Introduction to Boundary Element Method[1]

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Für Hinweise auf Tipp- und sonstige Fehler bin ich sehr dankbar! Für Rückfragen bitte einfach eine Mail an mich: `dirk.praetorius@tuwien.ac.at`[2345]

1. Representation Formula

As the very first start of our lecture, we prove the so-called *representation formula* (or *third Green's formula*) in the classical setting and introduce the basic integral operators. To that end, we define the **Newton kernel**

$$G(z) := \begin{cases} -\frac{1}{2\pi} \log |z|, & \text{for } d = 2, \\ +\frac{1}{4\pi} \frac{1}{|z|}, & \text{for } d = 3. \end{cases} \tag{1}$$

Note that there holds $|S_2^2| = 2\pi$ and $|S_2^3| = 4\pi$, where $|S_2^d|$ denotes the measure of the unit sphere in $\mathbb{R}^d$. Our first lemma easily follows from direct calculations and is left to the reader (The last two statements are obtained by use of polar coordinates).

Lemma 1. (i) There holds $G \in C^\infty(\mathbb{R}^d \backslash \{0\})$ with first and second derivatives

$$\partial_j G(z) = -\frac{1}{|S_2^d|} \frac{z_j}{|z|^d} \quad \text{and} \quad \partial_{jk} G(z) = -\frac{1}{|S_2^d|} \frac{\delta_{jk}|z|^2 - d z_j z_k}{|z|^{d+2}}. \tag{2}$$

(ii) There holds $-\Delta G(z) = 0$ for $z \neq 0$.
(iii) $G \in L^p_{\ell oc}(\mathbb{R}^d)$ for $d < 2p/(p-1)$, in particular $G \in L^2_{\ell oc}(\mathbb{R}^d)$.
(iv) $\partial_j G \in L^p_{\ell oc}(\mathbb{R}^d)$ for $d < p/(p-1)$, in particular $\partial_j G \in L^1_{\ell oc}(\mathbb{R}^d)$. ■

The main result of this section is the following representation formula. It states that the (smooth) solution of a Laplace problem $-\Delta u = f$ is uniquely determined by its **Cauchy data** $(u, \partial_n u)$, i.e. the trace and the normal derivative of u on the boundary Γ. Thus, we know u if we know the Dirichlet and Neumann data on the entire boundary Γ.

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Proposition 2 (Representation Formula). Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\Gamma := \partial\Omega$ and $u \in C^2(\overline{\Omega})$. With $f := -\Delta u \in C(\overline{\Omega})$, there holds

$$u(x) = \int_\Omega G(x-y)f(y)\,dy + \int_\Gamma G(x-y)\partial_{n(y)}u(y)\,ds_y - \int_\Gamma \partial_{n(y)}G(x-y)\,u(y)\,ds_y$$

for all $x \in \Omega$, where $n(y)$ denotes the outer normal vector at $y \in \Gamma$ and $\partial_{n(y)}$ is the associated normal derivative.

Proof. Fix $x \in \Omega$. We want to apply, for u and $v(y) = G(x-y)$, the second Green's formula which reads in classical terms

$$(-\Delta u\,;\,v)_{L^2(\Omega)} + (\partial_n u\,;\,v)_{L^2(\Gamma)} = (u\,;\,-\Delta v)_{L^2(\Omega)} + (u\,;\,\partial_n v)_{L^2(\Gamma)} \quad \text{for } u, v \in C^2(\overline{\Omega}). \tag{3}$$

As $v \notin C^2(\overline{\Omega})$, we cut-off the singularity for $y = x$ and consider (3) on $\Omega_\varepsilon := \Omega\backslash B(x,\varepsilon)$. Here, $\varepsilon > 0$ is chosen small enough so that $B(x,\varepsilon) \subset \Omega$. Then, with $\Gamma_\varepsilon := \partial\Omega_\varepsilon$, there holds $\Gamma_\varepsilon = \Gamma \cup S(x,\varepsilon)$ and $\Gamma \cap S(x,\varepsilon) = \emptyset$. The second Green's formula proves

$$(-\Delta u\,;\,v)_{L^2(\Omega_\varepsilon)} + (\partial_n u\,;\,v)_{L^2(\Gamma)} - (u\,;\,\partial_n v)_{L^2(\Gamma)} = -(\partial_n u\,;\,v)_{L^2(S(x,\varepsilon))} + (u\,;\,\partial_n v)_{L^2(S(x,\varepsilon))}.$$

It now remains to consider the convergence of the terms for $\varepsilon \to 0$, where the left-hand side tends to the right-hand side of the representation formula, cf. step 1, and where the right-hand side tends to $u(x)$, cf. step 2 and 3.

1. step. There holds $(-\Delta u\,;\,v)_{L^2(\Omega_\varepsilon)} \xrightarrow{\varepsilon\to 0} (-\Delta u\,;\,v)_{L^2(\Omega)}$ which follows obviously from the Lebesgue dominated convergence theorem as $-\Delta u = f \in L^2(\Omega)$ and $v \in L^2(\Omega)$.

2. step. There holds $(\partial_n u\,;\,v)_{L^2(S(x,\varepsilon))} \xrightarrow{\varepsilon\to 0} 0$: Note that, for $y \in S(x,\varepsilon)$, there holds

$$v(y) = G(x-y) = \frac{1}{|S_2^d|}\begin{cases} -\log\varepsilon & \text{for } d = 2, \\ 1/\varepsilon & \text{for } d = 3. \end{cases}$$

It therefore remains to consider

$$\int_{S(x,\varepsilon)} \partial_n u\,ds = -\int_{B(x,\varepsilon)} \Delta u\,dy = -|B(x,\varepsilon)| \fint_{B(x,\varepsilon)} \Delta u\,dy,$$

where here and in the following $\fint_B \cdot\,dy := |B|^{-1}\int_B \cdot\,dy$ denotes the integral mean. The surprising minus sign arises from integration by parts as n is the outer normal of Ω_ε and therefore the inner normal of $B(x,\varepsilon)$. There holds $|B(x,\varepsilon)| = |B_2^d|\varepsilon^d$, and the integral mean $\fint_{B(x,\varepsilon)} \Delta u\,dy$ converges to $\Delta u(x)$ as $\varepsilon \to 0$. Thus,

$$(\partial_n u\,;\,v)_{L^2(S(x,\varepsilon))} = -\frac{|B_2^d|}{|S_2^d|} \fint_{B(x,\varepsilon)} \Delta u\,dy \cdot \begin{cases} \varepsilon^2|\log\varepsilon| & \text{for } d = 2, \\ \varepsilon^2 & \text{for } d = 3, \end{cases}$$

vanishes with $\varepsilon \to 0$.

3. step. There holds $(u\,;\,\partial_n v)_{L^2(S(x,\varepsilon))} \xrightarrow{\varepsilon\to 0} u(x)$: We plug-in the formula for ∇G to obtain

$$\int_{S(x,\varepsilon)} u(y)\partial_n(y)G(x-y)\,ds_y = -\frac{1}{|S_2^d|\varepsilon^d}\int_{S(x,\varepsilon)} u(y)(y-x)\cdot n(y)\,ds_y.$$

With $f(y) := u(y)(y-x)$, there holds $\mathrm{div} f(y) = \nabla u(y) \cdot (y-x) + du(y)$. Thus, the Gauss divergence theorem implies

$$\int_{S(x,\varepsilon)} u(y)\partial_{n(y)} G(x-y)\, ds_y = -\frac{1}{|S_2^d|\varepsilon^d} \int_{S(x,\varepsilon)} f(y)\cdot n(y)\, ds_y = +\frac{1}{|S_2^d|\varepsilon^d} \int_{B(x,\varepsilon)} \mathrm{div} f\, dy,$$

where the change of the sign again arises since the outer normal n of Ω_ε is the inner normal of $B(x,\varepsilon)$. We now consider the two contributions of $\mathrm{div} f$ separately: With $|B(x,\varepsilon)| = |B_2^d|\varepsilon^d$, there holds

$$\frac{1}{|S_2^d|\varepsilon^d}\Big|\int_{B(x,\varepsilon)} \nabla u(y)\cdot(y-x)\, dy\Big| \le \frac{|B_2^d|\varepsilon}{|S_2^d|} \fint_{B(x,\varepsilon)} |\nabla u|\, dy \xrightarrow{\varepsilon\to 0} 0,$$

since the integral mean converges to $|\nabla u(x)|$. The second term from $\mathrm{div} f$ reads

$$\frac{d}{|S_2^d|\varepsilon^d} \int_{B(x,\varepsilon)} u\, dy = \underbrace{\frac{d|B_2^d|}{|S_2^d|}}_{=1} \fint_{B(x,\varepsilon)} u\, dy \xrightarrow{\varepsilon\to 0} u(x).$$

To see that the constant in front of the integral is one, recall that $|S_2^2| = 2\pi$ and $|B_2^2| = \pi$ as well as $|S_2^3| = 4\pi$ and $|B_2^3| = 4\pi/3$. ■

The representation formula from Proposition 2 allows to represent $u \in C^2(\overline{\Omega})$ in terms of the following three integral operators $\widetilde{N}$, $\widetilde{V}$, and $\widetilde{K}$, namely

- the **Newton potential** of $f : \Omega \to \mathbb{R}$

$$\widetilde{N} f(x) := \int_\Omega G(x-y) f(y)\, dy \quad \text{for } x \in \Omega, \tag{4}$$

- the **single layer potential** of $\phi : \Gamma \to \mathbb{R}$

$$\widetilde{V}\phi(x) := \int_\Gamma G(x-y)\phi(y)\, ds_y \quad \text{for } x \in \Omega, \tag{5}$$

- the **double layer potential** of $v : \Gamma \to \mathbb{R}$

$$\widetilde{K} v(x) := \int_\Gamma \partial_{n(y)} G(x-y)\, v(y)\, ds_y \quad \text{for } x \in \Omega. \tag{6}$$

Obviously, the operators $\widetilde{N}$, $\widetilde{V}$, and $\widetilde{K}$ are linear operators. Moreover, with this notation, the representation formula can simply be written as follows:

Corollary 3 (Representation Formula). For $u \in C^2(\overline{\Omega})$, there holds

$$u = \widetilde{N}(-\Delta u) + \widetilde{V}(\partial_n u) - \widetilde{K}(u) \quad \text{in } \Omega, \tag{7}$$

which is just the operator statement of Proposition 2. ■

2. Sobolev Spaces and Weak Solutions

From now on, we shall assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, for $d = 2, 3$. Under this technical assumption, the boundary $\Gamma = \partial\Omega$ is sufficiently smooth to define the outer normal vector n almost everywhere on Γ and to verify the integration by parts formula.

2.1. Sobolev Space $H^1(\Omega)$. As usual, $H^1(\Omega)$ denotes the Sobolev space of all weakly differentiable functions $u \in L^2(\Omega)$ whose gradient belongs to $L^2(\Omega)^d$ as well. Associated with the graph norm

$$\|u\|_{H^1(\Omega)} = \big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\big)^{1/2},$$

$H^1(\Omega)$ is a separable Hilbert space, and it is a well-known result due to Meyers-Serrin that $C^\infty(\overline{\Omega})$ is a dense subspace of $H^1(\Omega)$.

We shall show below that solutions of the Laplace equation exist uniquely in the so-called weak sense and belong to $H^1(\Omega)$. In the following, we develop the full functional setting for the weak form: Assume that $u \in C^2(\overline{\Omega})$ is a classical solution of the Laplace equation

$$-\Delta u = f. \tag{8}$$

We multiply this equation with a test function $v \in C^1(\overline{\Omega})$. Integration over Ω and integration by parts leads to the variational formulation

$$(f\,;\,v)_{L^2(\Omega)} = (\nabla u\,;\,\nabla v)_{L^2(\Omega)} - (\partial_n u\,;\,v)_{L^2(\Gamma)} \quad \text{for all } v \in C^1(\overline{\Omega}). \tag{9}$$

Note that every solution $u \in C^2(\overline{\Omega})$ of (8) necessarily is a solution of (9). We now study each of the three scalar products in (9) with respect to $H^1(\Omega)$.

2.2. Sobolev Space $\widetilde{H}^{-1}(\Omega)$. Note that, for any $f \in L^2(\Omega)$, the left-hand side of (9)

$$v \mapsto (f\,;\,v)_{L^2(\Omega)}$$

defines a functional in $H^1(\Omega)^*$. The following lemma shows that essentially all linear and continuous functionals on $H^1(\Omega)$ are of this type.

Lemma 4. Let X and Y be real Hilbert spaces with continuous inclusion $X \subseteq Y$, i.e. the identity $id : X \to Y$ is a well-defined and continuous linear operator. Then, the Riesz mapping $J_Y : Y \hookrightarrow Y^*$, $J_Y y := (y\,;\,\cdot)_Y$ is well-defined as operator $J_Y \in L(Y; X^*)$, and $J_Y(Y)$ is a dense subspace of X^*.

