Modelling with partial differential equations

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1 Traffic flow models – hyperbolic conservation laws

Aim of the lecture:
Introduction to several applied models involving differential equation: discussion of modelling and of analytic and numerical aspects.

1.1 Modelling

Prototypical question: How long should traffic light phases be so that during the green phase the traffic jam in front of the traffic light dissolves?

simplifying model assumptions:

- single-track road without possibility to overtake
- no entry/exit points or junctions
- busy road: no description of individual vehicles, but vehicle density \( \rho(x, t) \) (e.g. vehicles per km) at location \( x \in \mathbb{R} \) and time \( t > 0 \) instead

Number of vehicles in interval \((a, b)\) at time \( t\):

\[
\int_{a}^{b} \rho(x, t) \, dx
\]

- let \( v(x, t) \) be the speed of vehicles at \((x, t)\)

  \(\Rightarrow\) vehicles passing \( x \) at time \( t\): \( \rho(x, t)v(x, t) = J(x, t) \) ... flux density.

looking for equations of motion for density \( \rho \)

Balance equation \(\forall(a, b)\):

\[
\frac{d}{dt} \int_{a}^{b} \rho(x, t) \, dx = \rho(a, t)v(a, t) - \rho(b, t)v(b, t) = - \int_{a}^{b} \frac{\partial(\rho v)}{\partial x}(x, t) \, dx
\]
⇒ Continuity equation

\[ \rho_t + (\rho v)_x = 0, \quad x \in \mathbb{R}, \ t > 0 \quad (1.1) \]

with initial condition (IC): \( \rho(x, 0) = \rho_0(x), \ x \in \mathbb{R} \).

looking for (constitutive) equation for \( v \); includes modelling information on traffic dynamics and driving behaviour

Suppose \( v = v(\rho) \) with

- \( v(\rho) \) monotonically decreasing (lower velocity for denser traffic)
- \( v(\rho_{\text{max}}) = 0 \) (above some maximal vehicle density or below some minimum distance between vehicles, traffic stops)
- possibly: \( v(0) = v_{\text{max}} \) (maximum velocity on empty road)

1) Lighthill-Whitham-Richards (LWR) model (simplest model, \( v(\rho) \) linear):

\[ v(\rho) = v_{\text{max}} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right), \quad 0 \leq \rho \leq \rho_{\text{max}} \]

\[ \Rightarrow \text{(1.1) becomes } \rho_t + \left[ v_{\text{max}} \rho \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) \right]_x = 0, \ x \in \mathbb{R}, \ t > 0 \quad (1.2) \]

2) Greenberg model:

\[ v(\rho) = v_{\text{ref}} \ln \frac{\rho_{\text{max}}}{\rho}, \quad 0 < \rho \leq \rho_{\text{max}} \]

\[ \Rightarrow \rho_t - v_{\text{ref}} \left( \rho \ln \frac{\rho}{\rho_{\text{max}}} \right)_x = 0 \quad (1.3) \]

Drawback of Greenberg model: for density \( \rightarrow 0 \) velocity \( v(\rho) \) is unbounded – this is unrealistic

(1.2), (1.3) are conservation laws, as the total number of vehicles is conserved. Formal integration of (1.1) leads to:

\[ \frac{d}{dt} \int_{\mathbb{R}} \rho(x, t) \, dx = - \int_{\mathbb{R}} \frac{\partial}{\partial x} [\rho(x, t)v(\rho(x, t))] \, dx = 0. \]

(1.2), (1.3) are hyperbolic equations:

**Definition 1.1.** The system of equations

\[
\begin{align*}
    u_t + \partial_x f(u) &= 0, & x \in \mathbb{R}, \ t > 0 \\
    u(x, 0) &= u_0(x), & x \in \mathbb{R}
\end{align*}
\quad (1.4)
\]
with \( f : \mathbb{R}^m \to \mathbb{R}^m \) is called hyperbolic if \( f'(u) \in \mathbb{R}^{m \times m} \) is diagonalisable and has only real eigenvalues (\( \forall u \in \mathbb{R}^m \)).

A function \( u : \mathbb{R} \times [0, \infty) \to \mathbb{R}^m \) is called classical solution if \( u \in C^1(\mathbb{R} \times (0, \infty)) \cap C^0(\mathbb{R} \times [0, \infty)) \) and (1.4) holds pointwise.

Simplification of the LWR model:

Transform (1.2) into nondimensionalized form:

Let \( L \) and \( \tau \) be typical length and time scales such that \( L/\tau = v_{\text{max}} \).

scaled variables:

\[
  x_s := \frac{x}{L}, \quad t_s := \frac{t}{\tau}, \quad u := 1 - \frac{2 \rho}{\rho_{\text{max}}}
\]

\[
  \partial_t \rho = \frac{1}{\tau} \partial_{t_s} \left[ \frac{\rho_{\text{max}}}{2} (1 - u) \right] = -\frac{\rho_{\text{max}}}{2\tau} \partial_{t_s} u,
\]

\[
  \partial_x \left[ v_{\text{max}} \rho \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) \right] = \frac{1}{L} \partial_{x_s} \left[ v_{\text{max}} \frac{\rho_{\text{max}}}{2} (1 - u) \frac{1}{2} (1 + u) \right] = -\frac{\rho_{\text{max}}}{2\tau} \partial_{x_s} \left( \frac{u^2}{2} \right)
\]

\[
  \Rightarrow u_t + \left( \frac{u^2}{2} \right)_x = 0, \quad x \in \mathbb{R}, t > 0
\]

(1.5) is called inviscid Burgers’ equation.

\[
  \rho = 0 \quad \iff \quad u = 1; v = v_{\text{max}} \ldots \text{empty road}
\]

\[
  \rho = \rho_{\text{max}} \quad \iff \quad u = -1; v = 0 \ldots \text{traffic jam}
\]

Example 1.2.

\[
  u_0(x) = \begin{cases} 
  1, & x < 0 \\
  1 - x, & 0 \leq x < 1 \\
  0, & x \geq 1 
\end{cases}
\]

Method of characteristics for \( u_t + uu_x = 0 \):

\[
  \frac{dt}{ds} = 1, \quad \frac{dx}{ds} = u, \quad \frac{du}{ds} = 0,
\]
with \( t(0) = 0, x(0) = x_0, u(0) = u_0(x_0) \Rightarrow s = t \).

\[ \Rightarrow u(t) = u_0(x_0) \text{ (const.) along the characteristic } x(t) = u_0(x_0)t + x_0, \ t \geq 0 \]

\[ \Rightarrow \text{solution for } x \in \mathbb{R}, \ t < 1: \]

\[
u(x, t) = \begin{cases} 1, & x < t \\ \frac{1-x}{1-t}, & t \leq x < 1 \\ 0, & x \geq 1 \end{cases} \tag{1.6}
\]

Figure 1.1: characteristics: no trajectories (= paths of movement) of vehicles, but propagation of density values \( \rho(x, t) \)

Solution for \( t = 1 \) is discontinuous in \( x = 1 \) (a shock is created). This is the case as well for a (slightly) smoothened IC with \( u_0 \in C^1(\mathbb{R}) \): a classical solution exists only for a finite time in this case.

Questions:
- \( \exists \) solution for \( t \geq 1 \)?
- Which solution concept?

References: [Jü] §1,3; [LV] §1-3.
1.2 Scalar hyperbolic conservation laws

Consider the hyperbolic conservation law

\[ u_t + f(u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0 \]
\[ u(x,0) = u_0(x), \quad x \in \mathbb{R} \quad (1.7) \]

with \( f : \mathbb{R} \to \mathbb{R} \).

We generally assume that \( f''(u) > 0 \quad \forall u \in \mathbb{R} \) ("proper nonlinearity")

Motivation of weak solution: Multiply (1.7) with

\[ \Phi \in C^1_0(\mathbb{R}^2) := \{ \Phi \in C^1(\mathbb{R}^2) \mid \Phi \text{ has compact support} \}, \]

integrate over \( \mathbb{R}_x \times \mathbb{R}_t^+ \):

\[ 0 = \int_0^\infty \int_{\mathbb{R}} (u_t + f(u)_x) \Phi \, dx \, dt \]
\[ = -\int_0^\infty \int_{\mathbb{R}} (u\Phi_t + f(u)\Phi_x) \, dx \, dt - \int_{\mathbb{R}} u(x,0)\Phi(x,0) \, dx \]

For the two last integrals only "\( u \) integrable" needed.
Definition 1.3. Let $L^1_{\text{loc}} \ni u : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ with $f(u) \in L^1_{\text{loc}}$. $u$ is called weak solution of (1.7) if

$$\int_0^\infty \int_{\mathbb{R}} (u\Phi_t + f(u)\Phi_x) dx dt = - \int_{\mathbb{R}} u_0(x)\Phi(x,0) dx \quad \forall \Phi \in C^1_c(\mathbb{R}^2).$$

(1.8)

Every classical solution is a weak solution; the converse is not true in general.

Another weak formulation:

Integrate (1.7) over $(a,b) \times (s,t)$ for arbitrary $a,b \in \mathbb{R}; s,t > 0$:

$$\int_a^b u(x,t) dx - \int_a^b u(x,s) dx = - \int_s^t f(u(b,\tau)) d\tau + \int_s^t f(u(a,\tau)) d\tau.$$  (1.9)

One can show: each weak solution (as in Def. 1.3) satisfies (1.9).

Consider now conservation laws with discontinuous initial data; these appear e.g. in Ex. 1.2 at $t = 1$. Due to translation invariance of (1.7) in $x$ and $t$ we can assume that this discontinuity is situated in $(0,0)$.

Definition 1.4. Equation (1.7) with IC

$$u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x \geq 0 \end{cases}$$

(1.10)

with $u_l, u_r \in \mathbb{R}$ is called Riemann problem.

Let $u(x,t)$ be a solution of (1.7), (1.10).

$\Rightarrow u(\alpha x, \alpha t)$ also is a solution $\forall \alpha > 0$.

$\Rightarrow u$ depends only on $\xi = x/t$, i.e. $u = \tilde{u}(\xi)$.

Determination of $\tilde{u}(\xi)$:

$$0 = u_t + f(u)_x = -\frac{x}{t^2} \tilde{u}'(\xi) + f'(\tilde{u}(\xi))\tilde{u}'(\xi)\frac{1}{t} = \frac{1}{t} \tilde{u}'(\xi)[f'(\tilde{u}(\xi)) - \xi] \quad \forall \xi$$

$\Rightarrow$ 3 possibilities:

- $\tilde{u}'(\xi) = 0 \quad \Rightarrow \tilde{u}(\xi) = \text{const.}$
- $u$ is discontinuous along $\xi = x/t$, i.e., $\exists \tilde{u}'(\xi)$.
1.2 Scalar hyperbolic conservation laws

- \( f'(\tilde{u}(\xi)) = \xi \Rightarrow \tilde{u}(\xi) = (f')^{-1}(\xi) \); \( \exists \) inverse of \( f' \) (on \( f'(\mathbb{R}) \)) because \( f'' > 0 \) on \( \mathbb{R} \) (by assumption).

We consider 3 ICs corresponding to these possibilities:

Case 1, \( u_l = u_r \): \( u(x, t) = u_r = u_l \) \( \forall x \in \mathbb{R}, t \geq 0 \).

Case 2, \( u_l > u_r \):

Cf. Ex. 1.2 starting at \( t = 1 \): vehicle density for \( x > 0 \) greater than for \( x < 0 \).

\( \Rightarrow \) greater (positiv) speed for \( x < 0 \) than for \( x > 0 \).

\( \Rightarrow \) expect shock curve, i.e., discontinuity of the solution at \( x = \psi(t) \)

**Lemma 1.5.** The function

\[
u(x, t) := \begin{cases} u_l, & x < st \\ u_r, & x \geq st \end{cases}
\] (1.11)

is a weak solution of (1.7), (1.10) if and only if the shock speed \( s \) satisfies the Rankine-Hugoniot (RH) condition:

\[
s = \psi'(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r}.
\] (1.12)

(In this case it is even the unique “entropy solution”, see Theorem 1.13.)

**Proof.** Let \( \Phi \in C^1_0(\mathbb{R}^2). u = \text{const}, \) except on \( x = st \). \( \Rightarrow \)

\[
\int_0^\infty \int_\mathbb{R} u\Phi_t dx dt = \int_0^\infty \left( \int_{-\infty}^{st} u\Phi_t dx + \int_{st}^{\infty} u\Phi_t dx \right) dt
\]

\[
= \int_0^\infty \left( \partial_t \int_{-\infty}^{st} u\Phi dx - su(st - 0, t)\Phi(st, t) 
+ \partial_t \int_{st}^{\infty} u\Phi dx + su(st + 0, t)\Phi(st, t) \right) dt
\]

\[
= -\int_\mathbb{R} u(x, 0)\Phi(x, 0) dx - s(u_l - u_r)\int_0^\infty \Phi(st, t) dt.
\]
\[ \int_0^\infty \int_\mathbb{R} f(u)\Phi_x \, dx \, dt \begin{array}{c} \text{split,} \\ \text{by parts} \end{array} = \int_0^\infty \left( - \int_{-\infty}^{st} f(u) \Phi \, dx + f(u(st - 0, t))\Phi(st, t) \right. \\
\left. - \int_{st}^{\infty} f(u) \Phi \, dx - f(u(st + 0, t))\Phi(st, t) \right) \, dt \\
- f(u(st, t))\Phi(st, t) \right) \, dt. \]

Hence

\[ \int_0^\infty \int_\mathbb{R} (u\Phi_t + f(u)\Phi_x) \, dx \, dt = - \int_\mathbb{R} u_0(x)\Phi(x, 0) \, dx, \]

follows if and only if (1.12) holds.

\section*{Remark 1.6.} Weak solutions of (1.7), (1.10) are not unique! Additionally to (1.11) there are more, e.g. consisting of 3 shocks (see exercises; cf. also Theorem 1.13).

Generalised Rankine-Hugoniot condition for \( u \) not piecewise continuous and \( s \) not constant:

\[ s(t) = \psi'(t) = \frac{f(u_l(t)) - f(u_r(t))}{u_l(t) - u_r(t)} \quad (1.13) \]

with \( u_l(t) = \lim_{x \to \psi(t)} u(x, t), u_r(t) = \lim_{x \to \psi(t)} u(x, t) \).

\section*{Example 1.7.} Let \( f(u) = u^2/2, u_l = 0, u_r = -1 \).

\[ \Rightarrow s = \frac{1}{2} \frac{u_l^2 - u_r^2}{u_l - u_r} = -\frac{1}{2} \]

Characteristics see Figure 1.3

\[ \text{Case 3, } u_l < u_r: \text{ (1.11) is here as well one weak solution:} \]

\[ \square \]
1.2 Scalar hyperbolic conservation laws

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = 0, \quad \rho_t = \rho_{\text{max}}/2, \quad v = v_{\text{max}}/2, \quad u = -1, \quad \rho = \rho_{\text{max}}, \quad v = 0 \]

Figure 1.3: left end of traffic jam at \( x = st \). Characteristics are not vehicle trajectories.

Characteristics of (1.7) for \( f(u) = u^2/2, u_l = 0, u_r = 1, s = \frac{1}{2} \).

Solution is "instable" because characteristics begin in shock curve. From shock "newly generated" information which is not contained in \( u_0 \) is transported.

Further weak solution of (1.7), (1.10):

\[ u_2(x,t) := \begin{cases} 
    u_l & , \quad x < f'(u_l)t \\
    (f')^{-1} \left( \frac{x}{t} \right) & , \quad f'(u_l)t \leq x \leq f'(u_r)t \\
    u_r & , \quad x > f'(u_r)t 
\end{cases} \quad \text{ (1.14)} \]
Characteristics of rarefaction wave \( u_2 \) for \( f(u) = u^2/2, u_l = 0, u_r = 1, (f')^{-1}(\xi) = \xi \):
\[
  u_2(x,t) = \frac{x}{t} \text{ for } 0 \leq x \leq t.
\]

∃ even infinitely many weak solutions!

Solution concept is so weak that uniqueness was lost.

