Reproducing kernel almost Pontryagin spaces

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This presentation is based on:


H. Woracek. “Directing functionals and de Branges space completions in the almost Pontryagin space setting”. manuscript in preparation.

These slides are available from my website

http://asc.tuwien.ac.at/index.php?id=woracek
Outline

Almost Pontryagin Spaces
  Geometry
  Completions

Reproducing Kernel Spaces
  Continuity of point-evaluations
  Kernel Functions
  Reproducing kernel completions

Hamburger moment problem
  Review
  Indefinite version of the moment problem
  Significance of completions

Directing Functionals

Some Selected Literature
Almost Pontryagin Spaces
Definition of aPs

A triple $\langle \mathcal{A}, [\cdot, \cdot], \mathcal{O} \rangle$ is an almost Pontryagin space (aPs for short), if

- $\mathcal{A}$ is a linear space,
- $[\cdot, \cdot]_\mathcal{A}$ is an inner product on $\mathcal{A}$,
- $\mathcal{O}$ is a topology on $\mathcal{A}$,

such that the following axioms hold:
Definition of aPs

A triple $\langle A, [\cdot, \cdot]_A, \mathcal{O} \rangle$ is an *almost Pontryagin space* (*aPs* for short), if

(aPs1) The topology $\mathcal{O}$ is a Hilbert space topology on $A$ (i.e., it is induced by some inner product which turns $A$ into a Hilbert space).

(aPs2) The inner product $[\cdot, \cdot]_A$ is $\mathcal{O}$-continuous (i.e., it is continuous as a map of $A \times A$ into $\mathbb{C}$ where $A \times A$ carries the product topology $\mathcal{O} \times \mathcal{O}$ and $\mathbb{C}$ the euclidean topology).

(aPs3) There exists an $\mathcal{O}$-closed linear subspace $\mathcal{M}$ of $A$ with finite codimension in $A$, such that $\langle \mathcal{M}, [\cdot, \cdot]_A | \mathcal{M} \times \mathcal{M} \rangle$ is a Hilbert space.
Definition of aPs

A triple $\langle A, [\cdot, \cdot]_A, \mathcal{O} \rangle$ is an *almost Pontryagin space* (aPs for short), if

(aPs1) The topology $\mathcal{O}$ is a Hilbert space topology on $A$ (i.e., it is induced by some inner product which turns $A$ into a Hilbert space).

(aPs2) The inner product $[\cdot, \cdot]_A$ is $\mathcal{O}$-continuous (i.e., it is continuous as a map of $A \times A$ into $\mathbb{C}$ where $A \times A$ carries the product topology $\mathcal{O} \times \mathcal{O}$ and $\mathbb{C}$ the euclidean topology).

(aPs3) There exists an $\mathcal{O}$-closed linear subspace $\mathcal{M}$ of $A$ with finite codimension in $A$, such that $\langle \mathcal{M}, [\cdot, \cdot]_A |_{\mathcal{M} \times \mathcal{M}} \rangle$ is a Hilbert space.

If $\langle A, [\cdot, \cdot]_A, \mathcal{O} \rangle$ and $\langle B, [\cdot, \cdot]_B, \mathcal{T} \rangle$ are almost Pontryagin spaces, a map $\psi : A \to B$ is an *isomorphism*, if it is linear, isometric, and homeomorphic.
The role of the topology

Let \( \langle A, [.,.]_A \rangle \) be an inner product space. Denote \( A^\circ := \{ x \in A : [x, y]_A = 0, y \in A \} \).
The role of the topology

Let $\langle A, [\cdot, \cdot]_A \rangle$ be an inner product space. Denote $A^\circ := \{ x \in A : [x, y]_A = 0, y \in A \}$.

- Assume that $[\cdot, \cdot]_A$ is nondegenerated (i.e., $A^\circ = \{0\}$). Then being an aP is a property of the inner product alone: there exists at most one topology $\mathcal{O}$ s.t. $\langle A, [\cdot, \cdot]_A, \mathcal{O} \rangle$ is an aPs.
The role of the topology

Let $\langle \mathcal{A}, [.,.]_{\mathcal{A}} \rangle$ be an inner product space. Denote $\mathcal{A}^\circ := \{ x \in \mathcal{A} : [x, y]_{\mathcal{A}} = 0, y \in \mathcal{A} \}$.

- Assume that $[.,.]_{\mathcal{A}}$ is nondegenerated (i.e., $\mathcal{A}^\circ = \{0\}$). Then being an aPs is a property of the inner product alone: there exists at most one topology $\mathcal{O}$ s.t. $\langle \mathcal{A}, [.,.]_{\mathcal{A}}, \mathcal{O} \rangle$ is an aPs.

- If $[.,.]_{\mathcal{A}}$ is degenerated (i.e., $\mathcal{A}^\circ \neq \{0\}$), $\dim \mathcal{A} = \infty$, and $\mathcal{O}$ is a topology s.t. $\langle \mathcal{A}, [.,.]_{\mathcal{A}}, \mathcal{O} \rangle$ is an aPs, then there exists a topology $\mathcal{T}$, $\mathcal{T} \neq \mathcal{O}$, s.t. $\langle \mathcal{A}, [.,.]_{\mathcal{A}}, \mathcal{T} \rangle$ is an aPs.
The role of the topology

Let $\langle \mathcal{A}, [.,.]_\mathcal{A} \rangle$ be an inner product space. Denote $\mathcal{A}^\circ := \{ x \in \mathcal{A} : [x, y]_\mathcal{A} = 0, y \in \mathcal{A} \}$.

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- If $[.,.]_\mathcal{A}$ is degenerated (i.e., $\mathcal{A}^\circ \neq \{0\}$), $\dim \mathcal{A} = \infty$, and $\mathcal{O}$ is a topology s.t. $\langle \mathcal{A}, [.,.]_\mathcal{A}, \mathcal{O} \rangle$ is an aPs, then there exists a topology $\mathcal{T}$, $\mathcal{T} \neq \mathcal{O}$, s.t. $\langle \mathcal{A}, [.,.]_\mathcal{A}, \mathcal{T} \rangle$ is an aPs.

- $\langle \mathcal{A}, [.,.]_\mathcal{A}, \mathcal{O} \rangle$ is a nondegenerated aPs if and only if $\langle \mathcal{A}, [.,.]_\mathcal{A} \rangle$ is a Pontryagin space and $\mathcal{O}$ is its Pontryagin space topology.
The role of the topology

Let \( \langle A, [.,.]_A \rangle \) be an inner product space. Denote
\[ A^\circ := \{ x \in A : [x, y]_A = 0, y \in A \} \].

- Assume that \([.,.]_A\) is nondegenerated (i.e., \( A^\circ = \{0\} \)). Then being an aPs is a property of the inner product alone: there exists at most one topology \( \mathcal{O} \) s.t. \( \langle A, [.,.]_A, \mathcal{O} \rangle \) is an aPs.

- If \([.,.]_A\) is degenerated (i.e., \( A^\circ \neq \{0\} \)), \( \dim A = \infty \), and \( \mathcal{O} \) is a topology s.t. \( \langle A, [.,.]_A, \mathcal{O} \rangle \) is an aPs, then there exists a topology \( \mathcal{T}, \mathcal{T} \neq \mathcal{O}, \) s.t. \( \langle A, [.,.]_A, \mathcal{T} \rangle \) is an aPs.

- \( \langle A, [.,.]_A, \mathcal{O} \rangle \) is a nondegenerated aPs if and only if \( \langle A, [.,.]_A \rangle \) is a Pontryagin space and \( \mathcal{O} \) is its Pontryagin space topology.

- \( \langle A, [.,.]_A, \mathcal{O} \rangle \) is a nondegenerated and positive definite aPs if and only if \( \langle A, [.,.]_A \rangle \) is a Hilbert space and \( \mathcal{O} \) is its Hilbert space topology.
Example

For \( a > 0 \), the *Paley-Wiener space* is

\[
\mathcal{PW}_a := \left\{ F \text{ entire} : F \text{ exponential type } \leq a, \, F|_{\mathbb{R}} \in L^2(\mathbb{R}) \right\}
\]

\[
= \left\{ F : \exists f \in L^2([-a, a]) \, \text{s.t.} \, F(z) = \int_{[-a,a]} f(t)e^{-itz} \, dt \right\}.
\]
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\]

Set

\[
[F, G] := \int_\mathbb{R} F(t) \overline{G(t)} \, dt - \pi F(0) \overline{G(0)}, \quad F, G \in \mathcal{PW}_a,
\]

and let \( \mathcal{PW}_a \) be endowed with the subspace topology of \( L^2(\mathbb{R}) \).
Example

For \( a > 0 \), the \textit{Paley-Wiener space} is

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Set

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[F, G] := \int_{\mathbb{R}} F(t)\overline{G(t)} \, dt - \pi F(0)\overline{G(0)}, \quad F, G \in \mathcal{PW}_a,
\]

and let \( \mathcal{PW}_a \) be endowed with the subspace topology of \( L^2(\mathbb{R}) \).

Then

\[
\mathcal{PW}_a \text{ is } \begin{cases} 
\text{Hilbert space} , & a < \pi \\
\text{aPs} \ (\dim A^\circ = 1) , & a = \pi \\
\text{Pontryagin space} , & a > \pi
\end{cases}
\]
Equivalent definitions of aPs

Let there be given

- a linear space \( \mathcal{A} \),
- an inner product \([\cdot, \cdot]_{\mathcal{A}}\) on \( \mathcal{A} \),
- a topology \( \mathcal{O} \) on \( \mathcal{A} \).
Equivalent definitions of aPs

Let there be given

- a linear space $A$,
- an inner product $[\cdot, \cdot]_A$ on $A$,
- a topology $\mathcal{O}$ on $A$.