Proof. According to the assumptions, there holds $\|x\|_Y \le C\,\|x\|_X$ for all $x \in X$. Thus, the Cauchy inequality proves

$$(y\,;\,x)_Y \le \|y\|_Y\|x\|_Y \le C\|y\|_Y\|x\|_X.$$

Thus, $J_Y \in L(Y; X^*)$ is well-defined. Let $J_X : X \hookrightarrow X^*$ denote the Riesz mapping for X. Then, $J_Y(Y)$ is dense in X^* if and only if $V := J_X^{-1}(J_Y(Y))$ is dense in $X = \overline{V} \oplus \overline{V}^\perp$. Therefore, it remains to prove that $\overline{V}^\perp = \{0\}$. Let $x \in \overline{V}^\perp$. Then, for $y \in Y$, there holds

$$0 = (x\,;\,J_X^{-1}(J_Y y))_X = (J_Y y)(x) = (y\,;\,x)_Y.$$

Choose $y = x \in \overline{V}^{\perp} \subseteq Y$ to see $x = 0$ in $Y \supseteq X$, which concludes the proof. ■

We apply the lemma for $X = H^1(\Omega)$ and $Y = L^2(\Omega)$. We then obtain that $L^2(\Omega)$ is a dense subspace of $H^1(\Omega)^*$, and the duality brackets $\langle f\,;\,v\rangle$ coincide with the L^2-scalar product $(f\,;\,v)_{L^2(\Omega)}$ provided that $f \in L^2(\Omega)$. We denote by $\widetilde{H}^{-1}(\Omega)$ the dual space of $H^1(\Omega)$ with respect to the extended L^2-scalar product.

Remark. For $f \in \widetilde{H}^{-1}(\Omega)$, the left-hand side of (9) can be generalized to $\langle f\,;\,v\rangle$. Moreover, the first term $(\nabla u\,;\,\nabla v)_{L^2(\Omega)}$ on the right-hand side of (9) is well-defined for $u, v \in H^1(\Omega)$. ■

2.3. Sobolev Spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$.

The Sobolev space $H^{1/2}(\Gamma)$ is defined by

$$H^{1/2}(\Gamma) := \big\{ v \in L^2(\Gamma) \,\big|\, \|v\|_{H^{1/2}(\Gamma)} < \infty \big\}, \tag{10}$$

and associated with the norm $\|v\|_{H^{1/2}(\Gamma)} = \big(\|v\|^2_{L^2(\Gamma)} + |v|^2_{H^{1/2}(\Gamma)}\big)^{1/2}$, where

$$|v|_{H^{1/2}(\Gamma)} = \Big(\int_\Gamma \int_\Gamma \frac{|v(x) - v(y)|^2}{|x-y|^d}\, ds_y\, ds_x \Big)^{1/2}. \tag{11}$$

denotes the **Sobolev-Slobodeckij seminorm**.

It can be shown that $H^{1/2}(\Gamma)$ is a Hilbert space. Obviously, there holds $H^{1/2}(\Gamma) \subseteq L^2(\Gamma)$ even with continuous inclusion. Therefore, Lemma 4 applies with $X = H^{1/2}(\Gamma)$ and $Y = L^2(\Gamma)$. We then obtain that $L^2(\Gamma)$ is a dense subspace of $H^{1/2}(\Gamma)^*$, and the duality brackets $\langle \phi\,;\,v\rangle$ coincide with the L^2-scalar product $(\phi\,;\,v)_{L^2(\Gamma)}$ provided that $\phi \in L^2(\Gamma)$. We denote by $H^{-1/2}(\Gamma)$ the dual space of $H^{1/2}(\Gamma)$ with respect to the extended L^2-scalar product.

The following theorem states some fundamental relations between $H^1(\Omega)$ and $H^{1/2}(\Gamma)$.

Theorem 5. (i) There is a unique **trace operator** $\gamma_0 \in L\big(H^1(\Omega); H^{1/2}(\Gamma)\big)$ such that $\gamma_0 u = u|_\Gamma$ for all $u \in C^\infty(\overline{\Omega})$, i.e. $\gamma_0 u$ is the trace of a Sobolev function u on Γ.
(ii) There is a **lifting operator** $\mathcal{L} \in L\big(H^{1/2}(\Gamma); H^1(\Omega)\big)$ such that $\gamma_0 \mathcal{L} v = v$ for all $v \in H^{1/2}(\Gamma)$. ■

From these two results, we immediately obtain that $H^{1/2}(\Gamma)$ is the trace space of $H^1(\Omega)$.

Corollary 6. There holds the set equality $H^{1/2}(\Gamma) = \big\{ \gamma_0 \widehat{v} \,\big|\, \widehat{v} \in H^1(\Omega) \big\}$. Moreover, $\|v\| := \inf \big\{ \|\widehat{v}\|_{H^1(\Omega)} \,\big|\, \widehat{v} \in H^1(\Omega) \text{ with } \gamma_0 \widehat{v} = v \big\}$ is an equivalent norm on $H^{1/2}(\Gamma)$.

Proof. The inclusion $\supseteq$ follows from the existence of the trace operator, whereas $\subseteq$ follows from the existence of the lifting operator. We now prove $\|v\| \approx \|v\|_{H^{1/2}(\Gamma)}$ for all $v \in H^{1/2}(\Gamma)$:

1. step. The continuity of the lifting operator yields $\|v\| \le \|\mathcal{L}v\| \lesssim \|v\|_{H^{1/2}(\Gamma)}$.

2. step. To verify the converse inequality, let $\varepsilon > 0$. Given $v \in H^{1/2}(\Gamma)$, there is a $\widehat{v} \in H^1(\Omega)$ with $v = \gamma_0 \widehat{v}$ and $\|v\| + \varepsilon \ge \|\widehat{v}\|_{H^1(\Omega)}$. Therefore, the continuity of the trace operator yields $\|v\|_{H^{1/2}(\Gamma)} = \|\gamma_0 \widehat{v}\|_{H^{1/2}(\Gamma)} \lesssim \|\widehat{v}\|_{H^1(\Omega)} \le \|v\| + \varepsilon$. In the limit $\varepsilon \to 0$, we obtain $\|v\|_{H^{1/2}(\Gamma)} \lesssim \|v\|$.

3. step. It is left to the reader to prove that $\|\cdot\|$ is a norm on $H^{1/2}(\Gamma)$. We stress however that $\|\cdot\|$ is just the canonical norm on the quotient space $H^1(\Omega)/\ker(\gamma_0)$. ∎

2.4. Weak Form of Laplace Equation. We now may state the formal version of (9): For given $f \in \widetilde{H}^{-1}(\Omega)$, we say that a Sobolev function $u \in H^1(\Omega)$ solves

$$-\Delta u = f \in \widetilde{H}^{-1}(\Omega) \tag{12}$$

provided there is a functional $\gamma_1 u \in H^{-1/2}(\Gamma)$ such that

$$\langle f\,;\,v\rangle = (\nabla u\,;\,\nabla v)_{L^2(\Omega)} - \langle \gamma_1 u\,;\,\gamma_0 v\rangle \quad \text{for all } v \in H^1(\Omega). \tag{13}$$

This variational formulation is called the **weak form** of (8) and is, in fact, a generalization of (9):

Lemma 7. For $u \in C^2(\overline{\Omega})$ and $f := -\Delta u$ holds (13) with $\gamma_1 u = \partial_n u$.

Proof. Note that $f = -\Delta u \in C(\overline{\Omega}) \subseteq L^2(\Omega)$ as well as $\partial_n u \in L^\infty(\Gamma) \subseteq L^2(\Gamma)$. Therefore, the duality brackets in (13) are just L^2-scalar products, and it remains to show that, for all $v \in H^1(\Omega)$,

$$(f\,;\,v)_{L^2(\Omega)} = (\nabla u\,;\,\nabla v)_{L^2(\Omega)} - (\gamma_1 u\,;\,\gamma_0 v)_{L^2(\Gamma)}.$$

This variational form holds for all $v \in C^1(\overline{\Omega})$, c.f. Equation (9) above. Since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$ and all scalar products are continuous with respect to v, we conclude the proof. ∎

The next lemma shows that the **(generalized) normal derivative** $\gamma_1 u$ is uniquely determined.

Lemma 8. Assume that $u \in H^1(\Omega)$ solves (12). Then, $\gamma_1 u \in H^{-1/2}(\Gamma)$ from (13) is unique, and there holds the stability estimate

$$\|\gamma_1 u\|_{H^{-1/2}(\Gamma)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} + \|\nabla u\|_{L^2(\Omega)}. \tag{14}$$

Proof. For any $v \in H^{1/2}(\Gamma)$, the weak form (13) implies, in particular,

$$\langle \gamma_1 u\,;\,v\rangle = (\nabla u\,;\,\nabla \mathcal{L}v)_{L^2(\Omega)} - \langle f\,;\,\mathcal{L}v\rangle \quad \text{for all } v \in H^{1/2}(\Gamma).$$

Note that the right-hand side depends only on given data u and f. Therefore, $\gamma_1 u \in H^{-1/2}(\Gamma)$ is uniquely determined. Moreover, we may estimate the latter representation of $\gamma_1 u$ to see

$$\langle \gamma_1 u\,;\,v\rangle \le \big(\|\nabla u\|_{L^2(\Omega)} + \|f\|_{\widetilde{H}^{-1}(\Omega)}\big)\|\mathcal{L}v\|_{H^1(\Omega)} \lesssim \big(\|\nabla u\|_{L^2(\Omega)} + \|f\|_{\widetilde{H}^{-1}(\Omega)}\big)\|v\|_{H^{1/2}(\Omega)}.$$

Dividing by $\|v\|_{H^{1/2}(\Omega)}$ and taking the supremum, we prove (14). ∎

At the end of this section, we recall the unique existence of weak solutions. To that end, we have to specify some boundary conditions. For the ease of presentation, we consider the Laplace equation with pure Dirichlet conditions.

Proposition 9. For given $f \in \widetilde{H}^{-1}(\Omega)$ and $g \in H^{1/2}(\Gamma)$, there is a unique solution $u \in H^1(\Omega)$ of

$$-\Delta u = f \in \widetilde{H}^{-1}(\Omega) \quad \text{with Dirichlet boundary conditions} \quad \gamma_0 u = g \in H^{1/2}(\Gamma). \tag{15}$$

There holds the stability estimate

$$\|u\|_{H^1(\Omega)} + \|\gamma_1 u\|_{H^{-1/2}(\Gamma)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}. \tag{16}$$

Proof. If $u \in H^1(\Omega)$ solves (15), $u_0 := u - \mathcal{L}g \in H^1_0(\Omega) := \ker(\gamma_0)$ solves

$$(\nabla u_0\,;\,\nabla v_0)_{L^2(\Omega)} = \langle f\,;\,v_0\rangle - (\nabla \mathcal{L}g\,;\,\nabla v_0)_{L^2(\Omega)} \quad \text{for all } v_0 \in H^1_0(\Omega). \tag{17}$$

According to the Friedrichs inequality, the left-hand side of (17) defines an equivalent scalar product on the Hilbert space $H^1_0(\Omega)$. Moreover, the right-hand side is linear and continuous with respect to $v_0 \in H^1_0(\Omega)$. Therefore, the Riesz theorem proves the unique existence of u_0. In particular, $u = u_0 + \mathcal{L}g$ is unique and satisfies $\gamma_0 u = g$. It remains to define $\gamma_1 u \in H^{-1/2}(\Gamma)$ and to verify (15): Given $v \in H^{1/2}(\Gamma)$, let $\widehat{v} \in H^1(\Omega)$ with $v = \gamma_0 \widehat{v}$. In particular, $\mathcal{L}v - \widehat{v} \in H^1_0(\Omega)$ and thus (17) implies

$$(\nabla u\,;\,\nabla(\mathcal{L}v - \widehat{v}))_{L^2(\Omega)} = (\nabla u_0 + \nabla \mathcal{L}g\,;\,\nabla(\mathcal{L}v - \widehat{v}))_{L^2(\Omega)} = \langle f\,;\,\mathcal{L}v - \widehat{v}\rangle.$$

