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A Review on Results for the Derrida-Lebowitz-Speer-Spohn Equation

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A REVIEW ON RESULTS FOR THE DERRIDA-LEBOWITZ-SPEER-SPOHN EQUATION

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The Derrida-Lebowitz-Speer-Spohn equation is a nonlinear fourth-order parabolic equation arising, for example, in quantum semiconductor theory. Recent results on the existence and qualitative behavior of solutions are reviewed.

Keywords: Higher-order diffusion equation, quantum diffusion model, existence of solutions, nonuniqueness of solutions, entropy, exponential decay.

1. Introduction

Nonlinear diffusion equations of fourth and higher order have since long been of interest in various fields of mathematical physics. Applications range from fluid models for thin viscous films to statistical equations for quantum particles. However, while the theory of second-order diffusion equations is well understood and numerous tools have been developed to study the qualitative properties of their solutions, few mathematical results are available for higher-order equations.

In this short note, we collect and review old and new results for the fourth-order *Derrida-Lebowitz-Speer-Spohn (DLSS) equation*

$$\partial_t u + \operatorname{div}\Big(u\nabla\Big(\frac{\Delta\sqrt{u}}{\sqrt{u}}\Big)\Big) = 0, \quad u(\cdot,0) = u_0 \geq 0. \tag{1}$$

There are various motivations to study this equation, mainly from mathematical physics. Originally, (1) was deduced (on the positive half line) by Derrida, Lebowitz, Speer, and Spohn in the context of spin systems.[10] In their investigations of the Toom model, they derived an equation describing fluctuations of the interface between the regions of predominantly positive and negative spins, relative to the diagonal line. In a suitable scaling limit, a random quantity related to this deviation satisfies (1).

The multi-dimensional DLSS equation appears in the context of semiconductor modeling. In the simplified quantum drift-diffusion model,[2,13] the density u of electrons in a semiconductor device evolves according to

$$\partial_t u = \operatorname{div}\,(T\nabla u + u\nabla V), \quad V = V_{\mathrm{el}} - \frac{\varepsilon^2}{6}\frac{\Delta\sqrt{u}}{\sqrt{u}}. \tag{2}$$

Here $T > 0$ is the temperature, ε the Planck constant, and V is the potential felt by the electrons, which splits into the classical electric potential V_{el} and the Bohm potential, describing quantum effects. For vanishing temperature $T \to 0$ and no applied electric field, $V_{\mathrm{el}} = 0$, equation (2) turns into the DLSS equation (1). Since the Bohm potential describes an interaction force between electrons, which has a strongly diffusive effect on their density (independent of the temperature), one could refer to (1) also as a quantum diffusion equation.

At this point, we remark that one possible derivation of equation (2) is based on an expansion of the Wigner equation to order $O(\varepsilon^2)$.[8] By deriving an $O(\varepsilon^4)$ approximation, one finds a sixth-order diffusion model,[16] which is supposed to provide a more accurate description of the quantum effects. This is similar to lubrication theory, where sixth-order diffusion equations are used to improve the modeling of the fourth-order thin film equations.[12]

A further motivation to study the DLSS equation is the particular structure of its flow. It was shown recently that (1) constitutes the gradient flow of the Fisher information

$$F_1 = \int \left|\nabla\sqrt{u}\right|^2 dx \tag{3}$$

with respect to the Wasserstein metric.[14] We recall that a gradient-flow structure is a rare property, which is, for instance, not shared by the thin-film equations (except in the Hele-Shaw case). In fact, the situation is even more special. We recall that the entropy functional

$$E_1 = \int u \log u \, dx \tag{4}$$

is one of the most fundamental objects in information theory and the study of equilibration of many-particle systems. It is by now well-known that the heat equation is the Wasserstein gradient flow for the entropy (4), and its entropy production gives the Fisher information (3), i.e., the time derivative of E_1 along solutions to the heat equation is $-4F_1$. Loosely speaking, this coincidence makes the DLSS equation probably the most natural fourth-order extension of the heat equation.