Then the following statements are equivalent:
Equivalent definitions of aPs

• \( \langle A, [\cdot, \cdot]_A, O \rangle \) is an almost Pontryagin space.
Equivalent definitions of aPs

- \( \langle \mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \mathcal{O} \rangle \) is an almost Pontryagin space.
- \( \dim \mathcal{A}^\circ < \infty \). We have a decomposition

\[
\mathcal{A} = \mathcal{A}_+[\cdot\cdot] \mathcal{A}_- [\cdot\cdot] \mathcal{A}^\circ,
\]

with: \( \mathcal{A}_- \) finite dimensional and negative definite, \( \mathcal{A}_+ \) Hilbert space when endowed with \( [\cdot, \cdot]_{\mathcal{A}} \) and \( \mathcal{O} \)-closed.
Equivalent definitions of aPs

- \( \langle \mathcal{A}, [\cdot, \cdot]_\mathcal{A}, \mathcal{O} \rangle \) is an almost Pontryagin space.
- \( \dim \mathcal{A}^\circ < \infty \). We have a decomposition

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\]

with: \( \mathcal{A}_- \) finite dimensional and negative definite, \( \mathcal{A}_+ \) Hilbert space when endowed with \([\cdot, \cdot]_\mathcal{A}\) and \(\mathcal{O}\)-closed.
- There exists a Pontryagin space which (isometrically) contains \( \mathcal{A} \) as a closed subspace.
Equivalent definitions of aPs

- \( \langle \mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \mathcal{O} \rangle \) is an almost Pontryagin space.
- \( \dim \mathcal{A}^\circ < \infty \). We have a decomposition
  \[
  \mathcal{A} = \mathcal{A}_+ [\cdot] + \mathcal{A}_- [\cdot] \mathcal{A}^\circ,
  \]
  with: \( \mathcal{A}_- \) finite dimensional and negative definite, \( \mathcal{A}_+ \) Hilbert space when endowed with \( [\cdot, \cdot]_{\mathcal{A}} \) and \( \mathcal{O} \)-closed.
- There exists a Pontryagin space which (isometrically) contains \( \mathcal{A} \) as a closed subspace.
- There exists a Hilbert space inner product \( (\cdot, \cdot) \) on \( \mathcal{A} \), and \( G \) bounded selfadjoint in \( \langle \mathcal{A}, (\cdot, \cdot) \rangle \) s.t. \( (E \) spectral measure of \( G) \)
  \[
  [x, y]_{\mathcal{A}} = (Gx, y), \quad x, y \in \mathcal{A},
  \]
  \( \exists \varepsilon > 0 : \dim \text{ran } E((\infty, \varepsilon)) < \infty. \)
The dual space

Let $\langle \mathcal{A}, [\cdot, \cdot]_\mathcal{A}, \mathcal{O} \rangle$ be an aPs, and $\mathcal{A}'$ its topological dual space.
The dual space

Let \( \mathcal{A}, [\cdot, \cdot]_\mathcal{A}, \mathcal{O} \) be an aPs, and \( \mathcal{A}' \) its topological dual space.

- \( \{ [\cdot, y]_\mathcal{A} : y \in \mathcal{A} \} \) is a \( w^* \)-closed linear subspace of \( \mathcal{A}' \).
The dual space

Let \( \langle A, [\cdot, \cdot]_A, \mathcal{O} \rangle \) be an aPs, and \( A' \) its topological dual space.

- \( \{ [\cdot, y]_A : y \in A \} \) is a \( w^* \)-closed linear subspace of \( A' \).
- \( \dim \left( A'/\{ [\cdot, y]_A : y \in A \} \right) = \dim A^\circ. \)
The dual space

Let \( \langle \mathcal{A}, [,]_\mathcal{A}, \mathcal{O} \rangle \) be an aPs, and \( \mathcal{A}' \) its topological dual space.

- \( \{ [,]_\mathcal{A} : y \in \mathcal{A} \} \) is a \( w^* \)-closed linear subspace of \( \mathcal{A}' \).
- \( \dim \left( \mathcal{A}' / \{ [,]_\mathcal{A} : y \in \mathcal{A} \} \right) = \dim \mathcal{A}^\circ \).

Let \( \mathcal{F} \subseteq \mathcal{A}' \) be point separating on \( \mathcal{A}^\circ \), i.e. assume

\[
\mathcal{A}^\circ \cap \bigcap_{\varphi \in \mathcal{F}} \ker \varphi = \{0\},
\]

and denote by \( \pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ \) the canonical projection.
The dual space

Let $\langle \mathcal{A}, [, ,]_\mathcal{A}, \mathcal{O} \rangle$ be an aPs, and $\mathcal{A}'$ its topological dual space.

- $\{[\cdot, y]_\mathcal{A} : y \in \mathcal{A}\}$ is a $w^*$-closed linear subspace of $\mathcal{A}'$.
- $\dim \left( \mathcal{A}' / \{[\cdot, y]_\mathcal{A} : y \in \mathcal{A}\} \right) = \dim \mathcal{A}^\circ$.

Let $\mathcal{F} \subseteq \mathcal{A}'$ be point separating on $\mathcal{A}^\circ$, i.e. assume

$$\mathcal{A}^\circ \cap \bigcap_{\varphi \in \mathcal{F}} \ker \varphi = \{0\},$$

and denote by $\pi : \mathcal{A} \to \mathcal{A} / \mathcal{A}^\circ$ the canonical projection.

- $\mathcal{A}' = \{[\cdot, y]_\mathcal{A} : y \in \mathcal{A}\} + \text{span} \mathcal{F}$. 
The notion of a completion

Definition
Let $\langle \mathcal{L},[.,.]_\mathcal{L} \rangle$ be an inner product space. A pair $\langle \iota, \mathcal{A} \rangle$ is an $aPs$-completion of $\mathcal{L}$, if

$\iota$ is linear and isometric,
ran $\iota$ is dense in $\mathcal{A}$.

Two $aPs$-completions $\langle \iota_1, \mathcal{A}_1 \rangle$, $\langle \iota_2, \mathcal{A}_2 \rangle$, are isomorphic, if there exists an isomorphism $\phi: \mathcal{A}_1 \to \mathcal{A}_2$ with $\phi \circ \iota_1 = \iota_2$.

We speak of a Hilbert-space completion or a Pontryagin-space completion, if $\text{ind}^{-\mathcal{A}} = 0$, $\dim \mathcal{A} = 0$ or $\dim \mathcal{A} = 0$, resp.
Definition
Let \( \langle \mathcal{L}, [.,.]_\mathcal{L} \rangle \) be an inner product space. A pair \( \langle \iota, \mathcal{A} \rangle \) is an \textit{aPs-completion} of \( \mathcal{L} \), if

- \( \mathcal{A} \) is an aPs,
- \( \iota : \mathcal{L} \to \mathcal{A} \) is linear and isometric,
- \( \text{ran } \iota \) is dense in \( \mathcal{A} \).
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- $\text{ran } \iota$ is dense in $\mathcal{A}$.

Two aPs-completions $\langle \iota_i, \mathcal{A}_i \rangle$, $i = 1, 2$, are \textit{isomorphic}, if there exists an isomorphism $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ with $\varphi \circ \iota_1 = \iota_2$. 
The notion of a completion

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Let \( \langle \mathcal{L}, [.,.]_{\mathcal{L}} \rangle \) be an inner product space. A pair \( \langle \iota, \mathcal{A} \rangle \) is an \textit{aPs-completion} of \( \mathcal{L} \), if

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Two aPs-completions \( \langle \iota_i, \mathcal{A}_i \rangle, i = 1, 2 \), are \textit{isomorphic}, if there exists an isomorphism \( \varphi : \mathcal{A}_1 \to \mathcal{A}_2 \) with \( \varphi \circ \iota_1 = \iota_2 \).

We speak of a \textit{Hilbert-space completion} or a \textit{Pontryagin-space completion}, if

\[
\text{ind} \, \mathcal{A} = 0, \, \text{dim} \, \mathcal{A}^\circ = 0 \, \text{or} \, \text{dim} \, \mathcal{A}^\circ = 0, \, \text{resp.}
\]
Example

Consider $\mathcal{L} := \bigcup_{0 < a < \pi} \mathcal{P}W_a$ and set

$$[F, G] := \int_{\mathbb{R}} F(t)\overline{G(t)} \, dt - \pi F(0)\overline{G(0)}, \quad F, G \in \mathcal{L}. $$
Example

Consider $\mathcal{L} := \bigcup_{0 < a < \pi} \mathcal{P} W_a$ and set

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Then

- $\langle \mathcal{L}, [\cdot, \cdot] \rangle$ is positive definite.
Example
Consider $\mathcal{L} := \bigcup_{0 < a < \pi} \mathcal{P}W_a$ and set

$$[F, G] := \int_{\mathbb{R}} F(t)\overline{G(t)} \, dt - \pi F(0)\overline{G(0)}, \quad F, G \in \mathcal{L}.$$ 

Then

- $\langle \mathcal{L}, [, .] \rangle$ is positive definite.
- The norm $F \mapsto [F, F]^{\frac{1}{2}}$ is not equivalent to the $L^2(\mathbb{R})$-norm on $\mathcal{L}$. 
Example

Consider $\mathcal{L} := \bigcup_{0 < a < \pi} \mathcal{P}W_a$ and set

$$[F, G] := \int_{\mathbb{R}} F(t)\overline{G(t)} \, dt - \pi F(0)\overline{G(0)}, \quad F, G \in \mathcal{L}.$$ 

Then

- $\langle \mathcal{L}, \langle ., . \rangle \rangle$ is positive definite.
- The norm $F \mapsto [F, F]^{\frac{1}{2}}$ is not equivalent to the $L^2(\mathbb{R})$-norm on $\mathcal{L}$.
- $\langle \mathcal{P}W_\pi, \langle ., . \rangle \rangle$ with $\iota : F \mapsto F$ is an aPs completion of $\mathcal{L}$. 
Let $\langle \mathcal{L}, [.,.]_\mathcal{L} \rangle$ be an inner product space. Set

$$\text{ind} \mathcal{L} := \sup \left\{ \dim N : N \text{ negative definite subspace of } \mathcal{L} \right\}.$$
Completions: Existence

Let \( \langle \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}} \rangle \) be an inner product space. Set

\[
\text{ind}_- \mathcal{L} := \sup \{ \dim N : N \text{ negative definite subspace of } \mathcal{L} \}.
\]

**Proposition**

Let \( \mathcal{L} \) be an inner product space. The following are equivalent:

- \( \text{ind}_- \mathcal{L} < \infty \).
- \( \mathcal{L} \) has an aPs-completion.
- \( \mathcal{L} \) has an aPs-completion.
- \( \mathcal{L} \) has a Pontryagin-space completion.
Completions: Description?

