

Modal based hypocoercivity methods on the torus & real line with application to Goldstein-Taylor models

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Outline:

- 1 hypocoercive ODEs: sharp decay estimates
- 2 Goldstein-Taylor model (= 2 velocity BGK model) on torus
- 3 Goldstein-Taylor model on \mathbb{R}

Long-time decay for nonsymmetric ODEs

$$\dot{x} = -\mathbf{C}x, \quad t \geq 0, \quad x(t) \in \mathbb{C}^n \quad (1)$$

Definition: \mathbf{C} is *coercive* if $\bar{x}^T \mathbf{C} x \geq \kappa \|x\|^2 \forall x$ (for some $\kappa > 0$).

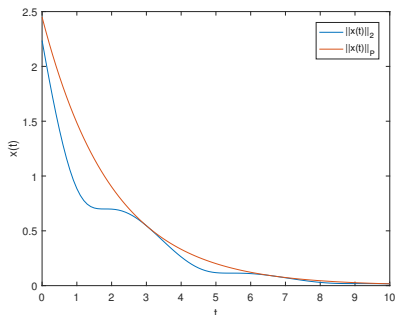
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ex: $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\lambda_{\mathbf{C}} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \Rightarrow$ decay rate = $\frac{1}{2}$ for (1).

- \mathbf{C} not coercive \Rightarrow no decay of $\|x(t)\|_2$ by trivial energy method!
- But decay of **modified norm** $\|x(t)\|_{\mathbf{P}} := \sqrt{\bar{x}^T \mathbf{P} x}$; $\mathbf{P} := [2 \ 1; 1 \ 2]$



How to find $\mathbf{P} > 0$ /
the Lyapunov functional?

hypo-coercive ODEs

$$\dot{x} = -\mathbf{C}x, \quad t \geq 0, \quad x(t) \in \mathbb{C}^n$$

Definition: \mathbf{C} is *hypo-coercive* (= positive stable) if $\exists \mu > 0$ such that:

$$\Re(\lambda_j) \geq \mu, \quad j = 1, \dots, n.$$

If all eigenvalues of \mathbf{C} are non-defective:

$$\exists c \geq 1 : \quad \|x(t)\|_2 \leq c \|x(0)\|_2 e^{-\mu t}, \quad t \geq 0.$$

- always: $\mu \geq \kappa := \max_x \frac{\bar{x}^T \mathbf{C}_H x}{\|x\|^2}$ (i.e. spectral gap \geq coercivity)

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Conditions for hypo-coercivity:

- 1 $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 \in \mathbb{C}^{n \times n}$; $\mathbf{C}_1^* = -\mathbf{C}_1$, $\mathbf{C}_2^* = \mathbf{C}_2 \geq 0$ (w.l.o.g.)
- 2 No (non-trivial) subspace of $\ker \mathbf{C}_2$ is invariant under \mathbf{C}_1

Choice of \mathbf{P} for $\|x\|_{\mathbf{P}}$ / Lyapunov matrix inequality

Lemma 1

Let $\mathbf{C} \in \mathbb{R}^{n \times n}$ be positive stable, i.e. $\mu := \min\{\Re \lambda_{\mathbf{C}}\} > 0$.

- ① If all $\lambda_{\mathbf{C}}^{\min} \in \{\lambda \in \sigma(\mathbf{C}) \mid \Re \lambda = \mu\}$ are *non-defective* (i.e. geometric = algebraic multiplicity)

$$\Rightarrow \exists \mathbf{P} \in \mathbb{R}^{n \times n}, \mathbf{P} > 0 : \mathbf{P}\mathbf{C} + \mathbf{C}^{\top}\mathbf{P} \geq 2\mu\mathbf{P}.$$

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- ② If (at least) one $\lambda_{\mathbf{C}}^{\min}$ is *defective* \Rightarrow

$$\forall \varepsilon > 0 \quad \exists \mathbf{P} = \mathbf{P}(\varepsilon) > 0 : \mathbf{P}\mathbf{C} + \mathbf{C}^{\top}\mathbf{P} \geq 2(\mu - \varepsilon)\mathbf{P}.$$

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Proof: \mathbf{P} can be constructed explicitly; e.g. for \mathbf{C} non-defective / diagonalizable:

$$\mathbf{P} := \sum_{j=1}^n z_j \otimes \bar{z}_j^{\top}; \quad z_j \dots \text{eigenvectors of } \mathbf{C}^{\top}$$

- \mathbf{P} not unique; but the decay rates μ (or $\mu - \varepsilon$) are independent of \mathbf{P} .
- For complex \mathbf{C} : $\mathbf{P} > 0$ Hermitian with $\mathbf{P}\mathbf{C} + \mathbf{C}^*\mathbf{P} \geq 2\mu\mathbf{P}$.