2. Basic Properties and Special Solutions

As (1) is a parabolic equation, its solutions $u(x,t)$ are classical, smooth and unique[3] — as long as the nonlinear term in (1) is well-defined, i.e. as long as $u(t)$ remains bounded and strictly positive. In contrast to second-order scalar diffusion equations, solutions to the fourth-order DLSS equation (1) *do not* obey a maximum principle in general, which would guarantee that $u(x,t)$ remains within the same absolute bounds as the initial (and boundary) conditions. In particular, it is unknown if solutions preserve strict positivity. Lacking a maximum principle, the classical solution u thus may break down at a finite time $T > 0$, either because of blow-up, $\sup u(T) = +\infty$, or because the profile touches zero, $\inf u(T) = 0$. Fortunately, this problem can be solved with a suitable concept of weak solutions, see Section 3 below.

A spatially periodic solution $u(x,t)$ violating the maximum principle is illustrated in Figure 1; see also Ref. 3. One observes the formation of two downward-pointing tips, which grow in height and merge into a single, even larger tip. From here, however, the solution monotonically retreats to homogeneity.

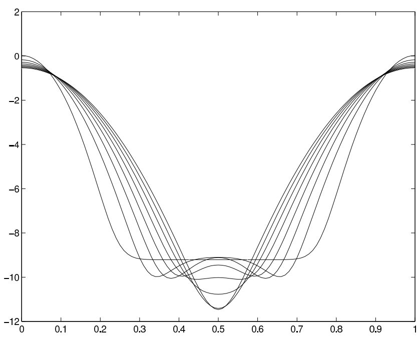

Fig. 1. Profiles of a solution violating the maximum principle.

A couple of explicit (formal) solutions to the DLSS equation are known, in particular in one spatial dimension.[3] We mention the traveling-wave solution $u(x,t) = \mathrm{Ai}^2(x-t)$, where Ai denotes an Airy function, and the stationary solution $u(x,t) = \sin^2 x$. As a d-dimensional extension of the latter example, one finds that

$$u_s(x) = \cos^2(n_1\pi x_1)\cdots\cos^2(n_d\pi x_d), \quad n_1,\ldots,n_d \in \mathbb{N}, \tag{5}$$

constitutes a stationary, spatially multi-periodic solution for $x \in \mathbb{R}^d$,[17] in the sense that the right-hand side of (1) vanishes on the set $\{u > 0\}$. Furthermore, (1) admits self-similar solutions on $\mathbb{R}^d$, which are multiples of

$$U(x,t) = \big(8\pi^2 t\big)^{-d/4} \exp\big(-|x|^2/\sqrt{8t}\big). \tag{6}$$

Recently, more general solutions of Gaussian type were found.[9] They read as $u(x,t) = r(t)\exp(-x^T P(t)x)$, with explicit formulas for the scaling factor $r(t) > 0$ and the symmetric, positive matrix $P(t)$.

3. Existence of Solutions

The first existence result is due to Bleher et al.[3] It provides the existence and uniqueness of *local-in-time* classical solutions to (1) on the d-dimensional torus $\mathbb{T}^d$ for strictly positive $W^{1,p}(\mathbb{T}^d)$ initial data with $p > d$. The existence result is obtained by means of classical semigroup theory applied to

$$2\partial_t\sqrt{u} + \Delta^2\sqrt{u} - \frac{(\Delta\sqrt{u})^2}{\sqrt{u}} = 0,$$

which is equivalent to (1) as long as u remains bounded away from zero. Lacking suitable a priori estimates, existence was proven only locally in time. In one spatial dimension, global existence of solutions can be related to strict positivity: If a classical solution breaks down at $t = t^*$, then the limit profile $\lim_{t\nearrow t^*} u(x,t)$ is in H^1 but vanishes at some point $x \in \mathbb{T}$.

