We have $\lim_{t\nearrow\sigma_1}\mathcal{T}_1(\upsilon(t)) = \omega(\sigma_-)$, $\omega(\sigma_1) = \omega(\sigma_-)\begin{pmatrix} 1 & lz \\ 0 & 1 \end{pmatrix}$, and $\lim_{t\nearrow\sigma_1}\tau_t = \tau$. Hence $\omega(\sigma_1)\star(\tau(z) - lz) = q_{\mathfrak{h}} - \frac{1}{z}$. This implies that $\tau(z) - lz$ is the Weyl coefficient of the maximal chain $\hat{\omega}(\sigma_1)^{-1}\hat{\omega}(t)|_{t\geq\sigma_1}$. Since this chain does not start with an indivisible interval of type 0, we conclude from [26, Theorem 5.7, Lemma 7.5, proof of Theorem 7.1, Lemma 5.2] that $\lim_{y\to+\infty}\frac{1}{iy}(\tau(iy) - liy) = 0$.

❑

5.2 Definition. For a triple $\mathfrak{e} = (e_1,e_2,e_3) \in \mathbb{R}\times\mathbb{R}\times[0,\infty)$ let us define a function $q_{\mathfrak{h}}^{\mathfrak{e}}(z)$ on $\mathbb{C}\setminus\mathbb{R}$ as

$$q_{\mathfrak{h}}^{\mathfrak{e}}(z) := M(z)\begin{pmatrix} 1 & (e_3 - l)z - e_1 - \frac{e_2}{z} \\ 0 & 1 \end{pmatrix} M(z)^{-1} \star q_{\mathfrak{h}}(z) \tag{5.6}$$

where the matrix function M is defined by (5.4).

The definition (5.6) of $q_{\mathfrak{h}}^{\mathfrak{e}}$ can be rewritten in two, sometimes more convenient, ways.

5.3 Proposition. *Denote by* $q_{\mathfrak{h},\sigma_1}$ *the intermediate Weyl coefficient* $q_{\mathfrak{h},\sigma_1}(z) := \lim_{t\nearrow\sigma_1} \upsilon(t,z) \star \infty$, *and let* $M_{21} = \lim_{t\nearrow\sigma_1} \frac{\upsilon_{21}(t,z)}{\alpha(t)}$ *be the left lower entry of* M. *Finally, put* $p_{\mathfrak{e}}(z) := (e_3 - l)z - e_1 - \frac{e_2}{z}$. *Then*

$$q_{\mathfrak{h}}^{\mathfrak{e}}(z) = \left(I + p_{\mathfrak{e}}(z) M_{21}(z)^2 \begin{pmatrix} q_{\mathfrak{h},\sigma_1}(z) \\ 1 \end{pmatrix} \begin{pmatrix} -1 & q_{\mathfrak{h},\sigma_1}(z) \end{pmatrix} \right) \star q_{\mathfrak{h}}(z) \tag{5.7}$$

and

$$q_{\mathfrak{h}}^{\mathfrak{e}}(z) = \frac{\big(q_{\mathfrak{h},\sigma_1}(z) - q_{\mathfrak{h}}(z)\big)^2 p_{\mathfrak{e}}(z) M_{21}(z)^2}{\big(q_{\mathfrak{h},\sigma_1}(z) - q_{\mathfrak{h}}(z)\big) p_{\mathfrak{e}}(z) M_{21}(z)^2 + 1}\,.$$

Proof. To see the first formula compute:

$$\begin{aligned} M \begin{pmatrix} 1 & p_{\mathfrak{e}} \\ 0 & 1 \end{pmatrix} M^{-1} &= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 1 & p_{\mathfrak{e}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} \\ &= \begin{pmatrix} M_{11} & p_{\mathfrak{e}} M_{11} + M_{12} \\ M_{21} & p_{\mathfrak{e}} M_{21} + M_{22} \end{pmatrix} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} \\ &= \begin{pmatrix} 1 - p_{\mathfrak{e}} M_{11} M_{21} & p_{\mathfrak{e}} M_{11}^2 \\ -p_{\mathfrak{e}} M_{21}^2 & 1 + p_{\mathfrak{e}} M_{11} M_{21} \end{pmatrix} \\ &= I + p_{\mathfrak{e}} \begin{pmatrix} M_{11} \\ M_{21} \end{pmatrix} \begin{pmatrix} -M_{21} & M_{11} \end{pmatrix} \\ &= I + p_{\mathfrak{e}} M_{21}^2 \begin{pmatrix} q_{\mathfrak{h},\sigma_1} \\ 1 \end{pmatrix} \begin{pmatrix} -1 & q_{\mathfrak{h},\sigma_1} \end{pmatrix}. \end{aligned}$$

In the last line it was used that

$$\frac{M_{11}(z)}{M_{21}(z)} = \frac{\lim_{t\nearrow\sigma_1} \frac{\upsilon_{11}(t,z)}{\alpha(t)}}{\lim_{t\nearrow\sigma_1} \frac{\upsilon_{21}(t,z)}{\alpha(t)}} = q_{\mathfrak{h},\sigma_1}(z).$$

In order to show the second formula, we compute further:

$$\left(I + p_{\mathfrak{e}} M_{21}^2 \begin{pmatrix} q_{\mathfrak{h},\sigma_1} \\ 1 \end{pmatrix} \begin{pmatrix} -1 & q_{\mathfrak{h},\sigma_1} \end{pmatrix} \right) = \begin{pmatrix} 1 - p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h},\sigma_1} & p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h},\sigma_1}^2 \\ -p_{\mathfrak{e}} M_{21}^2 & 1 + p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h},\sigma_1} \end{pmatrix}$$

and hence

$$q_{\mathfrak{h}}^{\mathfrak{e}} - q_{\mathfrak{h}} = \frac{\big(1 - p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h},\sigma_1}\big) q_{\mathfrak{h}} + p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h},\sigma_1}^2}{-p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h}} + \big(1 + p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h},\sigma_1}\big)} - q_{\mathfrak{h}} =$$

$$= \frac{\big(1 - p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h},\sigma_1}\big) q_{\mathfrak{h}} + p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h},\sigma_1}^2 + p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h}}^2 - \big(1 + p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h},\sigma_1}\big) q_{\mathfrak{h}}}{-p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h}} + 1 + p_{\mathfrak{e}} M_{21}^2 q_{\mathfrak{h},\sigma_1}} =$$

$$= \frac{p_{\mathfrak{e}} M_{21}^2 \big(q_{\mathfrak{h},\sigma_1} - q_{\mathfrak{h}}\big)^2}{p_{\mathfrak{e}} M_{21}^2 \big(q_{\mathfrak{h},\sigma_1} - q_{\mathfrak{h}}\big) + 1}\,.$$

❑

The functions $q_{\mathfrak{h}}^{\mathfrak{e}}$ can be used to describe the set of all Weyl coefficients $q_{\hat{\mathfrak{h}}}$ of indefinite Hamiltonians which differ from $\mathfrak{h}$ only in the scalar parameters d_0, d_1, $\ddot{o}$, b_1, and b_2 (in case $\ddot{o} = 1$). The following theorem is the main result of this paper.