Long-time decay of \mathbf{P} -norm

- Sharp decay estimate for $\dot{x} = -\mathbf{C}x$ (non-defective case, \mathbf{C} real):

Let $\|x\|_{\mathbf{P}}^2 := x^T \mathbf{P} x$.

$$\frac{d}{dt} \|x\|_{\mathbf{P}}^2 = -x^T \underbrace{(\mathbf{P}\mathbf{C} + \mathbf{C}^T \mathbf{P})}_{\geq 2\mu \mathbf{P}} x \leq -2\mu \|x\|_{\mathbf{P}}^2$$

$$\Rightarrow \|x(t)\|_{\mathbf{P}} \leq \|x(0)\|_{\mathbf{P}} e^{-\mu t}, \quad t \geq 0.$$

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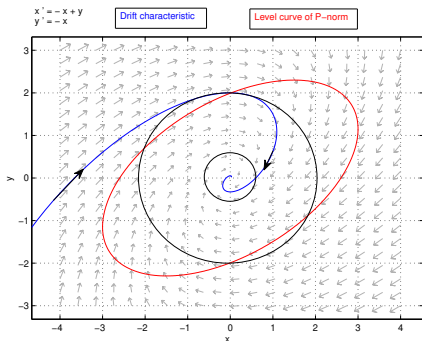
$$\Rightarrow \|x(t)\|_{\mathbf{P}} \leq \|x(0)\|_{\mathbf{P}} e^{-\mu t}, \quad t \geq 0.$$

- \mathbf{P} -norm can be used for entropy/energy methods of kinetic equations (e.g. relaxation/BGK, Fokker-Planck)

Decay of \mathbf{P} -norm (continued)

ex: $\dot{x} = -\mathbf{C}x$ with $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

- At x_2 -axis: trajectory $x(t)$ tangent to level curve of $|x|$:



- level curve of “distorted” vector norm $\sqrt{x^T \mathbf{P} x}$; $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
→ uniform decay with sharp rate $\frac{1}{2}$

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Goldstein-Taylor model

$$\begin{cases} \partial_t f_+(x, t) + \partial_x f_+(x, t) = \frac{\sigma(x)}{2}(f_- - f_+), \\ \partial_t f_-(x, t) - \partial_x f_-(x, t) = -\frac{\sigma(x)}{2}(f_- - f_+), \end{cases} \quad t > 0, x \in \mathbb{T} \text{ or } \mathbb{R}$$

- as BGK-toy model
- original applications: turbulent fluid motion, telegrapher's equation

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- as BGK-toy model
- original applications: turbulent fluid motion, telegrapher's equation
- mass density $u(x, t) := f_+ + f_-$, flux density $v(x, t) := f_+ - f_-$

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u = -\sigma(x)v \end{cases}$$

- goal (on \mathbb{T}):
sharp decay estimates $u(t) \rightarrow u_\infty = 2f_\infty := (f_{+,0} + f_{-,0})_{avg}$,
 $v(t) \rightarrow v_\infty = 0$; **ultimately for $\sigma(x)$**

Modal decomposition of (u, v) on torus for $\sigma = \text{const}$:

$$\frac{d}{dt} \begin{pmatrix} \widehat{u}(k) \\ \widehat{v}(k) \end{pmatrix} = - \begin{pmatrix} 0 & ik \\ ik & \sigma \end{pmatrix} \begin{pmatrix} \widehat{u}(k) \\ \widehat{v}(k) \end{pmatrix} =: -\mathbf{C}_k \begin{pmatrix} \widehat{u}(k) \\ \widehat{v}(k) \end{pmatrix}, \quad k \in \mathbb{Z}.$$

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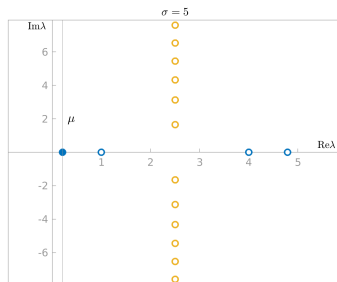
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Case I: $0 < |k| < \frac{\sigma}{2}$: $\lambda_{\mp, k} := \frac{\sigma}{2} \mp \sqrt{\frac{\sigma^2}{4} - k^2}$,

$$\mathbf{P}_k^{(I)} := \begin{pmatrix} 1 & -\frac{2ki}{\sigma} \\ \frac{2ki}{\sigma} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{2}{\sigma} \partial_x \\ \frac{2}{\sigma} \partial_x & 1 \end{pmatrix}$$