Task: describe the totality of completions of $\mathcal{L}$ (up to isomorphism).
Completions: Description?

Task: describe the totality of completions of $\mathcal{L}$ (up to isomorphism).

Proposition

*Let $\mathcal{L}$ be an inner product space with $\text{ind}_- \mathcal{L} < \infty$. Then each two Pontryagin-space completions of $\mathcal{L}$ are isomorphic.*
Completions: Description?

Task: describe the totality of completions of $\mathcal{L}$ (up to isomorphism).

Example

Let $\langle \mathcal{L}, (., .)\rangle$ be a Hilbert space, $f_1, \ldots, f_n : \mathcal{L} \to \mathbb{C}$ be linear with

$$\mathcal{L}' \cap \text{span}\{ f_1, \ldots, f_n \} = \{0\}.$$

Set

$$\mathcal{A} := \mathcal{L} \times \mathbb{C}^n, \quad \iota(x) := (x; (f_i(x))_{i=1}^n),$$

$$[(x; (\xi_i)_{i=1}^n), (y; (\eta_i)_{i=1}^n)]_{\mathcal{A}} := (x, y)\mathcal{L},$$

$$((x; (\xi_i)_{i=1}^n), (y; (\eta_i)_{i=1}^n))_{\mathcal{A}} := (x, y)\mathcal{L} + \sum_{i=1}^n \xi_i \bar{\eta}_i.$$

Then $\langle \iota, \mathcal{A} \rangle$ is an aPs-completion of $\mathcal{L}$ with $\dim \mathcal{A}^\circ = n$. 
The intrinsic dual

Let $\mathcal{L}$ be an inner product space with $\text{ind}^- \mathcal{L} < \infty$.

**Definition**

Let $\varphi : \mathcal{L} \to \mathbb{C}$ be linear. We write $\varphi \in \mathcal{L}^\wedge$, if

$$\forall (x_n)_{n \in \mathbb{N}}, x_n \in \mathcal{L} :$$

$$\left( [x_n, x_n]_\mathcal{L} \to 0, \ [x_n, x]_\mathcal{L} \to 0, x \in \mathcal{L} \right) \Rightarrow \varphi(x_n) \to 0.$$
The intrinsic dual

Let $\mathcal{L}$ be an inner product space with $\text{ind} \mathcal{L} < \infty$.

**Definition**

Let $\varphi : \mathcal{L} \to \mathbb{C}$ be linear. We write $\varphi \in \mathcal{L}^\wedge$, if

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$$\left( [x_n, x_n]_\mathcal{L} \to 0, \ [x_n, x]_\mathcal{L} \to 0, x \in \mathcal{L} \right) \Rightarrow \varphi(x_n) \to 0.$$

- $\mathcal{L}^\wedge$ can be interpreted as the topological dual w.r.t. a certain seminorm on $\mathcal{L}$.
Completions: Description

For an aPs-completion $\langle \iota, A \rangle$ of $\mathcal{L}$, set

$$\iota^*(A') := \{ f \circ \iota : f \in A' \}.$$
Completions: Description

For an aPs-completion \( \langle \iota, \mathcal{A} \rangle \) of \( \mathcal{L} \), set

\[
i^*(\mathcal{A}') := \{ f \circ \iota : f \in \mathcal{A}' \}.
\]

**Theorem**

The map \( \langle \iota, \mathcal{A} \rangle \mapsto i^*(\mathcal{A}') \) induces a bijection between

- the set of isomorphy classes of aPs-completions of \( \mathcal{L} \),

and

- the set of those linear subspaces of the algebraic dual \( \mathcal{L}^* \) of \( \mathcal{L} \) which contain \( \mathcal{L}^\perp \) with finite codimension.

For each aPs-completion it holds that

\[
\dim \left( i^*(\mathcal{A}') / \mathcal{L}^\perp \right) = \dim \mathcal{A}^\circ.
\]
Reproducing Kernel Spaces
Continuity of point evaluations

For a set $\Omega$ and $\eta \in \Omega$ denote by $\chi_{\eta} : \mathbb{C}^\Omega \rightarrow \mathbb{C}$ the \textit{point-evaluation functional} $\chi_{\eta} : f \mapsto f(\eta)$. 
Continuity of point evaluations

For a set $\Omega$ and $\eta \in \Omega$ denote by $\chi_\eta : \mathbb{C}^\Omega \to \mathbb{C}$ the point-evaluation functional $\chi_\eta : f \mapsto f(\eta)$.

**Definition**

Let $\Omega$ be a set. An aPs $A$ is a reproducing kernel aPs on $\Omega$, if

1. (rk1) $A \subseteq \mathbb{C}^\Omega$ (linear operations defined pointwise);
2. (rk2) $\forall \eta \in \Omega : \chi_\eta|_A \in A'$.

Continuity of point evaluations

For a set $\Omega$ and $\eta \in \Omega$ denote by $\chi_{\eta} : \mathbb{C}^{\Omega} \to \mathbb{C}$ the point-evaluation functional $\chi_{\eta} : f \mapsto f(\eta)$. 

**Definition**

Let $\Omega$ be a set. An aPs $A$ is a reproducing kernel aPs on $\Omega$, if

$(rk1)$ $A \subseteq \mathbb{C}^{\Omega}$ (linear operations defined pointwise);

$(rk2)$ $\forall \eta \in \Omega : \chi_{\eta}|_A \in A'$. 

Being a reproducing kernel aPs is a property of the inner product alone (regardless whether it is nondegenerated or degenerated):

**Proposition**

*If $\langle A, [, .]_A \rangle$ is an inner product space with $(rk1)$, then there exists at most one topology $\mathcal{O}$ on $A$ such that $\langle A, [, .]_A, \mathcal{O} \rangle$ is a reproducing kernel aPs.*
Continuity of point evaluations

Example
For each $a > 0$, the Paley-Wiener space $\mathcal{PW}_a$ endowed with the inner product

$$[F, G] := \int_{\mathbb{R}} F(t) \overline{G(t)} \, dt - \pi F(0) \overline{G(0)}, \quad F, G \in \mathcal{PW}_a,$$

and the subspace topology of $L^2(\mathbb{R})$ is a reproducing kernel aPs of entire functions.
Continuity of point evaluations

**Example**

For each \( a > 0 \), the Paley-Wiener space \( \mathcal{PW}_a \) endowed with the inner product

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\]

and the subspace topology of \( L^2(\mathbb{R}) \) is a reproducing kernel \( a\Ps \) of entire functions.

Remember

\[
\mathcal{PW}_a \text{ is } \begin{cases} 
\text{Hilbert space} & , \quad a < \pi \\
\text{aPs (dim } \mathcal{A}^{\circ} = 1) & , \quad a = \pi \\
\text{Pontryagin space} & , \quad a > \pi 
\end{cases}
\]
Kernel functions?

Let $\mathcal{A}$ be a reproducing kernel Pontryagin space (i.e., a nondegerated reproducing kernel aPs). Then

$$\exists! K : \Omega \times \Omega \to \mathbb{C} : \quad K(w,.) \in \mathcal{A}, w \in \Omega,$$

$$f(w) = [f,K(w,.)]_{\mathcal{A}}, f \in \mathcal{A}, w \in \Omega.$$ 

This function is called the reproducing kernel of $\mathcal{A}$. 

Example:

Let $a > 0$, $a \neq \pi$. The reproducing kernel of $\langle P_{\mathbb{W}^a},[.,.]\rangle$ is

$$K(w,z) := \sin[a(z-w)]\pi(z-w) + 1/\pi - a \cdot \sin[a w] \sin[a z].$$
Kernel functions?

- Let $\mathcal{A}$ be a reproducing kernel Pontryagin space (i.e., a nondegenerated reproducing kernel aPs). Then

$$\exists! K : \Omega \times \Omega \to \mathbb{C} : \quad K(w,.) \in A, w \in \Omega,$$

$$f(w) = [f, K(w,.)]_A, f \in \mathcal{A}, w \in \Omega.$$  

This function is called the reproducing kernel of $\mathcal{A}$.

- Let $\mathcal{A}$ be a degenerated reproducing kernel aPs. Then there cannot exist a function $K$ with these properties:

$$f(w) = [f, K(w,.)]_A = 0, \quad f \in \mathcal{A}^\circ, w \in \Omega.$$
Kernel functions?