This observation has motivated the study of *nonnegative weak* solutions instead of positive classical solutions. The first global existence results was proven in Ref. 18, and later generalized in Ref. 15. The DLSS equation was considered on $I = (0,1) \subset \mathbb{R}$, with physically motivated boundary conditions

$$u(x,t) = v(x) \quad \text{and} \quad u_x(x,t) = w(x) \quad \text{for } x \in \partial I,\ t > 0. \tag{7}$$

Global existence was proven in the class of functions with finite generalized entropy

$$E_0(u) = \int_I (u - \log u)\,dx. \tag{8}$$

The key ingredient for the proof is the following estimate on E_0,

$$\frac{dE_0}{dt} + \int_I (\log u)_{xx}^2\,dx \le 0.$$

This provides estimates for $\log u$ in $H^2(I)$, which are sufficient to conclude compactness. The restriction to one space dimension is essential, since E_0 is seemingly not a Lyapunov functional in higher dimensions.

For the multi-dimensional DLSS equation on a domain $\Omega \subset \mathbb{R}^d$, global existence of weak solutions was obtained only recently by two very different methods.[14,17] Whereas the framework of the first approach is that of mass transportation theory, the second approach is based on regularization techniques and a fixed-point theorem. Both proofs, however, rely at a crucial point on a compactness argument, that is a consequence of the following estimate of the entropy E_1 from (4),

$$\frac{dE_1}{dt} + \frac{1}{2}\int u\|D^2 \log u\|^2\,dx \leq 0. \tag{9}$$

Here $\|D^2 \log u\|$ is the matrix norm of the Hessian of $\log u$. Lacking a lower bound on u, inequality (9) does not yield an H^2 estimate for $\log u$. However, it is possible to show that

$$\int u\|D^2 \log u\|^2\,dx \geq \frac{4(4d-1)}{d(d+2)}\int \|D^2\sqrt{u}\|^2\,dx, \tag{10}$$

leading to an H^2 bound for $\sqrt{u}$. This motivates to rewrite the nonlinearity in (1) in terms of $\sqrt{u}$, giving, with the notation $\partial_i = \partial/\partial x_i$ etc.,

$$u_t + \sum_{i,j=1}^{d} \partial^2_{ij}\big(\sqrt{u}\partial^2_{ij}\sqrt{u} - \partial_i\sqrt{u}\partial_j\sqrt{u}\big) = 0. \tag{11}$$

In Ref. 14, the above estimates are used to prove that the subdifferential of the Fisher functional (3) is closed. From there, a variety of deep results from mass transportation theory[1] are employed. This eventually provides the existence of a global solution u to (11), with the natural regularity $\sqrt{u} \in L^2_{\mathrm{loc}}(0,\infty;H^2(\Omega))$. Here, Ω can be the whole space $\mathbb{R}^d$ or a bounded domain, equipped with variational boundary conditions.

The ideas in Ref. 17 are more elementary and straight-forward. The DLSS equation is written in logarithmic form, implicitly semi-discretized in time, and regularized by an additional bi-Laplacian term,

$$\delta_t u_\varepsilon + \frac{1}{2}\sum_{i,j=1}^{d} \partial^2_{ij}\big(u_\varepsilon \partial^2_{ij}\log u_\varepsilon\big) + \varepsilon\Delta^2 \log u_\varepsilon = 0.$$

Here, δ_t symbolizes the discrete time derivative, with some positive step size. By elliptic regularity, the logarithm of the regularized, time-discrete solution is bounded in modulus, $|\log u_\varepsilon| \leq M_\varepsilon < +\infty$. Hence, each u_ε is strictly positive, which justifies all formal manipulations. By the compactness that follows from estimates (9) and (10), we conclude the existence of solutions, with the same regularity as above, on the torus $\mathbb{T}^d$, where $d = 1, 2, 3$.