Case II: $|k| = \frac{\sigma}{2}$... defective: $\mathcal{O}((1+t)e^{-\frac{\sigma}{2}t})$



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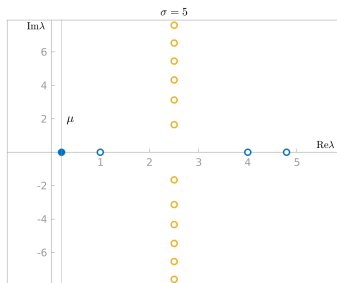
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Case II: $|k| = \frac{\sigma}{2}$... defective: $\mathcal{O}((1+t)e^{-\frac{\sigma}{2}t})$

Case III: $|k| > \frac{\sigma}{2}$: $\Re \lambda_{\pm, k} := \frac{\sigma}{2}$, $\mathbf{P}_k^{(III)} := \begin{pmatrix} 1 & -\frac{i\sigma}{2k} \\ \frac{i\sigma}{2k} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{\sigma}{2} \partial_x^{-1} \\ -\frac{\sigma}{2} \partial_x^{-1} & 1 \end{pmatrix}$



Modal Lyapunov functional

Situation A: $0 < \sigma < 2$: all modes $k \neq 0$ in Case III

$$\begin{aligned} \mathcal{E}(\widehat{u}, \widehat{v}) &:= \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\| \begin{pmatrix} \widehat{u}(k) \\ \widehat{v}(k) \end{pmatrix} \right\|_{\mathbf{P}_k^{(III)}}^2 + \left\| \begin{pmatrix} \widehat{u}(0) \\ \widehat{v}(0) \end{pmatrix} \right\|_2^2 \\ &= \underbrace{\|\tilde{u}\|_2^2 + \|v\|_2^2 - \frac{\sigma}{2\pi} \int_0^{2\pi} \Re(\partial_x^{-1} \tilde{u} \bar{v}) dx}_{=: E_\sigma(\tilde{u}, v)}, \quad \tilde{u}(x) := u(x) - u_{avg} \end{aligned}$$

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Situation B: $\sigma = 2$: defective

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Situation B: $\sigma = 2$: defective

Situation C: $\sigma > 2$: Low modes in Case I determine global decay rate, high modes in Case III.

\Rightarrow Mixture would yield a (complicated) Ψ DO with non-smooth symbol.

Mixed modes in Situation C ($\sigma > 2$)

remedy:

Only modes $k = \pm 1$ determine global decay rate $\mu = \frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}$.

\Rightarrow Higher modes gives some leeway to adjust the decay estimate.

\Rightarrow Suboptimal norms $\|\cdot\|_{\mathbf{P}_k}$ can be used for $|k| \geq 2$:

Lemma 2

Let $\sigma > 2$: $\mathbf{P}_k := \begin{pmatrix} 1 & -\frac{2i}{k\sigma} \\ \frac{2i}{k\sigma} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{2}{\sigma}\partial_x^{-1} \\ -\frac{2}{\sigma}\partial_x^{-1} & 1 \end{pmatrix}$ satisfies

$$\mathbf{C}_k^* \mathbf{P}_k + \mathbf{P}_k \mathbf{C}_k \geq 2\mu \mathbf{P}_k \quad \forall k \neq 0. \quad (2)$$

$\Rightarrow \{\mathbf{P}_k\}_{k \neq 0}$ is “good enough” for optimal decay, since $\mathbf{P}_1 = \mathbf{P}_1^{(l)}$, all modes satisfy (2) with uniform rate μ .

also: $\{\mathbf{P}_k\} \sim$ simple Ψ DO.

Decay of spatial Lyapunov functional – with symbol \mathbf{P}_k

$$E_\sigma(\tilde{u}, v) := \|\tilde{u}\|_2^2 + \|v\|_2^2 - \frac{\sigma}{2\pi} \int_0^{2\pi} \Re(\partial_x^{-1} \tilde{u} \bar{v}) dx, \quad \tilde{u}(x) := u(x) - u_{\text{avg}}$$

Proposition 1

- i) If $0 < \sigma < 2$: $E_\sigma(u(t) - u_{\text{avg}}, v(t)) \leq E_\sigma(u_0 - u_{\text{avg}}, v_0) e^{-\sigma t}$.
- ii) If $\sigma > 2$: $E_{\frac{4}{\sigma}}(u(t) - u_{\text{avg}}, v(t)) \leq E_{\frac{4}{\sigma}}(u_0 - u_{\text{avg}}, v_0) e^{-(\sigma - \sqrt{\sigma^2 - 4})t}$.