**Question:** which is the “correct” or physically relevant solution?

2 possibilities: first approach with entropy condition:

**Definition 1.8.** A weak solution \( u : \mathbb{R} \times (0,T) \to \mathbb{R} \) of (1.7), (1.10) satisfies Oleinik’s entropy condition if along every curve of discontinuity \( x = \psi(t) \) the following holds:

\[
  \frac{f(u_l(t)) - f(v)}{u_l(t) - v} \geq \psi'(t) \geq \frac{f(u_r(t)) - f(v)}{u_r(t) - v}
\]

\( \forall t \in (0,T), \forall v \text{ between } u_l(t) \text{ and } u_r(t). \)

**Rem:** Solutions without discontinuities satisfy (1.15) trivially. (1.15) is also used for non-convex \( f \).
RH-condition (1.13) implies

\[ \sigma(v) := \frac{f(u_l) - f(v)}{u_l - v} \geq \psi' = s \Rightarrow \frac{f(u_l) - f(u_r)}{u_l - u_r} \quad \forall v \text{ between } u_l \text{ and } u_r. \]

Due to monotony of \( \sigma(v) \), \( \sigma \) is maximal at \( v = u_r \) if \( u_l < u_r \) and minimal if \( u_l > u_r \).

\[ \Rightarrow u_l \geq u_r \quad \text{(for } f'' > 0) \]

Shock-solution (1.11) in case 3 does not \( (u_l < u_r) \) satisfy the entropy condition. For \( u_2 \) from (1.14) the entropy condition is trivial because \( u_2 \) is continuous.

For \( v \to u_{l,r} \) in (1.15): Propagation velocity of characteristics satisfies the Lax entropy condition:

\[ f'(u_l) \geq \frac{f(u_l) - f(u_r)}{u_l - u_r} \geq f'(u_r), \quad \text{da } f'' > 0. \]

Interpretation: Characteristics have to run into the shock from the left and right sides and stop there, i.e., “mathematical entropy” or “information” or range of \( u(.,t) \) decreases with time (cf. second law of thermodynamics; physical entropy \( \approx - \) mathematical entropy increases there).

Second approach with entropy function / viscosity solution:

Assumption: (1.7) is just an idealisation of the diffusion equation

\[ u_t + f(u)_x = \varepsilon u_{xx}, \quad x \in \mathbb{R}, t > 0 \quad (1.16) \]

with (small) \( \varepsilon > 0 \). (1.16) has unique smooth solution \( u^\varepsilon \).

Convention: the limit function \( u = \lim_{\varepsilon \to 0} u^\varepsilon \) shall be the physically relevant solution, viscosity solution.

Aim: find condition (only) on weak solution \( u \) such that it represents this limit.
**Definition 1.9.** The pair of functions $\eta \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$ are called entropy and (corresponding) entropy flux, if $\eta'' > 0$ and for all classical solutions $u$ of (1.7)

$$\eta(u)_t + \psi(u)_x = 0, \quad x \in \mathbb{R}, t > 0 \quad \text{(1.17)}$$

holds. Rem: this implies $\psi' = f' \eta'$.

**Assumptions for vanishing viscosity limit:**

- $u^\varepsilon \xrightarrow{\varepsilon \to 0} u$ pointwise a.e. in $\mathbb{R} \times (0, T)$,
- $u^\varepsilon \xrightarrow{\text{weak}} u$ in $L^1_{\text{loc}}(\mathbb{R} \times (0, T))$,
- $\|u^\varepsilon\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \text{const. \quad \forall 0 < \varepsilon < 1}$,
- $\|\eta''(u^\varepsilon)u^\varepsilon_x\|_{L^1(\mathbb{R} \times (0, T))} \leq \text{const. \quad \forall 0 < \varepsilon < 1}$.

Then (without proof): $u$ solves (1.7).

**Modification of entropy equation (1.17) for discontinuous $u$:**

Multiply (1.16) by $\eta'(u^\varepsilon)$; choose $\psi$ such that $\psi' = f' \eta'$:

$$\eta(u^\varepsilon)_t + \psi(u^\varepsilon)_x = \varepsilon \eta'(u^\varepsilon)u^\varepsilon_{xx} = \varepsilon (\eta'(u^\varepsilon)u^\varepsilon_x)_x - \varepsilon \eta''(u^\varepsilon)(u^\varepsilon_x)^2;$$

multiply by $\Phi \in C^1_0(\mathbb{R}^2)$, $\Phi \geq 0$, integrate over $\mathbb{R} \times (0, \infty)$:

$$\int_0^\infty \int_\mathbb{R} \left[\eta(u^\varepsilon)_t + \psi(u^\varepsilon)_x\right] \Phi \, dx \, dt \quad \text{(1.18)}$$

$$= -\varepsilon \int_0^\infty \int_\mathbb{R} \eta'(u^\varepsilon)u^\varepsilon_x \Phi_x \, dx \, dt - \varepsilon \int_0^\infty \int_\mathbb{R} \left(\frac{\eta''(u^\varepsilon)}{\varepsilon} \right) \frac{(u^\varepsilon_x)^2}{\varepsilon} \Phi \, dx \, dt$$

$$\leq \varepsilon \|\eta'(u^\varepsilon)u^\varepsilon_x\|_{L^1(\mathbb{R} \times (0,T))} \|\Phi_x\|_{L^\infty(\mathbb{R} \times (0,T))} \xrightarrow{\varepsilon \to 0} 0 \quad \text{with } T = T(\Phi).$$

As $\Phi \geq 0$ is arbitrary, the limit $u := \lim u^\varepsilon$ satisfies:

$$\Rightarrow \eta(u)_t + \psi(u)_x \leq 0 \quad \text{for smooth solutions}. \quad \text{(1.19)}$$

For weak solutions the following holds (from inequality (1.18) after integration by parts in $x, t$):

$$\int_0^\infty \int_\mathbb{R} \left[\eta(u)\Phi_t + \psi(u)\Phi_x\right] \, dx \, dt \geq - \int_\mathbb{R} \eta(u_0(x))\Phi(x,0) \, dx \quad \forall \Phi \in C^1_0(\mathbb{R}^2), \Phi \geq 0. \quad \text{(1.20)}$$

Rem: for the (direct) limit $\varepsilon \to 0$ on the left hand side of (1.18) our assumptions are not strong enough to obtain (1.19). One should therefore take the limit in the $\varepsilon$–analogon of (1.20). After reverting integration by parts one can conclude (1.19).
Definition 1.10. Let \( u : \mathbb{R} \times (0, T) \to \mathbb{R} \) be a weak solution of \((1.7)\). \( u \) is called an entropy solution if for all strictly convex entropies \( \eta \) and the corresponding entropy fluxes \( \psi \) inequality \((1.20)\) holds.

Rem: 1) For shock waves, the entropy inequality \((1.20)\) is equivalent to Oleinik’s entropy condition \((1.15)\) (see Th. II.1.1 in [LF]).
2) By [DeLellis-Otto-Westdickenberg, 2003] for this equivalence one strictly convex \( \eta \) suffices in Definition 1.10.
3) Rarefaction wave \( u_2 \) is an entropy solution, even satisfies the entropy equality \((1.17)\) a.e. (as \( u_2 \) is continuous, \( \exists \) weak derivative) resp. \((1.20)\) with “\(=\)”.
4) Entropy solutions are in general not reversible in time: a shock would become a rarefaction wave (and vice versa).

Example 1.11. Let \( f(u) = \frac{u^2}{2}, \eta(u) = u^2 \Rightarrow \psi(u) = \frac{2}{3}u^3 \) (as \( \psi' = f'\eta' \)). Let \( \Phi \in C_0^1(\mathbb{R}^2), \Phi \geq 0 \).

Shock wave \((1.11)\) for \( u_l < u_r \) is no entropy solution, as (with \( s = \frac{u_l + u_r}{2} \)) for \((1.11)\) we have:

\[
\int_0^\infty \int_0^\infty \left[ \frac{u^2}{\eta'(u)} \Phi_t + \frac{2}{3}u^3 \Phi_x \right] \mathrm{d}x \mathrm{d}t
\]

\[\begin{array}{c}
\text{“\(u = 0\)”,}
\int_0^\infty \left[ \partial_t \int_{-\infty}^{st} u^2 \Phi \mathrm{d}x - su_l^2 \Phi(st,t) + \partial_t \int_{st}^{\infty} u^2 \Phi \mathrm{d}x + su_r^2 \Phi(st,t) \\
+ \frac{2}{3}u_l^3 \Phi(st,t) - \frac{2}{3}u_r^3 \Phi(st,t) \right] \mathrm{d}t
\end{array}\]

\[= - \int_\mathbb{R} u_0(x)^2 \Phi(x,0) \mathrm{d}x - \frac{u_l + u_r}{2} (u_l^2 - u_r^2) \int_0^\infty \Phi(st,t) \mathrm{d}t
\]

\[+ \frac{2}{3}(u_l^3 - u_r^3) \int_0^\infty \Phi(st,t) \mathrm{d}t
\]

\[= - \int_\mathbb{R} u_0(x)^2 \Phi(x,0) \mathrm{d}x + \frac{1}{6}(u_l - u_r)^3 \int_0^\infty \Phi(st,t) \mathrm{d}t
\]

\[\geq - \int_\mathbb{R} \eta(u_0(x)) \Phi(x,0) \mathrm{d}x \quad \Leftrightarrow \quad u_l \geq u_r.
\]

Conclusion: \((1.11)\) satisfies the entropy inequality \((1.20)\) exactly for \( u_l \geq u_r \).
Similarly to the example for \( u_l < u_r \) we have: only rarefaction wave \( u_2 \) is entropy solution.

Summary:

**Theorem 1.12.** Let \( f \in C^2(\mathbb{R}) \) with \( f'' > 0 \) in \( \mathbb{R} \).

1. Let \( u_l > u_r \):
   \[
   u(x,t) = \begin{cases} 
   u_l & x < st \\
   u_r & x > st
   \end{cases}
   
   with \( s := \frac{f(u_l) - f(u_r)}{u_l - u_r} \)
   
   is a weak entropy solution of (1.7).

2. Let \( u_l < u_r \): \( u_2 \) from (1.14) is weak entropy solution of (1.7).

**Theorem 1.13** (Kruzkov, 1970). Let \( f \in C^2(\mathbb{R}), u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \).
\( \Rightarrow \exists! \) weak entropy solution of (1.7).

*Proof.** difficult, [LF], [Wa]; for \( f \) uniformly convex see also §3.4.2 in [Ev].

Summary for \( f(u) = \frac{u^2}{2} \):

\[
\begin{array}{c}
\text{\( u_t + uu_x = \varepsilon u_{xx} \)} \\
\Rightarrow \exists! \text{ class. solution for all } t > 0
\end{array}
\rightarrow
\begin{array}{c}
\text{\( \varepsilon \to 0 \)} \\
\text{\( u_t + uu_x = 0 \)}
\end{array}
\Rightarrow
\begin{array}{c}
\exists! \text{ class. solution for } t < t_0. \\
\exists \text{ weak solutions for } t \geq t_0 \text{ (satisfy RH-jump conditions); but not unique}
\end{array}
\]

\[
\begin{array}{c}
\text{\( u^\varepsilon(x,t) \)} \\
\Rightarrow \varepsilon \to 0
\end{array}
\rightarrow
\begin{array}{c}
\text{\( u(x,t) \)}
\end{array}
\Rightarrow
\begin{array}{c}
\exists \text{ unique weak solution}
\end{array}
\]

References: [Jü] §2, [LV] §3.
1.3 Traffic light problem

LWR-model for $u = 1 - \frac{2\rho}{\rho_{\text{max}}}$:

$$u_t + \left( \frac{u^2}{2} \right)_x = 0 \quad , \quad x \in \mathbb{R} \tag{1.21}$$

IC:

$$\rho_0(x) = \begin{cases} \bar{\rho} > 0 , & x < 0 \\ 0 , & x > 0 \end{cases}$$

$\bar{\pi} := 1 - 2\bar{\rho}/\rho_{\text{max}} \in (-1,1)$

Traffic light at $x = 0$ turns red at $t = 0$; traffic light phase has duration $\omega > 0$.

Question: Does the traffic jam dissolve in green phase $[\omega, 2\omega)$?

Step 1: Red phase $(0 \leq t \leq \omega)$

Solve (1.21) on $(-\infty, 0)$ with boundary condition (BC) $u(x = 0, t) = -1 \ldots$ models red traffic light.

Solution from (1.11) for $0 < t \leq \omega$:

$$u(x, t) = \begin{cases} \bar{\pi} , & x < st \\ -1 , & st < x < 0 \\ 1 , & x > 0 \end{cases} ; \quad s = \frac{u_l + u_r}{2} = \frac{\bar{\pi} - 1}{2} < 0$$

Step 2: Green phase $(t \geq \omega)$

Solve (1.21) on $\mathbb{R}$ with IC

$$u(x, \omega) = \begin{cases} \bar{\pi} , & x < s\omega \\ -1 , & s\omega < x < 0 \\ 1 , & x > 0 \end{cases}$$

i.e. 2 Riemann problems:

a) As $\bar{\pi} > -1$: shock $\psi(t) = st, s = \frac{\bar{\pi} - 1}{2}$

b) As $-1 < \pi$: rarefaction wave, originating in $(0, \omega)$

⇒ Solution for $t \geq \omega$:

$$u(x, t) = \begin{cases} \bar{\pi} , & x < st \\ -1 , & st < x < \omega - t \\ \frac{x}{t-\omega} , & \omega - t \leq x \leq t - \omega \\ 1 , & x > t - \omega \end{cases}$$
correct as long as \( st < \omega - t \) or \( t < t_1 := \frac{\omega}{s + 1} = \frac{2\omega}{u + 1} \) (\( t_1 \leq 2\omega \) as well as \( t_1 > 2\omega \) possible).

**Step 3: Green phase \( t > t_1 \)**

At \( t = t_1 \) shock and rarefaction wave interact.

Solve (1.21) on \( \mathbb{R} \) with IC \( u(x, t_1) \) and generalised RH-condition for shock starting from \((st_1, t_1)\):

\[
s(t) = \psi'(t) = \frac{1}{2}[u(\psi(t) + 0, t) + u(\psi(t) - 0, t)] = \frac{1}{2} \left( \frac{\psi(t)}{t - \omega} + \bar{u} \right), \quad t > t_1;
\]

i.e. linear ODE for \( \psi(t) \) with IC \( \psi(t_1) = st_1 = \omega \frac{\bar{u} - 1}{\bar{u} + 1} \)

Solution:

\[
\psi(t) = \bar{u}(t - \omega) - \sqrt{t - \omega} \sqrt{\omega(1 - \bar{u}^2)}, \quad t \geq t_1
\]

2 cases:

a) \( \bar{u} \leq 0 \) (high traffic density): \( \Rightarrow t_1 \geq 2\omega \), only relevant for longer green phases.

\[
\psi(t) \underset{t \to \infty}{\to} -\infty \Rightarrow \exists \text{ shock } \forall t.
\]

It moves to \( -\infty \) with speed \( \psi'(t) \underset{t \to \infty}{\to} \bar{u} \);

hence reduction of shock speed from \( s = \frac{\bar{u} - 1}{2} < 0 \) to \( \bar{u} \) with \( |\bar{u}| \leq |s| \).

Because \( \psi'(t) = \frac{u_l + u_r(t)}{2} = \frac{\bar{u} + u_r(t)}{2} \underset{t \to \infty}{\to} \bar{u} \): Jump distance \( u_l - u_r(t) \underset{t \to \infty}{\to} 0 \)

b) \( \bar{u} > 0 \) (low traffic density): \( \Rightarrow t_1 < 2\omega \)

\[
\psi(t) \underset{t \to \infty}{\to} \infty, \text{ i.e., shock curve } \psi(t) \text{ moves in positive } x\text{-direction.}
\]

\( \psi(t_2) = 0 \) has unique solution \( t_2 = \omega/\bar{u}^2 \):

Traffic jam or disturbance behind traffic light completely dissolved.

Traffic jam dissolves during green phase \([\omega, 2\omega) \iff t_2 \leq 2\omega \), i.e. \( \bar{u} \geq 1/\sqrt{2} \) or

\[
\bar{\rho} \leq \rho_0 := \frac{\rho_{\text{max}}}{2} \left( 1 - \frac{1}{\sqrt{2}} \right) \approx 0.146\rho_{\text{max}},
\]

independently of duration of green phase!
For $\overline{\rho} > \rho_0$: traffic jam grows with $t$.

Summary:

- $\overline{\rho} \geq \rho_{\text{max}}/2$: already one red phase disturbs traffic permanently, even if afterwards traffic light stays green forever.
- $(1 - 1/\sqrt{2})\rho_{\text{max}} - \overline{\rho} < \rho_{\text{max}}/2$: traffic jam accumulates with time, but vanishes after traffic light stays green.
- $\overline{\rho} \leq (1 - 1/\sqrt{2})\rho_{\text{max}}/2$: influence of red phase (behind traffic light) vanishes before end of green phase.