4. Entropies and Long-Time Behavior of Solutions

The relaxation of solutions to the DLSS equation to a steady state, and in particular estimates on the rate of convergence, were studied by a variety of authors. In all approaches, the key step is to determine suitable Lyapunov functionals of the evolution, which are typically of the form[a]

$$E_\alpha(u) = \frac{1}{\alpha(\alpha-1)} \int_\Omega u^\alpha \, dx, \quad F_\gamma(u) = \int_\Omega |\nabla u^{\gamma/2}|^2 \, dx,$$

with $\alpha, \gamma \geq 0$. Once convergence of one (or several) of these functionals is shown, a Sobolev or the Csiszár-Kullback inequality allow to conclude the relaxation of the solution u to the steady state u_∞ in L^1. We summarize the available results.

One dimension, periodic boundary conditions. Here, the steady state is obviously a constant function. It was already observed in Ref. 3, that the choices $0 \leq \alpha < \frac{3}{2}$ and $1 \leq \gamma \leq \frac{3}{2}$ give rise to Lyapunov functionals. However, no convergence results were derived.

About ten years later, exponential convergence of E_α to the respective values of the steady state was proven, first[4] on the smaller range $1 \leq \alpha \leq \frac{4}{3}$, and later[17] on the whole interval $1 \leq \alpha \leq \frac{3}{2}$. As remarked in Ref. 11, the known global decay rate for E_1 coincides with the decay rate of the linearized equation and thus, it is optimal.

The first decay result for a F_γ functional is also due to Ref. 4, where exponential decay of F_0 was shown, at least under a smallness condition on the initial value of F_0. Exponential convergence of the Fisher information F_1 along weak solutions was obtained in Ref. 11. It was observed later in Ref. 16 that actually all functionals F_γ with $\frac{2}{53}(25 - 6\sqrt{10}) < \gamma < \frac{2}{53}(25 + 6\sqrt{10})$ are nonincreasing, at least along smooth solutions. This observation was rigorously proven for weak solutions in Ref. 21, along with the exponential convergence to zero of those F_γ with $\gamma \leq 1$. The analytical results for E_1 and F_1 were verified in numerical experiments.[6]

[a]Here, $\alpha = 0$ and $\alpha = 1$ correspond to the functional (8) and entropy (4), respectively, $\gamma = 1$ gives the Fisher information (3), and we set $F_0 = \int |\nabla \log u|^2 \, dx$.

One dimension, Dirichlet-Neumann boundary conditions. For homogeneous Neumann boundary data, and Dirichlet data of the form $u(0,t) = u(1,t) = 1$, the situation is still comparatively simple. Also here, the steady state is a constant. Historically, the first decay estimate for the DLSS equation was actually proven in this setting,[18] stating that $\|\log u(t)\|_{L^2}$ converges to zero as $t \to \infty$. Later, convergence of the entropy E_1 to the steady state was shown.[20] This convergence is exponentially fast; however, with an exponential rate depending on the initial condition.

For general Dirichlet-Neumann boundary conditions (7), the steady state u_∞ is no longer spatially homogeneous. The existence and uniqueness of a smooth and positive steady state was shown in Ref. 15. This steady state was then proven to be exponentially attracting in terms of the relative entropy, however, only under the additional assumption that $\log u_\infty$ is concave.

Multiple dimensions, multi-periodic boundary conditions. Like in one dimension, the periodic boundary conditions imply homogeneity of the steady state, and allow for a variety of algebraic manipulations. It was shown in Ref. 17 that the functionals E_α are nonincreasing and exponentially converging to the respective values of the steady state for $1 \leq \alpha < (\sqrt{d}+1)^2/(d+2)$. The rates C_α of this exponential convergence depending on α and the dimension are plotted in Fig. 2.

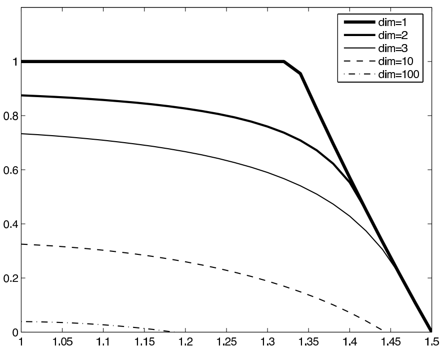

Fig. 2. Decay rates for the entropy E_α.