5.4 Theorem. *Let $\mathfrak{h} = (H, \mathfrak{b}, \mathfrak{d})$ be an indefinite Hamiltonian in limit point case with negative index 1 and Weyl coefficient $q_{\mathfrak{h}}$, where H is defined on $[\sigma_0, \sigma_1) \cup (\sigma_1, \sigma_2)$ with a singularity at σ_1. Assume without loss of generality that $\int_{\sigma_0}^{\sigma_1} (1,0)H(t)\binom{1}{0}\, dt < \infty$. Let $\upsilon(t,z)$ be the solution of the initial value problem*

$$\frac{\partial}{\partial t}\upsilon(t,z)J = z\upsilon(t,z)H(t),\ t \in [\sigma_0, \sigma_1), \qquad \upsilon(\sigma_0, z) = I\,, \tag{5.8}$$

and let M and $q_{\mathfrak{h}}^{\mathfrak{e}}$ be defined by (5.4) *and* (5.6). *Moreover, let $\mathcal{W}_H$ denote the set of all Weyl coefficients $q_{\hat{\mathfrak{h}}}$ of indefinite Hamiltonians $\hat{\mathfrak{h}} = (\hat{H}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}})$, $\operatorname{ind}_- \hat{\mathfrak{h}} = 1$, with $\hat{H} = H$.*

Case 1: *Assume that either for all $s_- \in [\sigma_0, \sigma_1)$ the interval (s_-, σ_1) is not indivisible, or that for all $s_+ \in (\sigma_1, \sigma_2)$ the interval (σ_1, s_+) is not invisible. Then the assignment $\mathfrak{e} \mapsto q_{\mathfrak{h}}^{\mathfrak{e}}$ maps $\mathbb{R} \times \mathbb{R} \times [0, \infty)$ bijectively onto $\mathcal{W}_H$.*

Case 2: *Assume that there exist $s_- \in [\sigma_0, \sigma_1)$ and $s_+ \in (\sigma_1, \sigma_2)$ such that both intervals (s_-, σ_1) and (σ_1, s_+) are maximal indivisible. Then the assignment $\mathfrak{e} \mapsto q_{\mathfrak{h}}^{\mathfrak{e}}$ is a bijection of*

$$\big(\mathbb{R} \times \mathbb{R} \times [0,\infty)\big) \setminus \begin{cases} \{-b_1\} \times (-\infty, d_0] \times \{0\} & \text{if } \ddot{o} = 0, \\ \{-b_2\} \times (-\infty, d_0] \times \{0\} & \text{if } \ddot{o} = 1, \end{cases}$$

onto $\mathcal{W}_H$.

If $\ddot{o} = 0$ and $\mathfrak{e} \in \{-b_1\} \times (-\infty, d_0] \times \{0\}$ or $\ddot{o} = 1$ and $\mathfrak{e} \in \{-b_2\} \times (-\infty, d_0] \times \{0\}$, then $q_{\mathfrak{h}}^{\mathfrak{e}}$ is the Weyl coefficient of the positive definite Hamiltonian

$$H_{\mathfrak{e}}(t) := \begin{cases} H(t+s_- - s_+ + (d_0 - e_2)), & t \in (\sigma_0 - s_- + s_+ - (d_0 - e_2), s_+ - (d_0 - e_2)), \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & t \in (s_+ - (d_0 - e_2), s_+), \\ H(t), & t \geq s_+. \end{cases}$$

The proof of this result is carried out in several steps. In the first three steps we deal with Case 1. Without loss of generality we assume that $\sigma_0 = 0$, $\sigma_2 = \infty$ and set $\sigma := \sigma_1$.

Step 1: Construction of chains with Weyl coefficient $q_{\mathfrak{h}}^{\mathfrak{e}}$ (Case 1)

Let $\mathfrak{e} \in \mathbb{R} \times \mathbb{R} \times [0, \infty)$ be given, and let S and $\omega_{\mathfrak{e}}$ be defined as in Definition 4.3. Define

$$\upsilon_{\mathfrak{e}}(t) := \mathfrak{T}_{-1}(\omega_{\mathfrak{e}}) \uplus (\upsilon_{St})_{t \in [S,\infty)}\,, \tag{5.9}$$

and assume that $\upsilon_{\mathfrak{e}}$ is parameterized such that

$$\upsilon_{\mathfrak{e}}(t) = \begin{cases} \upsilon(t), & t \in [0, \sigma), \\ \mathcal{T}_{-1}(\omega_{\mathfrak{e}})(t), & t \in (\sigma, S], \\ \mathcal{T}_{-1}(\omega_{\mathfrak{e}})(S)\upsilon_{St}, & t \in (S, \infty). \end{cases}$$

Let $\sigma < t \leq s \leq S$; then

$$\mathcal{T}_{-1}(\omega_{\mathfrak{e}})(t)^{-1}\mathcal{T}_{-1}(\omega_{\mathfrak{e}})(s) = \mathcal{T}_{-1}(\omega)(t)^{-1}\mathcal{T}_{-1}(\omega)(s) = \upsilon(t)^{-1}\upsilon(s)\,.$$

¿From the definition of $\upsilon_{\mathfrak{e}}$ it is now immediate that

$$\upsilon_{\mathfrak{e}}(t)^{-1}\upsilon_{\mathfrak{e}}(s) = \upsilon(t)^{-1}\upsilon(s), \quad \sigma < t \leq s < \infty\,.$$

Since υ is a maximal chain, the interval (σ,∞) is not indivisible. Since we assume that Case 1 prevails, in particular the singularity σ of $\mathfrak{T}_{-1}(\omega_{\mathfrak{e}})$ cannot lie in an indivisible interval with negative length. It follows that we can apply Remark 2.8, and conclude that $\upsilon_{\mathfrak{e}} \in \mathfrak{M}_1$.

Let q be the Weyl coefficient of the chain $\upsilon_{\mathfrak{e}}$, and let $\mathfrak{h}_{\mathfrak{e}}$ be the indefinite Hamiltonian with Weyl coefficient q. Since υ and $\upsilon_{\mathfrak{e}}$ have the same transfer matrices, they satisfy equation (1.1) with the same H between the singularities; hence $\mathfrak{h}_{\mathfrak{e}}$ is of the form $\mathfrak{h}_{\mathfrak{e}} = (H, \mathfrak{b}_{\mathfrak{e}}, \mathfrak{d}_{\mathfrak{e}})$.