References: [Jü] §3

1.4 Numerical Methods

1.4.1 Linear advection equation

- only finite difference methods, almost always explicit
- for linear advection equation (with $a > 0$):

$$u_t + au_x = 0, \quad x \in \mathbb{R}, t > 0$$
$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}.$$  (1.22)
For \( u_0 \in L^1_{\text{loc}}(\mathbb{R}) \) the explicit weak solution is
\[
  u(x,t) = u_0(x - at). \tag{1.23}
\]

- here: uniform mesh \((x_j, t_n)\) with
\[
x_j = jh \quad (j \in \mathbb{Z}), \quad t_n = nk \quad (n \in \mathbb{N}_0), \quad h, k > 0.
\]

Approximation \( u^n_j \sim u(x_j, t_n) \)

**Definition 1.14** (Difference quotients).

\[
\begin{align*}
  D^+_x v_j &= \frac{v_{j+1} - v_j}{h} \quad \ldots \quad \text{forward difference} \\
  D^-_x v_j &= \frac{v_j - v_{j-1}}{h} \quad \ldots \quad \text{backward difference} \\
  D^0_x v_j &= \frac{v_{j+1} - v_{j-1}}{2h} \quad \ldots \quad \text{central difference} \\
  D^2_x v_j &= \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} \quad \ldots \quad \text{second difference}
\end{align*}
\]

We have \( D^0_x v_j = \frac{1}{2}(D^+_x + D^-_x) v_j \) and by Taylor’s formula:

\[
\begin{align*}
  D^+_x v_j &= v'(x_j) + O(h) \quad \text{(for } v \in C^2(\mathbb{R})) \\
  D^0_x v_j &= v'(x_j) + O(h^2) \quad \text{(for } v \in C^3(\mathbb{R})).
\end{align*}
\]

Replacing derivatives in (1.22) by corresponding difference quotients gives finite difference scheme.

1st idea: central scheme:

\[
\frac{u_{j+1}^n - u_j^n}{k} = -a \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \quad n \geq 0, j \in \mathbb{Z};
\]

or

\[
u_{j+1}^n = u_j^n - \frac{ak}{2h}(u_{j+1}^n - u_{j-1}^n).
\]

→ explicit scheme with **numerical stencil**:
Disadvantage: method is unstable, i.e., develops (artificial) oscillations (→ Exercises).

2nd idea: **implicit scheme**:

\[
\frac{u_j^{n+1} - u_j^n}{k} = -a \frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2h}, \quad n \geq 0, j \in \mathbb{Z},
\]

or

\[
\frac{ak}{2h} u_{j+1}^{n+1} + u_j^{n+1} - \frac{ak}{2h} u_{j-1}^{n+1} = u_j^n.
\]

Disadvantage: in each time step a (tridiagonal) system of linear equations needs to be solved.

Numerical stencil:

\[
\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

3rd idea: **Lax-Friedrichs scheme**:

Approximation of \( t \)-derivative (first for \( u(x,t) \)):

\[
\frac{1}{k} \left( u(x,t+k) - \frac{1}{2} [u(x+h,t) + u(x-h,t)] \right),
\]

also

\[
u_j^{n+1} = \frac{1}{2} (u_{j+1}^n + u_{j-1}^n) - \frac{ak}{2h} (u_{j+1}^n - u_{j-1}^n), \quad n \geq 0, j \in \mathbb{Z}
\]

Numerical stencil:

Advantage: (conditionally) stable (for \( \frac{k}{h} \) small enough → Exercises)

Disadvantage: Solution is (strongly) smoothened out.

∀ schemes: exact solution (1.23) does not satisfy difference scheme. Hence:
Definition 1.15. Inserting the exact solution into difference scheme $U^{n+1} = \mathcal{H}_k U^n$ gives local truncation error – as residuum.

Notation: $U^n = \{u^n_j; j \in \mathbb{Z}\}$; the operator $\mathcal{H}_k$ is the propagator of the scheme for time step size $k$.

Example: local truncation error for Lax-Friedrichs scheme (1.24):

$$L_k(x,t) := \frac{1}{k} \left( u(x, t+k) - \mathcal{H}_k(u(.,t); x) \right)$$

$$= \frac{1}{k} \left( u(x, t+k) - \frac{1}{2} [u(x+h, t) + u(x-h, t)] \right)$$

$$+ \frac{a}{2h} [u(x+h, t) - u(x-h, t)].$$

Leading factor $\frac{1}{k}$ is important for the right order of the scheme; the global order is one below the local order.

$u(x, t+k)$ is the exact solution at time $t+k$; $\mathcal{H}_k(u(.,t); x)$ is the result of a numerical step, based on the exact Solution at time $t$.

Taylor expansion in $t, x$ around continuously varying argument $(x, t)$ for $u$ smooth enough:

$$\Rightarrow \quad L_k(x,t) = \frac{1}{k} \left[ \left( u + u_{tt} k + \frac{1}{2} u_{ttt} k^2 + O(k^3) \right) - \frac{1}{2} \left( 2u + u_{xx} h^2 + O(h^4) \right) \right]$$

$$+ \frac{a}{2h} \left( 2u_x h + O(h^3) \right)$$

$$= u_t + a u_x + \frac{1}{2} \left( u_{tt} k - u_{xx} \frac{h^2}{k^2} \right) + O(k^2) + O \left( \frac{h^4}{k} \right) + O(h^2) \quad (1.25)$$

From (1.22): $u_{tt} = -au_{xt} = a^2 u_{xx}$.

Let $\frac{k}{h} = \text{const}$ (henceforth our standard assumption).

$$\Rightarrow \quad L_k(x,t) = \frac{k}{2} \left( a^2 - \left( \frac{h}{k} \right)^2 \right) u_{xx}(x,t) + O(h^2) = O(k), \quad (1.26)$$

hence

$$|L_k(x,t)| \leq Ck \quad \forall k < k_0$$
∀(x, t), because C is determined by \((u_0)_{xx}\).

→ “First order method (in k)”; numerical solution gets better for smaller \(k > 0\).

**Definition 1.16.** Method is consistent if \(\|L_k(\cdot,t)\|_{L^1(\mathbb{R})} \to 0\) for \(k \to 0\) (∀ fixed \(t > 0\)).

2 approaches for better match between PDE and numerical scheme:

1. (different) scheme of higher order for the given PDE (see 6. idea);
2. same scheme (1.24) but modified PDE (depending on \(h\) and \(k\))!

From (1.26): Lax-Friedrichs is even method of second order for the modified equation:

\[
u_t + au_x = -\frac{k}{2} \left( a^2 - \left(\frac{h}{k}\right)^2 \right) u_{xx}, \quad x \in \mathbb{R}, \quad t > 0.
\]

(1.27)

Here we are looking for those modified equations which for the considered scheme are solved better than (1.22). Modified equations are not uniquely determined.

(1.27) is an advection-diffusion equation if \(D \geq 0\) (for \(D < 0\) it would be backwards parabolic and unstable!). Hence the following has to hold:

\[a^2 - \left(\frac{h}{k}\right)^2 \leq 0 \iff \frac{|a|k}{h} \leq 1 \ldots \text{stability condition}.
\]

Hence: (max.) numerical speed of propagation \(\frac{h}{k}\) has to be \(\geq\) real speed of propagation \(|a|\).

For \(k \to 0\) and \(\frac{h}{k} = \text{const} \) (1.27) formally converges to \(u_t + au_x = 0\) (cf. vanishing viscosity limit in (1.16)).

The Lax-Friedrichs scheme for (1.22) hence implies artificial diffusion (with constant \(D > 0\)) and thus prevents discontinuities and oscillations.

**Stability** means that error propagation remains bounded (for \(k \to 0\)).

**Definition 1.17.** For a given norm the numerical method \(\mathcal{H}_k\) is called stable if \(\forall T : \exists C > 0\) and \(k_0 > 0\) such that:

\[\|(\mathcal{H}_k)^n\| \leq C \forall nk \leq T, \ 0 < k < k_0\]

e.g. for \(\|\mathcal{H}_k\| \leq 1 + \alpha k\) \(\Rightarrow\) \(\|\mathcal{H}_k^n\| \leq (1 + \alpha k)^n \leq e^{\alpha kn} \leq e^{\alpha T}\.\)

**Definition 1.18.** Method is convergent, if \(u_j^{h,k} \xrightarrow{h,k \to 0} u(x_j, t_n)\).
Theorem 1.19 (Lax equivalence theorem; fundamental theorem of numerical analysis). For linear consistent difference methods, stabil $\Leftrightarrow$ convergent.

4th idea: Downwind scheme:
Aim: reduction of numerical diffusion (in comparison to Lax-Friedrichs schema)

$$u_j^{n+1} = u_j^n - \frac{ak}{h} (u_{j+1}^n - u_j^n) \quad \text{[for } a > 0\text{, otherwise exchange (1.28), (1.29)]} \quad (1.28)$$

Numerical stencil:

exakt solution: (1.23): wave travelling to the right
Disadvantage: scheme not useful (unstable), because information is transported into the wrong direction.

5th idea: Upwind scheme:

$$u_j^{n+1} = u_j^n - \frac{ak}{h} (u_j^n - u_{j-1}^n), \quad n \geq 0, j \in \mathbb{Z} \quad \text{[for } a > 0\text{]} \quad (1.29)$$

Numerical stencil:
1.4 Numerical Methods

Advantage: no oscillations; less artificial diffusion (smaller $D$) than Lax-Friedrichs.

Local truncation error:

$$L_k(x, t) := \frac{1}{k} \left( u(x, t + k) - u(x, t) + \frac{ak}{h} (u(x, t) - u(x - h, t)) \right)$$

Taylor:

$$\approx \frac{ak}{2} \left( a - \frac{h}{k} \right) u_{xx} + O(h^2) + O(k^2).$$

Modified equation of second order (with $k/h = \text{konst}$):

$$u_t + au_x = \frac{ak}{2} \left( a - \frac{h}{k} \right) u_{xx}$$

$$(1.30) \text{ well posed } \iff D \geq 0 \iff 0 \leq \frac{ak}{h} \leq 1. \quad (1.31)$$

This is an indicator for the stability of a numerical scheme, but no proof.

(1.31) is called Courant-Friedrichs-Levy (CFL) condition; here it is a stability condition, see slope of characteristic in numerical stencil.

typical value in practice: $\frac{ak}{h} \approx 0.8$

6. idea: Lax-Wendroff scheme:

Derivation via Taylor series:

$$u(x, t + k) = u(x, t) + ku_t(x, t) + \frac{k^2}{2} u_{tt}(x, t) + O(k^3);$$

use

$$u_t = -au_x, \quad u_{tt} = a^2 u_{xx}$$

and central and second difference approximations for $u_x, u_{xx}$:

$$\Rightarrow u_j^{n+1} = u^n_j - \frac{k}{2h} a(u_{j+1}^n - u_{j-1}^n) + \frac{k^2}{2h^2} a^2 (u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

Numerical stencil:

CFL condition: $\frac{|a|k}{h} \leq 1.$

Lax-Wendroff is second order scheme. The modified equation of third order is

$$u_t + au_x = \frac{h^2}{6} a \left( \frac{k^2}{h^2} a^2 - 1 \right) u_{xxx}.$$ 

(1.32)

which is a dispersive equation; no numerical diffusion.
numerical solution for discontinuous data:

e.g. $u_0(x) = \begin{cases} 
1 &, x < 0 \\
0 &, x > 0 
\end{cases}$

Phenomena:

- 1st order schemes smooth the discontinuity.
- 2nd order schemes develop oscillations (cf. Gibbs phenomenon).
- All (discussed) schemes calculate the correct "shock" speed.
- Order of convergences is reduced from 1 to $\frac{1}{2}$ resp. from 2 to $\frac{2}{3}$ (consider $L^1$-error, not $L^\infty$-error)


1.4.2 Nonlinear conservation laws

Consider the example: Burgers’ equation or LWR-model:

$$\begin{cases} 
  u_t + uu_x = 0, & x \in \mathbb{R}, t > 0 \\
  u(x, 0) = u_0(x) 
\end{cases} \quad (1.33)$$

1st idea: modified upwind scheme

e.g. for $u_0 \geq 0$:

$$u_j^{n+1} = u_j^n - \frac{k}{h} u_j^n (u_j^n - u_{j-1}^n), n \in \mathbb{N}_0, j \in \mathbb{Z} \quad (1.34)$$

For $u_0^j = \begin{cases} 
  1 &, j < 0 \\
  0 &, j \geq 0 
\end{cases}$, we have: $u_j^0 = u_j^1 = u_j^2 = \ldots \ \forall j \in \mathbb{Z}$.

$\Rightarrow$ numerical solution converges to $u(x, t) = u_0(x)$!
1.4 Numerical Methods

But this is not a weak solution of (1.33) or of \( u_t + \frac{1}{2}(u^2)_x = 0 \)!

For other Riemann problems: numerical method gives moving shock wave, but with wrong velocity!

⇒ method useless.

Problem: scheme (1.34) discretises (1.33), but not Burgers’ equation in conservation form:

\[ u_t + \frac{1}{2}(u^2)_x = 0 \]

See exercise: \( u_t + \frac{1}{2}(u^2)_x = 0, (u^2)_t + \frac{2}{3}(u^3)_x = 0 \) have different weak solutions.

Definition 1.20. (a) A difference scheme of the form

\[
u^n_{j+1} = u^n_j - \frac{k}{h}[F(u^n_{j-p}, \ldots, u^n_{j+q}) - F(u^n_{j-1-p}, \ldots, u^n_{j-1+q})]
\]

with a numerical flux function \( F: \mathbb{R}^{p+q+1} \to \mathbb{R} \) is called conservative.

(b) A conservative scheme is called consistent (with \( u_t + f(u)_x = 0 \)), if \( F \) is locally Lipschitz continuous and \( F(u, \ldots, u) = f(u) \forall u \in \mathbb{R} \).

Simple case: \( p = 0, q = 1 \)

\[
u^n_{j+1} = u^n_j - \frac{k}{h}[F(u^n_j, u^n_{j+1}) - F(u^n_{j-1}, u^n_j)]
\]

Conservative scheme ⇒ discrete conservation of mass (due to telescopic sum in \( j \)) ⇒ correct speed of (smoothed) shocks.

Interpretation of (1.36):

\[
\frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_{n+1})dx = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)dx \\
- \frac{k}{h} \left[ \frac{1}{k} \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dt \right] - \frac{1}{k} \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))dt
\]

with cell centers \( x_{j \pm \frac{1}{2}} := (j \pm \frac{1}{2})h \).

Interpret \( u^n_j \) as approximation for cell average of \( u(x, t) \):

\[
u^n_j \sim \bar{u}_j^n := \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)dx
\]
and $F(u^n_j, u^n_{j+1})$ as approximation of mean flow through $x_{j+1/2}$ during $(t_n, t_{n+1})$:

$$F(u^n_j, u^n_{j+1}) \sim \frac{1}{k} \int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t)) dt$$

$\Rightarrow$ scheme (1.36) follows from (1.37).

**Example 1.21.** Upwind-scheme for Burgers’ equation:

$$u^{n+1}_j = u^n_j - k \left[ \frac{1}{2}(u^n_j)^2 - \frac{1}{2}(u^n_{j-1})^2 \right], \quad n \geq 0, j \in \mathbb{Z}$$

for $u^n_j \geq 0 \quad \forall n, j$.

$F(u_j, u_{j-1}) = \frac{1}{2}u^n_j^2$; first order scheme.

**Example 1.22.** Lax-Friedrichs scheme:

$$u^{n+1}_j = \frac{1}{2}(u^n_{j-1} + u^n_{j+1}) - \frac{k}{2h}(f(u^n_{j+1}) - f(u^n_{j-1})),$$

$$F(u_j, u_{j+1}) = \frac{h}{2k}(u_j - u_{j+1}) + \frac{1}{2}(f(u_j) + f(u_{j+1})),$$

First order scheme, conservative, consistent.

**Example 1.23.** Lax-Wendroff scheme:

$$u^{n+1}_j = u^n_j - \frac{k}{2h} \left( f(u^n_{j+1}) - f(u^n_{j-1}) \right)$$

$$+ \frac{k^2}{2h^2} \left[ f'(u^n_{j+1/2})(f(u^n_{j+1}) - f(u^n_j)) - f'(u^n_{j-1/2})(f(u^n_j) - f(u^n_{j-1})) \right]$$

with $u^n_{j \pm 1} := (u^n_j + u^n_{j \pm 1})/2$.