Therefore,

$$\langle \phi\,;\,v\rangle := (\nabla u\,;\,\nabla \mathcal{L}v)_{L^2(\Omega)} - \langle f\,;\,\mathcal{L}v\rangle = (\nabla u\,;\,\nabla \widehat{v})_{L^2(\Omega)} - \langle f\,;\,\widehat{v}\rangle$$

defines a functional $\phi \in H^{-1/2}(\Gamma)$, where the right-hand does not depend on the precise lifting operator $\mathcal{L}$. In particular, we obtain

$$\langle f\,;\,\widehat{v}\rangle = (\nabla u\,;\,\nabla \widehat{v})_{L^2(\Omega)} - \langle \phi\,;\,\gamma_0 \widehat{v}\rangle \quad \text{for all } \widehat{v} \in H^1(\Omega),$$

which proves $-\Delta u = f \in \widetilde{H}^{-1}(\Gamma)$ and $\phi = \gamma_1 u$ according to the uniqueness of the normal derivative. It thus only remains to verify the stability estimate (16). The triangle inequality and $u = u_0 + \mathcal{L}g$ imply

$$\|u\|_{H^1(\Omega)} \le \|u_0\|_{H^1(\Omega)} + \|\mathcal{L}g\|_{H^1(\Omega)}$$

Choosing $v_0 = u_0$ in (17), a Friedrichs inequality yields

$$\|u_0\|^2_{H^1(\Omega)} \lesssim \|\nabla u_0\|^2_{L^2(\Omega)} \le \big(\|f\|_{\widetilde{H}^{-1}(\Omega)} + \|\mathcal{L}g\|_{H^1(\Omega)}\big)\, \|u_0\|_{H^1(\Omega)}.$$

With the continuity of the lifting operator $\mathcal{L}$, we obtain

$$\|u\|_{H^1(\Omega)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} + \|\mathcal{L}g\|_{H^1(\Omega)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}$$

Finally, the estimate $\|\gamma_1 u\|_{H^{-1/2}(\Gamma)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} + \|u\|_{H^1(\Omega)}$ of Lemma 8 concludes the proof. ■

3. Integral Operators

3.1. Newton Potential. We start with the analysis of the Newton potential

$$\widetilde{N}f(x) = \int_\Omega G(x-y)f(y)\,dy \quad \text{for all } x \in \Omega. \tag{18}$$

Note that the Newton potential is nothing but a convolution $\widetilde{N}f = G * f$ with the Newton kernel. Classical calculus thus shows

$$\widetilde{N}f \in C^1(\mathbb{R}^d) \quad \text{provided that} \quad f \in L^\infty(\Omega) \tag{19}$$

The first lemma proves that $\widetilde{N}$ is a left and right inverse of $-\Delta$ provided that the volume force is smooth.

Lemma 10. For $f \in \mathcal{D}(\Omega) := C_c^\infty(\Omega)$ holds $\widetilde{N}f \in C^\infty(\mathbb{R}^d)$ with $f = -\Delta(\widetilde{N}f) = \widetilde{N}(-\Delta f)$.

Proof. Classical calculus shows $\widetilde{N}f \in C^\infty(\mathbb{R}^d)$ with $\partial^\alpha(\widetilde{N}f) = \partial^\alpha(G * f) = G * \partial^\alpha f$. The equality $f = \widetilde{N}(-\Delta f)$ follows from the representation formula since the boundary integrals $\widetilde{V}(\partial_n f)$ and $\widetilde{K}(f)$ vanish. To prove $f = -\Delta(\widetilde{N}f)$, let $g \in \mathcal{D}(\Omega)$ as well. We use integration by parts to see

$$(-\Delta(\widetilde{N}f)\,;\, g)_{L^2(\Omega)} = (\widetilde{N}f\,;\, -\Delta g)_{L^2(\Omega)} = \int_\Omega (-\Delta g)(x) \int_\Omega G(x-y)f(y)\,dy\,dx.$$

With the Fubini theorem and symmetry $G(x-y) = G(y-x)$ of the kernel, we obtain

$$(-\Delta(\widetilde{N}f)\,;\, g)_{L^2(\Omega)} = \int_\Omega f(y) \int_\Omega G(y-x)(-\Delta g)(x)\,dx\,dy = (f\,;\, \widetilde{N}(-\Delta g))_{L^2(\Omega)} = (f\,;\, g)_{L^2(\Omega)},$$

where we have used $g = \widetilde{N}(-\Delta g)$ in the last step. We thus end up with

$$(f + \Delta(\widetilde{N}f)\,;\, g)_{L^2(\Omega)} = 0 \quad \text{for all } g \in \mathcal{D}(\Omega).$$

According to the fundamental theorem of calculus of variations, this equality implies that $f = -\Delta(\widetilde{N}f)$ almost everywhere in Ω. Since both sides are continuous functions, we thus get pointwise equality. ∎

The following theorem gathers together the most important properties of the operator $\widetilde{N}$.

Theorem 11. (i) $\widetilde{N}$ allows a unique extension $\widetilde{N} \in L\big(\widetilde{H}^{-1}(\Omega); H^1(\Omega)\big)$ from $\mathcal{D}(\Omega)$ to $\widetilde{H}^{-1}(\Omega)$.
(ii) $-\Delta(\widetilde{N}f) = f$ for all $f \in \widetilde{H}^{-1}(\Omega)$.
(iii) $\gamma_0\widetilde{N} \in L\big(\widetilde{H}^{-1}(\Omega); H^{1/2}(\Gamma)\big)$.
(iv) $\gamma_1\widetilde{N} \in L\big(\widetilde{H}^{-1}(\Omega); H^{-1/2}(\Gamma)\big)$.

Sketch of Proof. (i) We note that $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, and $L^2(\Omega)$ is dense in $\widetilde{H}^{-1}(\Omega)$. Consequently, $\mathcal{D}(\Omega)$ is dense in $\widetilde{H}^{-1}(\Omega)$ as well. For $f \in \mathcal{D}(\Omega)$ holds, in particular, $\widetilde{N}f \in C^\infty(\mathbb{R}^d) \subset H^1(\Omega)$. Clearly, $\widetilde{N} : \mathcal{D}(\Omega) \to H^1(\Omega)$ is linear. It thus only remains to prove that $\widetilde{N}$ is continuous from $\mathcal{D}(\Omega)$ associated with the $\widetilde{H}^{-1}$-norm to $H^1(\Omega)$, i.e.

$$\|\widetilde{N}f\|_{H^1(\Omega)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} \quad \text{for all } f \in \mathcal{D}(\Omega).$$

However, the proof of this estimate seems to be far beyond the scope of this introduction. We only remark that then abstract functional analysis applies to prove that $\widetilde{N}$ has a unique continuous and linear extension from the dense subspace $\mathcal{D}(\Omega)$ to the entire space $\widetilde{H}^{-1}(\Omega)$, which concludes the proof of (i). We stress that the mapping properties of $\gamma_0\widetilde{N}$ from (iii) are an immediate consequence of (i) and the mapping properties of the trace operator γ_0.

The remaining steps (ii) and (iv) are proven simultaneously:

1. step. There is a unique operator $N_1 \in L(\widetilde{H}^{-1}(\Omega); H^{-1/2}(\Gamma))$ with $N_1 = \gamma_1\widetilde{N}$ on $\mathcal{D}(\Omega)$: For $f \in \mathcal{D}(\Omega)$, there holds $\widetilde{N}f \in C^\infty(\overline{\Omega})$ with $-\Delta(\widetilde{N}f) = f \in \mathcal{D}(\Omega) \subset \widetilde{H}^{-1}(\Omega)$. By definition, $N_1 f := \gamma_1\widetilde{N}f \in H^{-1/2}(\Gamma)$ is thus well-defined. Moreover, Lemma 8 provides the stability estimate

$$\|N_1 f\|_{H^{-1/2}(\Gamma)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} + \|\widetilde{N}f\|_{H^1(\Omega)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)},$$

where we have used the continuity of $\widetilde{N}$ in the final estimate. Since $\mathcal{D}(\Omega)$ is a dense subspace of $\widetilde{H}^{-1}(\Omega)$, there is a unique extension of N_1 from $\mathcal{D}(\Omega)$ to an operator $N_1 \in L\big(\widetilde{H}^{-1}(\Omega); H^{-1/2}(\Gamma)\big)$.

2. step. There holds $-\Delta(\widetilde{N}f) = f$ for all $f \in \widetilde{H}^{-1}(\Omega)$: By definition, we have to show that there is an element $\phi \in H^{-1/2}(\Gamma)$ such that

$$\langle f\,;\,v\rangle = (\nabla(\widetilde{N}f)\,;\,\nabla v)_{L^2(\Omega)} - \langle \phi\,;\,\gamma_0 v\rangle \quad \text{for all } v \in H^1(\Omega). \tag{20}$$

We choose $\phi := N_1 f$ with N_1 the extended operator from step 1. Now, let (f_n) be a sequence in $\mathcal{D}(\Omega)$ with $\lim\limits_{n\to\infty} f_n = f \in \widetilde{H}^{-1}(\Omega)$. For each $n \in \mathbb{N}$, there holds

$$\langle f_n\,;\,v\rangle = (\nabla(\widetilde{N}f_n)\,;\,\nabla v)_{L^2(\Omega)} - \langle N_1 f_n\,;\,\gamma_0 v\rangle \quad \text{for all } v \in H^1(\Omega) \tag{21}$$

since $-\Delta(\widetilde{N}f_n) = f_n$ and $N_1 f_n = \gamma_1\widetilde{N}f_n$. Note that $\lim_{n\to\infty}\widetilde{N}f_n = \widetilde{N}f \in H^1(\Omega)$ according to (i) and $\lim_{n\to\infty} N_1 f_n = N_1 f \in H^{-1/2}(\Gamma)$ by continuity of N_1. Thus, the equality (21) implies the equality (20) in the continuous limit $n \to \infty$.

3. step. The operator N_1 from step 1 satisfies $N_1 f = \gamma_1\widetilde{N}f$ for all $f \in \widetilde{H}^{-1}(\Omega)$: From step 2, we derive that $\gamma_1\widetilde{N}f \in H^{-1/2}(\Gamma)$ is well-defined. Moreover, Equation (20) with $\phi = N_1 f$ and the uniqueness of $\gamma_1(\widetilde{N}f)$ prove $N_1 f = \gamma_1(\widetilde{N}f)$. ∎

3.2. Single-Layer Potential.

We next consider the single-layer potential

$$\widetilde{V}\phi(x) = \int_\Gamma G(x-y)\phi(y)\,ds_y \quad \text{for all } x \in \Omega. \tag{22}$$

According to elementary calculus, there holds

$$\widetilde{V}\phi \in C^\infty(\mathbb{R}^d\backslash\Gamma), \quad \text{where} \quad \partial^\alpha\widetilde{V}\phi(x) = \int_\Gamma \partial_x^\alpha G(x-y)\phi(y)\,ds_y, \quad \text{for all} \quad \phi \in L^1(\Gamma). \tag{23}$$

With some technical overhead, one even may show that

$$\widetilde{V}\phi \in C(\mathbb{R}^d) \quad \text{provided that} \quad \phi \in L^\infty(\Gamma). \tag{24}$$

The following theorem states the most important facts on the operator $\widetilde{V}$.

Theorem 12. (i) $\widetilde{V}$ allows a unique extension $\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^1(\Omega)\big)$ from $L^2(\Gamma)$ to $H^{-1/2}(\Gamma)$.
(ii) $-\Delta\widetilde{V}\phi = 0 \in \widetilde{H}^{-1}(\Omega)$ for all $\phi \in H^{-1/2}(\Gamma)$.
(iii) $\gamma_0\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$.
(iv) $\gamma_1\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$.

Sketch of Proof. Recall that $L^2(\Gamma)$ is a dense subspace of $H^{-1/2}(\Gamma)$ and that (23) holds, in particular, for $\phi \in L^2(\Gamma)$. To prove (i) it thus only remains to prove the inequality

$$\|\widetilde{V}\phi\|_{H^1(\Omega)} \lesssim \|\phi\|_{H^{-1/2}(\Gamma)} \quad \text{for all } \phi \in L^2(\Omega). \tag{25}$$

This is done below in two steps. According to functional analysis, $\widetilde{V}$ then allows a unique extension from the dense subspace $L^2(\Gamma)$ to an operator $\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^1(\Omega)\big)$. This proves (i) and consequently (iii). To prove (ii) and (iv) note that $-\Delta G = 0$ in $\mathbb{R}^d\backslash\{0\}$ and (23) yield $-\Delta\widetilde{V}\phi = 0$ in Ω and hence in $L^2(\Omega) \subseteq \widetilde{H}^{-1}(\Omega)$ for all $\phi \in L^2(\Gamma)$. With the same arguments as for the Newton potential, one now proves (ii) and (iv) simultaneously.