We will now show that $q = q_{\mathfrak{h}}^{\mathfrak{e}}$. Since $\upsilon_{\mathfrak{e},ts} = \upsilon_{ts}$ for $\sigma < t \leq s < \infty$, we have by Lemma 3.3 that

$$\mathcal{T}_1(\upsilon_{\mathfrak{e}}(t))^{-1}\mathcal{T}_1(\upsilon_{\mathfrak{e}}(s)) = A(t)^{-1}\mathcal{T}_1(\upsilon(t))^{-1}\mathcal{T}_1(\upsilon(s))A(s) \tag{5.10}$$

whenever all transforms are defined, and where $A(t)$ is equal to

$$\begin{pmatrix} \frac{\alpha(\mathcal{T}_1(\upsilon_{\mathfrak{e}}(t)),-1)}{\alpha(\mathcal{T}_1(\upsilon(t)),-1)} & \frac{\beta(\mathcal{T}_1(\upsilon_{\mathfrak{e}}(t)),-1)-\beta(\mathcal{T}_1(\upsilon(t)),-1)}{\alpha(\mathcal{T}_1(\upsilon(t)),-1)\alpha(\mathcal{T}_1(\upsilon_{\mathfrak{e}}(t)),-1)} - \frac{1}{z}\Big(\frac{1}{\alpha(\mathcal{T}_1(\upsilon_{\mathfrak{e}}(t)),-1)} - \frac{1}{\alpha(\mathcal{T}_1(\upsilon(t)),-1)}\Big) \\ 0 & \frac{\alpha(\mathcal{T}_1(\upsilon(t)),-1)}{\alpha(\mathcal{T}_1(\upsilon_{\mathfrak{e}}(t)),-1)} \end{pmatrix}$$

Let $\sigma < t \leq S$, then $\mathcal{T}_1(\upsilon_{\mathfrak{e}}(t)) = \omega_{\mathfrak{e}}(t)$ and $\mathcal{T}_1(\upsilon(t)) = \omega(t)$. Hence, by (4.6), (4.7) and (4.8), we have

$$A(t) = \begin{pmatrix} \frac{1}{\lambda(t)} & \varepsilon_1(t) + \frac{\varepsilon_2(t)}{z} \\ 0 & \lambda(t) \end{pmatrix}\,, \quad t \in (\sigma, S]\,,$$

where $\lambda, \varepsilon_1, \varepsilon_2$ are defined by (4.3). From their definition we see that

$$\lim_{t\searrow\sigma} A(t) = \begin{pmatrix} 1 & e_1 + \frac{e_2}{z} \\ 0 & 1 \end{pmatrix}\,. \tag{5.11}$$

Let $\sigma < t \leq s < \infty$. By (5.10) we have

$$\mathcal{T}_1(\upsilon_{\mathfrak{e}}(s)) = \mathcal{T}_1(\upsilon_{\mathfrak{e}}(t))A(t)^{-1}\mathcal{T}_1(\upsilon(t))^{-1}\mathcal{T}_1(\upsilon(s))A(s)\,.$$

Letting t tend to σ from above yields

$$\begin{aligned} \mathcal{T}_1(\upsilon_{\mathfrak{e}}(s)) &= \omega_{\mathfrak{e}}(\sigma)\begin{pmatrix} 1 & -e_1 - \frac{e_2}{z} \\ 0 & 1 \end{pmatrix}\omega(\sigma)^{-1}\mathcal{T}_1(\upsilon(s))A(s) \\ &= \omega_{\mathfrak{e}}(\sigma_-)W_{(e_3,0)}\begin{pmatrix} 1 & -e_1 - \frac{e_2}{z} \\ 0 & 1 \end{pmatrix}W_{(l,0)}^{-1}\omega(\sigma_-)^{-1}\mathcal{T}_1(\upsilon(s))A(s) \\ &= \omega_{\mathfrak{e}}(\sigma_-)\begin{pmatrix} 1 & (e_3 - l)z - e_1 - \frac{e_2}{z} \\ 0 & 1 \end{pmatrix}\omega(\sigma_-)^{-1}\mathcal{T}_1(\upsilon(s))A(s). \end{aligned}$$

It follows from (5.5) and the definition of $\mathcal{T}_1$ that

$$\begin{aligned} &\upsilon_{\mathfrak{e}}(s)\begin{pmatrix} \frac{1}{\alpha(\upsilon_{\mathfrak{e}}(s),1)} & \frac{\beta(\upsilon_{\mathfrak{e}}(s),1)}{\alpha(\upsilon_{\mathfrak{e}}(s),1)} + \frac{1}{z} \\ 0 & \alpha(\upsilon_{\mathfrak{e}}(s),1) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{z} \\ 0 & 1 \end{pmatrix}\mathcal{T}_1(\upsilon_{\mathfrak{e}}(s)) \\ &= \begin{pmatrix} 1 & \frac{1}{z} \\ 0 & 1 \end{pmatrix}\omega_{\mathfrak{e}}(\sigma_-)\begin{pmatrix} 1 & (e_3 - l)z - e_1 - \frac{e_2}{z} \\ 0 & 1 \end{pmatrix}\omega(\sigma_-)^{-1}\mathcal{T}_1(\upsilon(s))A(s) \\ &= M\begin{pmatrix} 1 & (e_3 - l)z - e_1 - \frac{e_2}{z} \\ 0 & 1 \end{pmatrix}M^{-1}\upsilon(s)\begin{pmatrix} \frac{1}{\alpha(\upsilon(s),1)} & \frac{\beta(\upsilon(s),1)}{\alpha(\upsilon(s),1)} + \frac{1}{z} \\ 0 & \alpha(\upsilon(s),1) \end{pmatrix}A(s). \end{aligned}$$

We conclude that

$$\upsilon_{\mathfrak{e}}(s)\star\infty = M\begin{pmatrix}1 & (e_3-l)z-e_1-\frac{e_2}{z}\\ 0 & 1\end{pmatrix}M^{-1}\upsilon(s)\star\infty \tag{5.12}$$

whenever $s\in(\sigma,\infty)$ is such that both of $\upsilon_{\mathfrak{e}}(s)$ and $\upsilon(s)$ belong to $\operatorname{dom}\mathcal{T}_1$. Let $a\in(\sigma,\infty]$ be such that (a,∞) is a maximal indivisible interval of type 0 of the chain υ, and thus also of the chain $\upsilon_{\mathfrak{e}}$. Then

$$q_{\mathfrak{h}}=\lim_{t\nearrow a}\upsilon(t)\star\infty,\qquad q=\lim_{t\nearrow a}\upsilon_{\mathfrak{e}}(t)\star\infty\,.$$

We have

$$\sup\big\{t\in(\sigma,a):\,\upsilon(t)\in\operatorname{dom}\mathcal{T}_1\big\}=\sup\big\{t\in(\sigma,a):\,\upsilon_{\mathfrak{e}}(t)\in\operatorname{dom}\mathcal{T}_1\big\}=a\,,$$

and hence we obtain from (5.12) that $q=q_{\mathfrak{h}}^{\mathfrak{e}}$. ❑

Step 2: Surjectivity (Case 1)

Let $\hat{\mathfrak{h}}$ be an indefinite Hamiltonian with $\operatorname{ind}_-\hat{\mathfrak{h}}=1$ which is of the form $\hat{\mathfrak{h}}=(H,\hat{c},\hat{d})$, and let $\hat{q}$ be its Weyl coefficient. Let $\hat{\upsilon}$ be the maximal chain whose Weyl coefficient is $\hat{q}$, and assume that $\hat{\upsilon}$ is parameterized such that $\operatorname{dom}\hat{\upsilon}=\operatorname{dom}\upsilon$, $\hat{\upsilon}(t)=\upsilon(t)$, $t\in[0,\sigma)$, and $\hat{\upsilon}_{ts}=\upsilon_{ts}$ for $\sigma<t\leq s<\infty$. Choose $\hat{L}\in(\sigma,L]$ such that $\alpha(\hat{\upsilon}(\hat{L}),1)<0$, and put $\hat{\omega}:=\mathfrak{T}_1(\hat{\upsilon}|_{[0,\hat{L}]\setminus\{\sigma\}})$. By Proposition 4.7 there exists $\mathfrak{e}\in\mathbb{R}\times\mathbb{R}\times[0,\infty)$ such that $\hat{\omega}=\omega_{\mathfrak{e}}$. Let $\upsilon_{\mathfrak{e}}$ be the maximal chain constructed in Step 1. We have

$$\upsilon_{\mathfrak{e}}(\hat{L})=\mathcal{T}_{-1}(\omega_{\mathfrak{e}}(\hat{L}))=\mathcal{T}_{-1}(\hat{\omega}(\hat{L}))=\hat{\upsilon}(\hat{L})\,.$$