Scheme conservative, consistent, second order.

Convergence:

Vague idea: numerical solution from Example 1.21-1.23 converges to weak solution of $u_t + f(u)_x = 0$ (for $h, k \to 0$).

Problem: weak solution is not unique in general!
1.4 Numerical Methods

Definition 1.24. Total variation of a function \( v : \mathbb{R} \to \mathbb{R} \):

\[
TV(v) := \sup \sum_{j=1}^{N} |v(\xi_j) - v(\xi_{j-1})|,
\]

Supremum over all subdivisions \(-\infty = \xi_0 < \xi_1 < \ldots < \xi_N = \infty \) of \( \mathbb{R} \).

For \( v \in C^1(\mathbb{R}) \) : \( TV(v) = \int_{\mathbb{R}} |v'(x)| dx \)

Necessary for \( TV(v) < \infty \) : \( \exists \lim_{x \to \pm\infty} v(x) \).

Theorem 1.25 (Lax-Wendroff). Let \( \{u_l(x,t)\} \) be a sequence of numerical solutions, calculated via a consistent and conservative method on a mesh sequence with \( h, k_l \to 0 \) . \( u_l \) is e.g. a constant extension of \( u^n_j \) on the cells.

Suppose there is a function \( u(x,t) \) such that:

1. \( u_l \xrightarrow{l \to \infty} u \) in \( L^1(\Omega) \) \( \forall \Omega = (a,b) \times (0,T) \),
2. \( \forall T > 0 : \exists R > 0 \) with \( TV(u_l(\cdot,t)) < R \) \( \forall 0 \leq t \leq T, \forall l \in \mathbb{N} \).

\( \Rightarrow u(x,t) \) is weak solution of \( u_t + f(u)_x = 0 \)

Proof. \([LV]\) §12.

Remark 1.26. Theorem 1.25 does not imply the convergence of the numerical approximation sequence \( u_l \); is also does not imply that \( u \) is the entropy solution.

Theorem 1.27. Additionally to the assumptions of Theorem 1.25 suppose: \( (\eta, \psi) \in C^2(\mathbb{R}) \times C^1(\mathbb{R}) \) with \( \eta'' > 0 \) is one entropy / entropy flux pair (see Def. 1.9). Let \( \Psi : \mathbb{R}^{p+q+1} \to \mathbb{R} \) be a numerical entropy flux function, consistent with \( \psi \) (i.e., \( \Psi(u, \ldots, u) = \psi(u) \forall u \in \mathbb{R} \)) and

\[
\eta(u_{j+1}^{n+1}) \leq \eta(u_j^n) - \frac{k}{h} \left[ \Psi(u_{j-p}^n, \ldots, u_j^n, u_{j+q}^n) - \Psi(u_{j-1-p}^n, \ldots, u_{j-1+q}^n) \right] \quad \forall j, n \quad (1.38)
\]

\( \Rightarrow u(x,t) \) (from Theorem 1.25) satisfies the (weak) entropy inequality (1.20):

\[
\int_0^\infty \int_{\mathbb{R}} [\eta(u)\Phi_t + \psi(u)\Phi_x] \, dx \, dt \geq - \int_{\mathbb{R}} \eta(u_0(x))\Phi(x,0) \, dx \quad \forall \Phi \in C^1_0(\mathbb{R}^2), \Phi \geq 0. \quad (1.39)
\]

Proof. \([LV]\) §12.
**Remark 1.28.** 1. cf. (1.38) with entropy inequality (1.19):

\[ \eta(u)_t + \psi(u)_x \leq 0. \]

2. By [DeLellis-Otto-Westdickenberg, 2003], already one strictly convex \( \eta \) is enough for entropy solutions in Def. 1.10.

3. Condition (1.38) holds e.g. for the Godunov scheme, a special version of the upwind method (details in [LV] §13, [Jü] §5).

2 Fluid mechanics

2.1 Euler equations

Consider the flow of a fluid (=liquid or gas) in the region $\Omega \subset \mathbb{R}^d$, $d = 2, 3$.

$\rho(x,t)$ . . . mass density
$u(x,t)$ . . . velocity (vector) field
$p(x,t)$ . . . pressure

• here: description by Euler coordinates, i.e., $x$ is a fixed point of space through which different fluid parcels flow.
• alternative description by Lagrange coordinates (mostly in §3): $X \in \Omega$ ist a fluid parcel (or particle), $t \mapsto x(t; X)$ with $x(t_0; X) = X$ its movement or trajectory.

**Aim:** derivation of the 3 Euler equations:

(a) conservation of mass:

consider (arbitrary) temporally fixed region $R \subset \Omega$ with smooth boundary $\partial R$ and outer normal vector $\nu$:

Balance equation:

$$\frac{d}{dt} \int_R \rho(x,t) \, dx = - \int_{\partial R} \rho u \cdot \nu dS$$
2 Fluid mechanics

\[
\text{divergence theorem } \Rightarrow \int_{R} \rho_t + \text{div} (\rho u) \, dx = 0 \quad \forall R \subset \Omega
\]

\[
\Rightarrow \rho_t + \text{div} (\rho u) = 0, \, x \in \Omega \quad \ldots \text{continuity equation} \quad (2.1)
\]

(b) conservation of momentum:

from Newton’s second law: \text{mass} \times \text{acceleration} = \text{force},

hence change of momentum through external/volume forces and surface forces

\( R \subset \Omega \ldots \) arbitrary (fixed) region

momentum of mass in \( R \):

\[
\int_{R} \rho u \, dx
\]

- external/volume forces:
  \[
  \int_{R} \rho f \, dx \quad \text{(e.g. gravitation, electromagnetic)}
  \]
- surface forces on \( \partial R \) with outer normal \( \nu \): \text{stress vector} \( \tau = \tau(x, t, \nu) \)

One can show:

1. \( \tau(x, -\nu) = -\tau(x, \nu) \ldots \) local equilibrium of stress (from Newton’s 3rd law)
2. \( \tau \) depends linearly on \( \nu \), so \( \tau(x, \nu) = T(x) \cdot \nu; \)
   matrix \( T \ldots \) stress tensor (from conservation of momentum)
3. \( T = T^\top \), rotation invariance (from conservation of angular momentum)
2.1 Euler equations

Total surface force:

$$\int_{\partial R} \tau(x,t,\nu) dS = \int_{\partial R} T(x,t) \cdot \nu dS \overset{\text{div. theorem}}{=} \int_{\partial R} \text{div } T dS = \int_{\partial R} \nabla \cdot T dS$$

$$\Rightarrow$$ force density on fluid: $\rho f + \nabla \cdot T$

Let $X \in \Omega$ be a particle; $x(t;X) = (x_1(t), x_2(t), x_3(t))$ its trajectory.

Speed of particle $X$: $\dot{x}(t) = u(x(t), t)$ [X skipped notationally]

Acceleration of particle $X$:

$$a(t) = \ddot{x}(t) = \frac{d}{dt} u(x(t), t)$$

$$= u_{x_1} \frac{\dot{x}_1}{u_1} + u_{x_2} \frac{\dot{x}_2}{u_2} + u_{x_3} \frac{\dot{x}_3}{u_3} + u_t = u_t + \left( u \cdot \nabla \right) u = \frac{Du}{Dt},$$

with material derivative $\frac{D}{Dt} := \partial_t + u \cdot \nabla$

It describes the temporal rate of change of an $x$- and $t$-dependent physical quantity (e.g. temperature) in a volume element which is transported in a flux field with speed $u$.

Hence describes the change in a frame of reference (which is transported with the flow).

Ex.: the temperature distribution (in 1D) is transported only by the flow, i.e., $\tilde{T}(x,t) = \tilde{T}_0(x - ut) \Rightarrow \frac{D\tilde{T}}{Dt} = 0$.

Newton’s second law $\Rightarrow$ balance equation for densities:

$$\rho \frac{D}{Dt} u = \rho f + \nabla \cdot T$$

add $u\rho_t + u \text{div}(\rho u) = 0$

$$\Rightarrow \partial_t (\rho u) + u \text{div}(\rho u) + \rho (u \cdot \nabla) u = \rho f + \nabla \cdot T$$

$$\Rightarrow \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u - T) = \rho f$$ \quad momentum balance equation \quad (2.2)

Special case: inviscid fluid $\rightarrow$ now shear stress

$$\tau(x,\nu) = -p(x)\nu, \quad p \ldots$$ pressures $\Rightarrow T = -p(x)I, \nabla \cdot T = -\nabla p$

$$\Rightarrow \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p = \rho f$$ \quad (2.3)
Rem.: no tangential forces ⇒ rotation cannot be started/stopped.

(c) conservation of energy

Balance: change of energy = power - heat loss

Energy density:

\[
\rho \left( \frac{|u|^2}{2} + e \right)
\]

\[
\frac{d}{dt} \int_R \rho \left( \frac{|u|^2}{2} + e \right) \, dx = - \int_{\partial R} \rho \left( \frac{|u|^2}{2} + e \right) u \cdot \nu \, dS
\]

+ \int_R \rho f \cdot u \, dx + \int_{\partial R} \left( \tau \cdot u - h \right) dS \quad \forall R \subset \Omega

\int_{\partial R} (\tau \cdot u - h) \, dS = \int_{\partial R} \nu \cdot (T \cdot u - q) \, dS;

with \( h(x,t) = \nu \cdot q(x,t) \); \( q \) . . . heat flux density (=vektor)

Divergence theorem ⇒ energy balance equation:

\[
\frac{\partial}{\partial t} \left[ \rho \left( \frac{|u|^2}{2} + e \right) \right] + \text{div} \left[ \rho u \left( \frac{|u|^2}{2} + e \right) + q - T \cdot u \right] = \rho f \cdot u
\]

\((2.4)\)

(2.1), (2.2), (2.4) . . . general balance equations; so far these do not incorporate physics resp. material properties, but they are the starting point for Euler (with \( T = -pI \)) and Navier-Stokes equations (in §2.2). In total we will examine \( 2 \times 2 \) models: inviscid / viscous × (in)compressible.

Special cases:

a) Fourier’s law of thermal conduction: \( q = -\kappa \nabla \hat{T} \); \( \kappa \) . . . thermal conductivity, \( \hat{T} \) . . . temperature

b) inviscid fluid: \( T = -pI \) ⇒ \( \text{div}(T \cdot u) = -\text{div}(pu) \)

c) inviscid ideal gas:

\[
T = -pI, p = \rho R \hat{T}, \quad R \ldots \text{gas constant}
\]

frequently: \( e = c_V \hat{T} + \text{const}, c_V \ldots \text{specific heat with constant volume} \)
d) inviscid ideal fluid with $f = 0, q = 0$:

compressible Euler equations (for inviscid ideal fluid):

$$
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0 \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p &= 0 \\
\partial_t \left[ \rho \left( \frac{|u|^2}{2} + e \right) \right] + \text{div} \left[ \rho u \left( \frac{|u|^2}{2} + e \right) + pu \right] &= 0
\end{align*}
$$

This is a hyperbolic conservation law: 5 equations for 6 variables ($\rho, u, p, e$). One needs one additional (physical) constitutive equation, e.g. $e = c_v T + \text{const} , p = \rho R T$.

e) inviscid incompressible fluid with $f = 0, q = 0$:

Flow $u(x,t)$ is incompressible if $\forall$ regions $R(t) \subset \Omega$ which are moved along the following holds:

$$
\text{vol} \left( R(t) \right) = \int_{R(t)} dx = \text{const in } t.
$$

This holds if and only if $\text{div} \ u = 0$ because

$$
0 = \frac{d}{dt} \text{vol} \left( R(t) \right) = \frac{d}{dt} \int_{R(t)} dx \overset{(*)}{=} \int_{\partial R(t)} u(x,t) \cdot \nu dS = \int_{R(t)} \text{div} \ u dx
$$

1D-illustration of $(*)$: $\frac{d}{dt} \int_{a(t)}^{b(t)} dx = \dot{b}(t) - \dot{a}(t) = u(b(t)) - u(a(t))$ 
(detailed proof of $(*)$: [CM] §1.1).

Incompressibility of good approximation for “small speeds” (e.g. mach number $Ma := \frac{|u|}{c} < 0.3$, with $c$ ... speed of sound).

Additionally suppose $\frac{De}{Dt} = 0$ (e.g. $e = \text{const}$):

incompressible Euler equations:

$$
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0 \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p &= 0 \\
\text{div} \ u &= 0
\end{align*}
$$

5 equations for 5 variables

energy equation is satisfied “automatically” (→ Exercises).

References: [CM] §1.1
2.2 Navier-Stokes equations

**Aim:** Derivation of NS equations

Shear stress in fluid (= gas or liquid) depends only on local changes of velocity \( u(x) \), i.e.,

\[
\frac{\partial u}{\partial x} = \left( \frac{\partial u_i}{\partial x_j} \right)_{i,j=1,2,3}
\]

Fluid at rest (i.e. \( u = 0 \)) or in homogeneous movement (i.e. \( u = \text{const} \)): no shear stress, \( \tau \) has only normal component:

\[
\tau(x,\nu) = -p(x)\nu, \quad p \ldots \text{pressure}, \quad \Rightarrow \quad T = -p(x)I
\]

in general:

\[
T = -pI + \sigma
\]

Matrix \( \sigma = (\sigma_{ij})_{i,j=1,2,3} \ldots \) viscous stress tensor (shear forces due to friction, viscosity)

Assumptions on \( \sigma \) — as function of \( \frac{\partial u}{\partial x} \):

1. \( \sigma \left( \frac{\partial u}{\partial x} \right) \) is linear, i.e. *Newtonian fluid* (Ex.: water, oil):

\[
\sigma_{ij}(x) = \sum_{k,l=1}^{3} C_{ijkl}(x) \frac{\partial u_k}{\partial x_l}(x) \quad (3^4 = 81 \text{ coefficients})
\]

   non-newtonian examples: ketchup, shampoo, blood, starch suspension (non-constant viscosity).

2. fluid is isotropic, i.e., \( \nabla \) distinguished direction

   \( \Rightarrow \) \( \sigma \) is invariant under (rigid body) rotations, i.e.

\[
\sigma \left( U \cdot \frac{\partial u}{\partial x} \cdot U^{-1} \right) = U \cdot \sigma \left( \frac{\partial u}{\partial x} \right) \cdot U^{-1} \quad \forall \text{ orthogonal matrices } U \quad (2.5)
\]

   fluid crystals are an example of anisotropic fluids.