1. step. There holds $(\widetilde{V}\phi\,;\,f)_{L^2(\Omega)} = (\phi\,;\,\widetilde{N}f)_{L^2(\Gamma)}$ for all $\phi \in L^2(\Gamma)$ and $f \in \mathcal{D}(\Omega)$, which follows again from the Fubini theorem and the symmetry of the Newton kernel:

$$(\widetilde{V}\phi\,;\,f)_{L^2(\Omega)} = \int_\Omega f(x)\int_\Gamma G(x-y)\phi(y)\,ds_y\,dx = \int_\Gamma \phi(y)\int_\Omega G(y-x)f(x)\,ds_y\,dx.$$

Note that the right-hand side could formally be written as $(\phi\,;\,\widetilde{N}f)_{L^2(\Gamma)} = (\phi\,;\,\gamma_0\widetilde{N}f)_{L^2(\Gamma)}$ with the trace of the Newton potential.

2. step. There holds (25): With the Hahn-Banach theorem, we have

$$\|\widetilde{V}\phi\|_{H^1(\Omega)} = \sup_{f\in\widetilde{H}^{-1}(\Omega)\backslash\{0\}} \frac{\langle\widetilde{V}\phi\,;\,f\rangle}{\|f\|_{\widetilde{H}^{-1}(\Omega)}} = \sup_{f\in\mathcal{D}(\Omega)\backslash\{0\}} \frac{(\widetilde{V}\phi\,;\,f)_{L^2(\Omega)}}{\|f\|_{\widetilde{H}^{-1}(\Omega)}} \tag{26}$$

since $\mathcal{D}(\Omega)$ is dense in $\widetilde{H}^{-1}(\Omega)$. From step 1 and the mapping properties of $\widetilde{N}$, we infer

$$(\widetilde{V}\phi\,;\,f)_{L^2(\Omega)} = (\phi\,;\,\gamma_0\widetilde{N}f)_{L^2(\Gamma)} \le \|\phi\|_{H^{-1/2}(\Gamma)}\|\gamma_0\widetilde{N}f\|_{H^{1/2}(\Gamma)} \lesssim \|\phi\|_{H^{-1/2}(\Gamma)}\|f\|_{\widetilde{H}^{-1}(\Omega)}$$

for all $f \in \mathcal{D}(\Omega)$. Together with (26), this proves (25). ■

3.3. Double-Layer Potential.

We finally consider the double-layer potential

$$\widetilde{K}v(x) = \int_\Gamma \partial_{n(y)}G(x-y)v(y)\,ds_y \quad \text{for all } x \in \Omega. \tag{27}$$

According to elementary calculus, there holds

$$\widetilde{K}\phi \in C^\infty(\mathbb{R}^d\backslash\Gamma), \quad \text{where} \quad \partial^\alpha\widetilde{K}\phi(x) = \int_\Gamma \partial_{n(y)}\partial_x^\alpha G(x-y)\phi(y)\,ds_y, \quad \text{for} \quad \phi \in L^1(\Gamma). \tag{28}$$

The fundamental theorem on the double-layer potential is the following:

Theorem 13. (i) There holds $\widetilde{K} \in L\big(H^{1/2}(\Gamma); H^1(\Omega)\big)$.
(ii) $-\Delta\widetilde{K}v = 0 \in \widetilde{H}^{-1}(\Omega)$ for all $v \in H^{1/2}(\Gamma)$.
(iii) $\gamma_0\widetilde{K} \in L\big(H^{1/2}(\Gamma); H^{1/2}(\Gamma)\big)$.
(iv) $\gamma_1\widetilde{K} \in L\big(H^{1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$.

Sketch of Proof. For $v \in H^{1/2}(\Gamma) \subseteq L^1(\Gamma)$ holds $\widetilde{K}v \in C^\infty(\Omega)$. Thus, we only need to prove that

$$\|\widetilde{K}v\|_{H^1(\Omega)} \lesssim \|v\|_{H^{1/2}(\Gamma)}.$$

This follows as in the proof for the single-layer potential by use of the Hahn-Banach theorem: For $\widetilde{K}$, the fundamental identity reads

$$(\widetilde{K}v\,;\,f)_{L^2(\Omega)} = (v\,;\,\gamma_1\widetilde{N}f)_{L^2(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma) \text{ and } f \in \mathcal{D}(\Omega),$$

where the normal derivative $\gamma_1\widetilde{N}f = \partial_n(\widetilde{N}f)$ is just classical. By this arguments, we prove (i) and (iii). The remaining proofs of (ii) and (iv) follow as for the Newton potential. ■

Contrary to the single-layer potential, the double-layer potential is not continuous on the entire space. Instead, one observes a jump of $\widetilde{K}v$ on Γ. To see this, we consider the simplest example with $v \equiv 1$ the constant function on Γ.

Lemma 14. There holds $(\widetilde{K}1)(x) = \int_\Gamma \partial_{n(y)}G(x-y)\,ds_y = \begin{cases} -1 & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \mathbb{R}^d\backslash\overline{\Omega}. \end{cases}$

Proof. The proof is only based on the representation formula. We consider the constant function $v \equiv 1$ in Ω. Then, the representation formula $v = \widetilde{N}(-\Delta v) + \widetilde{V}(\partial_n v) - \widetilde{K}v$ in Ω simplifies to

$$1 = -\widetilde{K}v(x) \quad \text{for all } x \in \Omega.$$

For $x \in \mathbb{R}^d\backslash\overline{\Omega}$, let $\varepsilon > 0$ with $B(x,\varepsilon) \subseteq \mathbb{R}^d\backslash\overline{\Omega}$ and consider the set $\Omega_\varepsilon := \Omega \cup B(x,\varepsilon)$. Applying the last result for Ω_ε and for $B(x,\varepsilon)$, we see

$$-1 = \int_{\partial\Omega_\varepsilon} \partial_{n(y)}G(x-y)\,ds_y = \widetilde{K}1(x) + \int_{S(x,\varepsilon)} \partial_{n(y)}G(x-y)\,ds_y = \widetilde{K}1(x) - 1,$$

whence $\widetilde{K}1(x) = 0$ for $x \in \mathbb{R}^d\backslash\overline{\Omega}$. ■

In fact, one can show that $K1(x) := \widetilde{K}1(x)$, for $x \in \Gamma$, takes the value $-1/2$ almost everywhere on Γ. Therefore, $K1(x) = -1/2 = (1/2 + \gamma_0\widetilde{K}1)(x)$, which implies $\gamma_0\widetilde{K}1 = (-1/2 + K)1$. It is quite technical but nevertheless elementary that this implies

$$\gamma_0(\widetilde{K}v) = (-1/2 + K)v \tag{29}$$

for all traces $v \in H^{1/2}(\Gamma)$.

4. Representation Theorem and Calderón System

In the analytical framework of the preceding sections, we can state the representation formula in its general form.

Theorem 15 (Representation Formula). For $u \in H^1(\Omega)$ with $-\Delta u = f \in \widetilde{H}^{-1}(\Omega)$ holds

$$u = \widetilde{N} f + \widetilde{V}(\gamma_1 u) - \widetilde{K}(\gamma_0 u). \tag{30}$$

In particular, the **Cauchy data** $(\gamma_0 u, \gamma_1 u) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ satisfy the **Calderón System**

$$\begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix} = \begin{pmatrix} -\gamma_0 \widetilde{K} & \gamma_0 \widetilde{V} \\ -\gamma_1 \widetilde{K} & \gamma_1 \widetilde{V} \end{pmatrix} \begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix} + \begin{pmatrix} \gamma_0 \widetilde{N} f \\ \gamma_1 \widetilde{N} f \end{pmatrix} \tag{31}$$

Proof. 1. step. We first prove (30) for $-\Delta u = f \in L^2(\Omega)$: So far, we have proven (30) only pointwise in Ω for $u \in C^2(\overline{\Omega})$,

$$u(x) = -\int_\Omega G(x-y)\Delta u(y)\, dy + \int_\Gamma G(x-y)\frac{\partial u}{\partial n_y}(y)\, ds_y - \int_\Gamma \frac{\partial_y}{\partial n_y} G(x-y)\, u(y)\, ds_y,$$

cf. Proposition 2. Note that the same proof works for $u \in H^1(\Omega)$ with $-\Delta u = f \in L^2(\Omega)$ since $f, u, \nabla u$ belong to $L^2(\Omega)$: The **Lebesgue differentiation theorem** states that, for any (representative of) Lebesgue function $v \in L^1_{\ell oc}(\Omega)$ holds

$$\lim_{\varepsilon \to 0} \fint_{B(x,\varepsilon)} |v(y) - v(x)|\, dy = 0 \quad \text{for almost all } x \in \Omega,$$

which then implies

$$v(x) = \lim_{\varepsilon \to 0} \fint_{B(x,\varepsilon)} v(y)\, dy.$$

With this new ingredient, you can use the old proof line by line.

2. step. We prove (30) for $-\Delta u = f \in \widetilde{H}^{-1}(\Omega)$: As $L^2(\Omega)$ is a dense subspace of $\widetilde{H}^{-1}(\Omega)$, we choose a sequence (f_n) in $L^2(\Omega)$ which converges to f in $\widetilde{H}^{-1}(\Omega)$. Then, let $u_n \in H^1(\Omega)$ be the unique weak solution of the Dirichlet problem

$$-\Delta u_n = f_n \text{ in } \Omega \quad \text{and} \quad \gamma_0 u_n = \gamma_0 u \text{ on } \Gamma.$$

According to Proposition 9, there holds $\|u - u_n\|_{H^1(\Omega)} + \|\gamma_1 u - \gamma_1 u_n\|_{H^{-1/2}(\Gamma)} \lesssim \|f - f_n\|_{\widetilde{H}^{-1}(\Omega)}$, whence (u_n) converges to u in $H^1(\Omega)$ and $(\gamma_1 u_n)$ converges to $\gamma_1 u$ in $H^{-1/2}(\Gamma)$. Since (30) is already proven for u_n, the continuity of the involved operators shows that (30) does even hold in the limit $n \to \infty$.

3. step. Taking the trace $\gamma_0 u$ and the normal derivative $\gamma_1 u$ in the representation formula (30), we prove (31). ■

The usual notation for the traces and normal derivatives of the three integral operators is the following:

- $N_0 := \gamma_0 \widetilde{N}$ $\qquad N_0 \in L\big(\widetilde{H}^{-1}(\Omega); H^{1/2}(\Gamma)\big)$
- $N_1 := \gamma_1 \widetilde{N}$ $\qquad N_1 \in L\big(\widetilde{H}^{-1}(\Omega); H^{-1/2}(\Gamma)\big)$
- $V := \gamma_0 \widetilde{V}$ **single-layer potential** $\qquad V \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$
- $K' := \gamma_1 \widetilde{V} - \frac{1}{2}$ **adjoint double-layer potential** $\qquad K' \in L\big(H^{-1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$
- $K := \gamma_0 \widetilde{K} + \frac{1}{2}$ **double-layer potential** $\qquad K \in L\big(H^{1/2}(\Gamma); H^{1/2}(\Gamma)\big)$
- $W := -\gamma_1 \widetilde{K}$ **hypersingular integral operator** $\qquad W \in L\big(H^{1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$

Note that, with this notation, the three operators with tilde, namely $\widetilde{N}$, $\widetilde{V}$, and $\widetilde{K}$, provide functions of $\widetilde{x} \in \Omega$, whereas the six operators without tilde provide functions (or functionals) defined on Γ. Don't get confused that $\widetilde{V}$ as well as V are named *single-layer potential* and $\widetilde{K}$ as well as K are named *double-layer potential* in the literature! With the new notation, the Calderón system reads

$$\begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - K & V \\ W & \frac{1}{2} + K' \end{pmatrix} \begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix} + \begin{pmatrix} N_0 f \\ N_1 f \end{pmatrix}$$

The operator matrix

$$C := \begin{pmatrix} \frac{1}{2} - K & V \\ W & \frac{1}{2} + K' \end{pmatrix} \tag{32}$$

is called **Calderón projector**. The first lemma provides some elementary further relations between the four operators V, K, K', W.

Lemma 16. The Calderón projector satisfies $C^2 = C$.

Proof. By definition, C is an operator from $\mathcal{H} := H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ to $\mathcal{H}$. We need to show that

$$C^2 \begin{pmatrix} v \\ \phi \end{pmatrix} = C \begin{pmatrix} v \\ \phi \end{pmatrix} \quad \text{for all } (v, \phi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma).$$

Let $(v, \phi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ and define $u := \widetilde{V}\phi - \widetilde{K}v \in H^1(\Omega)$. Then, $-\Delta u = 0 \in \widetilde{H}^{-1}(\Omega)$, and the Calderón system thus simplifies to

$$\begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix} = C \begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix}.$$

Moreover, we have $\gamma_0 u = V\phi + \big(\frac{1}{2} - K\big)v$ as well as $\gamma_1 u = \big(\frac{1}{2} + K'\big)\phi + Wv$ by definition of u. These equations for $\gamma_0 u$ and $\gamma_1 u$ can be written in the form

$$\begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix} = C \begin{pmatrix} v \\ \phi \end{pmatrix}.$$

Plugging this into the Calderón system, we conclude the proof. ■

Computing the entries of the matrix C^2 and comparing them to the entries of C, we obtain some elementary relations of the operators V, K, K', W. Beside these, we have some further *nontrivial* properties, which are now simply stated without a proof.