Since $\upsilon_{\mathfrak{e},\hat{L}t}=\upsilon_{\hat{L}t}=\hat{\upsilon}_{\hat{L}t}$ for all $t\in(\sigma,\infty)$, this shows that $\hat{\upsilon}=\upsilon_{\mathfrak{e}}$. ❑

Step 3: Injectivity (Case 1)

Let $\mathfrak{e}^1=(e_1^1,e_2^1,e_3^1),\mathfrak{e}^2=(e_1^2,e_2^2,e_3^2)\in\mathbb{R}\times\mathbb{R}\times[0,\infty)$, and assume that $q_{\mathfrak{h}}^{\mathfrak{e}^1}=q_{\mathfrak{h}}^{\mathfrak{e}^2}$.

For any $\mathfrak{e}\in\mathbb{R}\times\mathbb{R}\times[0,\infty)$ the number e_3 can be reconstructed from the Weyl coefficient $q_{\mathfrak{h}}^{\mathfrak{e}}$ as the limit

$$e_3=\lim_{y\to+\infty}\frac{1}{iy}\big(M^{-1}\star q_{\mathfrak{h}}^{\mathfrak{e}}(iy)\big)$$

by Proposition 5.1 and (5.6). We conclude that in the present situation $e_3^1=e_3^2$.

Let $\upsilon^1,\upsilon^2\in\mathfrak{M}_1$ be the corresponding maximal chains and assume that they are parameterized such that

$$\upsilon^1(t)=\upsilon(t)=\upsilon^2(t),\ t\in[0,\sigma)\,,$$

and

$$\upsilon^1_{ts}=\upsilon_{ts}=\upsilon^2_{ts},\ \sigma<t\leq s<\infty\,.$$

Since these chains have the same Weyl coefficient, there exists a continuous and increasing bijection ϕ of $[0,\sigma)\cup(\sigma,\infty)$ onto itself, such that $\upsilon^2=\upsilon^1\circ\varphi$. It follows that, for $\sigma<t\leq s<\infty$,

$$\upsilon_{ts}=\upsilon^2_{ts}=\upsilon^1_{\varphi(t)\varphi(s)}=\upsilon_{\varphi(t)\varphi(s)}\,. \tag{5.13}$$

In particular, this implies that

$$\mathfrak{t}(\upsilon(\varphi(s))) - \mathfrak{t}(\upsilon(\varphi(t))) = \mathfrak{t}(\upsilon(s)) - \mathfrak{t}(\upsilon(t))\,,$$

and hence the number

$$\gamma := \mathfrak{t}(\upsilon(\varphi(t))) - \mathfrak{t}(\upsilon(t))$$

does not depend on $t \in (\sigma, \infty)$.

Consider the case that $\gamma = 0$. Then it follows that $\upsilon(\varphi(t)) = \upsilon(t)$, and hence that $\varphi = \mathrm{id}$, i.e., $\upsilon^1 = \upsilon^2$. We see from (5.11) that this implies $e_1^1 = e_1^2$ and $e_2^1 = e_2^2$.

Assume now that $\gamma \neq 0$. We shall derive a contradiction. Assume without loss of generality that $\gamma > 0$. Then we always have $\varphi(t) > t$. Since H_1 satisfies the Hilbert–Schmidt condition, cf. [28, §2.3], there exists $\phi \in [0, \pi)$ such that $(\cos\phi, \sin\phi) H_1(t) (\cos\phi, \sin\phi)^T$ is integrable at σ. With N_α defined in (2.7) it follows that

$$\big(N_{\phi+\frac{\pi}{2}} \upsilon(t) N_{-\phi-\frac{\pi}{2}}\big)'_{21}(0)$$

remains bounded when t tends to σ. However, by (5.13),

$$\upsilon(\varphi^{-n}(0)) = \upsilon(0)\upsilon_{0,\varphi^{-1}(0)} = \dots$$

$$\dots = \upsilon(0)\upsilon_{0,\varphi^{-1}(0)} \cdots \upsilon_{\varphi^{-n+1}(0)\varphi^{-n}(0)} = \upsilon(0)\upsilon_{0\varphi(0)}^{-n}\,.$$

It follows that

$$\big(N_{\phi+\frac{\pi}{2}} \upsilon_{0\varphi(0)} N_{-\phi-\frac{\pi}{2}}\big)'_{21}(0) = 0\,,$$

and hence that $\upsilon_{0,\varphi(0)} = W_{(v,\phi+\frac{\pi}{2})}$ for some $v > 0$.

Let $\sigma < t \leq s < \infty$ be given. Since $\mathfrak{t}(\upsilon(\varphi^n(0))) = \mathfrak{t}(\upsilon(0)) + n\gamma$, this number tends to $\pm\infty$ if $n \to \pm\infty$, respectively. Hence there exist $n_-, n_+ \in \mathbb{Z}$ such that $\varphi^{n_-}(0) \leq t$ and $s \leq \varphi^{n_+}(0)$. Thus we have

$$W_{(n_+-n_-,\phi+\frac{\pi}{2})} = \upsilon_{\varphi^{n_-}(0)\varphi^{n_+}(0)} = \upsilon_{\varphi^{n_-}(0)t}\upsilon_{ts}\upsilon_{s\varphi^{n_+}(0)}\,,$$

where all three factors belong to $\mathcal{M}_0$. This, however, implies that each of these factors, in particular υ_{ts}, is of the form $W_{(u,\phi+\frac{\pi}{2})}$ with some $u \geq 0$. We have reached a contradiction, since the whole interval (σ, ∞) cannot be indivisible. ❑

In order to settle Case 2, we first start with a particular case which is accessible to explicit computation.

