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Sommersemester 2019
03.04.2019

# Übungsaufgaben zur VU Computermathematik
## Serie 4

We use the packages `plots` and `LinearAlgebra`.

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**Exercise 4.1:** ***Visualization of linear mappings.***

**a)** With `plots[arrow]` you can draw arrows. *Use this to visualize the behavior of a linear mapping* $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ *represented by its coefficient matrix, by drawing the parallelogram spanned by the images of the unit vectors* $(1,0)$ *and* $(0,1)$ *under the mapping. Produce a nice plot.*

**b)** *Analogous to* **a)***, but for* $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$.

**Exercise 4.2:** ***Data-sparse representation of a linear mapping.***

A data-sparse representation of a low-rank linear mapping or of its coefficient matrix $A$ is a procedural representation which does not involve the explicit form of the matrix. Example:

$$A\,x \,=\, \prod_{k=1}^{m} (I - u_k v_k^T) \cdot x \,=\, (I - u_m v_m^T) \cdots (I - u_1 v_1^T) \cdot x, \quad u_k, v_k \in \mathbb{R}^n$$

with $m \ll n$. Here, the column vectors $u_k, v_k \in \mathbb{R}^n$ contain the full information about the mapping.

*Design a procedure* `mvmul(U::Matrix,V::Matrix,x::Vector)` *which computes* $A\,x$ *without explicitly building the matrix* $A$*. (Here, the columns of* `U,V` $\in \mathbb{R}^{n\times m}$ *represent the vectors* $u_k, v_k$*,* $k = 1 \ldots m$*.) Use inner products and scalar · vector multiplications only. Verify the correctness of your code for at least one example.*
Hint: Since matrix multiplication is associative, we have $(uv^T)x = u(v^T x) = (v^T x)u$.

**Exercise 4.3:** ***Sherman-Morrison-Woodbury (SMW).***

Let $A \in \mathbb{R}^{n\times n}$ be invertible, and $U, V \in \mathbb{R}^{n\times m}$. Then, the Sherman-Morrison-Woodbury-formula holds true: $A + UV^T \in \mathbb{R}^{n\times n}$ is invertible iff $I + V^T A^{-1} U \in \mathbb{R}^{m\times m}$ is invertible, and

$$(A + UV^T)^{-1} \,=\, A^{-1} - A^{-1} U \,(I + V^T A^{-1} U)^{-1}\, V^T A^{-1}\,.$$

This identity (which is not very difficult to prove) can be used to compute the inverse $(A + UV^T)^{-1}$, assuming $A^{-1}$ is already known. The additional effort involves only computing the small inverse $(I + V^T A^{-1} U)^{-1} \in \mathbb{R}^{m\times m}$; thus, using the SMW formula is more efficient than direct inversion of $(A + UV^T)^{-1}$, if $m \ll n$.

*Implement this formula in form of a procedure*

```
SMW(AI::Matrix,U::{Matrix,Vector[column]},V::{Matrix,Vector[column]})
```

*and test.*

**Exercise 4.4:** ***A matrix depending on two parameters.***

Consider the matrix

$$A \,=\, \begin{pmatrix} 0 & \alpha & 1 & 0 & \beta \\ 0 & 1 & 0 & \alpha & 0 \\ \alpha & 0 & 1 & 0 & 0 \\ \beta & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha\beta & 0 \end{pmatrix}$$

depending on two parameters $\alpha, \beta \in \mathbb{C}$.

**a)** *What is the generic rank of A?* [1]

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[1] Recall from lecture what is to be understood by ‘generic’.