(i) The operator V is symmetric, i.e.

$$\langle V\phi\,;\,\psi\rangle = \langle \phi\,;\,V\psi\rangle \quad \text{for all } \phi,\psi \in H^{-1/2}(\Gamma). \tag{33}$$

(ii) Provided that $\mathrm{diam}(\Omega) < 1$ in case of $d = 2$, the operator V is $H^{-1/2}(\Gamma)$-elliptic, i.e.

$$\langle V\phi\,;\,\phi\rangle \gtrsim \|\phi\|_{H^{-1/2}(\Gamma)} \quad \text{for all } \phi \in H^{-1/2}(\Gamma). \tag{34}$$

(iii) The operator W is symmetric, i.e.

$$\langle Wv\,;\,w\rangle = \langle v\,;\,Ww\rangle \quad \text{for all } v,w \in H^{1/2}(\Gamma). \tag{35}$$

(iv) The operator W is $H^{1/2}_*(\Gamma)$-elliptic, i.e.

$$\langle Wv\,;\,v\rangle \gtrsim \|v\|_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}_*(\Gamma) := \big\{ w \in H^{1/2}(\Gamma) \,\big|\, \textstyle\int_\Gamma w\,ds = 0 \big\} \tag{36}$$

(v) The operator K' is the adjoint of K in the functional analytic sense, i.e.

$$\langle K'\phi\,;\,v\rangle = \langle \phi\,;\,Kv\rangle \quad \text{for all } \phi \in H^{-1/2}(\Gamma) \text{ and } v \in H^{1/2}(\Gamma). \tag{37}$$

Remark. We assume that $\mathrm{diam}(\Omega) < 1$ in case of $d = 2$. As an immediate consequence of (i) and (ii), we then observe that

$$\langle\!\langle \phi\,;\psi \rangle\!\rangle := \langle V\phi\,;\,\psi\rangle \quad \text{for } \phi,\psi \in H^{-1/2}(\Gamma) \tag{38}$$

defines a scalar product on $H^{-1/2}(\Gamma)$. With the continuity of V, one observes that the induced norm

$$|\!|\!|\phi|\!|\!| := \langle\!\langle \phi\,;\phi \rangle\!\rangle^{1/2} \quad \text{for } \phi \in H^{-1/2}(\Gamma) \tag{39}$$

is an equivalent norm on $H^{-1/2}(\Gamma)$. ■

Corollary 17. The operator $V \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$ is an isomorphism, where we additionally assume that $\mathrm{diam}(\Omega) < 1$ in case of $d = 2$.

Proof. Since V is a continuous operator from the Banach space $H^{-1/2}(\Gamma)$ to the Banach space $H^{1/2}(\Gamma)$, it only remains to prove that V is bijective. Note that ellipticity implies $\ker(V) = \{0\}$, whence injectivity. To prove surjectivity, let $F \in H^{1/2}(\Gamma)$. According to the Hahn-Banach theorem

$$V\phi = F \tag{40}$$

is equivalent to

$$\langle\!\langle \phi\,;\psi \rangle\!\rangle := \langle V\phi\,;\,\psi\rangle = \langle F\,;\,\psi\rangle \quad \text{for all } \psi \in H^{-1/2}(\Gamma). \tag{41}$$

Since the right-hand side defines a continuous linear functional on $H^{-1/2}(\Gamma)$, the Riesz theorem on Hilbert spaces proves the unique existence of $\phi \in H^{-1/2}(\Gamma)$ with (41) resp. (40). ■

Remark. The reader may want to prove as an exercise that $W \in L\big(H_*^{1/2}(\Gamma); H_*^{-1/2}(\Gamma)\big)$ is an isomorphism. To that end, one has to prove that

- $Wv \in H_*^{-1/2}(\Gamma) := \big\{\psi \in H^{-1/2}(\Gamma) \,\big|\, \langle \psi\,;\,1\rangle = 0\big\}$ for all $v \in H^{1/2}(\Gamma)$.
- $H_*^{-1/2}(\Gamma)$ is the dual space of $H_*^{1/2}(\Gamma)$ with respect to the extended L^2-scalar product.

From this, we may deduce that $\ker W = \mathbb{R}$, i.e. the kernel of W are the constant functions. ■

5. Galerkin Discretization

5.1. PDE vs. Integral Equation. From now on, we consider the model problem

$$-\Delta u = f \in \widetilde{H}^{-1}(\Omega) \quad \text{with Dirichlet boundary conditions} \quad \gamma_0 u = v \in H^{1/2}(\Gamma). \tag{42}$$

The (unknown) weak solution $u \in H^1(\Omega)$ is written by the representation formula

$$u = \widetilde{N}f + \widetilde{V}\phi - \widetilde{K}v \tag{43}$$

with the (unknown) Neumann data $\phi = \gamma_1 u \in H^{-1/2}(\Gamma)$. From the first equality of the Calderón system, we obtain

$$v = \gamma_0 u = \gamma_0 \widetilde{N}f + \gamma_0 \widetilde{V}\phi - \gamma_0 \widetilde{K}v = N_0 f + V\phi - (K - 1/2)v.$$

This yields **Symm's integral equation**

$$V\phi = (K + 1/2)v - N_0 f, \tag{44}$$

which is a first-kind integral equation. We stress the following elementary observations, which state equivalency of the partial differential equation (42) and the integral equation (44).

Remark. If $u \in H^1(\Omega)$ solves (42), then $\phi := \gamma_1 u$ solves Symm's integral equation (44). Conversely, if $\phi \in H^{-1/2}(\Gamma)$ solves (44), the function $u \in H^1(\Omega)$ defined by (43) solves the Dirichlet problem (42). Throughout, we shall assume that V is an elliptic isomorphism, so that the normal derivative $\phi = \gamma_1 u$ is in fact the unique solution of Symm's integral equation (44). ■

Usually, it is impossible to solve (44) analytically. Therefore, we aim to compute an approximation ϕ_h of ϕ by use of a Galerkin scheme.

5.2. Galerkin Method. To simplify notation, we assume that $F \in H^{1/2}(\Gamma)$ is given. We aim to approximate the unique solution $\phi \in H^{-1/2}(\Gamma)$ of

$$V\phi = F. \tag{45}$$

According to the Hahn-Banach theorem, this is equivalently written by

$$\langle\!\langle \phi\,;\psi \rangle\!\rangle = \langle F\,;\,\psi \rangle \quad \text{for all } \psi \in H^{-1/2}(\Gamma), \tag{46}$$

where $\langle\!\langle \phi\,;\psi \rangle\!\rangle := \langle V\phi\,;\,\psi \rangle$. We have already noted that the left-hand side of (46) defines a scalar product on $H^{-1/2}(\Gamma)$, and the induced **energy norm**

$$|\!|\!|\psi|\!|\!| := \langle\!\langle \psi\,;\psi \rangle\!\rangle^{1/2} \quad \text{for } \psi \in H^{-1/2}(\Gamma) \tag{47}$$

is an equivalent norm on $H^{-1/2}(\Gamma)$. We therefore proceed as in the context of the finite element method: Let X_h be a finite dimensional subspace of $H^{-1/2}(\Gamma)$ with basis $\{\chi_1, \dots, \chi_N\}$. Then, the Lax-Milgram lemma applies for X_h and provides a unique **Galerkin approximation** $\phi_h \in X_h$ such that

$$\langle\!\langle \phi_h\,;\psi_h \rangle\!\rangle = \langle F\,;\,\psi_h \rangle \quad \text{for all } \psi_h \in X_h. \tag{48}$$

If we write $\phi_h = \sum_{j=1}^N x_j\chi_j$ with respect to the fixed basis, the coefficient vector $x \in \mathbb{R}^N$ solves the linear system

$$Ax = b, \tag{49}$$

where the Galerkin matrix $A \in \mathbb{R}^{N\times N}$ and the right-hand side $b \in \mathbb{R}^N$ are defined by

$$A_{jk} = \langle\!\langle \chi_j\,;\chi_k \rangle\!\rangle \quad \text{and} \quad b_k = \langle F\,;\chi_k\rangle \quad \text{for all } j,k = 1,\dots,N.$$

Note that the Galerkin matrix is symmetric and positive definite. In particular, the linear system (49) is equivalent to the Galerkin equations (48). We stress that ϕ_h is in fact characterized by the **Galerkin orthogonality**

$$\langle\!\langle \phi - \phi_h\,;\psi_h \rangle\!\rangle = 0 \quad \text{for all } \psi_h \in X_h. \tag{50}$$

There holds the following well-known bestapproximation property of the Galerkin solution.

Proposition 18 (Céa-Lemma). There holds $|\!|\!| \phi - \phi_h |\!|\!| = \min_{\psi_h\in X_h} |\!|\!| \phi - \psi_h |\!|\!|$.

Proof. For any $\psi_h \in X_h$, the Galerkin orthogonality proves

$$|\!|\!| \phi - \phi_h |\!|\!|^2 = \langle\!\langle \phi - \phi_h\,;\phi - \psi_h \rangle\!\rangle \le |\!|\!| \phi - \phi_h |\!|\!| |\!|\!| \phi - \psi_h |\!|\!|.$$

Taking the infimum over all $\psi_h \in X_h$, we conclude the proof. ■

Remark. For given $\psi \in H^{-1/2}(\Gamma)$, the Lax-Milgram lemma provides a unique element $\mathbb{G}_h\psi \in X_h$ such that

$$\langle\!\langle \mathbb{G}_h\psi\,;\psi_h \rangle\!\rangle = \langle\!\langle \psi\,;\psi_h \rangle\!\rangle \quad \text{for all } \psi_h \in X_h.$$

This provides the so-called **Galerkin projection**

$$\mathbb{G}_h : H^{-1/2}(\Gamma) \to X_h. \tag{51}$$

According to the Galerkin orthogonality (50), elementary functional analysis proves that $\mathbb{G}_h$ is the orthogonal projection with respect to the energy scalar product $\langle\!\langle \cdot\,;\cdot \rangle\!\rangle$. Therefore,

$$|\!|\!| \mathbb{G}_h\psi |\!|\!| \le |\!|\!| \psi |\!|\!|, \tag{52}$$

which also follows directly from the Pythagoras theorem $|\!|\!| \psi |\!|\!|^2 = |\!|\!| \psi - \mathbb{G}_h\psi |\!|\!|^2 + |\!|\!| \mathbb{G}_h\psi |\!|\!|^2$. ■

From now on, $\phi \in H^{-1/2}(\Gamma)$ denotes the exact solution of (48), whereas $\phi_h = \mathbb{G}_h\phi \in X_h$ denotes the Galerkin approximation with respect to X_h.

6. A Priori Error Analysis

For the Galerkin boundary element method, we choose piecewise polynomial spaces X_h: Let $\mathcal{T}_h$ be a triangulation of Γ, i.e.

- $\mathcal{T}_h = \{T_1, \dots, T_N\}$ is a finite set of subsets $T_j \subseteq \Gamma$,
- each $T_j \in \mathcal{T}_h$ is (relatively) open and connected with positive surface measure $|T_j| > 0$,
- for $T_j, T_k \in \mathcal{T}_h$ with $j \neq k$ holds $T_j \cap T_k = \emptyset$,
- $\Gamma = \bigcup \{\overline{T} \mid T \in \mathcal{T}_h\}$, i.e. $\mathcal{T}_h$ is a covering of Γ.

For the ease of presentation, we additionally assume that the elements $T \in \mathcal{T}_h$ are flat, i.e. there is an open and connected set $V_T \subset \mathbb{R}^{d-1}$ and an affine bijection $\Phi_T : \mathbb{R}^d \to \mathbb{R}^d$ such that $\Phi_T(V_T) = T$. Moreover, we assume that the elements are convex — this clearly holds in 2D and is the common case in 3D, where the elements usually are flat triangles or rectangles. With χ_T the characteristic function of a set T, we consider the space

$$\mathcal{P}^0(\mathcal{T}_h) := \operatorname{span}\{\chi_T \mid T \in \mathcal{T}_h\} \tag{53}$$

of all $\mathcal{T}_h$-piecewise constant functions. We define the **local mesh-width**

$$h \in \mathcal{P}^0(\mathcal{T}_h), \quad h|_{T_j} = h_{T_j} := \operatorname{diam}(T_j) := \sup_{x,y \in T_j} |x - y| \tag{54}$$

as well as the **maximal mesh-width**

$$h_{\max} := \|h\|_{L^\infty(\Gamma)} = \max_{T \in \mathcal{T}_h} h_T. \tag{55}$$

Moreover, we define the **shape regularity constant**

$$\sigma(\mathcal{T}_h) := \max_{T \in \mathcal{T}_h} \frac{h_T^{d-1}}{|T|} \tag{56}$$

to measure the degeneracy of the elements in $\mathcal{T}_h$.