Step 4: Case 2 and $[0, \sigma)$ *indivisible*

Assume that $[0, \sigma)$ is indivisible. Then the chain $\upsilon(t)$ is given on the interval $[0, \sigma)$ as

$$\upsilon(t) = \begin{pmatrix} 1 & 0 \\ -\gamma(t) z & 1 \end{pmatrix}$$

with some increasing function $\gamma(t)$ with $\gamma(0) = 0$ and $\lim_{t\nearrow\sigma} \gamma(t) = +\infty$. It follows that

$$\alpha(t) = 1 + \gamma(t),\ \beta(t) = 0$$

and

$$M = \lim_{t\nearrow\sigma} \begin{pmatrix} 1 & 0 \\ -\gamma(t)z & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1+\gamma(t)} & \frac{1}{z} \\ 0 & 1+\gamma(t) \end{pmatrix}$$
$$= \lim_{t\nearrow\sigma} \begin{pmatrix} \frac{1}{1+\gamma(t)} & \frac{1}{z} \\ -\frac{\gamma(t)}{1+\gamma(t)}z & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{z} \\ -z & 1 \end{pmatrix}. \tag{5.14}$$

This yields (with $p_{\mathfrak{e}}(z) := (e_3 - l)z - e_1 - \frac{e_2}{z}$)

$$q_{\mathfrak{h}}^{\mathfrak{e}} = M \begin{pmatrix} 1 & p_{\mathfrak{e}} \\ 0 & 1 \end{pmatrix} M^{-1} \star q_{\mathfrak{h}} = \begin{pmatrix} 1 & 0 \\ -z^2 p_{\mathfrak{e}} & 1 \end{pmatrix} \star q_{\mathfrak{h}}$$
$$= \frac{q_{\mathfrak{h}}}{-z^2 p_{\mathfrak{e}} q_{\mathfrak{h}} + 1} = \frac{-1}{-\frac{1}{q_{\mathfrak{h}}} + [(e_3 - l)z^3 - e_1 z^2 - e_2 z]}. \tag{5.15}$$

Now we use the assumption that also to the right of σ there is an indivisible interval. Let s_+ be the right endpoint of the maximal indivisible interval to the right of σ, i.e., $s_+ = \sup\{s > \sigma : (\sigma, s) \text{ indivisible}\} > \sigma$. Then, by the definition of the maximal chain associated with an indefinite Hamiltonian (see [29]),

$$\upsilon(s_+) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -zd_0 + z^2 b_1 & 1 \end{pmatrix}, & \ddot{o} = 0, \\ \begin{pmatrix} 1 & 0 \\ -zd_0 + z^2 b_2 + z^3 b_1 & 1 \end{pmatrix}, & \ddot{o} = 1. \end{cases}$$

Moreover, the chain $\upsilon(t)$ is given on the interval $(\sigma, s_+]$ as

$$\upsilon(t) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -z(d_0 + \hat{\gamma}(t)) + z^2 b_1 & 1 \end{pmatrix}, & \ddot{o} = 0, \\ \begin{pmatrix} 1 & 0 \\ -z(d_0 + \hat{\gamma}(t)) + z^2 b_2 + z^3 b_1 & 1 \end{pmatrix}, & \ddot{o} = 1, \end{cases}$$

with some increasing function $\hat{\gamma}(t)$ with $\hat{\gamma}(s_+) = 0$ and $\lim_{t\searrow\sigma} \hat{\gamma}(t) = -\infty$.

We compute from (3.3):

$$\mathfrak{t}\big(\mathcal{T}_1(\upsilon(t))\big) = \frac{\gamma(t)}{1+\gamma(t)},$$

for $t < \sigma$ and

$$\mathfrak{t}\big(\mathcal{T}_1(\upsilon(t))\big) = \begin{cases} \dfrac{-b_1^2 + d_0 + \hat{\gamma}(t)}{1 + d_0 + \hat{\gamma}(t)}, & \ddot{o} = 0, \\ \dfrac{-b_2^2 + d_0 + \hat{\gamma}(t)}{1 + d_0 + \hat{\gamma}(t)} - b_1, & \ddot{o} = 1 \end{cases}$$

for $t > \sigma$. It follows from this, (5.2), $\lim_{t\nearrow\sigma} \gamma(t) = \infty$ and $\lim_{t\searrow\sigma} \hat{\gamma}(t) = -\infty$ that

$$l = \begin{cases} 0, & \ddot{o} = 0, \\ -b_1, & \ddot{o} = 1. \end{cases}$$

Let $q \in \mathcal{N}_0$ be the Weyl coefficient of the positive definite maximal chain $(\upsilon_{s_+,t})_{t\geq s_+}$, and put

$$p(z) := \begin{cases} d_0 z - b_1 z^2, & \ddot{o} = 0, \\ d_0 z - b_2 z^2 - b_1 z^3, & \ddot{o} = 1. \end{cases}$$

Then

$$q_{\mathfrak{h}} = \frac{q}{-pq+1} = \frac{-1}{\frac{-1}{q} + p},$$

and using (5.15) we get

$$q_{\mathfrak{h}}^{\mathfrak{e}} = \frac{-1}{\frac{-1}{q} + p + [(e_3 - l)z^3 - e_1 z^2 - e_2 z]}.$$

Since

$$p + [(e_3 - l)z^3 - e_1 z^2 - e_2 z] = \begin{cases} e_3 z^3 - (e_1 + b_1)z^2 + (d_0 - e_2)z & , \ddot{o} = 0 \\ e_3 z^3 - (e_1 + b_2)z^2 + (d_0 - e_2)z & , \ddot{o} = 1 \end{cases}$$

we conclude that

$$q_{\mathfrak{h}}^{\mathfrak{e}} \in \mathcal{N}_0 \iff \begin{cases} e_3 = 0, e_1 = -b_1, e_2 \leq d_0 & , \ddot{o} = 0 \\ e_3 = 0, e_1 = -b_2, e_2 \leq d_0 & , \ddot{o} = 1 \end{cases}$$

Note here that, since s_+ is not left endpoint of an indivisible interval of type $\frac{\pi}{2}$, we have $\lim_{y\to+\infty} \frac{1}{y}\frac{-1}{q(iy)} = 0$.

Consider the case that $q_{\mathfrak{h}}^{\mathfrak{e}} \in \mathcal{N}_0$. Then

$$q_{\mathfrak{h}}^{\mathfrak{e}} = \frac{-1}{\frac{-1}{q} + (d_0 - e_2)z} = \frac{q}{-(d_0 - e_2)zq + 1} = \begin{pmatrix} 1 & 0 \\ -(d_0 - e_2)z & 1 \end{pmatrix} \star q.$$

Hence $q_{\mathfrak{h}}^{\mathfrak{e}}$ is the Weyl coefficient of the positive definite Hamiltonian

$$H_{\mathfrak{e}}(t) := \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & t \in (s_+ - (d_0 - e_2), s_+), \\ H(t), & t \geq s_+. \end{cases}$$

Next consider the case that $q_{\mathfrak{h}}^{\mathfrak{e}} \in \mathcal{N}_1$. Again the maximal chain whose Weyl coefficient is equal to $q_{\mathfrak{h}}^{\mathfrak{e}}$ can be guessed easily. We have

$$q_{\mathfrak{h}}^{\mathfrak{e}} = \begin{pmatrix} 1 & 0 \\ -(p + [(e_3 - l)z^3 - e_1 z^2 - e_2 z]) & 1 \end{pmatrix} \star q,$$

which implies that $p + [(e_3 - l)z^3 - e_1 z^2 - e_2 z] \in \mathcal{N}_1$. Hence the maximal chain with Weyl coefficient $q_{\mathfrak{h}}^{\mathfrak{e}}$ is given by

$$\upsilon_{\mathfrak{e}} = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\gamma(t)z & 1 \end{pmatrix}, & t \in [0, \sigma), \\ \begin{pmatrix} 1 & 0 \\ -\big(p + [(e_3 - l)z^3 - e_1 z^2 - e_2 z]\big) - \hat{\gamma}(t)z & 1 \end{pmatrix}, & t \in (\sigma, s_+], \\ \begin{pmatrix} 1 & 0 \\ -\big(p + [(e_3 - l)z^3 - e_1 z^2 - e_2 z]\big) & 1 \end{pmatrix} \cdot \upsilon_{s_+ t}, & t \geq s_+ \end{cases}$$

