Mathematical Theory of Finite Element Methods and Iterative Solvers

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Office Hours: right after class, or by appointment (please contact me by email.)

Prerequisites: This is a graduate level class for Master students of CES, Mathematics, as well as Simulation Sciences. The intended audience should be familiar with basics concepts of PDE's theory, functional analysis, and linear algebra.

Course Description: This is a course at the intersection of functional analysis, Finite Element methods, and numerical linear algebra. Its objective is to show how all three areas are essential to modern computational techniques. We begin by reviewing the theory of elliptic PDE's, including non-coercive problems, and introduce the concept of a Continuous Galerkin discretization. We discuss stability, optimality, and convergence, as well as asymptotic stability in the context of non-coercive problems. The second part of the course focuses on iterative solvers as efficient tools to tackle the linear systems that arise from the discretizations. We discuss applications that originate from fluid dynamics: the convection-diffusion equation, along with stabilization through bubbles, and the Stokes equations.

Workload: 4SWS | 6ECTS

Textbook: The notes by the instructor found under *RWTHMoodle* are intended as the main course material; additional reading material might also be posted as the semester progresses. The following is an incomplete bibliography:

- H. Elman, D. Silvester, and A. Wathen, *Finite Elements and Fast Iterative Solvers with Applications in Incompressible Fluid Dynamics*, Oxford University Press, 2005;
- L. Demkowicz and J.T. Oden, *Applied Functional Analysis*, CRC Press, 2010;
- L. Demkowicz, *Computing with hp-adaptive Finite Elements*, Volume 1, Chapman & Hall/CRC Press, 2007;
- Y. Saad, *Iterative Methods for Sparse Linear Systems*;
- G.H. Golub and C.F. Van Loan, *Matrix Computations*, The John Hopkins University Press, 1996;
- Ph.G. Ciarlet, *The Finite Element Method for Elliptic Problems*;
- G. Strang, *Computational Science and Engineering*, Wellesley Cambridge Press, 2012;
- L.N. Trefethen and D. Bau, *Numerical Linear Algebra*, SIAM, 1997.

Assignments: Exercises from the notes, that complement the material, will be assigned during the lectures. Students are expected to discuss their solutions in class, during exercise sessions. Although the assignments are neither collected nor graded, the assigned exercises constitute the bulk of the final exam.

Attendance: It is in students best interest to attend lectures. If you are unable to attend, you are personally responsible for the material covered in class.

Grading/Exam: To be decided, possibly dependent upon the size of class.

Tentative Course Outline:

1. Course presentation; derivation of equations of fluid dynamics.
2. The Laplace equation and harmonic functions.
3. Regularity of solutions of the Poisson equation; introduction of weak solutions.
4. Variational formulations, Sobolev embedding theorem, elliptic regularity.
5. Lax-Milgram lemma.
6. Generalized Lax-Milgram lemma; Sobolev spaces; Diffusion-Convection-Reaction equation.
7. Weak coercivity, pivot space, compact perturbation of identity.
8. Fredholm alternative.
9. Discussion of exercises.
10. Galerkin approximation, Ceá lemma, Babuska theorem.
11. Asymptotic stability, Mikhlin theorem and its extension.
12. Forced vibrations of a string, characterization of inf-sup constant.
13. Projection operators, improvement of Babuska estimate.
14. FE error estimates.
15. FE error estimates (cont'd).
16. FE error estimates (cont'd).
17. Discussion of exercises.
18. Review of numerical linear algebra.
19. Iterative methods, Krylov subspaces, and Conjugate Gradient (CG) method.
20. The Conjugate Gradient (CG) method (cont'd).
21. The GMRES and MINRES methods.
22. Advection-Diffusion equation; stabilization through bubble functions.

23. Banach Closed Range theorem and its implications.
24. A historical prospective on the Stokes problem.
25. FE discretization of Stokes problem.
26. FE discretization of Stokes problem (cont'd).
27. Discussion of exercises.

The material covered in each individual lecture might slightly change, as it depends on the progress of the class.

Miscellanea: The lectures on numerical linear algebra and iterative solvers closely follow the book "Computational Science and Engineering" by Gilbert Strang; this is the text book of course 18.086, offered at MIT, and the lectures are available online at the following link:
`https://ocw.mit.edu/courses/mathematics/18-086-mathematical-methods-for-engineers-ii-spring-2006/index.htm`
If you are interested in implementation and computational aspects of finite elements, a great resource is the homepage of the `dealii` project, see `http://www.dealii.org/`; you can also find video lectures at `http://www.math.colostate.edu/b̃angerth/videos.html` about the code and general theory and practice of finite elements.

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