3. \( \sigma \) is symmetric (follows from conservation of angular momentum)
From (2.) we deduce:

\[ \sigma = \sigma(D) \quad \text{with} \quad D := \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}^\top \right) \quad \ldots \text{deformation tensor} \]

**Proof.** \( \sigma = 0 \) for rotations with constant angular velocity; e.g. rotation around \( x_3 \)-axis:

\[ \tilde{u} = \omega(-x_2, x_1, 0) \top, \quad \frac{\partial \tilde{u}}{\partial x} = \omega\left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \]

\[ \sigma\left( \frac{\partial \tilde{u}}{\partial x} \right) = 0 \Rightarrow C_{ij21} = C_{ij12} \quad ; \quad i, j = 1, 2, 3 \]

analogously for \( x_1 \)-, \( x_2 \)-axis: \( C_{ij23} = C_{ij32}, C_{ij13} = C_{ij31} \).

\[ \Rightarrow \sigma_{ij} = C_{ij11}(u_1)_x + C_{ij22}(u_2)_x + C_{ij33}(u_3)_x \\
+ C_{ij12}((u_1)_x + (u_2)_x) + C_{ij13}((u_1)_x + (u_3)_x) \\
+ C_{ij23}((u_2)_x + (u_3)_x), \]

hence \( \sigma = \sigma(D) \). \[\blacksquare\]

- \( \sigma = \sigma^T \) is a linear, isotropic (i.e. satisfying (2.5)) function of \( D \). One can show that \( \sigma, D \) commute \(^1\). (cf. theorem of Rivlin-Ericksen, [EGK] §5.9)

\[ \Rightarrow \sigma, D \text{ simultaneously diagonalisable} \]

\[ \Rightarrow \sigma_i \ (=\text{eigenvalues of } \sigma) \text{ are linear functions of } d_i \ (=\text{eigenvalues of } D) \]

Due to rotation invariance (2.): \( \sigma_i \) is symmetric function with respect to index permutations

\[ \Rightarrow \sigma_i = \lambda(d_1 + d_2 + d_3) + 2\mu d_i \quad ; \quad i = 1, 2, 3. \]

Transforming back to basis of \( \sigma, D \):

\[ \Rightarrow \boxed{\sigma = \lambda(\text{div } u)I + 2\mu D} \quad \text{(2.6)} \]

only 2 coefficients left; interpretation of \( \lambda, \mu \):

**Example 2.1.** isotropic expansion: \( u = cx, c > 0 \)

\[ \text{div } u = 3c, D = cI \]

stress tensor:

\[ T = -pI + \sigma = -pI + \lambda(\text{div } u)I + 2\mu D = -(p - (3\lambda + 2\mu)c)I \]

effective pressure
\[ \mu_d := \lambda + \frac{2}{3} \mu \geq 0 \ldots \text{pressure viscosity resp. 2. viscosity coefficient} \]

\[ \to \text{effective pressure is lower than thermodynamic pressure.} \]

**Example 2.2.** Shear flow \( u = (\kappa x_2, 0, 0)^\top, \kappa = \text{const}, p = 0 \)

\[ \Rightarrow \text{div} \, u = 0, \ D = \frac{1}{2} \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \Rightarrow T = \lambda (\text{div} \, u) I + 2\mu D = \mu \kappa \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

Stress vector \( \tau = T \cdot \nu = \mu \kappa (\nu_2, \nu_1, 0)^\top \)

\( \tau \) is pure shear force

\( \mu \geq 0 \ldots \text{shear viscosity, 1st viscosity coefficient} \)

\[ ^1 \text{M.E. Gurtin, A short proof of the representation theorem for isotropic, linear stress-strain relations; J. of Elasticity 4, 1974} \]
2.2 Navier-Stokes equations

\[ \nu = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ \tau \]

\[ \nu = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ \Rightarrow \sigma = \mu_d (\text{div } u) I + 2 \mu \left( D - \frac{1}{3} \text{(div } u) I \right) \]

\[ Sp(D - \frac{1}{3} (\text{div } u) I) = \text{div } u - \frac{1}{3} \text{div } u \cdot 3 = 0 \]

Inserting \( T = -pI + \sigma \) in balance equation of momentum (2.2):

\[ \nabla \cdot T = -\nabla p + \nabla (\lambda \text{div } u) + 2 \nabla \cdot (\mu D) \]

\[ \Rightarrow \text{compressible Navier-Stokes equations:} \]

\[ \begin{cases} 
\rho_t + \text{div}(\rho u) = 0 \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u - 2 \mu D) + \nabla (p - \lambda \text{div } u) = \rho f \\
\partial_t \left[ \rho \left( \frac{|u|^2}{2} + e \right) \right] + \text{div} \left[ \rho u \left( \frac{|u|^2}{2} + e \right) + q - T \cdot u \right] = \rho f \cdot u 
\end{cases} \]

5 equations for 9 variables \((\rho, p, u, e, q)\)

special cases:

a) \( \lambda = \mu = 0 \Rightarrow \text{compressible Euler equations} \)

b) \( \lambda = \text{const}, \mu = \text{const} \) (henceforth assumed):

\[ (2 \nabla \cdot D)_i = \left( \nabla \cdot \left( \frac{\partial u}{\partial x} + \left( \frac{\partial u}{\partial x} \right)^\top \right) \right)_i = \sum_{j=1}^3 \frac{\partial}{\partial x_j} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) = \partial_{x_i} \text{(div } u) + \Delta u_i \]

\[ \Rightarrow \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla (p - (\lambda + \mu) \text{div } u) = \mu \Delta u + \rho f \]

c) \text{incompressible homogeneous fluid: (e.g. water, oil)}

\[ \text{div } u = 0, \rho(x,t) = \rho_0 = \text{const} \Rightarrow \text{continuity equation trivially satisfied} \]
⇒ incompressible Navier-Stokes equations for homogeneous fluid:

\[
\begin{aligned}
\rho_0 [u_t + \nabla \cdot (u \otimes u)] + \nabla p &= \mu \Delta u + \rho_0 f \quad \text{(parabolic for } u) \\
\text{div } u &= 0
\end{aligned}
\]  

(2.7)

4 equations for 4 variables \((u, p)\) → closed system

possible boundary conditions: \(u(x, t) = 0, x \in \partial \Omega\) (no slip condition)

If \(\mu = 0\) (i.e. shear forces, viscosity negligible) ⇒

incompressible homogeneous Euler equations:

\[
\begin{aligned}
\rho_0 [u_t + \nabla \cdot (u \otimes u)] + \nabla p &= \rho_0 f \quad \text{(hyperbolic for } u) \\
\text{div } u &= 0
\end{aligned}
\]  

(2.8)

possible boundary conditions: \(u(x, t) \cdot \nu = 0, x \in \partial \Omega\)

Solution theory:

in \(\mathbb{R}^2\): \(\exists \! \! \! ! \) solution \(\forall t \geq 0\) for (2.7) resp. (2.8)

in \(\mathbb{R}^3\): \(\exists \! \! \! ! \) solution for “small times” for (2.7) resp. (2.8). It is not clear whether a solution exists \(\forall t \geq 0\).

Problem in \(\mathbb{R}^3\): there can be turbulences or “chaotic behaviour”; but not in \(\mathbb{R}^2\).

d) ideal compressible gas: (e.g. air, thin gases)

constant shear viscosity \(\mu \geq 0\)

vanishing pressure viscosity: \(\mu_d = \lambda + \frac{2}{3} \mu = 0\)

\(\Rightarrow\) \(\sigma = 2\mu[D - \frac{1}{3} \text{div } u]I\)

The rest analogously to Euler equations
e) homogeneous incompressible “slow” flow:

Let \(f = 0\). If the nonlinear term \((u \cdot \nabla)u\) in (2.7) is negligible:

\[
\nabla \cdot (u \otimes u) = (\text{div } u)u + (u \cdot \nabla)u \approx 0
\]

\(\Rightarrow\) Stokes equations (linear for \(u, p\)):

\[
\begin{aligned}
\begin{cases}
\rho_0 [u_t - \frac{1}{\rho_0} \nabla p + \nu_0 \Delta u, & \nu_0 := \mu/\rho_0 \ldots \text{kinematic viscosity} \\
\text{div } u &= 0
\end{cases}
\end{aligned}
\]  

(2.9)
Motivation: let $\tilde{x} := x/L$, $\tilde{u} := u/U$ with typical reference length $L$ and reference velocity $U$.

$$\Rightarrow \ (u \cdot \nabla_x)u = \frac{U^2}{L} (\tilde{u} \cdot \nabla_{\tilde{x}})\tilde{u}, \ \nu_0 \Delta_x u = \nu_0 \frac{U}{L^2} \Delta_{\tilde{x}} \tilde{u}$$

Disregard OK for $\frac{U^2}{L} \ll \nu_0 \frac{U}{L^2}$ resp. $Re := \frac{LU}{\nu_0} \ll 1 \ldots \textit{Reynolds number}$ (dimensionless)

Rem:

- Typical scales of $\nabla u$, $\Delta u$ still missing;
- Only $(u \cdot \nabla)u$ and $\Delta u$ are compared because these “drive” the flow; $\nabla p$ is only the response to the constraint $\text{div} \ u = 0$, see (2.14).

Flows with equivalent Reynolds numbers allow for scaled wind tunnel experiments.

\textbf{Example 2.3.} incompressible, homogeneous, stationary flow between 2 parallel moving plates:

Assumptions: $f = 0$, $2D$-flow, infinite plates, no pressure drop in $x$, hence $p = p(y)$.

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (6,0) node[right] {$x$};
\draw[->] (0,0) -- (0,3) node[left] {$y$};
\draw[thick,red] (0,0) -- (6,0) node[above] {$U$};
\draw[thick,red] (0,0) -- (0,3) node[below left] {$d$};
\end{tikzpicture}
\end{center}

$$\Rightarrow \ \text{div} \ u = 0, \ \rho = \rho_0, \ \frac{\partial u}{\partial t} = 0$$

$$\begin{align*}
\rho_0 \nabla \cdot (u \otimes u) + \nabla p &= \mu \Delta u \\
\text{div} \ u &= 0
\end{align*}$$

(2.10)

Look for special $x$-independent solution because problem is $x$-independent:

$u(y) = (u_1(y), u_2(y))^\top$, $p = p(y)$

$$\begin{align*}
\text{div} \ u &= \partial_x u_1 + \partial_y u_2 = 0 \Rightarrow u_2 = 0 \ (\text{due to boundary condition} \ u(x, 0) = 0) \\
\Rightarrow \ \nabla \cdot (u \otimes u) &= (\text{div} \ u) u + \left( \nabla \cdot \begin{array}{c} u \\ u_1 \partial_x + u_2 \partial_y = 0 \end{array} \right) u = 0
\end{align*}$$

$$\Rightarrow \begin{cases} 
0 = \mu \Delta u_1 = \mu \partial^2_y u_1 \\
p_y = 0
\end{cases}$$
Fluid mechanics

\[ p = \text{const} = p_0, \partial_y^2 u_1 = 0 \]

No-slip condition: \( u_1(0) = 0, u_1(d) = U \)

\[ \Rightarrow u = u(y) = \left( \frac{Uy}{d}, 0 \right) \top \]

This is pure shear flow, “planar Couette flow”.

Force on (lower) plate at rest:

\[ \tau(\nu) = T \cdot \nu = -p_0 \nu + \mu \frac{U}{d} (\nu_2, \nu_1) \top \quad (\text{cf Ex. 2.2}) \]

For \( \nu = (0, 1) \top \): \( \tau = (\frac{U}{d}, -p_0) \top \)

**Example 2.4.** like Ex. 2.3; both plates at rest (i.e. \( U = 0 \)) with pressure drop \( p_x = -c < 0 \). (2.10) is still \( x \)-independent. \( \Rightarrow \) look for \( x \)-independent solution. \( \Rightarrow \) 1. line of (2.10):

\[ \begin{aligned} p_x = \mu (u_1)_{yy} & \Rightarrow (u_1)_{yy} = -\frac{c}{\mu} \\ u_1(0) = u_1(d) = 0 \end{aligned} \]

\[ \Rightarrow u(y) = \left( \frac{c}{2\mu} y(d - y), 0 \right) \top, p(x) = -cx + p_0 = \text{const} \]

This is a “planar Poiseuille flow” (balance between pressure drop and friction)
2.2 Navier-Stokes equations

Application: measurement of viscosity (in practice: viscometer with 2 concentric cylinders):

transported mass per time per length = \[ \int_0^d \rho_0 u_1(y)dy = \frac{\rho_0 cd^3}{12\mu} \]

Poiseuille flow is unstable for large Reynolds numbers; transition to turbulent flow.

References: [CM] §1.3, [EGK] §5.9

Figure 2.1: tube flow for increasing Re: transition from laminar to turbulent flow
2 Fluid mechanics

2.2.1 Helmholtz-Hodge decomposition

Aim: interpretation of incompressible (Navier-)Stokes equations as evolution equations for $u$ with $p$ as Lagrange multiplier with constraint $\text{div} \, u = 0$.

\[
\begin{align*}
\begin{cases}
u_t + (u \cdot \nabla)u &= -\frac{1}{\rho_0}\nabla p + \nu_0 \Delta u \
\text{div} \, u &= 0 \
\text{div} \, u &= 0 \\
\text{div} \, u &= 0 \\
\text{div} \, u &= 0 \\
\text{div} \, u &= 0 \\
\text{div} \, u &= 0
\end{cases}
\end{align*}
\tag{2.11}
\]

Physically the stricter constraint $u = 0$ would be better, for the following (purely analytical) lemma $\text{div} \, u = 0$ suffices.

Lemma 2.5 (Helmholtz-Hodge decomposition). Let $\Omega \subset \mathbb{R}^d, d \geq 2$ be bounded with $\partial \Omega \in C^{2,\alpha}(0 < \alpha < 1), w \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^d)$.

$\Rightarrow \exists u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^d), p \in C^{2,\alpha}(\overline{\Omega}) : (p \text{ scalar; unique up to additive constant})$

\[
w = u + \nabla p
\]

(2.12)

with $\text{div} \, u = 0$ in $\Omega$, $u \cdot \nu = 0$ on $\partial \Omega$. $(u, w \ldots \text{vector fields})$

Proof. 1. Show:

$\forall u$ with $\text{div} \, u = 0, u \cdot \nu = 0$ on $\partial \Omega$ holds $\int_\Omega u \cdot \nabla p dx = 0$ (i.e. $u \perp \nabla p$ in $L^2(\Omega)$)

because:

$\text{div}(pu) = (\text{div} \, u)p + u \cdot \nabla p = u \cdot \nabla p$

$\Rightarrow 0 = BC \int_{\partial \Omega} pu \cdot \nu ds = \int_\Omega \text{div}(pu) dx = \int_\Omega u \cdot \nabla p dx \quad \checkmark$

hence: (2.12) is orthogonal decomposition in $L^2(\Omega)$.

2. Uniqueness: let $w = u_1 + \nabla p_1 = u_2 + \nabla p_2$

$\Rightarrow 0 = (u_1 - u_2) + \nabla (p_1 - p_2), (u_1 - u_2) \cdot \nu |_{\partial \Omega} = 0, \text{div} (u_1 - u_2) = 0 \quad (2.13)$

$\Rightarrow (\text{due to 1.}) \quad (u_1 - u_2) \perp \nabla (p_1 - p_2) \text{ in } L^2(\Omega)$ and

$0 \overset{(2.13)}{=} \int_\Omega [(u_1 - u_2) + \nabla (p_1 - p_2)] \cdot (u_1 - u_2) dx = \int_\Omega |u_1 - u_2|^2 dx$

$\Rightarrow u_1 = u_2 \Rightarrow \nabla p_1 = \nabla p_2 \quad \checkmark$

$^2$Hölder seminorm: $|f|_{C^{0,\alpha}(\Omega)} := \sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha}, 0 \leq \alpha \leq 1$;

Hölder norm: $\|f\|_{C^{0,\alpha}} := \|f\|_{C^0} + \max_{|\beta| = n} |D^\beta f|_{C^{0,\alpha}}, n \in \mathbb{N}_0$. 

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2.2 Navier-Stokes equations

3. **Existence**: Rem.: \(w = u + \nabla p \Rightarrow \text{div} \ w = \text{div} \ u + \nabla \nabla p = \Delta p\) and on \(\partial \Omega : w \cdot \nu = \nabla \nabla p \cdot \nu\). Thus solve for \(p\):

\[
\Delta p = \text{div} \ w \quad \text{in} \ \Omega, \ \nabla \nabla p \cdot \nu = w \cdot \nu \quad \text{on} \ \partial \Omega \quad (= \text{Neumann problem for Poisson equation})
\]

Due to \(\text{div} \ w \in C^{0,\alpha}(\Omega)\) and \(w \cdot \nu \in C^{1,\alpha}(\partial \Omega)\) there exists \(p \in C^{2,\alpha}(\overline{\Omega})\) (see PDE course). Let \(u := w - \nabla p \in C^{1,\alpha}(\overline{\Omega})\)

\[
\Rightarrow \ \text{div} \ u = \text{div} \ w - \Delta p = 0
\]

\[
\nu \cdot \nu|_{\partial \Omega} = w \cdot \nu|_{\partial \Omega} - \nabla \nabla p \cdot \nu|_{\partial \Omega} = 0.
\]

**Definition 2.6.** *Projection operator:* \(Pw := u\), where \(w = u + \nabla p\), \(\text{div} \ u = 0\) and \(u \cdot \nu|_{\partial \Omega} = 0\).

\(P : C^{1,\alpha}(\overline{\Omega}) \rightarrow C^{1,\alpha}(\overline{\Omega})\) is well defined due to above Lemma.

**Properties:**

(a) \(P\) is linear

(b) \(w = u + \nabla p = Pw + \nabla p\)

(c) \(P(u) = u\) \quad \forall u \text{ with } \text{div} u = 0, u \cdot \nu|_{\partial \Omega} = 0

(d) \(P(\nabla p) = 0\).

Apply \(P\) to (2.11):

\[
P \left( \partial_t u + \frac{1}{\rho_0} \nabla p \right) = P(- (u \cdot \nabla) u + \nu_0 \Delta u)
\]
Due to $\text{div} \partial_t u = \partial_t \text{div} u = 0$ and $(\partial_t u) \cdot \nu = \partial_t (u \cdot \nu) = 0$:

$$\mathbb{P}(\partial_t u) = \partial_t u \quad \text{(lt. (c))}.$$ 

From (d): $\mathbb{P}(\nabla p) = 0$

$$\Rightarrow \partial_t u = \mathbb{P}(-(u \cdot \nabla)u + \nu_0 \Delta u) \quad (2.14)$$

This is an evolution equation only for $u$; $p$ eliminated!