because s_+ is not left endpoint of an indivisible interval of type $\frac{\pi}{2}$. Moreover, the indefinite Hamiltonian $\mathfrak{h}_{\mathfrak{e}}$ corresponding to this chain is given by $\mathfrak{h}_{\mathfrak{e}} = (H, \mathfrak{b}_{\mathfrak{e}}, \mathfrak{d}_{\mathfrak{e}})$ where

$$d_{\mathfrak{e},0} = d_0 - e_2, \qquad d_{\mathfrak{e},1} = 0, \qquad \ddot{o}_{\mathfrak{e}} = \begin{cases} 0, & e_3 = 0, \\ 1, & e_3 > 0. \end{cases}$$

$$b_{\mathfrak{e},1} = \begin{cases} e_1 + b_1, & \ddot{o}_{\mathfrak{e}} = 0, \ddot{o} = 0, \\ e_1 + b_2, & \ddot{o}_{\mathfrak{e}} = 0, \ddot{o} = 1, \\ -e_3, & \ddot{o}_{\mathfrak{e}} = 1, \end{cases} \qquad b_{\mathfrak{e},2} = \begin{cases} e_1 + b_1, & \ddot{o}_{\mathfrak{e}} = 1, \ddot{o} = 0, \\ e_1 + b_2, & \ddot{o}_{\mathfrak{e}} = 1, \ddot{o} = 1. \end{cases}$$

We see that, if $\mathfrak{e}$ runs through the set

$$\mathbb{R} \times \mathbb{R} \times [0, \infty) \setminus \begin{cases} \{-b_1\} \times (-\infty, d_0] \times \{0\}, & \ddot{o} = 0, \\ \{-b_2\} \times (-\infty, d_0] \times \{0\}, & \ddot{o} = 1, \end{cases}$$

then $\mathfrak{h}_{\mathfrak{e}}$ runs through all possible indefinite Hamiltonians of the form $(H, \hat{\mathfrak{b}}, \hat{\mathfrak{d}})$. ❑

We will use the following general observation to reduce Case 2 to the just treated situation.

Step 5: $q \mapsto q^{\mathfrak{e}}$ is compatible with cutting off

Assume that $s_- \in [0, \sigma)$ is not inner point of an indivisible interval. Then we can consider the maximal chain

$$\tilde{\upsilon}(t) := \upsilon_{s_-,t}, \quad t \in [s_-, \sigma) \cup (\sigma, \infty)$$

and the corresponding indefinite Hamiltonian $\tilde{\mathfrak{h}}$. Its Weyl coefficient $q_{\tilde{\mathfrak{h}}}$ is equal to $\upsilon(s_-)^{-1} \star q_{\mathfrak{h}}$. We shall prove that

$$q^{\mathfrak{e}}_{\tilde{\mathfrak{h}}} = \upsilon(s_-)^{-1} \star q^{\mathfrak{e}}_{\mathfrak{h}}, \quad \mathfrak{e} \in \mathbb{R} \times \mathbb{R} \times [0, \infty)\,.$$

Let α, β, M be defined by (5.3) and (5.4), respectively, and let $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{M}$ be defined correspondingly for the chain $\tilde{\upsilon}$ instead of υ.

We compute

$$\upsilon(t) \begin{pmatrix} \frac{1}{\alpha(t)} & \frac{\beta(t)}{\alpha(t)} + \frac{1}{z} \\ 0 & \alpha(t) \end{pmatrix} = \upsilon(s_-)\tilde{\upsilon}(t) \begin{pmatrix} \frac{1}{\alpha(t)} & \frac{\beta(t)}{\alpha(t)} + \frac{1}{z} \\ 0 & \alpha(t) \end{pmatrix} =$$

$$= \upsilon(s_-)\tilde{\upsilon}(t) \begin{pmatrix} \frac{1}{\tilde{\alpha}(t)} & \frac{\tilde{\beta}(t)}{\tilde{\alpha}(t)} + \frac{1}{z} \\ 0 & \tilde{\alpha}(t) \end{pmatrix} \begin{pmatrix} \frac{\tilde{\alpha}(t)}{\alpha(t)} & [\frac{\tilde{\alpha}(t)}{\alpha(t)}\beta(t) - \frac{\alpha(t)}{\tilde{\alpha}(t)}\tilde{\beta}(t)] + \frac{1}{z}[\tilde{\alpha}(t) - \alpha(t)] \\ 0 & \frac{\alpha(t)}{\tilde{\alpha}(t)} \end{pmatrix}.$$

We see that the last matrix on the right-hand side of this relation possesses a limit B for $t \nearrow \sigma$, and that $M = \upsilon(s_-)\tilde{M}B$.

We have $\alpha(t) = \tilde{\alpha}(t) - \upsilon(s_-)'_{21}(0)$. Since $\lim_{t \nearrow \sigma} \alpha(t) = +\infty$, the relation

$$\lim_{t \nearrow \sigma} \frac{\alpha(t)}{\tilde{\alpha}(t)} = 1$$

holds. It follows that B is of the form

$$B = \begin{pmatrix} 1 & \gamma + \delta\frac{1}{z} \\ 0 & 1 \end{pmatrix}$$

with some $\gamma, \delta \in \mathbb{C}$. Put again $p_{\mathfrak{e}}(z) = (e_3 - l)z - e_1 - \frac{e_2}{z}$. We obtain

$$q_{\mathfrak{h}}^{\mathfrak{e}} = M \begin{pmatrix} 1 & p_{\mathfrak{e}} \\ 0 & 1 \end{pmatrix} M^{-1} \star q_{\mathfrak{h}} = \upsilon(s_-)\tilde{M}B \begin{pmatrix} 1 & p_{\mathfrak{e}} \\ 0 & 1 \end{pmatrix} B^{-1}\tilde{M}^{-1}\upsilon(s_-)^{-1} \star q_{\mathfrak{h}} =$$

$$= \upsilon(s_-)\tilde{M} \begin{pmatrix} 1 & p_{\mathfrak{e}} \\ 0 & 1 \end{pmatrix} \tilde{M}^{-1} \star \big(\underbrace{\upsilon(s_-)^{-1} \star q_{\mathfrak{h}}}_{=q_{\tilde{\mathfrak{h}}}} \big) = \upsilon(s_-) \star q_{\tilde{\mathfrak{h}}}^{\mathfrak{e}} .$$

❑

Step 6: Finishing Case 2

Assume that $s_- := \inf\{s \in [0, \sigma) : (s, \sigma) \text{ indivisible}\} > 0$ and let $\tilde{\mathfrak{h}}$ be as in Step 5. Let $\mathfrak{h}_1$ be a general Hamiltonian with negative index 1 of the form $\mathfrak{h}_1 = (H, \mathfrak{b}_1, \mathfrak{d}_1)$. Then, by Step 4, the Weyl coefficient $q_{\tilde{\mathfrak{h}}_1}$ of $\tilde{\mathfrak{h}}_1 := (H|_{[s_-,\sigma)\cup(\sigma,\infty)}, \mathfrak{b}_1, \mathfrak{d}_1)$ can be written as $q_{\tilde{\mathfrak{h}}}^{\mathfrak{e}}$ with a unique triple $\mathfrak{e}$. It follows from Step 5 that

$$q_{\mathfrak{h}_1} = \upsilon(s_-) \star q_{\tilde{\mathfrak{h}}_1} = \upsilon(s_-) \star q_{\tilde{\mathfrak{h}}}^{\mathfrak{e}} = q_{\mathfrak{h}}^{\mathfrak{e}} ;$$

in particular $q_{\tilde{\mathfrak{h}}_1}$ must belong to $\mathcal{N}_1$.