- Let \( \mathcal{A} \) be a reproducing kernel Pontryagin space (i.e., a nondegerated reproducing kernel aPs). Then
  \[ \exists! K : \Omega \times \Omega \rightarrow \mathbb{C} : \quad K(w,.) \in \mathcal{A}, w \in \Omega, \]
  \[ f(w) = [f, K(w,.)]_\mathcal{A}, f \in \mathcal{A}, w \in \Omega. \]

This function is called the reproducing kernel of \( \mathcal{A} \).

- Let \( \mathcal{A} \) be a degenerated reproducing kernel aPs. Then there cannot exist a function \( K \) with these properties:
  \[ f(w) = [f, K(w,.)]_\mathcal{A} = 0, \quad f \in \mathcal{A}^\circ, w \in \Omega. \]

Example

Let \( a > 0, a \neq \pi \). The reproducing kernel of \( \langle \mathcal{P}W_a, [.,.] \rangle \) is

\[ K(w, z) := \frac{\sin[a(z - \overline{w})]}{\pi(z - \overline{w})} + \frac{1}{\pi - a} \cdot \frac{\sin[a\overline{w}]}{\overline{w}} \cdot \frac{\sin[a\overline{z}]}{z}. \]
Almost reproducing kernels

Definition
Let $\mathcal{A}$ be a reproducing kernel aPs. A function $K : \Omega \times \Omega \rightarrow \mathbb{C}$ is an *almost reproducing kernel of* $\mathcal{A}$, if
Almost reproducing kernels

Definition

Let \( \mathcal{A} \) be a reproducing kernel aPs. A function \( K : \Omega \times \Omega \rightarrow \mathbb{C} \) is an almost reproducing kernel of \( \mathcal{A} \), if

(aRK1) \( K \) is a hermitian kernel on \( \Omega \), i.e.,
\[
K(z, w) = \overline{K(w, z)}, \quad z, w \in \Omega,
\]

(aRK2) \( K(w, .) \in \mathcal{A}, \quad w \in \Omega, \)

(aRK3) There exists data \( \delta = ((w_i)_{i=1}^n; (\gamma_i)_{i=1}^n) \in \Omega^n \times \mathbb{R}^n \) where \( n := \dim \mathcal{A}^\circ \), such that

\[
\forall f \in \mathcal{A}, w \in \Omega : \quad f(w) = [f, K(w, .)]_{\mathcal{A}} + \sum_{i=1}^n \gamma_i \cdot \chi_{w_i}(f) \overline{\chi_{w_i}(K(w, .))}.
\]
Almost reproducing kernels: Existence

**Theorem**

Let $\mathcal{A}$ be a reproducing kernel aPs, set $n := \dim \mathcal{A}^\circ$, and let 
\[(w_i)_{i=1}^n \in \Omega^n \text{ be such that} \]
\[\mathcal{A}^\circ \cap \bigcap_{i=1}^n \ker \chi_{w_i} = \{0\}.\]
Almost reproducing kernels: Existence

Theorem

Let \( A \) be a reproducing kernel aPs, set \( n := \dim A^\circ \), and let \( (w_i)_{i=1}^n \in \Omega^n \) be such that

\[
A^\circ \cap \bigcap_{i=1}^n \ker \chi_{w_i} = \{0\}.
\]

Then there exists a closed and nowhere dense exceptional set \( E \subseteq \mathbb{R}^n \), such that for each \( (\gamma_i)_{i=1}^n \in \mathbb{R}^n \setminus E \) there exists an almost reproducing kernel of \( A \) with data \( \delta := ((w_i)_{i=1}^n; (\gamma_i)_{i=1}^n) \).
Almost reproducing kernels: Existence

**Theorem**

Let $\mathcal{A}$ be a reproducing kernel $\mathcal{A}P$s, set $n := \dim \mathcal{A}^\circ$, and let $(w_i)_{i=1}^n \in \Omega^n$ be such that

$$\mathcal{A}^\circ \cap \bigcap_{i=1}^n \ker \chi_{w_i} = \{0\}.$$ 

Then there exists a closed and nowhere dense exceptional set $E \subseteq \mathbb{R}^n$, such that for each $(\gamma_i)_{i=1}^n \in \mathbb{R}^n \setminus E$ there exists an almost reproducing kernel of $\mathcal{A}$ with data $\delta := ((w_i)_{i=1}^n; (\gamma_i)_{i=1}^n)$.

- Such choices of $(w_i)_{i=1}^n \in \Omega^n$ certainly exist since $\{\chi_w : w \in \Omega\}$ is point separating.
Almost reproducing kernels: Properties

For a hermitian kernel $K$ we denote by $\text{ind}_- K \in \mathbb{N}_0 \cup \{\infty\}$ the supremum of the numbers of negative squares of quadratic forms

$$\sum_{i,j=1}^{n} K(w_j, w_i)\xi_i \bar{\xi}_j$$

where $n \in \mathbb{N}$, $w_1, \ldots, w_n \in \Omega$. 
Almost reproducing kernels: Properties

For a hermitian kernel $K$ we denote by $\text{ind}_- K \in \mathbb{N}_0 \cup \{\infty\}$ the supremum of the numbers of negative squares of quadratic forms

$$\sum_{i,j=1}^{n} K(w_j, w_i)\xi_i\bar{\xi}_j \quad \text{where} \quad n \in \mathbb{N}, \ w_1, \ldots, w_n \in \Omega.$$

Theorem

*Let $A$ be a reproducing kernel aPs, set $n := \dim A^\circ$, and let $
\delta = ((w_i)_{i=1}^{n}; (\gamma_i)_{i=1}^{n}) \in \Omega^n \times \mathbb{R}^n.$*
Almost reproducing kernels: Properties

For a hermitian kernel $K$ we denote by $\operatorname{ind}_- K \in \mathbb{N}_0 \cup \{\infty\}$ the supremum of the numbers of negative squares of quadratic forms

$$\sum_{i,j=1}^{n} K(w_j, w_i) \bar{\xi}_i \bar{\xi}_j \quad \text{where} \quad n \in \mathbb{N}, \ w_1, \ldots, w_n \in \Omega.$$

**Theorem**

*Let $A$ be a reproducing kernel $aPs$, set $n := \dim A^\circ$, and let $\delta = ((w_i)_{i=1}^{n}; (\gamma_i)_{i=1}^{n}) \in \Omega^n \times \mathbb{R}^n$. Assume $K$ is an almost reproducing kernel of $A$ with data $\delta$.***
Almost reproducing kernels: Properties

For a hermitian kernel $K$ we denote by $\text{ind}_- K \in \mathbb{N}_0 \cup \{\infty\}$ the supremum of the numbers of negative squares of quadratic forms

$$\sum_{i,j=1}^{n} K(w_j, w_i) \xi_i \overline{\xi}_j$$

where $n \in \mathbb{N}$, $w_1, \ldots, w_n \in \Omega$.

**Theorem**

*Let $\mathcal{A}$ be a reproducing kernel aPs, set $n := \dim \mathcal{A}^\circ$, and let $\delta = ((w_i)_{i=1}^{n}; (\gamma_i)_{i=1}^{n}) \in \Omega^n \times \mathbb{R}^n$. Assume $K$ is an almost reproducing kernel of $\mathcal{A}$ with data $\delta$. Then*

- $\mathcal{A}^\circ \cap \bigcap_{i=1}^{n} \ker \chi w_i = \{0\}$,
- $\text{ind}_- K < \infty$,
- $\gamma_i \neq 0$, $i, i = 1, \ldots, n$,
- $K(w_i, w_j) = \delta_{ij} \frac{1}{\gamma_i}$, $i, i = 1, \ldots, n$. 
Almost reproducing kernels: Uniqueness

Theorem

Let $A$ be a reproducing kernel $apS$, set $n := \dim A^\circ$, and let $K_1$ and $K_2$ be almost reproducing kernels for $A$ with corresponding data $\delta_1$ and $\delta_2$, respectively.
Almost reproducing kernels: Uniqueness

Theorem

Let $A$ be a reproducing kernel $aPs$, set $n := \dim A^\circ$, and let $K_1$ and $K_2$ be almost reproducing kernels for $A$ with corresponding data $\delta_1$ and $\delta_2$, respectively.

Non-uniqueness: If the data $\delta_1$ and $\delta_2$ has the same points $(w_i)_{i=1}^n$ but different weights $(\gamma_i)_{i=1}^n$, then $K_1 \neq K_2$. 
Almost reproducing kernels: Uniqueness

Theorem

Let \( \mathcal{A} \) be a reproducing kernel \( aPs \), set \( n := \dim \mathcal{A}^\circ \), and let \( K_1 \) and \( K_2 \) be almost reproducing kernels for \( \mathcal{A} \) with corresponding data \( \delta_1 \) and \( \delta_2 \), respectively.

Non-uniqueness: If the data \( \delta_1 \) and \( \delta_2 \) has the same points \( (w_i)_{i=1}^n \) but different weights \( (\gamma_i)_{i=1}^n \), then \( K_1 \neq K_2 \).

Uniqueness: If \( \delta_1 = \delta_2 \), then \( K_1 = K_2 \).
Almost reproducing kernels: Uniqueness

Theorem

Let $\mathcal{A}$ be a reproducing kernel aPs, set $n := \dim \mathcal{A}^\circ$, and let $K_1$ and $K_2$ be almost reproducing kernels for $\mathcal{A}$ with corresponding data $\delta_1$ and $\delta_2$, respectively.