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Lemma 3

For $0 < \theta < 2$:

$$\left(1 - \frac{\theta}{2}\right) \left(\|f\|_2^2 + \|g\|_2^2\right) \leq E_\theta(f, g) \leq \left(1 + \frac{\theta}{2}\right) \left(\|f\|_2^2 + \|g\|_2^2\right).$$

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Remark: The sharp rate from Proposition 1 can also be recovered from [Dolbeault-Mouhot-Schmeiser 2015] when optimizing a parameter $\delta(k)$.

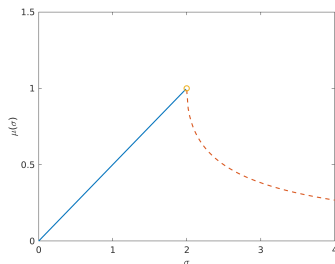
⇒ Decay in L^2 with sharp rate $\mu(\sigma)$:

Theorem 1 (AA-Einav-Signorello-Wöhner, JSP 2021)

For $\sigma = \text{const} \neq 2$:

$$\left\| \begin{pmatrix} f_+(t) \\ f_-(t) \end{pmatrix} - \begin{pmatrix} f_{\infty} \\ f_{\infty} \end{pmatrix} \right\|_2 \leq C_{\sigma} \left\| \begin{pmatrix} f_{+,0} \\ f_{-,0} \end{pmatrix} - \begin{pmatrix} f_{\infty} \\ f_{\infty} \end{pmatrix} \right\|_2 e^{-\mu(\sigma)t}, \quad t \geq 0;$$

$$C_{\sigma} := \begin{cases} \sqrt{\frac{2+\sigma}{2-\sigma}}, & 0 < \sigma < 2 \\ \sqrt{\frac{\sigma+2}{\sigma-2}}, & \sigma > 2 \end{cases}, \quad \mu(\sigma) := \begin{cases} \frac{\sigma}{2}, & 0 < \sigma < 2 \\ \frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}, & \sigma > 2 \end{cases}$$



Extension to $\sigma(x)$ on torus

- modal decomposition unfeasible
- But $E_\theta(\tilde{u}, v) := \|\tilde{u}\|_2^2 + \|v\|_2^2 - \frac{\theta}{2\pi} \int_0^{2\pi} \Re(\partial_x^{-1} \tilde{u} \bar{v}) dx$ is still a Lyapunov functional:

Theorem 2 (AA-Einav-Signorello-Wöhler, JSP 2021)

For $0 < \sigma_{\min} \leq \sigma_{\max} < \infty$, let $\theta^* := \min\left(\sigma_{\min}, \frac{4}{\sigma_{\max}}\right)$. Then

$$E_{\theta^*}(u(t) - u_{\text{avg}}, v(t)) \leq E_{\theta^*}(u_0 - u_{\text{avg}}, v_0) e^{-\alpha^* t}, \quad t \geq 0$$

for an explicit $\alpha^*(\sigma_{\min}, \sigma_{\max}) > 0$.

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for an explicit $\alpha^*(\sigma_{\min}, \sigma_{\max}) > 0$.

- pros: Strategy is applicable to BGK models with n velocities; $n = 3$ in [AESW '21].
- cons: Decay rate α^* is not optimal. Sharp rate for $\sigma(x)$ in [Bernard-Salvarani 2013] is based on equivalence to telegrapher's equation; but only for $n = 2$.

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Goldstein-Taylor on \mathbb{R} with $\sigma \equiv 1$: modal decomposition

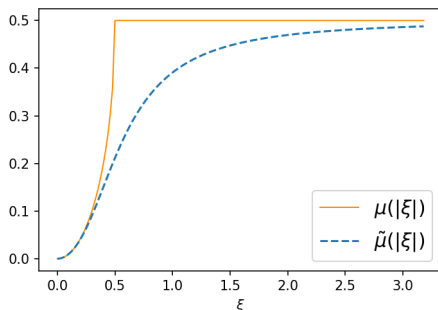
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$$\frac{d}{dt} \begin{pmatrix} \widehat{u}(\xi) \\ \widehat{v}(\xi) \end{pmatrix} = - \begin{pmatrix} 0 & i\xi \\ i\xi & 1 \end{pmatrix} \begin{pmatrix} \widehat{u}(\xi) \\ \widehat{v}(\xi) \end{pmatrix} =: -\mathbf{C}_\xi \begin{pmatrix} \widehat{u}(\xi) \\ \widehat{v}(\xi) \end{pmatrix}, \quad \xi \in \mathbb{R}.$$

Goldstein-Taylor on \mathbb{R}

3 problems:

- Decay rate $\mu(\xi)$ of small modes vanishes at $\xi = 0$.
 \Rightarrow Global decay is only algebraic.