Caution: $\text{div}(\Delta u) = \Delta(\text{div} u) = 0$, but in general $\left. (\Delta u) \cdot \nu \right|_{\partial \Omega} \neq 0$.

$$\Rightarrow \text{in general } \mathbb{P}(\Delta u) \neq \Delta u$$

$(2.14)$ also useful for numerical algorithms.

Determination of pressure $p$ from $u$:

$$(2.11) : \quad \nabla p = -\rho_0[u_t + (u \cdot \nabla)u - \nu_0 \Delta u]$$

$$\overset{(2.14)}{=} \rho_0(\mathbb{I} - \mathbb{P})[-(u \cdot \nabla)u + \nu_0 \Delta u]$$

References: [CM] §1.3

### 2.2.2 Rotation

**Definition 2.7.** $\omega := \text{rot} u := \nabla \times u$ is called rotation oder “vorticity field” of the 3D velocity field $u$.

In 2D $\omega$ is scalar: $\omega := \partial_{x_1} u_2 - \partial_{x_2} u_1$ (embedded in in $\mathbb{R}^3$).

**Example 2.1** (continuation). $u(x) = cx, c \in \mathbb{R}$

$$\omega = c \begin{pmatrix} \frac{\partial x_1}{\partial x_1} \\ \frac{\partial x_2}{\partial x_2} \\ \frac{\partial x_3}{\partial x_3} \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
Example 2.8. \( u(x) = \Omega(-x_2, x_1, 0)\top \Rightarrow \omega = \text{rot } u = (0, 0, 2\Omega)\top \)

\[ \omega_3 = \text{double angular velocity around } x_3\text{-axis (= axis of rotation).} \]

In general: direction of \( \omega \) defines (as normal vector) local plane of rotation, its length the local intensity of vorticity.

local decomposition of flow:

Movement \( \approx \) (rigid) translation + deformation + (rigid) rotation

Lemma 2.9. Let \( u(x) \) be a smooth 3D vectorfield.

\[ u(y) = u(x) + D(x) \cdot (y - x) + \frac{1}{2} \omega(x) \times (y - x) + O(||y - x||^2) \quad \forall x, y \in \mathbb{R}^3 \]

Proof. From Taylor’s theorem:

\[ u(y) = u(x) + \frac{\partial u}{\partial x}(x) \cdot (y - x) + O(||y - x||^2) \]

Moreover:

\[ \left( D \cdot (y - x) + \frac{1}{2} \omega \times (y - x) \right)_1 = \partial_1 u_1(y_1 - x_1) \]

\[ + \frac{1}{2} (\partial_1 u_2 + \partial_2 u_1)(y_2 - x_2) + \frac{1}{2} (\partial_1 u_3 + \partial_3 u_1)(y_3 - x_3) \]

\[ + \frac{1}{2} \left[ (\partial_3 u_1 - \partial_1 u_3)(y_3 - x_3) - (\partial_1 u_2 - \partial_2 u_1)(y_2 - x_2) \right] = \omega_2 \]

\[ = \partial_1 u_1(y_1 - x_1) + \partial_2 u_1(y_2 - x_2) + \partial_3 u_1(y_3 - x_3) \]

\[ = \sum_{j=1}^{3} \partial_j u_1(y_j - x_j) = \left[ \frac{\partial u}{\partial x} \cdot (y - x) \right]_1 \]

other components analogously.

References: [CM] §1.3, [MP] §1.2
2.3 Vorticity models

Aim: Vorticity formulation for homogeneous incompressible Euler equation

2.3.1 Vector fields from sources and vortices

Let $G \subset \mathbb{R}^d; d = 2, 3$; simply connected region; let $u \in C^1(G, \mathbb{R}^d)$ (in this chapter).

Definition 2.10. $u$ is called irrotational or curl-free if $\text{rot } u = 0$ in $G$.

$u$ is called conservative if $\int_C u \, ds$ is path-independent.

Theorem 2.11. $u$ curl-free $\iff$ $u$ conservative $\iff \exists$ potential $\varphi : u = \nabla \varphi$

Proof. Analysis course.

Lemma 2.12. Let $u \in C^1(G)$ be a vector field. $\Rightarrow$

(i) $\text{rot } u = 0 \iff \exists \varphi : u = \nabla \varphi$

(ii) $\text{div } u = 0 \iff \exists$ vector potential $A : u = \text{rot } A$ (only in 3D)

(2D-interpretation only via embedding in 3D): $\exists A = (0, 0, A_3) \top : u = \text{rot } A = (\partial_2 A_3, -\partial_1 A_3, 0) \top$, bzw. $u = (\partial_2 A_3, -\partial_1 A_3) \top =: \nabla \perp A_3$

Aim: solution $u \in C^1(G)$ of system

\[
\left\{ \begin{array}{ll}
\text{rot } u = \omega & \text{in } G \quad (\text{vortex of } u: \omega \in C^1(G)) \\
\text{div } u = f & \text{in } G \quad (\text{source of } u: f \in C^0(G))
\end{array} \right.
\tag{2.15}
\tag{2.16}
\]

Solution of (2.15):

Because $\text{div rot } u = 0$: (2.15) is solvable $\iff$ $\text{div } \omega = 0$:

First look for special solution $u_0 = (u_1, u_2, u_3) \top$ with $u_3 = 0$.

$\text{rot } u_0 = \omega \iff$

\[
\left\{ \begin{array}{ll}
-\partial_3 u_2 & = \omega_1 \\
\partial_3 u_1 & = \omega_2 \\
\partial_1 u_2 - \partial_2 u_1 & = \omega_3
\end{array} \right.
\]

Choose special solution

\[
u_1 := \int \omega_2(x_1, x_2, x_3) \, dx_3
\]

\[
u_2 := -\int \omega_1(x_1, x_2, x_3) \, dx_3 + g(x_1, x_2)
\]
with $\partial_1 g = \partial_1 \int \omega_1 dx_3 + \partial_2 \int \omega_2 dx_3 + \omega_3$.

General solution according to Lemma 2.12(i):

$$u = u_0 + \nabla \varphi, \quad \forall \varphi \in C^2(G)$$

in 2D: via embedding into $\mathbb{R}^3$:

$$\text{rot } u = \begin{pmatrix} \partial_1 \\ \partial_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}$$

$\Rightarrow \omega_1 = \omega_2 = 0$ (necessary condition on data) $\Rightarrow u_0 = (0, \int \omega_3 dx_1)^\top$

Rest analogously.

Solution of (2.16):

General solution according to Lemma 2.12 (ii):

$$u = \left( \int f dx_1, 0, 0 \right)^\top + \text{rot } A, \quad \forall A \in C^2(G)$$

in 2D:

With $\psi = A_3$.

$$u = \left( \int f dx_1 \right)^\top + \text{rot } \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix} = \begin{pmatrix} \int f dx_1 + \partial_2 \psi \\ -\partial_1 \psi \\ 0 \end{pmatrix}, \quad \forall \psi \in C^2(G).$$

System (2.15), (2.16): Let $f, \omega$ be given with $\text{div } \omega = 0$

Strategy: Decomposition $u = u_q + u_w$ where $u_q$ is divergence-free and $u_w$ curl-free.

**Lemma 2.13.** Let $\text{div } \omega = 0$. Solution $u$ of (2.15), (2.16) has the general form $u = u_q + u_w$, where $u_q := \text{rot } A$, $u_w := \nabla \varphi$, and $A, \varphi$ solve:

$$-\Delta A = \omega \quad \text{and} \quad \text{div } A = 0$$

resp. $\Delta \varphi = f$. 

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Proof. First solve
\[
\begin{cases}
\text{div } u_q = 0 \\
\text{rot } u_q = \omega 
\end{cases}
\] (solvable because \(\text{div } \omega = 0\))

According to Lemma 2.12 (ii): \(u_q = \text{rot } A\)

\(\Rightarrow \omega = \text{rot } u_q = \text{rot } \text{rot } A = \nabla (\text{div } A) - \Delta A\)

Let for example \(\text{div } A = 0\).

\(\Rightarrow -\Delta A = \omega\); is compatible with \(\text{div } A = 0\) because:

\[-\text{div}(\Delta A) = -\Delta(\text{div } A) = 0 = \text{div } \omega.\]

Now solve
\[
\begin{cases}
\text{div } u_w = f \\
\text{rot } u_w = 0
\end{cases}
\]

According to Lemma 2.12 (i): \(u_w = \nabla \varphi\)

\(\Rightarrow \text{div } u_w = \text{div } \nabla \varphi = f\), hence \(\Delta \varphi = f\)

\(\Rightarrow u_0 := u_q + u_w\) is special solution of (2.15), (2.16).

General solution: \(u = u_0 + \nabla \tilde{\varphi}\), \(\forall \tilde{\varphi}\) with \(\Delta \tilde{\varphi} = 0\). \(\blacksquare\)

Remark 2.14. A function \(u = \nabla \varphi\) with \(\Delta \varphi = 0\) is called Laplace field. It is divergence-free and curl-free because

\(\text{rot } u = \text{rot } \nabla \varphi = 0\), \(\text{div } u = \Delta \varphi = 0\).

In fluid dynamics \(u\) describes an incompressible potential flow.

2.3.2 The vorticity equation

Homogeneous, incompressible Euler equation in \(\mathbb{R}^2, \mathbb{R}^3\):

\[
\begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho_0} \nabla p \\
\text{div } u = 0
\end{cases}
\]

(2.18)

We have: \((u \cdot \nabla) u = \frac{1}{2} \nabla |u|^2 - u \times \omega\) with \(\omega := \text{rot } u\)

\(\text{rot } \text{of (2.18)} \Rightarrow \)

\[
\frac{\partial}{\partial t} \text{rot } u + \frac{1}{2} \text{rot}(\nabla |u|^2) - \text{rot}(u \times \omega) = -\frac{1}{\rho_0} \text{rot } \nabla p
\]
2.3 Vorticity models

\[ \text{rot}(u \times \omega) = (\omega \cdot \nabla) u - \omega \text{div } u - (u \cdot \nabla) \omega + u \frac{\text{div } \omega}{=} = 0 \]

\[ = (\omega \cdot \nabla) u - (u \cdot \nabla) \omega \]

\[ \Rightarrow \partial_t \omega - (\omega \cdot \nabla) u + (u \cdot \nabla) \omega = 0 \quad \text{Here we have eliminated } p. \]

\[ \Rightarrow \frac{D\omega}{Dt} = (\omega \cdot \nabla) u \quad \text{with } \text{rot } u = \omega, \text{div } u = 0 \ldots \text{ vorticity equation in } \mathbb{R}^3 \]

\((\omega \cdot \nabla) u\) describes the vortex dilation in 3D (with simultaneous thinning out of the vortex and increase of vortex intensity).

Simplification in 2D:

\[ \omega = (0,0,\partial_1 u_2 - \partial_2 u_1)^T, \nabla = (\partial_1, \partial_2,0) \Rightarrow (\omega \cdot \nabla) u = 0 \]

\[ \Rightarrow \frac{D\omega}{Dt} = 0 \quad \text{resp. } \frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = 0 \ldots \text{vorticity equation in } \mathbb{R}^2 \]

\(\omega\) here is a scalar function! The vorticity equation is nonlinear because \(u = u[\omega]\).

A-priori estimates of \(\omega\):

**Lemma 2.15.** For the vorticity equation in \(D \subset \mathbb{R}^2\) with \(u \cdot \nu = 0\) on \(\partial D\) the following holds:

\[ \Rightarrow \|\omega(\cdot,t)\|_{L^p(D)} = \|\omega_0\|_{L^p(D)}, 1 \leq p \leq \infty, \forall t \geq 0. \quad (2.19) \]

**Proof.** For \(1 \leq p < \infty\) multiply the vorticity equation by \(|\omega|^{p-1} \text{sign}(\omega)\):

\[ (\partial_t |\omega|^{p-1} \text{sign}(\omega) + (u \cdot \nabla |\omega|^{p-1} \text{sign}(\omega) = 0, \]

\[ \Rightarrow \partial_t |\omega|^p + u \cdot \nabla |\omega|^p = 0 \]

\[ \frac{d}{dt} \int_D |\omega|^p dx = - \int_D u \cdot \nabla |\omega|^p dx = - \int_D \text{div}(u|\omega|^p) dx + \int_{\partial D} (\text{div } u)|\omega|^p ds = 0 \]

This gives (2.19) for \(1 \leq p < \infty\). The case \(p = \infty\) follows from \(\|\omega(\cdot,t)\|_{L^\infty(D)} = \lim_{p \to \infty} \|\omega(\cdot,t)\|_{L^p(D)}\).
This is an important estimate for the proof of existence in $\mathbb{R}^2$ (much more difficult in $\mathbb{R}^3$).

**Rekonstruktion of $u$ from $\omega$:**

1st case: Let $D \subset \mathbb{R}^2$ be simply connected and bounded

\[
\begin{align*}
\partial_1 u_2 - \partial_2 u_1 &= \omega, & D \\
\partial_1 u_1 + \partial_2 u_2 &= 0, & D \\
\mathbf{u} \cdot \mathbf{v} &= 0, & \partial D
\end{align*}
\]

(2.20)

\[
\text{div} \, \mathbf{u} = 0 \Rightarrow \exists \mathbf{A} = (0, 0, \psi)^\top : \mathbf{u} = \text{rot} \, \mathbf{A}, \text{i.e.} \, u_1 = \partial_2 \psi, u_2 = -\partial_1 \psi, \text{resp.} \, \mathbf{u} = \nabla^\perp \psi \text{ with } \\
\nabla^\perp := \left(\begin{array}{c}
\partial_2 \\
-\partial_1
\end{array}\right)
\]

**Definition 2.16.** $\psi$ with $\mathbf{u} = \nabla^\top \psi$ is called stream function; for given $\mathbf{u}$ unique up to an additive constant.

**Definition 2.17.** The integral curves $x(s) = \left(\begin{array}{c} x_1(s) \\
x_2(s) \end{array}\right)$, $s \in \mathbb{R}$ of $\mathbf{u}(x, t)$ for $t$ fixed are called stream lines / flow lines. They solve $\frac{d \mathbf{x}}{ds} = \mathbf{u}(x; t)$.

Caution: stream lines $\neq$ particle trajectories (except in stationary flow).

**Interpretation:**

Stream lines are contour lines of $\psi(x, t)$ for $t$ fixed because:

\[
\frac{d}{ds} \psi(x(s); t) = \partial_1 \psi \cdot \dot{x}_1 + \partial_2 \psi \cdot \dot{x}_2 = -u_2 u_1 + u_1 u_2 = 0
\]

Integration of tangential vector field $\mathbf{u}$ along $\partial D$ gives $x(s)$.

Due to $\mathbf{u} \cdot \mathbf{v} = 0$ on $\partial D$: $\partial D$ (with suitable parameterisation) is stream line $\Rightarrow \psi = \text{const}$ on $\partial D$.

Convention: choose the additive constant for $\psi$ such that $\psi = 0$ on $\partial D$.

This way $\psi$ is uniquely determined:
Lemma 2.18. \( u = u[\omega] = \nabla^\perp \psi \) is the unique solution of (2.20), where \( \psi \) solves the potential problem
\[
\begin{cases}
-\Delta \psi = \omega, & D \\
\psi = 0, & \partial D.
\end{cases}
\] (2.21)

Proof. Existence: follows from (2.21) and \( \omega = -\Delta \psi = \text{rot}(\nabla^\perp \psi) \)
\[
\text{div} \ u = \text{div}(\nabla^\perp \psi) = 0 \checkmark
\]
\[
\psi|_{\partial D} = 0 \Rightarrow \nabla \psi \perp \partial D \Rightarrow u = \nabla^\perp \psi \big|_{\partial D} \checkmark
\]

Uniqueness: let \( v := u - \tilde{u} \) be the difference between two solutions, hence
\[
\text{rot} \ v = \text{div} \ v = 0 \text{ in } D; \ v \cdot \nu = 0, \partial D
\]
According to Lemma 2.12 (i): \( v = \nabla \varphi \)
\[
\Rightarrow 0 = \text{div} \ v = \text{div} \nabla \varphi = \Delta \varphi \text{ in } D, \ \nabla \varphi \cdot \nu = 0 \text{ on } \partial D
\]
\[
\Rightarrow \varphi = \text{const} \Rightarrow v = \nabla \varphi \equiv 0 \quad \blacksquare
\]

With Lemma 2.18 the “coefficient function” \( u[\omega] \) in the 2D vorticity equation
\[
\frac{\partial \omega}{\partial t} + u[\omega] \cdot \nabla \omega = 0
\]
is defined. For proof of well-posedness of this evolution problem the a-priori estimate (2.19) is essential (see §2.3 in [MP]; §3.2.3, 3.3, 4.2 in [MB]).