Non-uniqueness: If the data $\delta_1$ and $\delta_2$ has the same points $(w_i)_{i=1}^n$ but different weights $(\gamma_i)_{i=1}^n$, then $K_1 \neq K_2$.

Uniqueness: If $\delta_1 = \delta_2$, then $K_1 = K_2$.

• Due to the Existence Theorem, $\mathcal{A}$ has many different almost reproducing kernels.
Almost reproducing kernels: Description

Theorem
Let $K$ be a hermitian kernel, let $((w_i)_{i=1}^n, (\gamma_i)_{i=1}^n) \in \Omega^n \times \mathbb{R}^n$, and assume that

- $\text{ind}_- K < \infty$,
- $\gamma_i \neq 0$, $i, i = 1, \ldots, n$,
- $K(w_i, w_j) = \delta_{ij} \frac{1}{\gamma_i}$, $i, i = 1, \ldots, n$. 

Then there exists a unique reproducing kernel $a_{Ps}$, such that $K$ is the almost reproducing kernel of $A$ with data $\delta = ((w_i)_{i=1}^n, (\gamma_i)_{i=1}^n)$.
Almost reproducing kernels: Description

Theorem

Let $K$ be a hermitian kernel, let $((w_i)_{i=1}^n, (\gamma_i)_{i=1}^n) \in \Omega^n \times \mathbb{R}^n$, and assume that

- $\text{ind}_- K < \infty$,
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- $K(w_i, w_j) = \delta_{ij} \frac{1}{\gamma_i}$, $i, i = 1, \ldots, n$.

Then there exists a unique reproducing kernel $aPs$, such that $K$ is the almost reproducing kernel of $A$ with data $\delta = ((w_i)_{i=1}^n; (\gamma_i)_{i=1}^n)$. 

Reproducing kernel space completions?

Let $\mathcal{L}$ be an inner product space whose elements are functions.

*Does there exist a reproducing kernel $aP$ which contains $\mathcal{L}$ isometrically and densely?*
Reproducing kernel space completions?

Does there exist a reproducing kernel aPs which contains \( \mathcal{L} \) isometrically and densely?

Example

Consider the space \( \mathcal{L} := \bigcup_{0 < a < \pi} \mathcal{P}W_a \) endowed with

\[
[F, G] := \int_{\mathbb{R}} F(t)G(t) \, dt - \pi F(0)\overline{G(0)}, \quad F, G \in \mathcal{L}.
\]
Reproducing kernel space completions?

*Does there exist a reproducing kernel aPs which contains \( \mathcal{L} \) isometrically and densely?*

**Example**

Consider the space \( \mathcal{L} := \bigcup_{0 < a < \pi} \mathcal{P}W_a \) endowed with

\[
[F, G] := \int_{\mathbb{R}} F(t) \overline{G(t)} \, dt - \pi F(0) \overline{G(0)}, \quad F, G \in \mathcal{L}.
\]

Then \( \mathcal{L} \) is positive definite, and

- \( \mathcal{L} \) is isometrically and densely contained in the (degenerated) reproducing kernel aPs \( \langle \mathcal{P}W_\pi, [\cdot, \cdot] \rangle \).
- There does not exist a reproducing kernel Pontryagin space which contains \( \mathcal{L} \) isometrically and densely.
- There does not exist a reproducing kernel Hilbert space which contains \( \mathcal{L} \) isometrically.
Reproducing kernel space completions?

Does there exist a reproducing kernel \( aPs \) which contains \( \mathcal{L} \) isometrically and densely?

Example

Let \( \mu \) be a positive Borel measure on the real line which is compactly supported and not discrete, and consider the space \( \mathcal{L} \) of all polynomials endowed with

\[
[p, q] := \int_{\mathbb{R}} p\overline{q} \, d\mu, \quad p, q \in \mathcal{L}.
\]
Reproducing kernel space completions?

Does there exist a reproducing kernel $\mathcal{aPs}$ which contains $\mathcal{L}$ isometrically and densely?

Example

Let $\mu$ be a positive Borel measure on the real line which is compactly supported and not discrete, and consider the space $\mathcal{L}$ of all polynomials endowed with

$$[p, q] := \int_{\mathbb{R}} p\overline{q} \, d\mu, \quad p, q \in \mathcal{L}.$$ 

Then there does not exist a reproducing kernel $\mathcal{aPs}$ which contains $\mathcal{L}$ isometrically.
Proposition

Let $\mathcal{L}$ be an inner product space with $\text{ind} - \mathcal{L} < \infty$. Then, for each aPs-completion $\langle \iota, \mathcal{A} \rangle$ of $\mathcal{L}$, it holds that

$$\mathcal{L}^\perp = \iota^* \left( \{ [., y]_{\mathcal{A}} : y \in \mathcal{A} \} \right) = \{ x \mapsto [\iota x, y]_{\mathcal{A}} : y \in \mathcal{A} \}.$$
Topologising the intrinsic dual

Proposition

Let $\mathcal{L}$ be an inner product space with $\text{ind}_- \mathcal{L} < \infty$. Then, for each aPs-completion $\langle \iota, \mathcal{A} \rangle$ of $\mathcal{L}$, it holds that

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- The map $\iota^*|_{\mathcal{A}'}$ is injective since $\iota(\mathcal{L})$ is dense in $\mathcal{A}$. 
Topologising the intrinsic dual

**Proposition**

Let $\mathcal{L}$ be an inner product space with $\text{ind}_{-} \mathcal{L} < \infty$. Then, for each aPs-completion $\langle \iota, \mathcal{A} \rangle$ of $\mathcal{L}$, it holds that

$$\mathcal{L}^{\wedge} = \iota^* (\{ [, y]_\mathcal{A} : y \in \mathcal{A} \}) = \{ x \mapsto [\iota x, y]_\mathcal{A} : y \in \mathcal{A} \}.$$

- The map $\iota^* |_{\mathcal{A}'}$ is injective since $\iota(\mathcal{L})$ is dense in $\mathcal{A}$.

**Definition**

Let $\mathcal{T}^{\wedge}$ be the topology induced by the norm

$$\| \phi \|^{\wedge} := \| (\iota^* |_{\mathcal{A}'} )^{-1} \phi \|_{\mathcal{A}'} , \quad \phi \in \mathcal{L}^{\wedge}.$$
Existence Theorem

**Theorem**

*Let $\mathcal{L}$ be an inner product space whose elements are functions.*
Existence Theorem

Theorem
Let \( \mathcal{L} \) be an inner product space whose elements are functions.

There exists a reproducing kernel \( \text{aPs} \) which contains \( \mathcal{L} \) isometrically, if and only if

(A) \( \text{ind} \, \mathcal{L} < \infty \),

and
Existence Theorem

Theorem
Let \( \mathcal{L} \) be an inner product space whose elements are functions. There exists a reproducing kernel \( \alpha \) which contains \( \mathcal{L} \) isometrically, if and only if

(A) \( \text{ind} \, \mathcal{L} < \infty \),

and

(B) \( \dim \left( \left[ \mathcal{L}^\wedge + \text{span}\{\chi_w|\mathcal{L} : w \in \Omega\}\right] / \mathcal{L}^\wedge \right) < \infty \),

(C) \( \mathcal{L}^\wedge \cap \text{span}\{\chi_w|\mathcal{L} : w \in \Omega\} \) is \( \mathcal{J}^{\wedge} \)-dense in \( \mathcal{L}^\wedge \).
Existence Theorem

Theorem

Let $\mathcal{L}$ be an inner product space whose elements are functions.
There exists a reproducing kernel $\alpha P$ which contains $\mathcal{L}$ isometrically, if and only if

(A) $\text{ind}_- \mathcal{L} < \infty$,

and

(B) $\dim \left( \left[ \mathcal{L}^\wedge + \text{span}\{\chi_w | \mathcal{L} : w \in \Omega\} \right] / \mathcal{L}^\wedge \right) < \infty$,

(C) $\mathcal{L}^\wedge \cap \text{span}\{\chi_w | \mathcal{L} : w \in \Omega\}$ is $\mathcal{T}^\wedge$-dense in $\mathcal{L}^\wedge$.

These conditions can be reformulated in a concrete way. It holds that

(B) $\Leftrightarrow$ (B')
(C) $\Rightarrow$ (C')
(B) $\land$ (C') $\Rightarrow$ (C)
(B’) There exist $N \in \mathbb{N}$ and $(w_i)_{i=1}^N \in M^N$, such that the following implication holds. If $(f_n)_{n \in \mathbb{N}}$ is a sequence of elements of $\mathcal{L}$ with

$$\lim_{n \to \infty} [f_n, f_n]_{\mathcal{L}} = 0, \quad \lim_{n \to \infty} [f_n, g]_{\mathcal{L}} = 0, \quad g \in \mathcal{L},$$

$$\lim_{n \to \infty} \chi_{w_i}(f_n) = 0, \quad i = 1, \ldots, N,$$

then $\lim_{n \to \infty} \chi_w(f_n) = 0, \quad w \in \Omega$. 
(B′) There exist $N \in \mathbb{N}$ and $(w_i)_{i=1}^N \in M^N$, such that the following implication holds. If $(f_n)_{n \in \mathbb{N}}$ is a sequence of elements of $\mathcal{L}$ with

$$\lim_{n \to \infty} [f_n, f_n]_\mathcal{L} = 0, \quad \lim_{n \to \infty} [f_n, g]_\mathcal{L} = 0, \quad g \in \mathcal{L},$$

$$\lim_{n \to \infty} \chi_{w_i}(f_n) = 0, \quad i = 1, \ldots, N,$$

then $\lim_{n \to \infty} \chi_w(f_n) = 0$, $w \in \Omega$.