**b)** *Determine all possible values of the parameters $\alpha, \beta$ such that $A$ is a singular matrix, and determine the rank of $A$ for these special cases.*

**Exercise 4.5:** ***Projection in 3D.***

Let $\mathcal{U}$ be a linear subspace of $\mathbb{R}^3$ of dimension 2 (i.e., a plane containing the point 0). We wish to determine the matrix representation of the projector $P$, a linear mapping which projects points $x \in \mathbb{R}^3$ onto $\mathcal{U}$ in the direction of a given vector $0 \neq w \notin \mathcal{U}$. It is not difficult to see (*check this*) that $P$ is a rank 2 matrix uniquely determined by the requirements

$$P\,u = u, \quad P\,v = v, \quad P\,w = 0,$$

where $u, v \in \mathcal{U}$ are given linearly independent vectors spanning $\mathcal{U}$.

**a)** Design a procedure

```
projector(u::Vector,v::Vector,w::Vector)
```

which returns the matrix $P \in \mathbb{R}^{3\times 3}$ in form of an object of type `Matrix`. Use `LinearSolve`.

**b)** It is easy to see that $P^2 = P$. *Check this by an example.*

**c)** If $w \perp \mathcal{U}$, the outcome $P$ is the so-called orthogonal projector onto $\mathcal{U}$. In this case not only $P^2 = P$ holds, but also $P = Q$. *What is $Q$ ?* [2]

**Exercise 4.6:** (*) ***Similar matrices.*** [3]

A pair of matrices $A, B \in \mathbb{R}^{n\times n}$ is called similar if there exists a regular matrix $X \in \mathbb{R}^{n\times n}$ such that $B = X\,A\,X^{-1}$. For given $A, B$ we want to find $X$ (if it exists).

**a)** *Reformulate the problem in a way such that the inverse of the unknown matrix $X$ is not involved.*

**b)** *Try to solve this problem for the special case $n = 2$, at least for a simple numerical example.*

**Exercise 4.7:** ***Derivation of a formula for the numerical approximation of a second derivative.***

Let $h > 0$, and consider

$$\varphi(x,h) \;=\; c_{-2}\,f(x-2h) + c_{-1}\,f(x-h) + c_0\,f(x) + c_1\,f(x+h) + c_2\,f(x+2h),$$

where the constants $c_i$ are to be determined in a such a way that $\varphi(x,h)$ approximates the second derivative $f''(x)$ of some function $f$ at the point $x$.

*Find the $c_i$ such that as many as possible terms in the Taylor expansion of $\varphi(x,h) - f''(x)$ about $h = 0$ vanish.*

Hint: Use `taylor` and solve a system of linear equations.

**Exercise 4.8:** ***Polynomial derivatives and matrix representation.***

We consider the $(n+1)$-dimensional vector space $\mathcal{P}_n$ of polynomials $p$ of degree $\leq n$,

$$p(t) \;=\; a_0 + a_1\,t + a_2\,t^2 + \ldots + a^n\,t^n\,,$$

and the linear operation $D\colon \mathcal{P}_n \to \mathcal{P}_{n-1}$ defined by $D(p) = p'$ (first derivative w.r.t. $t$).

**a)** Computation of the polynomial $D(p) = p'$ is equivalent to a linear transformation which maps the coefficient vector $(a_0, a_1, a_2, \ldots, a_n)$ of $p(t)$ to the corresponding coefficient vector of $p'(t) = a_1 + 2\,a_2\,t + \ldots$.

*Design a procedure which, for given $n$, returns the matrix representation $A \in \mathbb{R}^{(n-1)\times n}$ of this linear transformation.*

**b)** *How can you use* **a)** *to generate the analogous matrix representation of $D^{(2)}(p) = p''$?*

Validate your findings using numerical examples (i.e., for some concrete values of $n$ and the $a_k$).

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[2] This is a nice exercise in linear algebra. If you do not already know the answer, you may test examples, and such an experiment maybe helpful for grabbing the idea of the proof.

[3] This problem looks simple at first sight, but it is not straightforward to solve (see the course Linalg 2). Do not worry too much about it – it is not mandatory. On the other hand, it is a realistic example in the following sense: If not sure what is going on with some problem, one may use computer algebra for trying to get an idea.