Multiple dimensions, whole space and variational boundary conditions. A very general result on the relaxation properties of the DLSS equation was recently proven in Ref. 14. Both for the entropy E_1 and the Fisher information F_1, exponential convergence was obtained (as a consequence of the gradient-flow structure) in a variety of relevant situations. These situations include relaxation to homogeneity on any smoothly bounded domain, equipped with variational boundary conditions, as well as convergence to a nontrivial steady state in the whole space $\mathbb{R}^d$ in the presence of a (sufficiently convex) confinement potential. Decay estimates are also available for the unconstrained whole-space case; they are naturally only algebraic in time.

We conclude this section with an argument on the intermediate asymptotics of solutions to the DLSS equation on the real line detailed in Ref. 7. By a suitable rescaling $u(x,t) \mapsto v(y,s)$, which makes the self-similar solution U from (6) stationary, (1) is brought into the form

$$\partial_s v = -\big(v(\log v)_{yy}\big)_{yy} + (yv)_y = -(vh_{yy})_{yy} + (vh_y)_y, \tag{12}$$

where $h = \log v + \frac{1}{2}y^2$. The crucial observation is that the relative entropy functional

$$H[v] = \int_{\mathbb{R}} vh\,dy = \int_{\mathbb{R}} \Big(v\log v + \frac{1}{2}y^2 v\Big)\,dy \tag{13}$$

converges to its terminal value with the rate $\exp(-2s)$, at least along strictly positive solutions v. In the original variables, one concludes convergence of the solution u to self-similarity at an algebraic rate,

$$\|u(t) - U(t)\|_{L^1} \le C(1+t)^{-1/4}. \tag{14}$$

The rate in (14) can be improved by adjusting the first and second moments of the self-similar solution U used for comparison.

5. Open Problems

The existence theory for the DLSS equation (1) is more or less completed. Moreover, as already mentioned in Section 2, this equation is by now resonably well understood in the regime of positive solutions. As long as the spatial profile of $u(t)$ remains strictly positive for all $t > 0$, the solution is classical, smooth and unique, it dissipates a variety of Lyapunov functionals, and it converges to a uniquely defined stationary profile as $t \to \infty$. On the contrary, little is known about how to treat the situations in which $u(t)$ touches zero.

Naturally, one expects that solutions to the DLSS equation stay strictly positive if the initial datum is positive. In fact, numerical experiments[3,6,18] indicate that the solution is positive for all $t > 0$ even if the initial datum vanishes at some point. However, so far it is unclear how to turn this intuition into a mathematical proof. Basically, only two results for positivity are known:[3] First, by continuity, a strictly positive profile remains strictly positive for some small time. And second, on the one-dimensional torus, the decay of the Fisher information (3) implies H^1-closedness of $u(t)$ to the homogeneous steady state, and thus strict positivity of $u(t)$ for $t > T_*$. The second argument was re-investigated in Ref. 21 for H^1 initial data, leading to an explicit estimate on T_*.

As a concluding remark, we illustrate the difficulties introduced by non-strictly positive initial data for (1). Observe that the smooth function u_s given in (5) constitutes a stationary solution to (1), at least on a formal level. However, by the recent existence results,[17] there is a transient weak solution $u(x,t)$ of the DLSS equation on the torus $\mathbb{T}^d$, which attains u_s as initial condition, but converges to a homogeneous stationary state, $u(t) \to u_\infty = \text{const}$. So there exist two distinct solutions for the same initial data.

There is a certain hope to decouple the question of uniqueness from that of strict positivity. To this end, we remark that the nonlinear operator in (1) is formally monotone with respect to the square root $\sqrt{u}$. The derivation of a rigorous argument is currently work in progress.

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