Conversely, let $\mathfrak{e}$ be in the set of parameters described in Step 4, so that $q_{\tilde{\mathfrak{h}}}^{\mathfrak{e}} \in \mathcal{N}_1$. Then the general Hamiltonian whose Weyl coefficient equals $q_{\tilde{\mathfrak{h}}}^{\mathfrak{e}}$ is of the form $(H|_{[s_-,\sigma)\cup(\sigma,\infty)}, \mathfrak{b}_{\mathfrak{e}}, \mathfrak{d}_{\mathfrak{e}})$. Since s_- is not inner point of an indivisible interval in H, the maximal chain with Weyl coefficient $\upsilon(s_-) \star q_{\tilde{\mathfrak{h}}}^{\mathfrak{e}} = q_{\mathfrak{h}}^{\mathfrak{e}}$ corresponds to the general Hamiltonian $(H, \mathfrak{b}_{\mathfrak{e}}, \mathfrak{d}_{\mathfrak{e}})$.

If $\mathfrak{e}$ is a parameter such that $q_{\tilde{\mathfrak{h}}}^{\mathfrak{e}} \in \mathcal{N}_0$, then clearly $\upsilon(s_-) \star q_{\tilde{\mathfrak{h}}}^{\mathfrak{e}} \in \mathcal{N}_0$, and it is the Weyl coefficient of the positive definite Hamiltonian given in Theorem 5.4.

❑

All assertions of Theorem 5.4 are proved ✌

Let us point out one particular case.

5.5 Corollary. *Let $\mathfrak{h}$ be as in Theorem 5.4 and assume that $[\sigma_0, \sigma_1)$ is indivisible. Then the function $q_{\mathfrak{h}}^{\mathfrak{e}}$ can be written as*

$$q_{\mathfrak{h}}^{\mathfrak{e}} = \frac{-1}{-\frac{1}{q_{\mathfrak{h}}} + [(e_3 - l)z^3 - e_1 z^2 - e_2 z]} ,$$

and

$$l = \lim_{y \to +\infty} \frac{1}{iy^3 q_{\mathfrak{h}}(iy)} .$$

Proof. The first formula is just (5.15). ¿From Proposition 5.1 and (5.14) we get

$$\tau = M^{-1} \star q_{\mathfrak{h}} = \begin{pmatrix} 1 & -\frac{1}{z} \\ z & 0 \end{pmatrix} \star q_{\mathfrak{h}} = \frac{q_{\mathfrak{h}} - 1/z}{z q_{\mathfrak{h}}} = \frac{1}{z} - \frac{1}{z^2 q_{\mathfrak{h}}}$$

and hence

$$l = \lim_{y \to +\infty} \frac{1}{iy} \tau(iy) = \lim_{y \to +\infty} \frac{1}{iy^3 q_{\mathfrak{h}}(iy)} .$$

by Proposition 5.1.

❑

We want to illustrate the above results with two examples.

5.6 Example. Consider the Bessel equation, this is the equation

$$-y''(t) + \frac{\nu^2 - \frac{1}{4}}{t^2} y(t) = \lambda y(t), \quad t \in (0, \infty), \tag{5.16}$$

where ν is a nonnegative parameter; a classical and well-studied object. For a discussion of this equation and corresponding integral transforms, see, e.g. [36, 12, 11, 34]. Recently, also some attempts were made to use indefinite inner product structures in its study, see [10, 14, 15, 31].

At the point ∞ always limit point case prevails. At the point 0 we have limit circle case if and only if $\nu < 1$, and for such values of ν the Weyl coefficient $m(\lambda)$ is given by

$$m(\lambda) = -\frac{1}{c} \lambda^{\nu} ,$$

where $c := 2^{2\nu-1}\pi^{-1}\Gamma(\nu)^2 \sin\nu \cdot e^{i\nu\pi}$. Moreover, it is known that the self-adjoint realizations of (5.16) show a nice behaviour, regardless whether the equation is in the limit circle or limit point case at 0.

For $\nu < 1$, the Bessel equation can be transformed into a canonical system (1.1). In fact, if we put

$$x_1(t) = \frac{1}{z} t^{-\frac{\alpha}{2}} \Big(y'(t) + \frac{\alpha}{2t} y(t) \Big), \quad x_2(t) = t^{\frac{\alpha}{2}} y(t), \quad z^2 = \lambda , \tag{5.17}$$

then we obtain a canonical system with Hamiltonian

$$H_\alpha(t) = \begin{pmatrix} t^\alpha & 0 \\ 0 & t^{-\alpha} \end{pmatrix} \tag{5.18}$$

where $\alpha = 2\nu - 1$.

Consider now the case that $\nu \geq 1$. Following our general rule how to rewrite a Sturm–Liouville equation that is limit point at both endpoints as an indefinite canonical system, we should use a general Hamiltonian which has only one singularity, namely 0, and whose Hamiltonian function is defined to the right of 0 by the potential and to the left of 0 just as one indivisible interval. This gives

$$H_\alpha(t) := \begin{cases} \frac{1}{t^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & t \in (-1, 0), \\ \begin{pmatrix} t^\alpha & 0 \\ 0 & t^{-\alpha} \end{pmatrix}, & t \in (0, \infty). \end{cases}$$

In [33] it is shown that this function actually qualifies for being the Hamiltonian function of a general Hamiltonian. Moreover, for a certain choice of parameters $\mathfrak{b}_0$ and $\mathfrak{d}_0$, the corresponding maximal chain of matrices and its Weyl coefficient is computed. It is shown that for $\alpha \in (0, \infty) \setminus (2\mathbb{N} - 1)$ the function $\omega_\alpha(t, z)$ defined as

$$\omega_\alpha(t, z) := \begin{pmatrix} 1 & 0 \\ (1 + \frac{1}{t})z & 1 \end{pmatrix}, \quad t \in (-1, 0),$$

$$\omega_\alpha(t, z) := \begin{pmatrix} 2^{\nu-1}\Gamma(\nu) z^{-\nu+1} t^{-\nu+1} J_{\nu-1}(zt) & 2^{\nu-1}\Gamma(\nu) z^{-\nu+1} t^{\nu} J_{\nu}(zt) \\ -2^{-\nu}\Gamma(1-\nu) z^{\nu} t^{-\nu+1} J_{-\nu+1}(zt) & 2^{-\nu}\Gamma(1-\nu) z^{\nu} t^{\nu} J_{-\nu}(zt) \end{pmatrix},$$
$$t \in (0, \infty),$$

is a maximal chain of matrices with negative index $\kappa = [\frac{\alpha+1}{2}]$ whose corresponding general Hamiltonian $\mathfrak{h}_\alpha$ consists of the Hamiltonian function H_α and some parameters $\mathfrak{b}_0$, $\mathfrak{d}_0$, and whose Weyl coefficient $q_{\mathfrak{h}_\alpha}$ is equal to

$$q_{\mathfrak{h}_\alpha}(z) = cz^{-\alpha}, \quad \operatorname{Im} z > 0, \tag{5.19}$$

where

$$c := \frac{2^\alpha}{\pi}\Big(\Gamma\Big(\frac{\alpha+1}{2}\Big)\Big)^2 \sin\Big(\frac{\alpha+1}{2}\Big) e^{i\frac{\alpha+1}{2}\pi}\,.$$

Here the power $z^{-\alpha}$ is defined such that there is a cut at the negative real axis and $z^{-\alpha}$ is positive for positive z.