Representation of \( \omega \) from (2.21):

Theorem 2.19.
\[
\psi(x) = \int_D G_D(x, x') \omega(x') dx';
\]
the Green’s function \( G_D \) solves
\[
\begin{align*}
\Delta_x G_D(x, x') &= -\delta(x - x') \text{ in } D, \\
G_D(x, x') &= 0 \quad \forall x \in \partial D \text{ or } x' \in \partial D.
\end{align*}
\]

We have:
\[
G_D(x, x') = G(x, x') + \gamma(x, x') \text{ with } G(x, x') = -\frac{1}{2\pi} \log |x - x'|, \ \Delta_x \gamma = \Delta_x \omega \gamma = 0, + \ BC \text{ for } \gamma.
\]
\[
\Rightarrow u(x) = \nabla^\perp \psi(x) = \int_D \nabla^\perp G_D(x, x') \omega(x') dx'.
\]
Remark 2.20. Under which condition does one obtain a stationary flow?

We have

\[ u \cdot \nabla \omega = u_1 \partial_1 \omega + u_2 \partial_2 \omega = \partial_2 \psi \cdot \partial_1 \omega - \partial_1 \psi \cdot \partial_2 \omega \]

\[ = \det \begin{pmatrix} \partial_1 \omega & \partial_2 \omega \\ \partial_1 \psi & \partial_2 \psi \end{pmatrix} =: \det J(\omega, \psi) \]

Hence:

\[ \frac{\partial \omega}{\partial t} = 0 \iff \det J(\omega, \psi) = 0 \quad \forall x \in D. \]

Then \( \omega(x), \psi(x) \) are (functionally) dependent, i.e., \( \omega = f(\psi) \) or \( \psi = g(\omega) \).

2nd case: \( D = \mathbb{R}^2 \)

Therefore solve:

\[
\begin{cases}
\text{div } u = 0 \\
\text{rot } u = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}
\end{cases}
\] (2.22)

Analogously to Lemma 2.18 \( u \) can be determined from \( u = \nabla^\perp \psi \) and \(-\Delta \psi = \omega \) in \( \mathbb{R}^2 \).

Special solution can be given by means of Green's function for the Poisson equation in \( \mathbb{R}^2 \),

\[ G(x, x') = -\frac{1}{2\pi} \log |x - x'|; x, x' \in \mathbb{R}^2 \]

\[ \psi(x) = \int_{\mathbb{R}^2} G(x, x') \omega(x') dx', \]

\[ u_0(x) = \nabla^\perp \psi(x) = \int_{\mathbb{R}^2} K(x - x') \omega(x') dx' \] (2.23)

with

\[ K(x - x') = -\frac{1}{2\pi} \frac{(x - x')^\perp}{|x - x'|^2}; \quad x^\perp = \left( \begin{array}{c} x_2 \\ -x_1 \end{array} \right) \text{ for } x = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right). \]
Remark for analysis regarding existence of $u$: The *generalised Young inequality*

$$\|u\|_{L^r(\mathbb{R}^2)} \leq C\|\omega\|_{L^p(\mathbb{R}^2)}; 1 < p < 2, 2 < r < \infty \text{ with } \frac{1}{r} = \frac{1}{p} - \frac{1}{2}; \frac{1}{|x|} \in L_w^2(\mathbb{R}^2).$$

holds.

**Interpretation of $K(x - x')$:**

Let $\omega(x) = \delta(x - x'), x' \in \mathbb{R}^2$ be given.

$\Rightarrow K(x - x') = \text{velocity vector field } u(x), \text{“produced” by unit vortex } \omega(x) = \delta(x - x') \text{ at } x'$:

**General solution of (2.22) (cf. Rem. 2.14):**

$$u = \nabla \perp \psi + \nabla \varphi \quad \text{with } \Delta \varphi = 0 \text{ in } \mathbb{R}^2$$


Laplace field

Without “boundary condition at infinity” the solution is not unique.

Possible boundary conditions:

$$u(x) \xrightarrow{|x|\to\infty} u_\infty (= \text{const}) \quad (\text{i.e., uniform flow at infinity})$$

(2.24)

$\Rightarrow$ unique solution of (2.22), (2.24) (e.g. for $\omega$ with compact support):

$$u = \nabla \perp \psi + u_\infty$$

References: [MP] §1.2

**2.3.3 Motion of point vortices in $\mathbb{R}^2$**

We first consider the vorticity equation in $D \subset \mathbb{R}^2$ simply connected and bounded:

$$\omega_t = -u \cdot \nabla \omega \quad ,$$

(2.25)

and $u$ satisfies: $\text{div } u = 0 \text{ in } D, u \cdot \nu = 0 \text{ on } \partial D.$
Aim: Reduce the PDE “vorticity equation” to a system of ODEs.

Consider the initial condition (linear combination of point vortices):

$$\omega_0(x) := \sum_{i=1}^{N} a_i \delta(x - x_i), \quad x \in D \subset \mathbb{R}^2$$  \hspace{1cm} (2.26)

with given positions $x_i \in D \subset \mathbb{R}^2$ and intensities $a_i \in \mathbb{R}$.

In euler equations the conservation of $N$ point vortices for $t > 0$ is plausible because the model contains no diffusion/viscosity.

Problems: a distributional formulation of PDE (2.25) is “delicate” because already the coefficient function $u[\omega_0]$ is singular at $x_i$, hence is not usable in weak formulation. ⇒

regularisation needed:

The following step function approximates $\omega_0$:

$$\omega_\varepsilon^0(x) := \frac{1}{\varepsilon^2 \pi} \sum_{i=1}^{N} a_i \chi_{K(x_i, \varepsilon)}(x)$$

We have $\chi_{K(x_i, \varepsilon)} \xrightarrow{\varepsilon \to 0} \delta(x - x_i)$ in $\mathcal{D}'(D)$.

For the reformulation we consider $\forall f \in C^1(\bar{D})$ (and sufficiently smooth $u, \omega$):

$$\frac{d}{dt} \int_D \omega f dx = - \int_D (u \cdot \nabla \omega) f dx \overset{\text{div } u = 0}{=} - \int_D \text{div}(fu\omega) dx + \int_D \omega u \cdot \nabla f dx$$

$$\overset{\text{div.thm}}{=} - \int_{\partial D} f \omega u \cdot \nu ds + \int_D \omega u \cdot \nabla f dx.$$

⇒ this motivates the weak formulation of the vorticity equation:

look for $\omega \in C^1([0,T], L^1(D))$ with

$$\frac{d}{dt} \langle \omega(t), f \rangle = \langle \omega(t), u(t) \cdot \nabla f \rangle \quad \forall f \in C^1(\bar{D}), \omega(t = 0) = \omega_0,$$  \hspace{1cm} (2.27)

and $\langle f, g \rangle := \int_D f(y)g(y)dy$. Same form for $D = \mathbb{R}^2$.

Properties of solution $\omega^\varepsilon(t)$ of (2.27) in $D = \mathbb{R}^2$ with IC $\omega_0^\varepsilon$:

**Theorem 2.21.** For $D = \mathbb{R}^2$ we have:

$$\lim_{\varepsilon \to 0} \langle \omega^\varepsilon(t), f \rangle = \sum_{i=1}^{N} a_i f(x_i(t)) = \langle \omega(t), f \rangle, \quad \forall f \in C^1(\mathbb{R}^2),$$

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with \( \omega(x,t) = \sum_{i=1}^{N} a_i \delta(x - x_i(t)) \).

\( x_i(t) \) solves the ODE ("discrete vorticity model"):

\[
\begin{align*}
\frac{d}{dt} x_i(t) &= u(x_i(t), t) = \sum_{j \neq i} K(x_i(t) - x_j(t)) a_j \\
x_i(0) &= x_i; \quad i = 1, \ldots, N.
\end{align*}
\]  

(2.28)

\(^{(*)} \) ... velocity field of the "other" vortices ⇒ one (single) vortex is stationary.

**Proof.** [MP] Th. 4.2.3.

**Remark 2.22.** Is \( \omega(x,t) \) (measure valued) solution of (2.27) with IC \( \omega_0 \) from (2.26)?

Almost, because \( \omega \) solves

\[
\frac{d}{dt} \langle \omega(t), f \rangle = \langle \omega(t), u_r \cdot \nabla f \rangle \quad \forall f \in C^1(\mathbb{R}^2)
\]  

(2.29)

with the "regularised velocity"

\[
\begin{align*}
u_r(x,t) &:= \int_{\mathbb{R}^2} \nabla \perp G(x,x') \chi_{\{x \neq x'\}} \omega(x',t) \, dx' \\
&= 0 \text{ for } x = x'
\end{align*}
\]  

(2.30)

(in mathematically sloppy notation).

\( \chi_{\{x \neq x'\}} \) prevents the "self interaction" of the point vortices; this is used now as additional physical assumption. (2.30) with singularities of the integral kernel at positions of the deltas would not even be defined. That one single point vortex has to be stationary is also seen from the fact that in this case there is no distinguished direction.

Source of problem: weak formulation (2.27) with velocity field \( u \) from (2.23) is not defined for distributional solution.

Solution \( \omega(t) \) with \( x_i(t) \) solves the PDE (2.29)-(2.30) by reduction to a system of ODEs (2.28).

**Vorticity model as Hamiltonian system:**

(2.28) ist is equivalent to

\[
\begin{align*}
a_i \frac{d}{dt} x_i^1 &= \frac{\partial}{\partial x_i^2} H, \quad i = 1, \ldots, N, \\
a_i \frac{d}{dt} x_i^2 &= -\frac{\partial}{\partial x_i^1} H,
\end{align*}
\]  

(2.31)

with Hamiltonian ("energy")

\[
H := -\frac{1}{4\pi} \sum_{j \neq i} a_i a_j \ln |x_i - x_j|;
\]
2 Fluid mechanics

Notation \( x_i = (x_i^1, x_i^2) \)\(^\top\).

Compare to Hamilton’s equations of point mechanics:

a particle with mass \( m \), kinetic energy \( \frac{p^2}{2m} \), momentum \( p = mu \) and potential energy \( V(x) \).

With Hamiltonian \( H(x, p) := \frac{p^2}{2m} + V(x) \) we have:

\[
\begin{align*}
\frac{d}{dt} x &= \frac{\partial}{\partial p} H = \frac{p}{m} = u, \\
\frac{d}{dt} p &= -\frac{\partial}{\partial x} H = -\frac{dV}{dx} = F \ldots \text{Newton’s second law.}
\end{align*}
\]

Hamiltonian sytems always have the following property:

“Energie” is constant in time, hence: \( H(t) = \text{const} \ \forall t \).

Moreover:

Center of vortex \( B(t) := \sum_{i=1}^{N} a_i x_i(t) \)\( \sum_{i=1}^{N} a_i \) is constant in time (for \( \sum_{i=1}^{N} a_i \neq 0 \) \( \Rightarrow M(t) := \sum_{i=1}^{N} a_i x_i(t) = \text{const} \).

Inertia is constant in time, i.e.: \( I(t) := \sum_{i=1}^{N} a_i |x_i(t)|^2 = \text{const} \)

Hence: 4 (scalar) first integrals of the motion \( \Rightarrow \) ODE system (2.31) for max. 3 point vortices \( (N \leq 3) \) is explicitly solvable (because a Hamiltonian system in \( \mathbb{R}^{2N} \) with \( N + 1 \) Poisson-commuting conserves quantities is completely integrable [V.I. Arnold, Dynamical Systems III]).

Example 2.23. \( N=2 \):

\[
H = -\frac{1}{2\pi} a_1 a_2 \ln |x_1 - x_2| = \text{const} \ \Rightarrow \ |x_1(t) - x_2(t)| = \text{const}
\]

Vortex center \( \ldots B := \frac{M}{a_1 + a_2} = \text{const} \) and is on line connecting \( x_1(t) \) and \( x_2(t) \).

1st case: \( \text{sign } a_1 = \text{sign } a_2 \)
2.3 Vorticity models

2nd case: \( \text{sign } a_1 \neq \text{sign } a_2 \) and \(|a_1| \neq |a_2|\)

3rd case: \( a_1 = -a_2 \)

Question: Does system (2.28) have a global (in time) solution? 2 possible Problems:

a) \( |x_i(t) - x_j(t)| \to 0 \) for \( t \to T^* \), which means 2 “particles” at one place and the right hand side of (2.28) is not well-defined anymore.

b) \( |x_i| \to \infty \) for \( t \to T^* \).

Solution: Global solvability depends on \( \{\text{sign } a_i\} \).
2 Fluid mechanics

Theorem 2.24. Let \( \text{sign} a_i = \text{sign} a_1 \neq 0, \forall i = 2, \ldots, N \) be solution of (2.28) \( \exists \) for \( 0 \leq t < \infty \).

Proof. 1st claim: system stays in finite region, i.e., \( |x_i(t)| \leq \text{const} \forall t \), because:

\[
|x_i(t)|^2 \leq \frac{1}{|a_i|} \sum_j |a_j||x_j|^2 = \frac{|I(t)|}{|a_i|} = \text{const} \quad \checkmark
\] (2.32)

2nd claim: all pairs \( k \neq l \) have fixed minimal distance \( |x_k(t) - x_l(t)| \), i.e., velocity is always finite \( \forall t < \infty \), because:

\[
- a_k a_l \ln |x_k(t) - x_l(t)| = 4\pi H(t) + \sum_{i,j \neq (k,l)} a_i a_j \ln |x_i(t) - x_j(t)| \leq |x_i| + |x_j|
\]

\[
\overset{(2.32)}{\leq} 4\pi H(t) + \sum_{i,j \neq (k,l)} a_i a_j \ln \left( \sqrt{\frac{|I|}{|a_i|}} + \sqrt{\frac{|I|}{|a_j|}} \right) =: C = \text{const} \quad \forall t.
\]

\[
\Rightarrow |x_k(t) - x_l(t)| \geq \exp \left( -\frac{C}{a_k a_l} \right) > 0 \quad \forall t \checkmark
\]

For different signs of \( a_i \) and \( N \geq 3 \) a “collapse” (i.e. \( x_i(T^*) = x_j(T^*) \)) is possible in finite time.

Example 2.25. \( N = 3, a_1 = a_2 = 2, a_3 = -1; \)
\( x_1 = (-1,0)^\top, x_2 = (1,0)^\top, x_3 = (1,\sqrt{2})^\top \)
\( B = \left( -\frac{1}{3}, -\frac{\sqrt{2}}{3} \right)^\top \) ... center of vortex,
self-similar evolution, collapse at \( T^* = 3\sqrt{2}\pi \).
After collapse: continues as (stationary) 1-vortex-flow; is not time-reversible!

Due to 2.24 the system stays in finite region for \( \text{sign } a_i = \text{sign } a_1 \); velocities also stay finite.

Generalisation:

**Theorem 2.26.** Suppose:

\[
\forall J \subset \{1, \ldots, N\} : \sum_{i \in J} a_i \neq 0. \tag{2.33}
\]

Then \( \forall R > 0, T > 0: \exists \tilde{R} = \tilde{R}(a_i, R, N, T) \) (independently of \( x_1, \ldots, x_N \)) with

\[
x_1, \ldots, x_N \in K_R(0) \implies x_i(t) \in K_{\tilde{R}}(0) \quad \forall i = 1, \ldots, N; \forall 0 \leq t \leq T
\]

(if trajectory exists up to that time).

*Proof. Corollary 4.2.1 [MP]*
Rem: Condition (2.33) is necessary – see Ex. 2.23, 3rd case with $x_1 \to x_2$.

Using this one can show:

**Theorem 2.27.** Suppose that $\forall J \subset \{1, \ldots, N\}$: $\sum_{i \in J} a_i \neq 0, N \geq 3$.

$\Rightarrow$ for almost all initial conditions $(x_1, \ldots, x_N) \in \mathbb{R}^{2N}$:

$\exists$ global solution $(x_1(t), \ldots, x_N(t))$ of (2.28); i.e.: let $A \subset \mathbb{R}^{2N}$ be bounded and $B \subset A$ the set of initial conditions which lead to a collapse in finite time. Then:

$\mu(B) = 0$.