Remark. In the finite element analysis, the shape regularity constant from (56) involves h_T^d instead of h_T^{d-1}. We stress that in the context of boundary elements h_T^{d-1} coincides to the fact that we are dealing with $(d-1)$-dimensional manifolds. ■

Theorem 19 (Approximation Theorem). Let $\Pi_h : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_h)$ denote the L^2-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_h)$. For $\psi \in L^2(\Gamma) \cap H^1(\mathcal{T}_h)$, holds

$$\|\psi - \Pi_h \psi\|_{H^{-1/2}(\Gamma)} \lesssim \|h^{3/2} \nabla_{\mathcal{T}} \psi\|_{L^2(\Gamma)}, \tag{57}$$

where the constant only depends on the shape regularity constant $\sigma(\mathcal{T}_h)$. Here, $\psi \in H^1(\mathcal{T}_h)$ means that $\psi \in H^1(T)$ for all $T \in \mathcal{T}_H$, and $\nabla_{\mathcal{T}}$ thus denotes the $\mathcal{T}_h$-elementwise gradient.

Proof. The elementary proof of (57) is split into four steps.

1. step. The L^2-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_h)$ can explicitly be written as

$$(\Pi_h v)|_T = \frac{1}{|T|}\int_T v\,ds \quad \text{for all } v \in L^2(\Gamma) \text{ and all } T \in \mathcal{T}_h. \tag{58}$$

This follows from the orthogonality property

$$0 = (v - \Pi_h v\,;\,\chi_T)_{L^2(\Gamma)} = \int_T v\,ds - \int_T (\Pi_h v)\,ds = \int_T v\,ds - |T|\,(\Pi_h v)|_T \quad \text{for all } T \in \mathcal{T}_h.$$

2. step. According to the Poincaré inequality, there holds, for any $T \in \mathcal{T}_h$,

$$\|\psi - \Pi_h\psi\|_{L^2(T)} \le \frac{1}{\pi}\,h_T\|\nabla\psi\|_{L^2(T)},$$

where convexity of $T \in \mathcal{T}_h$ provides the Poincaré constant $1/\pi$.

3. step. For any $v \in H^{1/2}(\Gamma)$ and $T \in \mathcal{T}_h$ holds

$$\|v - \Pi_h v\|_{L^2(T)} \le \sigma(\mathcal{T}_h)^{1/2}\,h_T^{1/2}\,|v|_{H^{1/2}(T)}.$$

We recall that the $H^{1/2}$-Sobolev-Slobodecky seminorm is defined by

$$|v|_{H^{1/2}(T)} = \Big(\int_T\int_T \frac{|v(x)-v(y)|^2}{|x-y|^d}\,ds_y\,ds_x\Big)^{1/2}$$

For fixed $x \in T$, the closed form of $\Pi_h v$ from step 1 and the Cauchy inequality prove

$$\begin{aligned}
|v(x) - \Pi_h v(x)|^2 &= \frac{1}{|T|^2}\Big(\int_T v(x) - v(y)\,ds_y\Big)^2 \\
&\le \frac{1}{|T|^2}\Big(\int_T \frac{|v(x)-v(y)|^2}{|x-y|^d}\,ds_y\Big)\Big(\int_T |x-y|^d\,ds_y\Big) \\
&\le \frac{h_T^{d-1}}{|T|}\,h_T\int_T \frac{|v(x)-v(y)|^2}{|x-y|^d}\,ds_y.
\end{aligned}$$

Integration over T now yields

$$\|v - \Pi_h v\|_{L^2(T)}^2 \le \sigma(\mathcal{T}_h)\,h_T\,|v|_{H^{1/2}(T)}^2.$$

4. step. Finally, we estimate the dual norm

$$\|\psi - \Pi_h\psi\|_{H^{-1/2}(\Gamma)} = \sup_{v\in H^{1/2}(\Gamma)\setminus\{0\}} \frac{\langle \psi - \Pi_h\psi\,;\,v\rangle}{\|v\|_{H^{1/2}(\Gamma)}}.$$

Let $v \in H^{1/2}(\Gamma)$. We stress that the duality brackets are just the L^2-scalar product since both $\psi - \Pi_h\psi, v \in L^2(\Gamma)$. Orthogonality of Π_h provides

$$\langle \psi - \Pi_h\psi\,;\,v\rangle = (\psi - \Pi_h\psi\,;\,v - \Pi_h v)_{L^2(\Gamma)} = \sum_{T\in\mathcal{T}_h}(\psi - \Pi_h\psi\,;\,v - \Pi_h v)_{L^2(T)}.$$

For fixed $T \in \mathcal{T}_h$ holds

$$(\psi - \Pi_h\psi\,;\, v - \Pi_h v)_{L^2(T)} \le \|\psi - \Pi_h\psi\|_{L^2(T)}\|v - \Pi_h v\|_{L^2(T)}$$
$$\le \frac{\sigma(\mathcal{T}_h)^{1/2}}{\pi}\, h_T^{3/2}\, \|\nabla\psi\|_{L^2(T)}|v|_{H^{1/2}(T)}.$$

Therefore, the Cauchy inequality proves

$$\langle \psi - \Pi_h\psi\,;\, v\rangle \le \frac{\sigma(\mathcal{T}_h)^{1/2}}{\pi}\Big(\sum_{T\in\mathcal{T}} h_T^3\|\nabla\psi\|^2_{L^2(T)}\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}} |v|^2_{H^{1/2}(T)}\Big)^{1/2}$$
$$\le \frac{\sigma(\mathcal{T}_h)^{1/2}}{\pi}\,\|h^{3/2}\nabla_{\mathcal{T}}\psi\|_{L^2(\Gamma)}\,|v|_{H^{1/2}(\Gamma)}.$$

This concludes the proof. ■

Remark. The proof of Proposition 19 is suboptimal in the sense that Estimate (57) holds without dependence on the shape regularity constant $\sigma(\mathcal{T}_h)$. However, the proof of which in Carstensen-Praetorius 2006 relies on deeper mathematical techniques. ■

Remark. We stress that the same techniques as in the preceding proof yield

$$\|\psi - \Pi_h\psi\|_{H^{-1/2}(\Gamma)} \lesssim \|h^{1/2}(\psi - \Pi_h\psi)\|_{L^2(\Gamma)} \le \|h^{1/2}\psi\|_{L^2(\Gamma)} \quad \text{for all } \psi \in L^2(\Gamma). \tag{59}$$

Here, the second inequality follows from the Pythagoras theorem (i.e. the L^2-orthogonality)

$$\|\psi - \Pi_h\psi\|^2_{L^2(T)} = \|\psi\|^2_{L^2(T)} - \|\Pi_h\psi\|^2_{L^2(T)} \le \|\psi\|^2_{L^2(T)}.$$

This $\mathcal{T}_h$-elementwise estimate is simply weighted by h_T and then added over all $T \in \mathcal{T}_h$. ■

The combination of Céa-Lemma and approximation theorem provides an a priori error estimate.

Corollary 20 (A Priori Estimate for Galerkin Error). Provided that the exact solution $\phi \in H^{-1/2}(\Gamma)$ of (45) satisfies $\phi \in L^2(\Gamma) \cap H^1(\mathcal{T}_h)$, there holds

$$|||\phi - \phi_h||| \lesssim \|h^{3/2}\nabla\phi\|_{L^2(\Gamma)}, \tag{60}$$

where the constant only depends on Γ and the shape regularity constant $\sigma(\mathcal{T}_h)$.

Proof. First recall that the energy norm $|||\cdot|||$ is an equivalent norm on $H^{-1/2}(\Gamma)$, i.e.

$$C_{\text{lower}}\|\psi\|_{H^{-1/2}(\Gamma)} \le |||\psi||| \le C_{\text{upper}}\|\psi\|_{H^{-1/2}(\Gamma)} \quad \text{for all } \psi \in H^{-1/2}(\Gamma).$$

The lower constant $C_{\text{lower}} > 0$ is just the square-root of the ellipticity constant of V, whereas $C_{\text{upper}} > 0$ is the square-root of the operator norm of V, i.e. both constants depend on Γ only. With the L^2-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_h)$, the Céa-Lemma proves

$$|||\phi - \phi_h||| \le |||\phi - \Pi_h\phi||| \approx \|\phi - \Pi_h\phi\|_{H^{-1/2}(\Gamma)} \lesssim \|h^{3/2}\nabla\phi\|_{L^2(\Gamma)},$$

where we have used that the energy norm $\|\cdot\|$ is an equivalent norm on $H^{-1/2}(\Gamma)$. ■

The preceeding corollary proves that

$$\|\phi - \phi_h\| = \mathcal{O}(h_{\max}^{3/2})$$

in the case that ϕ is sufficiently regular and that the shape regularity constant remains bounded. Finally, we prove that — even without any further regularity assumptions on the exact solution $\phi \in H^{-1/2}(\Gamma)$ — the sequence of Galerkin solutions ϕ_h converges to ϕ. To that end, we consider a sequence $\mathcal{T}_h^{(n)}$ of triangulations with

$$\mathcal{P}^0(\mathcal{T}_h^{(n)}) \subseteq \mathcal{P}^0(\mathcal{T}_h^{(n+1)}),$$

i.e. $\mathcal{T}_h^{(n+1)}$ is obtained from certain refinements of $\mathcal{T}_h^{(n)}$. Let $\phi_h^{(n)} \in \mathcal{P}^0(\mathcal{T}_h^{(n)})$ the sequence of corresponding Galerkin solutions.

Corollary 21 (Convergence of Galerkin Method). Provided that

$$\sigma := \sup_{n\in\mathbb{N}} \sigma(\mathcal{T}_h^{(n)}) < \infty \quad \text{and} \quad \lim_{n\to\infty} h_{\max}^{(n)} = 0,$$

there holds convergence $\lim_{n\to\infty} \|\phi - \phi_h^{(n)}\| = 0$.

Proof. Note that $H^1(\Gamma)$ is dense in $H^{-1/2}(\Gamma)$. Given $\varepsilon > 0$, we therefore find $\psi \in H^1(\Gamma)$ such that $\|\phi - \psi\| \le \varepsilon$. According to the a priori error estimate, there holds $\|\psi - \mathbb{G}_h^{(n)}\psi\| = \mathcal{O}((h_{\max}^{(n)})^{3/2})$. Therefore, there is $n_0 \in \mathbb{N}$ such that

$$\|\psi - \mathbb{G}_h^{(n)}\psi\| \le \varepsilon$$

for any $n \ge n_0$. The triangle inequality now proves

$$\|\phi - \phi_h^{(n)}\| \le \|\phi - \psi\| + \|\psi - \mathbb{G}_h^{(n)}\psi\| + \|\mathbb{G}_h^{(n)}\psi - \phi_h^{(n)}\| \le 3\varepsilon \quad \text{for all } n \ge n_0,$$

where we have used $\mathbb{G}_h^{(n)}\psi - \phi_h^{(n)} = \mathbb{G}_h^{(n)}(\psi - \phi)$ as well as $\|\mathbb{G}_h^{(n)}(\psi - \phi)\| \le \|\psi - \phi\|$. This proves convergence. ■

Remark. Since the step functions are dense in $L^2(\Gamma)$, one can prove that

$$\mathcal{P}^0(\mathcal{T}_h^{(n)}) \subseteq \mathcal{P}^0(\mathcal{T}_h^{(n+1)}) \quad \text{and} \quad \lim_{n\to\infty} h_{\max}^{(n)} = 0$$

implies that $X := \bigcup_{n\in\mathbb{N}} \mathcal{P}^0(\mathcal{T}_h^{(n)})$ is dense in $L^2(\Gamma)$ as well. Recalling that $L^2(\Gamma)$ is dense in $H^{-1/2}(\Gamma)$, we derive that X is dense in $H^{-1/2}(\Gamma)$ as well. In particular, this proves convergence of the Galerkin boundary element method without the additional assumption of

$$\sigma := \sup_{n\in\mathbb{N}} \sigma(\mathcal{T}_h^{(n)}) < \infty.$$

We stress, however, that this is a special observation for piecewise constant ansatz functions and negative-order Sobolev spaces. The proof of Corollary 21 even applies for the finite element method and positive-order Sobolev spaces, e.g., $H^1(\Omega)$. ■

7. A Posteriori Error Analysis

The a priori error analysis from the previous section provides an estimate of the form

$$|||\phi - \phi_h||| \le C_{\mathrm{apriori}}\,\eta(\phi)$$

with a right-hand side $\eta(\phi)$ which depends on the (regularity of the) exact solution $\phi \in H^{-1/2}(\Gamma)$ and given data but not on ϕ_h, i.e. we can estimate the error before the computation of ϕ_h. However, in general ϕ is unknown and thus $\eta(\phi)$ cannot be computed in practice. Moreover, in general, ϕ does not satisfy the necessary regularity assumptions. Nevertheless, the a priori analysis provides the convergence analysis for the numerical schemes and is thus a necessary part of the numerical analysis.