(C′) If $(f_n)_{n \in \mathbb{N}}$ is a sequence of elements of $\mathcal{L}$ with

$$\lim_{n,m \to \infty} [f_n - f_m, f_n - f_m]_\mathcal{L} = 0, \quad \lim_{n \to \infty} [f_n - f_m, g]_\mathcal{L} = 0, \quad g \in \mathcal{L},$$

$$\lim_{n \to \infty} \chi_w(f_n) = 0, \quad w \in \Omega,$$

then $\lim_{n \to \infty} [f_n, g]_\mathcal{L} = 0$, $g \in \mathcal{L}$. 
Uniqueness

Theorem

Let $\mathcal{L}$ be an inner product space whose elements are functions, and assume that (A), (B), (C) hold. Then there exists a unique reproducing kernel $\mathfrak{aP}s$ which contains $\mathcal{L}$ isometrically and densely.
Uniqueness

Theorem

Let $\mathcal{L}$ be an inner product space whose elements are functions, and assume that (A), (B), (C) hold. Then there exists a unique reproducing kernel $aPs$ which contains $\mathcal{L}$ isometrically and densely.

• We call this unique space the reproducing kernel completion of $\mathcal{L}$. 
Uniqueness

Theorem

Let \( \mathcal{L} \) be an inner product space whose elements are functions, and assume that (A), (B), (C) hold. Then there exists a unique reproducing kernel aPs which contains \( \mathcal{L} \) isometrically and densely.

- We call this unique space the reproducing kernel completion of \( \mathcal{L} \).
- The number \( \Delta(\mathcal{L}) := \dim \mathcal{A}^\circ \) where \( \mathcal{A} \) is the reproducing kernel completion of \( \mathcal{L} \) is an important geometric invariant of \( \mathcal{L} \).
A motivating example: The Hamburger power moment problem
The Hamburger power moment problem

Given \((s_n)_{n=0}^\infty, s_n \in \mathbb{R}\), does there exist a positive Borel measure on \(\mathbb{R}\) with

\[ s_n = \int_{\mathbb{R}} t^n \, d\mu(t), \quad n = 0, 1, 2, \ldots \]
Existence of solutions

Theorem

There exists a solution $\mu$ if and only if

$$\forall N \in \mathbb{N}_0 : \det \left[ \left( s_i + j \right)_{i,j} \right] \geq 0$$

Consider the inner product $\left[ \sum_i \alpha_i t^i, \sum_j \beta_j t^j \right] := \sum_{i,j} (s_i + j) \cdot \alpha_i \beta_j$ on the space $C[z]$ of all polynomials. Then $\langle C[z], .,. \rangle$ is positive semidefinite.
Existence of solutions

**Theorem**

There exists a solution $\mu$ if and only if

$$\forall N \in \mathbb{N}_0 : \det [(s_{i+j})_{i,j=0}^N] \geq 0$$
Existence of solutions

**Theorem**

*There exists a solution \( \mu \) if and only if*

\[
\forall N \in \mathbb{N}_0 : \quad \det \left[ (s_{i+j})_{i,j=0}^N \right] \geq 0
\]

Consider the inner product

\[
\left[ \sum_i \alpha_i t^i, \sum_j \beta_j t^j \right] := \sum_{i,j} s_{i+j} \cdot \alpha_i \bar{\beta}_j
\]

on the space \( \mathbb{C}[z] \) of all polynomials. Then \( \langle \mathbb{C}[z], [\cdot, \cdot] \rangle \) is positive semidefinite.
Existence of solutions

**Theorem**

*Assume the moment problem is solvable. Then one of the following alternatives must occur.*
Existence of solutions

**Theorem**

*Assume the moment problem is solvable. Then one of the following alternatives must occur.*

- The solution $\mu$ is unique (*determinate* case).
- There exist infinitely many solutions (*indeterminate* case).
Existence of solutions

Theorem

Assume the moment problem is solvable. Then one of the following alternatives must occur.

- The solution $\mu$ is unique (determinate case).
- There exist infinitely many solutions (indeterminate case).

Let $S$ be the multiplication operator $Sp(z) := zp(z)$ on $\mathbb{C}[z]$. Let $\mathcal{H}$ be the Hilbert space completion of $\langle \mathbb{C}[z], [.,.] \rangle$, and let $T$ be the closure of $S$ in $\mathcal{H}$. Then one of the following holds.

- $T$ is selfadjoint (determinate case).
- $T$ is symmetric with defect index $(1, 1)$ (indeterminate case).
The Nevanlinna parameterisation

Theorem

Assume the moment problem is indeterminate.
The Nevanlinna parameterisation

**Theorem**

Assume the moment problem is indeterminate.

There exist four entire functions $A, B, C, D$, such that

$$
\int_{\mathbb{R}} \frac{d\mu(t)}{t - z} = \frac{A(z)\tau(z) + B(z)}{C(z)\tau(z) + D(z)}
$$

establishes a bijection between $\{\mu : \text{solution}\}$ and $\mathcal{N}_0 := \{\tau : \text{analytic in } \mathbb{C}^+, \text{Im} \tau(z) \geq 0\}$. 
The Nevanlinna parameterisation

**Theorem**

*Assume the moment problem is indeterminate.*

There exist four entire functions $A, B, C, D$, such that

$$
\int_{\mathbb{R}} \frac{d\mu(t)}{t-z} = \frac{A(z)\tau(z) + B(z)}{C(z)\tau(z) + D(z)}
$$

establishes a bijection between \( \{\mu : \text{solution}\} \) and \( \mathcal{N}_0 := \{\tau : \text{analytic in } \mathbb{C}^+, \text{Im } \tau(z) \geq 0\} \).

The operator $T$ is entire with respect to the gauge $u := 1$. The matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the $u$-resolvent matrix of $T$. 
The three term recurrence

Given $\mu$ with all power moments, let $p_n, n \in \mathbb{N}_0$, be the polynomials with degree $n$ and positive leading coefficient, such that $\{p_n : n \in \mathbb{N}_0\}$ is orthonormal w.r.t. $[p,q] := \int_{\mathbb{R}} p\overline{q} \, d\mu$. 
The three term recurrence

Given $\mu$ with all power moments, let $p_n$, $n \in \mathbb{N}_0$, be the polynomials with degree $n$ and positive leading coefficient, such that $\{p_n : n \in \mathbb{N}_0\}$ is orthonormal w.r.t. $[p, q] := \int_{\mathbb{R}} p q \, d\mu$.

**Theorem**

There exist unique $a_n > 0$ and $b_n \in \mathbb{R}$, s.t. ($p_{-1} := 0$)

$$zp_n(z) = a_{n+1} p_{n+1}(z) + b_n p_n(z) + a_n p_{n-1}(z), \quad n \in \mathbb{N}_0$$
The three term recurrence

Given $\mu$ with all power moments, let $p_n$, $n \in \mathbb{N}_0$, be the polynomials with degree $n$ and positive leading coefficient, such that $\{p_n : n \in \mathbb{N}_0\}$ is orthonormal w.r.t. $[p, q] := \int_{\mathbb{R}} p \overline{q} \, d\mu$.

**Theorem**

There exist unique $a_n > 0$ and $b_n \in \mathbb{R}$, s.t. $(p_{-1} := 0)$

$$zp_n(z) = a_{n+1}p_{n+1}(z) + b_np_n(z) + a_np_{n-1}(z), \quad n \in \mathbb{N}_0$$

The operator $T$ is unitarily equivalent to the operator in $\ell^2$ defined by the Jacobi matrix

$$J := \begin{pmatrix} b_0 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$
(In-)determinate measures

Definition
Let $\mu$ be a positive measure with all power moments. Then $\mu$ is called *determinate* if it is uniquely determined by the sequence of its power moments, and *indeterminate* otherwise.
(In-)determinate measures

Definition
Let \( \mu \) be a positive measure with all power moments. Then \( \mu \) is called \textit{determinate} if it is uniquely determined by the sequence of its power moments, and \textit{indeterminate} otherwise.

Theorem
\( \mu \) is determinate if and only the polynomials are dense in \( L^2(\mu) \).
(In-)determinate measures

Definition
Let $\mu$ be a positive measure with all power moments. Then $\mu$ is called determinate if it is uniquely determined by the sequence of its power moments, and indeterminate otherwise.

Theorem
\(\mu\) is determinate if and only the polynomials are dense in \(L^2(\mu)\).

Being (in-)determinate means that the moment problem for

\[ s_n := \int_{\mathbb{R}} t^n \, d\mu(t), \quad n = 0, 1, 2, \ldots, \]

which is by definition solvable, is actually (in-)determinate.
The index of determinacy

Definition
For $\mu$ determinate and $w \in \mathbb{C}$ set

$$\text{ind}_w(\mu) := \sup \left\{ k \in \mathbb{N}_0 : \left| t-w \right|^{2k} d\mu(t) \text{ determinate} \right\} \in \mathbb{N}_0 \cup \{\infty\}.$$
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Theorem
Let $\mu$ be determinate.

- If $\text{ind}_w(\mu) = \infty$ for some $w \in \mathbb{C}$, then $\text{ind}_w(\mu) = \infty$ for all $w \in \mathbb{C}$.
- Assume $\text{ind}_w(\mu) < \infty$ for some $w \in \mathbb{C}$. Then $\mu$ is discrete and $\text{ind}_w(\mu)$ is constant on $\mathbb{C} \setminus \text{supp } \mu$; denote this constant by $\text{ind}(\mu)$.
- Assume $\text{ind}_w(\mu) < \infty$ for some $w \in \mathbb{C}$. Then $\text{ind}_w(\mu) = \text{ind}(\mu) + 1$, $w \in \text{supp } \mu$. 
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For each $k \in \mathbb{N}$ the (infinite, still well-defined) matrix $J^k$ defines a linear operator $V_k$ on $\ell^2$ by taking the closure of the operator defined by the action of $J^k$ on the subspace of finite sequences.
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**Theorem**

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Let $\mu$ be determinate. Then the following are equivalent.