If $\alpha \in 2\mathbb{N}-1$, naturally, formulae have to be modified and are getting more complicated. However, also for this case a maximal chain whose corresponding general Hamiltonian has Hamiltonian function H_α is given explicitly in [33]:

$$\omega_\alpha(t,z) = \begin{pmatrix} 2^{\nu-1}\Gamma(\nu)z^{-\nu+1}t^{-\nu+1}J_{\nu-1}(zt) & 2^{\nu-1}\Gamma(\nu)z^{-\nu+1}t^{\nu}J_\nu(zt) \\ \frac{2^{-\nu}}{\Gamma(\nu)}z^\nu t^{-\nu+1}\Big(-\pi Y_{\nu-1}(zt) + 2\log(z)J_{\nu-1}(zt)\Big) & \\ & \frac{2^{-\nu}}{\Gamma(\nu)}z^\nu t^\nu\Big(-\pi Y_\nu(zt) + 2\log(z)J_\nu(zt)\Big) \end{pmatrix}$$

where α and ν are again related by $\alpha = 2\nu - 1$. Its negative index is equal to $\frac{\alpha+1}{2}$, and its Weyl coefficient is

$$q_{\mathfrak{h}_\alpha}(z) = \frac{\hat{c}z^{-\alpha}}{\log(-iz)}, \quad \operatorname{Im} z > 0\,, \tag{5.20}$$

where

$$\hat{c} := 2^{\alpha-1}\Big(\Big(\frac{\alpha-1}{2}\Big)!\Big)^2\,.$$

We see that our present results, Theorem 5.4 and Corollary 5.5, will cover the cases $\alpha \in [1,3)$. For such values of α, the parameters $\mathfrak{b}_0$ and $\mathfrak{d}_0$ leading to the above Weyl coefficients (5.19) and (5.20), are actually given as

$$E := \{-1, t_0\},\ \ddot{o} = 0,\ b_1 = d_1 = 0,\ d_0 = \begin{cases} \frac{1}{1-\alpha}t_0^{1-\alpha}, & \alpha \in (1,3), \\ \ln\frac{t_0}{2} - \gamma, & \alpha = 1, \end{cases}$$

where t_0 is an arbitrary number in $(0,\infty)$ and γ denotes the Euler–Mascheroni constant.

An application of Corollary 5.5 yields that, for $\alpha \in [1,3)$, all possible Titchmarsh–Weyl coefficients of general Hamiltonians with negative index 1 which are of the form $\mathfrak{h} = (H_\alpha, \mathfrak{b}, \mathfrak{d})$, are given by

$$q_{\mathfrak{h}_\alpha}(z) = \begin{cases} \dfrac{1}{-e_3z^3 + e_1z^2 + e_2z + \frac{1}{c}z^\alpha} & \text{if } 1 < \alpha < 3, \\ \dfrac{1}{-e_3z^3 + e_1z^2 + e_2z + z\log(-iz)} & \text{if } \alpha = 1, \end{cases}$$

where $(e_1, e_2, e_3) \in \mathbb{R} \times \mathbb{R} \times [0,\infty)$.

5.7 Example. Consider the equation of Sturm–Liouville type

$$-y''(t) + \frac{2}{(t-1)^2} y(t) = \lambda y(t), \quad t \in [0, \infty) .$$

This equation appeared in [32] in connection with an extension problem of positive definite functions. Apparently the potential has a singularity at the point 1 and is not integrable at this point.

If we considered this equation only on the interval $[0,1)$, then we would have a Sturm–Liouville problem which is regular at 0 and in limit point case at 1. Using a similar transformation like (5.17), this problem could be rewritten as a canonical system (1.1) with Hamiltonian ($t \in [0,1)$)

$$H(t) = \begin{pmatrix} (t-1)^2 & 0 \\ 0 & \frac{1}{(t-1)^2} \end{pmatrix} . \tag{5.21}$$

Let us consider the equation on the whole interval $[0, \infty)$ and proceed according to our method how to associate a general Hamiltonian with a singular potential. Thus we should choose a general Hamiltonian $\mathfrak{h}$ which has one singularity, namely 1, and whose Hamiltonian function is obtained by applying the same transformations as used above for $t \in [0,1)$ also to the right of the singularity. In this way we obtain that the Hamiltonian function of $\mathfrak{h}$ is simply given by the formula (5.21) for all $t \in [0, \infty) \setminus \{1\}$.

Of course it is now unclear how to choose the parameters $\mathfrak{b}$ and $\mathfrak{d}$. For a certain choice, namely for

$$E = \{0, t_0\},\ \ddot{o} = 0,\ b_1 = d_1 = 0,\ d_0 = \frac{t_0}{1-t_0}$$

with $t_0 \in (1, \infty)$, the corresponding maximal chain of matrices $\omega(t,z)$, $t \in [0,1) \cup (1, \infty)$, and its Weyl coefficient $q(z)$ have been computed in [32]. There it is shown that

$$\omega(t,z) = \begin{pmatrix} \dfrac{\sin zt - z \cos zt}{z(t-1)} & \Big(\dfrac{1}{z^2} - (t-1)\Big) \sin zt - \dfrac{t \cos zt}{z} \\ \dfrac{\sin zt}{t-1} & \dfrac{\sin zt}{z} - (t-1) \cos zt \end{pmatrix} \tag{5.22}$$

for $t \in [0,1) \cup (1, \infty)$, and that

$$q_{\mathfrak{h}}(z) = i + \frac{1}{z} .$$

Moreover, it is seen that the negative index of the chain ω is equal to 1.

Next we have to compute the data needed for an application of Theorem 5.4. From (5.22), however, we easily get

$$M(z) = \begin{pmatrix} \cos z - \dfrac{\sin z}{z} & \sin z + \dfrac{\cos z}{z} \\ -\sin z & \cos z \end{pmatrix}$$

and

$$\tau(z) = i,\ l = 0, \quad q_{\mathfrak{h},1}(z) = \frac{1}{z} - \cot z .$$

Hence the totality of all Weyl coefficients of general Hamiltonians with negative index 1 which are of the form $\mathfrak{h} = (H, \mathfrak{b}, \mathfrak{d})$ with H as in (5.21) is

$$q_{\mathfrak{h}}^{\mathfrak{e}}(z) = i + \frac{1}{z} + \frac{\left(e_3 z - e_1 - \frac{e_2}{z}\right)(1 + i \tan z)}{1 - \left(i + e_3 z - e_1 - \frac{e_2}{z}\right) \tan z}$$

where $(e_1, e_2, e_3) \in \mathbb{R} \times \mathbb{R} \times [0, \infty)$.

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