- Mixed mode-behavior, like Situation C on torus \Rightarrow non-smooth Ψ DO
- Defective case $|\xi| = \frac{\sigma}{2}$ is always present, cannot be skipped.
 \Rightarrow Global approximation by $\tilde{\mu}(\xi)$ is one possible remedy.

Approximated modal decay function

- $\tilde{\mathbf{P}}(\xi) := \begin{pmatrix} 1 & -\frac{2i\xi}{1+4\xi^2} \\ \frac{2i\xi}{1+4\xi^2} & 1 \end{pmatrix}$ is a good approximation of the optimal scaling matrices $\mathbf{P}_\xi^{(I)}$ as $\xi \rightarrow 0$ and of $\mathbf{P}_\xi^{(III)}$ as $|\xi| \rightarrow \infty$.
- \Rightarrow sharp modal spectral gap is replaced by $\tilde{\mu}(\xi) = \frac{1}{2} - \frac{1}{2\sqrt{1+4\xi^2(1+4\xi^2)}}$, avoids the defective case $|\xi| = \frac{\sigma}{2}$ at the kink of $\mu(\xi)$.

Approximated modal decay function

- $\tilde{\mathbf{P}}(\xi) := \begin{pmatrix} 1 & -\frac{2i\xi}{1+4\xi^2} \\ \frac{2i\xi}{1+4\xi^2} & 1 \end{pmatrix}$ is a good approximation of the optimal scaling matrices $\mathbf{P}_\xi^{(I)}$ as $\xi \rightarrow 0$ and of $\mathbf{P}_\xi^{(III)}$ as $|\xi| \rightarrow \infty$.
- \Rightarrow sharp modal spectral gap is replaced by $\tilde{\mu}(\xi) = \frac{1}{2} - \frac{1}{2\sqrt{1+4\xi^2(1+4\xi^2)}}$, avoids the defective case $|\xi| = \frac{\sigma}{2}$ at the kink of $\mu(\xi)$.
- global Lyapunov functional on $L^2(\mathbb{R})$:

$$\tilde{\mathcal{E}}(\hat{u}, \hat{v}) := \int_{\mathbb{R}} \left\| \begin{pmatrix} \hat{u}(\xi) \\ \hat{v}(\xi) \end{pmatrix} \right\|_{\tilde{\mathbf{P}}(\xi)}^2 d\xi = \|u\|^2 + \|v\|^2 - 4 \int_{\mathbb{R}} u \partial_x (1 - 4 \partial_x^2)^{-1} v dx$$

- Decay proof uses modal decay estimates for $\xi \neq 0$:

$$\left\| \begin{pmatrix} \hat{u}(\xi, t) \\ \hat{v}(\xi, t) \end{pmatrix} \right\|_2^2 \leq \text{cond}(\tilde{\mathbf{P}}(\xi)) e^{-2\tilde{\mu}(\xi)t} \left\| \begin{pmatrix} \hat{u}(\xi, 0) \\ \hat{v}(\xi, 0) \end{pmatrix} \right\|_2^2, \quad t \geq 0.$$

algebraic decay

Theorem 3 (AA-Dolbeault-Schmeiser-Wöhler 2021)

Let $y := \begin{pmatrix} u \\ v \end{pmatrix}$. Then:

$$\begin{aligned} & \|y(x, t)\|_{L^2(\mathbb{R})}^2 \\ & \leq \inf_{0 < R \leq \frac{\sqrt{5}-1}{4}} \left(\frac{1+2R+4R^2}{1-2R+4R^2} \min \left\{ 2R, \sqrt{\frac{\pi}{2t}} \right\} \|y_0\|_{L^1(\mathbb{R})}^2 + 3e^{-2\tilde{\mu}(R)t} \|y_0\|_{L^2(\mathbb{R})}^2 \right) \end{aligned}$$

- High modes (with $|\xi| > R$) decay exponentially in $L^2(\mathbb{R})$.
- Low modes (with $|\xi| < R$) decay algebraically, using $\|\widehat{y}(\cdot, 0)\|_{L^\infty(\mathbb{R}_\xi)}^2$.

Conclusion

- Modal decomposition to obtain sharp decay estimates for modal ODEs
- Modal scaling matrices \mathbf{P}_k hint to pseudo-differential Lyapunov functionals to be used with non-constant coefficients.
- Hypocoercivity on \mathbb{R} : splitting of low and high modes \rightarrow algebraic decay.

Conclusion

- Modal decomposition to obtain sharp decay estimates for modal ODEs
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References

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