- $\mu$ has finite index of determinacy.
- There exists $N \in \mathbb{N}$ such that $V, \ldots, V_N$ are selfadjoint, but $V_{N+1}$ is not.
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If $\mu$ has finite index of determinacy, then $N = \text{ind}(\mu) + 1$. 
A class of distributions

Definition

1. Let \( \mu \) be a distribution on \( \mathbb{R} \). We write \( \mu \in \mathcal{D}_{<\infty} \), if

\[ \exists N \in \mathbb{N}_0, c_1, \ldots, c_N \in \mathbb{R}, \mu \text{ positive measure on } \mathbb{R} \setminus \{c_1, \ldots, c_N\} : \]

\[ \mu(f) = \int_{\mathbb{R} \setminus \{c_1, \ldots, c_N\}} f \, d\mu, \quad f \in C^\infty_{00}(\mathbb{R}), \supp f \subseteq \mathbb{R} \setminus \{c_1, \ldots, c_N\} \]
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(3) Let $\mathcal{R}_{<\infty}$ be the set of formal expressions $\rho := \sum_{i=1}^m \sum_{l=0}^{k_i} a_{il} \delta_{w_i}^{(l)}$ where $w_i \in \mathbb{C}^+$ pairwise different, $k_i \in \mathbb{N}_0$, $a_{il} \in \mathbb{C}$ with $a_{iki} \neq 0$. 
A class of distributions

Definition

Let \((\mu, \rho) \in D_{<\infty} \times R_{<\infty}\) and assume that \(\mu\) has all power moments. For \(f\) which is \(C^\infty(\mathbb{R})\) with \(f(t) = O(|t|^n), \ t \to \infty\), and locally holomorphic at \(w_i\), define

\[
(\mu, \rho)(f) := \mu(f) + \sum_{i=1}^m \sum_{l=0}^{k_i} \left( a_{il} \cdot [f]^{(l)}(w_i) + \overline{a_{il}} \cdot [f]^{(l)}(\overline{w_i}) \right), \quad n \in \mathbb{N}_0
\]
The indefinite moment problem

Given \((s_n)_{n=0}^{\infty}, s_n \in \mathbb{R}\), does there exist
\((\mu, \rho) \in \mathcal{D}_\infty \times \mathcal{R}_\infty \) with
\(s_n = (\mu, \rho)(t^n), n \in \mathbb{N}_0\) ?
Existence of solutions

For a sequence \((s_n)_{n=0}^{\infty}\) of real numbers, set \(\mathcal{L} := \mathbb{C}[z]\) and

\[
\left[ \sum_i \alpha_i t^i, \sum_j \beta_j t^j \right] := \sum_{i,j} s_{i+j} \cdot \alpha_i \beta_j.
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\exists N \in \mathbb{N}_0 : \quad \text{sgn} \det [(s_{i+j})_{i,j=0}^{n}] \text{ constant for } n \geq N
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The inner product space \(\langle \mathbb{C}[z], [,..] \rangle\) has finite negative index.
Existence of solutions

Theorem

Assume the indefinite moment problem is solvable.
Existence of solutions

**Theorem**

Assume the indefinite moment problem is solvable.

*Then there exists a number $\Delta \in \mathbb{N}_0 \cup \{\infty\}$, such that*

$(\kappa_0 := \text{ind}_- \mathcal{L})$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>$\cdots$</th>
<th>$\kappa_0$</th>
<th>$\kappa_0 + 1$</th>
<th>$\cdots$</th>
<th>$\kappa_0 + \Delta$</th>
<th>$\kappa_0 + \Delta$</th>
<th>$\cdots$</th>
</tr>
</thead>
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<td>0</td>
<td>$\cdots$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>ind$_- = n$</td>
<td>0</td>
<td>$\cdots$</td>
<td>1</td>
<td>0</td>
<td>$\cdots$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

*This includes the extremal case as follows:*

- *If $\Delta = 0$, the number of solutions is $\infty$ for all $n \geq \kappa_0$;*
- *If $\Delta = \infty$, the number of solutions is 0 for all $n > \kappa_0$.***
Parameterization of solutions

Let $\mathcal{K}^\Delta_\kappa$ be the set of all function $\tau$ meromorphic in $\mathbb{C}^+$, such that the maximal number of quadratic forms

$$Q(\xi_1, \ldots, \xi_m; \eta_0, \ldots, \eta_{\Delta-1}) := \sum_{i,j=1}^{m} \frac{\tau(w_i) - \overline{\tau(w_j)}}{w_i - w_j} \xi_i \xi_j + \sum_{k=0}^{\Delta-1} \sum_{i=1}^{m} \text{Re} (z_i^k \xi_i \overline{\eta_k})$$

where $m \in \mathbb{N}_0$, $w_1, \ldots, w_m \in \mathbb{C}^+$, equals $\kappa$. 
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**Theorem**

Assume the indefinite moment problem has $\Delta < \infty$. 
Parameterization of solutions

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Theorem

Assume the indefinite moment problem has $\Delta < \infty$.

There exist four entire functions $A, B, C, D$, such that

$$(\mu, \rho) \left( \frac{1}{t - z} \right) = \frac{A(z)\tau(z) + B(z)}{C(z)\tau(z) + D(z)}$$

establishes a bijection between $\{(\mu, \rho) : \text{ind}_-(\mu, \rho) = \kappa, \text{solution}\}$ and $K_{\kappa-\kappa_0}^{\Delta}$. 
The significance of completions

The positive definite case:

- The moment problem is solvable and indeterminate if and only if $\mathcal{L}$ has a reproducing kernel Hilbert space completion.
- Assume the moment problem is solvable and determinate, and let $\mu$ be its unique solution. Then $\text{ind}(\mu) < \infty$ if and only if $\mathcal{L}$ has a reproducing kernel aPs-completion. If $\text{ind}(\mu) < \infty$, then $\text{ind}(\mu) = \Delta(\mathcal{L}) - 1$. 

The indefinite case:

- Assume the indefinite moment problem is solvable. Then $\Delta < \infty$ if and only if $\mathcal{L}$ has a reproducing kernel aPs-completion. If $\Delta < \infty$, then $\Delta = \Delta(\mathcal{L})$. 

- Assume the indefinite moment problem is solvable with $\Delta < \infty$. The functions $A, B, C, D$ occur from (an aPs-version) of Krein’s resolvent matrix.
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The positive definite case:

- The moment problem is solvable and indeterminate if and only if \( \mathcal{L} \) has a reproducing kernel Hilbert space completion.
- Assume the moment problem is solvable and determinate, and let \( \mu \) be its unique solution. Then \( \text{ind}(\mu) < \infty \) if and only if \( \mathcal{L} \) has a reproducing kernel aPs-completion. If \( \text{ind}(\mu) < \infty \), then \( \text{ind}(\mu) = \Delta(\mathcal{L}) - 1 \).

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- Assume the indefinite moment problem is solvable with \( \Delta < \infty \). The functions \( A, B, C, D \) occur from (an aPs-version) of Krein’s resolvent matrix.
Directing Functionals
Let \( \mathcal{L} \) be an inner product space whose elements are analytic functions.

- Can one improve the general conditions for existence of a reproducing kernel aPs-completion of \( \mathcal{L} \) due to analyticity?
- If there exists a reproducing kernel aPs-completion, are its elements again analytic?
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- If there exists a reproducing kernel aPs-completion, are its elements again analytic?

An answer is obtained from an aPs-version of Krein's method of directing functionals.
Sets of semi-$\Phi$-regularity

**Definition**

Let $\mathcal{L}$ be an inner product space, let $S$ be a linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$ and $M \subseteq \Omega$, and $\Phi : \mathcal{L} \times \Omega \rightarrow \mathbb{C}$. 

$\mathcal{R}(S, \Phi) := \{ \eta \in \Omega : \text{ran}(S - \eta) \subseteq \ker \Phi(\cdot, \eta) \}$

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$\mathcal{R}(S, \Phi) := \mathcal{R}(S, \Phi) \cap \mathcal{R}(S, \Phi)$

$\mathcal{R}_{\text{app}}(S, \Phi; M) := \{ \eta \in \Omega : \forall x \in \ker \Phi(\cdot, \eta) \exists (x_n)_{n \in \mathbb{N}} \text{ s.t. } x_n \in \text{ran}(S - \eta), \lim_{n \to \infty}[x_n, x_n]_X = [x, x]_X, \lim_{n \to \infty}[x_n, y]_X = [x, y]_X, y \in \mathcal{L}, \lim_{n \to \infty} \Phi(x_n, w) = \Phi(x, w), w \in M \}$. 

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\[
\begin{align*}
    r_{\subseteq}(S, \Phi) &:= \{ \eta \in \Omega : \text{ran}(S - \eta) \subseteq \ker \Phi(\cdot, \eta) \} \\
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$$x_n \in \text{ran}(S - \eta),$$

$$\lim_{n \to \infty} [x_n, x_n]_\chi = [x, x]_\chi, \quad \lim_{n \to \infty} [x_n, y]_\chi = [x, y]_\chi, \quad y \in \mathcal{L},$$

$$\lim_{n \to \infty} \Phi(x_n, w) = \Phi(x, w), \quad w \in M \right\}.$$
Directing functionals in aPs

Definition
Let $\mathcal{L}$ be an inner product space, let $\mathcal{S}$ be a symmetric linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$, and let $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$. 