Since it is important to control the error $|||\phi - \phi_h|||$ in numerical computations, one aims to provide so-called **a posteriori error estimates**, which are divided in **reliability**

$$|||\phi - \phi_h||| \le C_{\mathrm{rel}}\eta(\phi_h)$$

and **efficiency**

$$\eta(\phi_h) \le C_{\mathrm{eff}}|||\phi - \phi_h|||.$$

Here, the **error estimator** $\eta = \eta(\phi_h)$ is a computable quantity that only depends on some given data and an already computed Galerkin solution ϕ_h, i.e. we estimate the error after the computation of ϕ_h. For practical purposes, one is interested in the localization of the error estimator to get information where one has to refine the triangulation $\mathcal{T}_h := \{T_1, \dots, T_N\}$ to improve the computation. To that end, one usually aims to provide an a posteriori error estimator η which satisfies

$$\eta = \Big(\sum_{j=1}^{N} \eta_j^2\Big)^{1/2}$$

with some so-called **refinement indicators** η_j which (heuristically) reflect the error $\phi - \phi_h$ on the surface panel T_j.

7.1. Residual-Based Error Estimators. We consider the residual $F - V\phi_h = V(\phi - \phi_h) \in H^{1/2}(\Gamma)$. Since V is an isomorphism between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, norm equivalency on $H^{-1/2}(\Gamma)$ proves

$$|||\phi - \phi_h||| \approx \|\phi - \phi_h\|_{H^{-1/2}(\Gamma)} \approx \|V(\phi - \phi_h)\|_{H^{1/2}(\Gamma)} = \|F - V\phi_h\|_{H^{1/2}(\Gamma)} \qquad (61)$$

In theory, the right-hand side provides a reliable and efficient a posteriori error estimator. However, we stress that the $H^{1/2}$-norm is very hard (if not impossible) to compute in practice. Moreover, the $H^{1/2}$-norm is nonlocal, i.e. for a generic function $v \in H^{1/2}(\Gamma)$ [which does not satisfy $\mathrm{supp}(v) \subseteq \overline{T}_j$ for some $T_j \in \mathcal{T}_h$] holds

$$\|v\|^2_{H^{1/2}(\Gamma)} \gneqq \sum_{j=1}^{N} \|v\|^2_{H^{1/2}(T_j)},$$

and the converse inequality fails to hold for any constant $C > 0$. Therefore, $\|F - V\phi_h\|_{H^{1/2}(\Gamma)}$ does not provide any information where to refine the triangulation $\mathcal{T}_h$. To obtain a local bound, note that the residual satisfies

$$\int_{T_j} (F - V\phi_h)\, ds = \langle V(\phi - \phi_h)\,;\, \chi_{T_j}\rangle = \langle\!\langle \phi - \phi_h\,;\chi_{T_j}\rangle\!\rangle = 0 \quad \text{for all } j = 1, \dots, N$$

according to the Galerkin orthogonality. The following theorem now provides a localization.

Theorem 22 (Carstensen, Maischak, Stephan 2001). Provided that $R \in H^1(\Gamma)$ satisfies $\int_T R\, ds = 0$ for all $T \in \mathcal{T}_h$, there holds

$$\|R\|_{H^{1/2}(\Gamma)} \lesssim \|h^{1/2}\nabla R\|_{L^2(\Gamma)}, \tag{62}$$

where the constant only depends on Γ and the shape regularity constant $\sigma(\mathcal{T}_h)$. ■

Corollary 23 (Carstensen, Maischak, Stephan 2001). Provided $F \in H^1(\Gamma)$, there holds

$$|\!|\!|\phi - \phi_h|\!|\!| \lesssim \eta_R := \Big(\sum_{j=1}^N \eta_{R,j}^2\Big)^{1/2} \quad \text{where} \quad \eta_{R,j} := h_{T_j}^{1/2}\|\nabla(F - V\phi_h)\|_{L^2(T_j)}, \tag{63}$$

where the reliability constant only depends on Γ and the shape regularity constant $\sigma(\mathcal{T}_h)$.

Proof. One can prove that $V \in L\big(L^2(\Gamma); H^1(\Gamma)\big)$ is a well-defined and continuous operator. According to $F \in H^1(\Gamma)$ and $\mathcal{P}^0(\mathcal{T}_h) \subset L^2(\Gamma)$, we thus obtain that the residual $F - V\phi_h$ belongs to $H^1(\Gamma)$. Therefore, the reliability (63) follows from (62). ■

Remark. Efficiency of the residual-based error estimator η_R is observed numerically. The analytical proof is still open in general. Carstensen 1996 proved efficiency of η_R in 2D for uniform mesh-refinement, which is the only result available. This does not cover, however, the more important case of adaptive mesh-refinement. ■

7.2. Error Estimators based on h-h/2-Strategy. The next technique is a rather classical idea to derive an a posteriori error estimate. We assume that $\mathcal{T}_h = \{T_1, \dots, T_N\}$ is a given triangulation and $\mathcal{T}_{h/2}$ is a uniform refinement of $\mathcal{T}_h$. In particular, there holds

$$\mathcal{P}^0(\mathcal{T}_h) \subseteq \mathcal{P}^0(\mathcal{T}_{h/2}). \tag{64}$$

We then compute the corresponding Galerkin solutions $\phi_h \in \mathcal{P}^0(\mathcal{T}_h)$ and $\phi_{h/2} \in \mathcal{P}^0(\mathcal{T}_{h/2})$. Note that the Galerkin orthogonality for $\mathcal{P}^0(\mathcal{T}_{h/2})$ and the Pythagoras theorem prove

$$|\!|\!|\phi - \phi_h|\!|\!|^2 = |\!|\!|(\phi - \phi_{h/2}) + (\phi_{h/2} - \phi_h)|\!|\!|^2 = |\!|\!|\phi - \phi_{h/2}|\!|\!|^2 + |\!|\!|\phi_{h/2} - \phi_h|\!|\!|^2 \tag{65}$$

since $\phi_{h/2} - \phi_h \in \mathcal{P}^0(\mathcal{T}_{h/2})$. In particular, we obtain the efficiency estimate

$$|\!|\!|\phi_{h/2} - \phi_h|\!|\!| \le |\!|\!|\phi - \phi_h|\!|\!|. \tag{66}$$

Under the **saturation assumption**

$$\|\phi - \phi_{h/2}\| \le q\,\|\phi - \phi_h\| \quad \text{for some uniform constant } q < 1 \tag{67}$$

Equation (65) implies the reliability estimate

$$\|\phi - \phi_h\| \le \frac{1}{\sqrt{1-q^2}}\,\|\phi_{h/2} - \phi_h\|. \tag{68}$$

Therefore, $\|\phi_{h/2} - \phi_h\|$ may be used for error estimation.

Remark. Note that the space inclusion (64) and the Céa-Lemma imply $\|\phi - \phi_{h/2}\| \le \|\phi - \phi_h\|$, i.e. the saturation assumption (67) is a uniformly stronger version of the Céa-Lemma. We assume that uniform refinement always leads to a uniform and strict improvement of the Galerkin error. ■

Remark. The quantity $\|\phi_{h/2} - \phi_h\|$ is easy to compute with the help of the Galerkin orthogonality: Note that $\phi_h = \mathbb{G}_h\phi_{h/2}$ so that the Pythagoras theorem yields $\|\phi_{h/2} - \phi_h\|^2 = \|\phi_{h/2}\|^2 - \|\phi_h\|^2$. The computation of the discrete energies, e.g. $\|\phi_h\|^2$, is a byproduct of our computation: To compute ϕ_h, we assemble the Galerkin matrix $A \in \mathbb{R}^{N\times N}_{\rm sym}$ and the right-hand side $b \in \mathbb{R}^N$. Solving the linear system $Ax = b$ provides the coefficient vector $x \in \mathbb{R}^N$ of ϕ_h. Then, $\|\phi_h\|^2 = \langle\!\langle \phi_h\,;\phi_h \rangle\!\rangle = x\cdot Ax = x\cdot b$. ■

Since the energy norm $\|\cdot\|$ in case of Symm's integral equation is nonlocal, we again do not get any information, where to refine the mesh. This drawback is overcome by the following theorem.

Theorem 24 (Ferraz-Leite, Praetorius 2008). The error estimator

$$\eta_H := \Big(\sum_{j=1}^N \eta_{H,j}^2\Big)^{1/2} \quad \text{with} \quad \eta_{H,j} = h_{T_j}^{1/2}\|\phi_{h/2} - \phi_h\|_{L^2(T_j)} \tag{69}$$

is always efficient, i.e.

$$\eta_H \lesssim \|\phi - \phi_h\|. \tag{70}$$

Under the saturation assumption (67), there even holds reliability

$$\|\phi - \phi_h\| \lesssim \eta_H. \tag{71}$$

The efficiency constant depends only on Γ and $\sigma(\mathcal{T}_h)$, whereas the reliability constant additionally depends on the saturation constant $q < 1$ from (67).

Proof. The proof is split into four steps.

1. step. Efficiency of η_H: It is a result of GRAHAM-HACKBUSCH-SAUTER 2005 that

$$\|h^{1/2}\psi_h\|_{L^2(\Gamma)} \lesssim \|\psi_h\| \quad \text{for all } \psi_h \in \mathcal{P}^0(\mathcal{T}_h), \tag{72}$$

where the constant only depends on the polynomial degree $p = 0$ and the shape-regularity constant $\sigma(\mathcal{T}_h)$. Since $\phi_{h/2} - \phi_h \in \mathcal{P}^0(\mathcal{T}_{h/2})$, the inverse estimate (72) implies

$$\eta_H = \|h^{1/2}(\phi_{h/2} - \phi_h)\|_{L^2(\Gamma)} \lesssim \sqrt{2}\,\|\phi_{h/2} - \phi_h\|.$$

This proves $\eta_H \lesssim \|\phi - \phi_h\|$.

2. step. The Galerkin projection satisfies $\|\psi - \mathbb{G}_h\psi\| \lesssim \|h^{1/2}\psi\|_{L^2(\Gamma)}$ for all $\psi \in L^2(\Gamma)$: Let Π_h be the L^2-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_h)$. According to the Céa-Lemma and the approximation property (59) of Π_h, there holds

$$\|\psi - \mathbb{G}_h\psi\| \leq \|\psi - \Pi_h\psi\| \approx \|\psi - \Pi_h\psi\|_{H^{-1/2}(\Gamma)} \lesssim \|h^{1/2}\psi\|_{L^2(\Gamma)}.$$

3. step. The Galerkin projection satisfies $\|\psi - \mathbb{G}_h\psi\| \lesssim \|h^{1/2}(\psi - \mathbb{G}_h\psi)\|_{L^2(\Gamma)}$ for all $\psi \in L^2(\Gamma)$: The projection property $\mathbb{G}_h^2 = \mathbb{G}_h$ yields $\Psi := \psi - \mathbb{G}_h\psi = (1 - \mathbb{G}_h)\psi = (1 - \mathbb{G}_h)^2\psi = \Psi - \mathbb{G}_h\Psi$. We now simply apply step 2 to the function $\Psi \in L^2(\Gamma)$ to obtain

$$\|\psi - \mathbb{G}_h\psi\| = \|\Psi - \mathbb{G}_h\Psi\| \lesssim \|h^{1/2}\Psi\|_{L^2(\Gamma)} = \|h^{1/2}(\psi - \mathbb{G}_h\psi)\|_{L^2(\Gamma)}.$$

4. step. Reliability of η_H: The inclusion $\mathcal{P}^0(\mathcal{T}_h) \subseteq \mathcal{P}^0(\mathcal{T}_{h/2})$ implies $\mathbb{G}_h\phi_{h/2} = \phi_h$. Therefore, step 3 applies and proves

$$\|\phi_{h/2} - \phi_h\| = \|\phi_{h/2} - \mathbb{G}_h\phi_{h/2}\| \lesssim \|h^{1/2}(\phi_{h/2} - \mathbb{G}_h\phi_{h/2})\|_{L^2(\Gamma)} = \eta_H.$$