We call $\Phi$ a directing functional for $\mathcal{S}$, if it satisfies the following axioms.

(DF1) For each $w \in \Omega$ the function $\Phi(\cdot, w) : \mathcal{L} \to \mathbb{C}$ is linear.

(DF2) The set $\Omega$ is open. For each $x \in \mathcal{L}$ the function $\Phi(x, \cdot) : \Omega \to \mathbb{C}$ is analytic.

(DF3) There is no nonempty open subset $O$ of $\Omega$, such that $\Phi|_{\mathcal{L} \times O} = 0$.

(DF4) The set $r \subseteq (\mathcal{S}, \Phi)$ has accumulation points in each connected component of $\Omega \setminus \mathcal{R}$.

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Directing functionals in aPs

Example
Let \((s_n)_{n=0}^{\infty}, s_n \in \mathbb{R}\), be given and consider:

- \(\mathcal{L} := \mathbb{C}[z] \text{ with } [.,.]\);
- \(S := \{(p(z); zp(z)) : p \in \mathbb{C}[z]\}\);
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- \(\forall w \in \mathbb{C} : \text{ran}(S - w) = \{p \in \mathbb{C}[z] : p(w) = 0\} = \ker \Phi(\cdot, w)\), hence \(r(S, \Phi) = \mathbb{C}\).
aPs-completion of spaces of analytic functions

**Theorem**

Let $\mathcal{L}$ be an inner product space with $\text{ind}_{\mathcal{L}} \mathcal{L} < \infty$, let $S$ be a symmetric linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$, and let $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$ be a directing functional for $S$. 
aPs-completion of spaces of analytic functions

Theorem
Let \( \mathcal{L} \) be an inner product space with \( \text{ind} \mathcal{L} < \infty \), let \( S \) be a symmetric linear relation in \( \mathcal{L} \), let \( \Omega \subseteq \mathbb{C} \), and let \( \Phi : \mathcal{L} \times \Omega \rightarrow \mathbb{C} \) be a directing functional for \( S \).

Assume that

- \( \exists M \subseteq r_{\subseteq}(S, \Phi) \) s.t. \( M \) has accumulation points in each connected component of \( \Omega \setminus \mathbb{R} \), and

\[
\dim \left( \left[ \mathcal{L}^\perp + \text{span}\{\Phi(\cdot, w) : w \in M\} \right] / \mathcal{L}^\perp \right) < \infty;
\]
Theorem

Let $\mathcal{L}$ be an inner product space with $\text{ind}_{\mathcal{L}} \mathcal{L} < \infty$, let $S$ be a symmetric linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$, and let $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$ be a directing functional for $S$. Assume that

- $\exists M \subseteq r_{\subseteq}(S, \Phi)$ s.t. $M$ has accumulation points in each connected component of $\Omega \setminus \mathbb{R}$, and
  \[
  \text{dim} \left( \left[ \mathcal{L}^\wedge + \text{span}\{\Phi(\cdot, w) : w \in M\} \right] / \mathcal{L}^\wedge \right) < \infty;
  \]

- Either $\mathcal{L}^\wedge \cap \text{span}\{\Phi(\cdot, w) : w \in r_{\subseteq}(S, \Phi), \Phi(\cdot, w) \in \mathcal{L}^\wedge + \text{span}\{\Phi(\cdot, w) : w \in M\}\}$
  or $\mathcal{L}^\wedge \cap \text{span}\{\Phi(\cdot, w) : w \in r_{\supseteq}^{\text{app}}(S, \Phi; \Omega \setminus \mathbb{R}) \setminus \mathbb{R}\}$
  is dense in $\mathcal{L}^\wedge$ w.r.t. $\mathcal{T}^\wedge$.  

reproducing kernel spaces

Hamburger moment problem

Directing Functionals

Almost Pontryagin Spaces

Reproducing Kernel Spaces

Hamburger moment problem

Directing Functionals
aPs-completion of spaces of analytic functions

Theorem

Let \( \mathcal{L} \) be an inner product space with \( \text{ind}_- \mathcal{L} < \infty \), let \( S \) be a symmetric linear relation in \( \mathcal{L} \), let \( \Omega \subseteq \mathbb{C} \), and let \( \Phi : \mathcal{L} \times \Omega \to \mathbb{C} \) be a directing functional for \( S \).

Under these assumptions:

- There exists a unique reproducing kernel aPs \( \mathcal{B} \), such that \( \Phi_{\mathcal{L}} : x \mapsto \Phi(x, \cdot) \) maps \( \mathcal{L} \) isometrically onto a dense subspace of \( \mathcal{B} \).
- The elements of \( \mathcal{B} \) are analytic on \( \Omega \).
- \( \text{Clos}_\mathcal{B} ( (\Phi_{\mathcal{L}} \times \Phi_{\mathcal{L}})(S) ) = S(\mathcal{B}) \).

Here \( S(\mathcal{B}) \) is the multiplication operator in \( \mathcal{B} \).
aPs-completion of spaces of analytic functions

Theorem

Let $\mathcal{L}$ be an inner product space with $\text{ind}_- \mathcal{L} < \infty$, let $S$ be a symmetric linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$, and let $\Phi : \mathcal{L} \times \Omega \rightarrow \mathbb{C}$ be a directing functional for $S$.

Under these assumptions:

Concerning the geometry of $\mathcal{B}$, we have

- $\Phi^*_L(\mathcal{B}') = \mathcal{L}^\wedge + \text{span} \{ \Phi(\cdot, w) : w \in M \} = \mathcal{L}^\wedge + \text{span} \{ \Phi(\cdot, w) : w \in \Omega \}$

- $\text{ind}_0 \mathcal{B} = \dim \left( \left[ \mathcal{L}^\wedge + \text{span}\{ \Phi(\cdot, w) : w \in \Omega \} \right] / \mathcal{L}^\wedge \right)$

- The set $\{ w \in \Omega : \vartheta_\mathcal{B}(w) > 0 \}$ is discrete.
  Here $\vartheta_\mathcal{B}(w)$ is the minimal multiplicity of $w$ as a zero of some element of $\mathcal{B} \setminus \{0\}$.
aPs-completion of spaces of analytic functions

Theorem

Let $\mathcal{L}$ be an inner product space with $\text{ind}_- \mathcal{L} < \infty$, let $S$ be a symmetric linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$, and let $\Phi : \mathcal{L} \times \Omega \rightarrow \mathbb{C}$ be a directing functional for $S$.

Under these assumptions:

Concerning the operator theory of $S(\mathcal{B})$, we have

- $S(\mathcal{B})$ is of defect $(1, 1)$;
- $\Omega \subseteq r(S(\mathcal{B}))$
- $\text{ran}(S(\mathcal{B}) - w) = \ker \chi_{w}(\partial^{\mathcal{B}(w)})|_{\mathcal{B}}$, $w \in \Omega$
De Branges space completions

Definition

An inner product space $\mathcal{L}$ whose elements are entire functions is called \textit{algebraic de Branges space}, if

- If $f \in \mathcal{L}$, $w \in \mathbb{C} \setminus \mathbb{R}$ with $f(w) = 0$, then $\frac{f(z)}{z-w} \in \mathcal{L}$. We have

$$\begin{bmatrix} \frac{z-w}{z-w} f(z), \frac{z-w}{z-w} g(z) \end{bmatrix}_\mathcal{L} = [f(z), g(z)]_\mathcal{L},$$

$$f, g \in \mathcal{B}, f(w) = g(w) = 0.$$

- If $f \in \mathcal{L}$ then $f^\#(z) := \overline{f(\overline{z})} \in \mathcal{L}$. We have

$$[f^\#, g^\#]_\mathcal{L} = [g, f]_\mathcal{L}, \quad f, g \in \mathcal{L}.$$
De Branges space completions

Definition

An inner product space \( \mathcal{L} \) whose elements are entire functions is called \textit{algebraic de Branges space}, if

- If \( f \in \mathcal{L}, w \in \mathbb{C} \setminus \mathbb{R} \) with \( f(w) = 0 \), then \( \frac{f(z)}{z-w} \in \mathcal{L} \). We have

\[
\begin{bmatrix}
    z - \overline{w} f(z), & z - \overline{w} g(z)
\end{bmatrix}_{\mathcal{L}} = \begin{bmatrix} f(z), g(z) \end{bmatrix}_{\mathcal{L}},
\]

\( f, g \in \mathcal{B}, f(w) = g(w) = 0 \).

- If \( f \in \mathcal{L} \) then \( f^\#(z) := \overline{f(\overline{z})} \in \mathcal{L} \). We have

\[
\begin{bmatrix} f^\#, g^\# \end{bmatrix}_{\mathcal{L}} = \begin{bmatrix} g, f \end{bmatrix}_{\mathcal{L}}, \quad f, g \in \mathcal{L}.
\]

If in addition \( \mathcal{L} \) is a reproducing kernel aPs, then \( \mathcal{L} \) is called a \textit{de Branges aPs}. 
Theorem

Let $\mathcal{L}$ be an algebraic de Branges space. If $\mathcal{L}$ has a reproducing kernel aPs-completion, then this completion is a de Branges aPs.
Selected Literature

**INDEFINITE INNER PRODUCTS:**


Selected Literature

**Reproducing kernel spaces:**

Selected Literature

**Moment problems:**


Selected Literature

**DIRECTING FUNCTIONALS:**