With the help of the saturation assumption, we conclude $\|\phi - \phi_h\| \lesssim \|\phi_{h/2} - \phi_h\| \lesssim \eta_H$. ■

Remark. Reliability of η_H depends crucially on the saturation assumption (67), which is always observed to hold in practice. Nevertheless, we stress that (67) has not been proven for the boundary element method, yet. For the finite element method, however, Dörfler-Nochetto show that the saturation assumption holds up to some oscillation terms of higher order. ■

Remark. One remarkable advantage of the h-h/2-error estimation strategy is that there is no implementational overhead. Said differently, this is an easy-to-implement adaptive strategy to check whether the implementation is right or not. An adaptive algorithm, of course, returns $\phi_{h/2}$ after the final step since this is usually the better approximation than ϕ_h. Moreover, the nonlocal error estimator $\|\phi_{h/2} - \phi_h\|$ appears to be very accurate to estimate the error in adaptive computations in the sense that error $\|\phi - \phi_h\|$ and estimator $\|\phi_{h/2} - \phi_h\|$ almost coincide. ■

7.3. Error Estimators based on p-(p+1)-Strategy[6]. The next technique is also a classical idea to derive a posteriori error estimates. We assume that $\mathcal{T}_h = \{T_1, \ldots, T_N\}$ is a given triangulation. We then consider the boundary element spaces $\mathcal{P}^0(\mathcal{T}_h)$ and $\mathcal{P}^1(\mathcal{T}_h)$ which consist of $\mathcal{T}_h$-piecewise constant functions and $\mathcal{T}_h$-piecewise affine but discontinuous functions, respectively. Let $\phi_{h,0} \in \mathcal{P}^0(\mathcal{T}_h)$ and $\phi_{h,1} \in \mathcal{P}^1(\mathcal{T}_h)$ be the corresponding Galerkin solutions. As in (64), we have nestedness of the discrete spaces

$$\mathcal{P}^0(\mathcal{T}_h) \subseteq \mathcal{P}^1(\mathcal{T}_h). \tag{73}$$

We therefore adopt the observations of the previous section, namely

$$\|\phi - \phi_{h,0}\|^2 = \|\phi - \phi_{h,1}\|^2 + \|\phi_{h,1} - \phi_{h,0}\|^2. \tag{74}$$

[6] This section was not part of my lecture, but answers a question of one of the participants. Thank you, David!

Under the saturation assumption

$$\|\phi - \phi_{h,1}\| \le q\,\|\phi - \phi_{h,0}\| \quad \text{for some uniform constant } q < 1, \tag{75}$$

we thus obtain efficiency and reliability

$$\|\phi_{h,1} - \phi_{h,0}\| \le \|\phi - \phi_{h,0}\| \le \frac{1}{\sqrt{1-q^2}}\,\|\phi_{h,1} - \phi_{h,0}\|, \tag{76}$$

where only the reliability estimate depends on q. The reader may copy the proof of Theorem 24 to verify the following result.

Theorem 25. The error estimator

$$\eta_P := \Big(\sum_{j=1}^N \eta_{P,j}^2\Big)^{1/2} \quad \text{with} \quad \eta_{P,j} = h_{T_j}^{1/2}\|\phi_{h,1} - \phi_{h,0}\|_{L^2(T_j)} \tag{77}$$

is always efficient, i.e. $\eta_P \lesssim \|\phi - \phi_{h,0}\|$. Under the saturation assumption (75), there even holds reliability $\|\phi - \phi_{h,0}\| \lesssim \eta_H$. The efficiency constant depends only on Γ and $\sigma(\mathcal{T}_h)$, whereas the reliability constant additionally depends on the saturation constant $q < 1$ from (75).

7.4. Error Estimators based on Averaging. The drawback in the preceeding Theorem 25 is that one has to compute the higher-order Galerkin solution corresponding to $\mathcal{P}^1(\mathcal{T}_h)$, which yields an additional implementational effort. In this section, we give some kind of remedy: We consider two meshes $\mathcal{T}_H = \{T_1, \dots, T_N\}$ and $\mathcal{T}_h$ such that $\mathcal{T}_h$ is a certain refinement of $\mathcal{T}_H$, i.e.

$$\mathcal{P}^1(\mathcal{T}_H) \subset \mathcal{P}^1(\mathcal{T}_h). \tag{78}$$

We shall need the approximation property of the Galerkin projection onto $\mathcal{P}^0(\mathcal{T}_h)$, namely

$$\|\psi - \mathbb{G}_h\psi\| \le C_{\rm approx}\|h^{1/2}\psi\|_{L^2(\Gamma)} \quad \text{for all } \psi \in L^2(\Gamma), \tag{79}$$

as well as the inverse estimate of Graham-Hackbusch-Sauter 2005

$$\|h^{1/2}\psi_h\|_{L^2(\Gamma)} \le C_{\rm inv}\|\psi_h\| \quad \text{for all } \psi_h \in \mathcal{P}^1(\mathcal{T}_h), \tag{80}$$

cf. proof of Theorem 24. We then assume that

$$\lambda := C_{\rm approx}C_{\rm inv}\|(h/H)^{1/2}\|_{L^\infty(\Gamma)} < 1, \tag{81}$$

i.e. the local mesh-size of $\mathcal{T}_h$ is fine enough when compared to the local mesh-size of $\mathcal{T}_H$.

Remark. We stress that we just know the existence of the constants $C_{\mathrm{approx}}, C_{\mathrm{inv}} > 0$ and that both depend on the boundary Γ as well as on the shape regularity constant $\sigma(\mathcal{T}_h)$. In particular, both constants could be rather large. In practice, however, it empirically suffices to use $h = H/2$, i.e. $\mathcal{T}_h$ is a uniform refinement of $\mathcal{T}_H$, to observe $\lambda < 1$. ■

Theorem 26 (Carstensen, Praetorius 2006). Let $\mathbb{G}_H$ and Π_H denote the Galerkin and the L^2-orthogonal projection onto $\mathcal{P}^1(\mathcal{T}_H)$, respectively. Define the error estimator

$$\eta_A := \Big(\sum_{j=1}^N \eta_{A,j}^2\Big)^{1/2} \quad \text{where} \quad \eta_{A,j} = H_{T_j}^{1/2}\|\phi_h - \Pi_H\phi_h\|_{L^2(T_j)}. \tag{82}$$

Provided that $\lambda < 1$ for the constant from (81), there holds

$$\eta_A - |||\phi - \mathbb{G}_H\phi||| \lesssim |||\phi - \phi_h||| \lesssim \eta_A + |||\phi - \mathbb{G}_H\phi|||, \tag{83}$$

i.e. the error estimator η_A is reliable and efficient up to the higher-order Galerkin error $|||\phi - \mathbb{G}_H\phi|||$. The efficiency constant depends only on Γ, $\sigma(\mathcal{T}_h)$, and $\|(H/h)^{1/2}\|_{L^\infty(\Gamma)}$, whereas the reliability constant additionally depends on $\lambda < 1$ from (81).

Proof. The proof is again split in several steps.

1. step. There holds $\eta_A \lesssim |||\phi - \phi_h||| + |||\phi - \mathbb{G}_H\phi|||$: Note that Π_H is the $\mathcal{T}_H$-elementwise L^2-projection onto $\mathcal{P}^1(T_j)$ since we are dealing with discontinuous polynomials. Therefore,

$$\|\phi_h - \Pi_H\phi_h\|_{L^2(T_j)} \le \|\phi_h - \mathbb{G}_H\phi_h\|_{L^2(T_j)} \quad \text{for all } T_j \in \mathcal{T}_H.$$

As above, this yields

$$\eta_A = \|H^{1/2}(\phi_h - \Pi_H\phi_h)\|_{L^2(\Gamma)} \le \|H^{1/2}(\phi_h - \mathbb{G}_H\phi_h)\|_{L^2(\Gamma)}.$$

Since $\phi_h - \mathbb{G}_H\phi_h \in \mathcal{P}^1(\mathcal{T}_h)$, the inverse estimate (80) proves

$$\begin{aligned}\|H^{1/2}(\phi_h - \mathbb{G}_H\phi_h)\|_{L^2(\Gamma)} &\le \|(H/h)^{1/2}\|_{L^\infty(\Gamma)}\|h^{1/2}(\phi_h - \mathbb{G}_H\phi_h)\|_{L^2(\Gamma)}\\ &\lesssim \|(H/h)^{1/2}\|_{L^\infty(\Gamma)}|||\phi_h - \mathbb{G}_H\phi_h|||.\end{aligned}$$

The right-hand side is further estimated by a triangle inequality and the Céa-Lemma, so that

$$|||\phi_h - \mathbb{G}_H\phi_h||| \le |||(1 - \mathbb{G}_H)(\phi - \phi_h)||| + |||(1 - \mathbb{G}_H)\phi||| \le |||\phi - \phi_h||| + |||\phi - \mathbb{G}_H\phi|||.$$

2. step. The Galerkin error $e := \phi - \phi_h$ satisfies $|||\mathbb{G}_He||| \le \lambda|||e|||$: With the symmetry of the Galerkin projection $\mathbb{G}_H$ and the Galerkin orthogonality, we are led to

$$|||\mathbb{G}_He|||^2 = \langle\!\langle e\,;\mathbb{G}_He\rangle\!\rangle = \langle\!\langle e\,;(1 - \mathbb{G}_h)\mathbb{G}_He\rangle\!\rangle \le |||e|||\,|||(1 - \mathbb{G}_h)\mathbb{G}_He|||.$$

The last term on the right-hand side is estimated by (79)–(80) to obtain

$$\begin{aligned}|||(1 - \mathbb{G}_h)\mathbb{G}_He||| \le C_{\mathrm{approx}}\|h^{1/2}\mathbb{G}_He\|_{L^2(\Gamma)} &\le C_{\mathrm{approx}}\|(h/H)^{1/2}\|_{L^\infty(\Gamma)}\|H^{1/2}\mathbb{G}_He\|_{L^2(\Gamma)}\\ &\le C_{\mathrm{approx}}C_{\mathrm{inv}}\|(h/H)^{1/2}\|_{L^\infty(\Gamma)}|||\mathbb{G}_He|||,\end{aligned}$$

where the constant on the right-hand side is just λ. The combination of both estimates thus yields

$$|||\mathbb{G}_H e|||^2 \le \lambda |||e||| |||\mathbb{G}_H e|||.$$

3. step. There holds $|||\phi - \phi_h||| \lesssim \frac{1}{\sqrt{1-\lambda^2}} \big(\eta_A + |||\phi - \mathbb{G}_H \phi||| \big)$: With the Galerkin error $e = \phi - \phi_h$, the Pythagoras theorem yields

$$|||e|||^2 = |||(1-\mathbb{G}_H)e|||^2 + |||\mathbb{G}_H e|||^2 \le |||(1-\mathbb{G}_H)e|||^2 + \lambda^2 |||e|||^2$$

according to step 2. Therefore, we may use the triangle inequality to derive

$$\sqrt{1-\lambda^2}\, |||e||| \le |||(1-\mathbb{G}_H)e||| \le |||(1-\mathbb{G}_H)\phi||| + |||(1-\mathbb{G}_H)\phi_h|||.$$

The Céa-Lemma implies $|||(1-\mathbb{G}_H)\phi_h||| \le |||(1-\Pi_H)\phi_h|||$. The same arguments as in the proof of the Approximation Theorem 19 prove that

$$|||\psi - \Pi_H \psi||| \approx \|\psi - \Pi_H \psi\|_{H^{-1/2}(\Gamma)} \lesssim \|H^{1/2}(\psi - \Pi_H \psi)\|_{L^2(\Gamma)} \quad \text{for all } \psi \in L^2(\Gamma)$$

since $\mathcal{P}^0(T_j) \subseteq \mathcal{P}^1(T_j)$. Altogether, we thus obtain

$$\sqrt{1-\lambda^2}\, |||\phi - \phi_h||| \le |||\phi - \mathbb{G}_H \phi||| + |||\phi_h - \Pi_H \phi_h||| \lesssim |||\phi - \mathbb{G}_H \phi||| + \eta_A,$$

which concludes the proof. ∎

Remark. In numerical experiments, one empirically observes a saturation estimate

$$|||\phi - \mathbb{G}_H \phi||| \le q\, |||\phi - \phi_h||| \quad \text{with a uniform constant } q < 1,$$

i.e. it pays to use higher-order elements on a coarser mesh if compared to lower-order elements on a finer mesh. Therefore, the theoretical term $|||\phi - \mathbb{G}_H \phi|||$ can be neglected for the implementation. Moreover, for adaptive mesh-refinement, one even observes $q \to 0$. We stress that both statements are not proven, yet. Anyhow, this — at least empirically — verifies that η_A is efficient and reliable. ∎