*Proof. Th. 4.2.2 [MP]*

**Example 2.28** (von Karman street).

- $\infty$ many vortices of intensity $\pm a$

System is subject to rigid translation with constant $v, \forall t > 0$.

- Application: Flow around rigid body $\Rightarrow$ viscosity (only important near surface) produces contra-rotating vortices, then: transport of vortices by Euler flow for (quite) long time

- vortex street for suitable $a, b, h, l$ linearly stable.

*References: [MP] §4.1-3*
2.4 Boundary layers for Navier-Stokes equations

Consider incompressible, homogeneous (scaled) Navier-Stokes equations with no-slip boundary conditions:

\[
\begin{align*}
\begin{cases}
    u_t + (u \cdot \nabla) u + \nabla p &= \frac{1}{Re} \Delta u, &\Omega \\
    \text{div} u &= 0, &\Omega \\
    u &= 0, &\partial \Omega \\
    u(x, 0) &= u_0(x), &\Omega
\end{cases}
\end{align*}
\]

(2.34)

In “interior” of $\Omega$: friction term $\nu_0 \Delta u$ often negligible in contrast to convective term $(u \cdot \nabla) u \to$ Euler equations (easier to solve).

In Boundary layer at $\partial \Omega$: friction term essential, because $u$ “small” and influence of boundary conditions.

Aim: coupling of Euler equations in interior of $\Omega$ with boundary layer equations.

Model problem for Method of asymptotic expansion:

\[
\begin{align*}
\begin{cases}
    \varepsilon y'' + 2y' + 2y &= 0, & 0 < x < 1, \quad \varepsilon \ll 1, \\
    y(0) = 0, & y(1) = 1
\end{cases}
\end{align*}
\]

(2.35)

Exact solution: $y_\varepsilon(x) = \frac{1}{\varepsilon \lambda_1 - \varepsilon \lambda_2} \left( e^{\lambda_1 x} - e^{\lambda_2 x} \right) \approx e \left( e^{-x} - e^{-\frac{2x}{\varepsilon}} \right) e^{-\frac{2x}{\varepsilon}} e^{1-x},$

\[
\lambda_1 = \frac{-1 + \sqrt{1 - 2\varepsilon}}{\varepsilon} \approx -1, \quad \lambda_2 = \frac{-1 - \sqrt{1 - 2\varepsilon}}{\varepsilon} \approx -\frac{2}{\varepsilon} + 1
\]

$y_0 = e^{1-x}$ solves reduced equation

\[
\begin{align*}
\begin{cases}
    2y' + 2y &= 0, \\
    y(1) &= 1;
\end{cases}
\end{align*}
\]

(2.36)

is for $x \gg \varepsilon$ good approximation for $y_\varepsilon$, but not for $x \approx 0$. 

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Idea of asymptotic expansion: approximation of solution of (2.35), separately on $(0, \delta(\varepsilon))$ and $(\delta(\varepsilon), 1)$; here $\delta(\varepsilon) = O(\varepsilon)$.

**Step 1 (outer expansion):**

Formal ansatz for solution on $(\delta(\varepsilon), 1)$:

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \ldots$$

Rem.: Convergence of this “series” does not matter as it is always truncated after a few terms.

Plug into (2.35) $\Rightarrow$

$$\varepsilon^0(2y''_0 + 2y_0) + \varepsilon^1(y''_1 + 2y'_1 + 2y_1) + \varepsilon^2(y''_2 + 2y'_2 + 2y_2) + \cdots = 0$$

$$\varepsilon^0 y_0(1) + \varepsilon^1 y_1(1) + \varepsilon^2 y_2(1) + \cdots = 1$$

Equating the coefficients suggests:

\begin{align*}
2y''_0 + 2y_0 &= 0, \quad y_0(1) = 1 \\
y''_1 + 2y'_1 + 2y_1 &= 0, \quad y_1(1) = 0 \\
y''_2 + 2y'_2 + 2y_2 &= 0, \quad y_2(1) = 0 \\
&\vdots
\end{align*}

$\Rightarrow y_0(x) = e^{1-x}, \ldots$

**Step 2 (inner expansion):**

The inner expansion should approximate the solution on $(0, \delta(\varepsilon))$.

Let $\xi := \frac{x}{\varepsilon}$ (rapid variable), $Y(\xi) := y(\varepsilon \xi)$.

$Y(\xi)$ satisfies

$$\frac{1}{\varepsilon} y'' + \frac{2}{\varepsilon} y' + 2Y = 0, \quad Y(0) = 0.$$  \hspace{1cm} (2.38)

Expansion ansatz:

$$Y(\xi) = Y_0(\xi) + \varepsilon Y_1(\xi) + \varepsilon^2 Y_2(\xi) + \ldots$$

Plug into (2.38) $\Rightarrow$

$$\varepsilon^{-1}[Y''_0 + 2Y'_0] + \varepsilon^0[Y''_1 + 2Y'_1 + 2Y_0] + \varepsilon[Y''_2 + 2Y'_2 + 2Y_1] + \cdots = 0$$

Equating coefficients suggests:

\begin{align*}
Y''_0 + 2Y'_0 &= 0, \quad Y_0(0) = 0 \\
Y''_1 + 2Y'_1 + 2Y_0 &= 0, \quad Y_1(0) = 0 \\
Y''_2 + 2Y'_2 + 2Y_1 &= 0, \quad Y_2(0) = 0
\end{align*} \hspace{1cm} (2.39)
\[ Y_0(\xi) = a(1 - e^{-2\xi}) \] for some \( a \in \mathbb{R} \).

Step 3 (matching):
Compatibility condition for \( y_0 \) and \( Y_0 \) for \( \varepsilon \to 0 \) (\( Y_0 \) gives boundary layer transition between boundary condition at \( x = 0 \) and \( y_0(\delta(\varepsilon)) \); for \( \varepsilon > 0 \) it is still discontinuous):
\[
\lim_{\xi \to \infty} Y_0(\xi) \overset{1}{\Rightarrow} \lim_{x \to 0} y_0(x) \Rightarrow a = e.
\]

Step 4 (composite solution):
\[
\tilde{y}(x) := \begin{cases} 
Y_0(\frac{x}{\varepsilon}), & x \in (0, \delta(\varepsilon)) \\
y_0(x), & x \in (\delta(\varepsilon), 1)
\end{cases}
\]
is discontinuous and not an approximation of order \( O(\varepsilon) \) to the exact solution \( y \) (compare for \( \delta(\varepsilon) = \varepsilon : y_0(\varepsilon) = e^{1-\varepsilon}, y_\varepsilon(\varepsilon) \)).

Step 5 (uniform approximation):
\[
\hat{y}(x) := Y_0\left(\frac{x}{\varepsilon}\right) + y_0(x) - \lim_{x \to 0} y_0(x) = \cdots = e\left(e^{-x} - e^{-\frac{2x}{\varepsilon}}\right)
\]
cf. Taylor expansion of \( y_\varepsilon \)
is uniform approximation \( w.r.t. x \in [0,1] \) of order \( O(\varepsilon) \) (follows from Taylor expansion of \( y_\varepsilon \)).

Rem.: In \( \tilde{y} \) the sum of the last two terms vanishes for \( x \to 0 \), as well as the first and third term for \( x \to \infty \). For small and large \( x \) one thus obtains \( \tilde{y}(x) \approx \hat{y}(x) \).

Remark 2.29.
1) In the outer expansion \( \varepsilon y'' \) plays no role, but in the inner expansion - because of rescaling to \( \xi = \frac{x}{\varepsilon} \).
2) The further expansion terms \( y_1(x), Y_1(\xi) \) can be calculated from the inhomogeneous ODEs in (2.37) resp. (2.39).
3) General inner expansion with \( \xi := \frac{x}{\varepsilon} \) and
\[
Y(\xi) = Y_0(\xi) + \varepsilon^\beta Y_1(\xi) + \varepsilon^{2\beta} Y_2(\xi) + \ldots
\]
Plausible values for \( \alpha > 0, \beta > 0 \) can be found by inserting into ODE and balancing dominant \( \varepsilon \)-terms (i.e. smallest \( \varepsilon \)-exponents). Aim: as many such terms as possible.
Prandtl’s boundary layer equations (1904)

Consider 2D Navier-Stokes equations on flat plane, \( u = (v, w) \in \mathbb{R}^2; (x, y) \in \Omega := \mathbb{R} \times \mathbb{R}^+ \); let \( \varepsilon = \frac{1}{Re} \ll 1 \) (but fixed):

\[
\begin{align*}
\partial_t v + v \partial_x v + w \partial_y v + \partial_x p &= \varepsilon \Delta v \\
\partial_t w + v \partial_x w + w \partial_y w + \partial_y p &= \varepsilon \Delta w \\
\partial_x v + \partial_y w &= 0 \\
v|_{y=0} = w|_{y=0} &= 0 \\
v(0, x, y) = v_I(x, y); w(0, x, y) = w_I(x, y)
\end{align*}
\]

(2.40)

Step 1 (outer expansion): Ansatz (“away from \( \{y = 0\}\)“):

\[
\begin{align*}
v &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots \\
w &= w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots \\
p &= p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots
\end{align*}
\]

Plugging into (2.40) gives in lowest order (\( \varepsilon^0 \)) the Euler equations:

\[
\begin{align*}
\partial_t v_0 + v_0 \partial_x v_0 + \partial_x v_0 + \partial_y v_0 &= 0 \\
\partial_t w_0 + v_0 \partial_x w_0 + \partial_x w_0 + \partial_y w_0 &= 0 \\
\partial_x v_0 + \partial_y w_0 &= 0 \\
v_0(0, x, y) = v_I(x, y); w_0(0, x, y) = w_I(x, y)
\end{align*}
\]

(2.41)

(but no BC at \( y = 0 \))

Step 2 (inner Expansion):

We expect large changes of the solution in \( y \)-direction, but not in \( x \)-direction \( \Rightarrow \) Scaling ansatz near \( \{ y = 0 \} \):

\[
T := t, X := x, Y := \frac{y}{\varepsilon^\alpha}, \text{ with to be determined } \alpha > 0;
\]

\[
\begin{align*}
V(T, X, Y) &:= v(t, x, \varepsilon^\alpha Y), \\
W(T, X, Y) &:= w(t, x, \varepsilon^\alpha Y), \\
P(T, X, Y) &:= p(t, x, \varepsilon^\alpha Y).
\end{align*}
\]

Einsetzen in (2.40):

\[
\begin{align*}
\partial_t V + V \partial_X V + \varepsilon^{-\alpha} W \partial_Y V + \partial_X P &= \varepsilon \partial_X^2 V + \varepsilon^{1-2\alpha} \partial_Y^2 V \\
\partial_t W + V \partial_X W + \varepsilon^{-\alpha} W \partial_Y W + \varepsilon^{-\alpha} \partial_Y P &= \varepsilon \partial_X^2 W + \varepsilon^{1-2\alpha} \partial_Y^2 W \\
\partial_X V + \varepsilon^{-\alpha} \partial_Y W &= 0 \\
V|_{Y=0} = W|_{Y=0} &= 0
\end{align*}
\]

(2.42)
Expansion approach for $V$, $W$, $P$:

\[
\begin{align*}
V &= V_0 + \varepsilon \beta V_1 + \varepsilon^2 \beta V_2 + \ldots \\
W &= W_0 + \varepsilon \beta W_1 + \varepsilon^2 \beta W_2 + \ldots \\
P &= P_0 + \varepsilon \beta P_1 + \varepsilon^2 \beta P_2 + \ldots
\end{align*}
\] (2.43)

with $\beta > 0$ to be determined.

Plugging into 3rd equation of (2.42) \Rightarrow

\[
\left[ \partial X V_0 + \varepsilon \beta \partial X V_1 + \varepsilon^2 \beta \partial X V_2 + \ldots \right] + \varepsilon^{-\alpha} \left[ \partial Y W_0 + \varepsilon \beta \partial Y W_1 + \varepsilon^2 \beta \partial Y W_2 + \ldots \right] = 0
\]

Leading $\varepsilon$-power is $\partial Y W_0$; this suggests:

\[
\partial Y W_0 = 0, \quad W_0(T, X, 0) = 0, \quad \forall T, X \Rightarrow \quad W_0 \equiv 0.
\]

Hence the vertical velocity in the boundary is of order at most $O(\varepsilon^\beta)$.

Balance of next higher $\varepsilon$-power suggests $\alpha = \beta$, hence

\[
\left[ \partial X V_0 + \partial Y W_1 = 0. \right]
\]

Inserting (2.43) into 1st equation of (2.42) \Rightarrow

\[
\begin{align*}
\left[ \partial T V_0 + \varepsilon \partial T V_1 + \ldots \right] &+ \left[ V_0 + \varepsilon \partial T V_1 + \ldots \right] \cdot \left[ \partial X V_0 + \varepsilon \partial X V_1 + \ldots \right] \\
&+ \varepsilon^{-\alpha} \left[ 0 + \varepsilon \partial Y W_1 + \ldots \right] \cdot \left[ \partial Y V_0 + \varepsilon \partial Y V_1 + \ldots \right] + \left[ \partial X P_0 + \varepsilon \partial X P_1 + \ldots \right] \\
&= \varepsilon \left[ \partial X^2 V_0 + \varepsilon \partial X^2 V_1 + \ldots \right] + \varepsilon^{1-2\alpha} \left[ \partial X^2 P_0 + \varepsilon \partial X^2 P_1 + \ldots \right].
\end{align*}
\]

If $1 - 2\alpha < 0$, there was only one leading term: $\partial T^2 V_0 = 0$.

The choice $1 - 2\alpha = 0$ gives the maximal number of leading terms:

\[
\left[ \partial T V_0 + V_0 \partial X V_0 + W_1 \partial Y V_0 + \partial X P_0 = \partial T^2 V_0, \right.
\]

and $\alpha = \beta = \frac{1}{2}$ gives the boundary layer thickness $\delta(\varepsilon) = O(\varepsilon^{\frac{1}{2}})$.

Inserting (2.43) into 2nd equation of (2.42) \Rightarrow

\[
\begin{align*}
\left[ \partial T W_0 + \varepsilon \frac{1}{2} \partial T W_1 + \ldots \right] &+ \left[ V_0 + \varepsilon \frac{1}{2} V_1 + \ldots \right] \cdot \left[ 0 + \varepsilon \frac{1}{2} \partial X W_1 + \ldots \right] \\
&+ \varepsilon^{-\frac{1}{2}} \left[ 0 + \varepsilon \frac{1}{2} W_1 + \ldots \right] \cdot \left[ 0 + \varepsilon \frac{1}{2} \partial Y W_1 + \ldots \right] + \varepsilon^{-\frac{1}{2}} \left[ \partial T P_0 + \varepsilon \frac{1}{2} \partial T P_1 + \ldots \right] \\
&= \varepsilon \left[ 0 + \varepsilon \frac{1}{2} \partial X^2 W_1 + \ldots \right] + \left[ 0 + \varepsilon \frac{1}{2} \partial X^2 W_1 + \ldots \right].
\end{align*}
\]

For the leading order we have \( \partial T P_0 = 0. \)

Step 3 (matching):
2 Fluid mechanics

Conditions:

\[
\lim_{Y \to \infty} V_0(T, X, Y) = \lim_{y \to 0} v_0(t, x, y)
\]
\[
\lim_{Y \to \infty} W_0(T, X, Y) = \lim_{y \to 0} w_0(t, x, y)
\]
\[
\lim_{Y \to \infty} P_0(T, X, Y) = \lim_{y \to 0} p_0(t, x, y)
\]

2nd row and \( W_0 \equiv 0 \) lead to: \( w_0(t, x, 0) = 0 \) (typical Euler BC \( u_0 \cdot \nu = 0 \)).

Solution step 1: Solve the euler equations (2.41) for \( v_0, w_0, p_0 \) with \( w_0(t, x, 0) = 0 \) in exterior region (for \( y > 0 \)).

The 3rd row of the coupling conditions and \( \partial_Y P_0 = 0 \) give \( P_0(T, X) = p_0(t, x, 0), \forall T = t, X = x \).

Hence: pressure in boundary layer = pressure of outer flow at boundary (\( y = 0 \)).

Solution step 2: Using the functions \( p_0|_{y=0}, v_0|_{y=0} \) which are known from the outer flow, solve the Prandtl boundary layer equations in the boundary layer. (for \( V_0, W_1; X \in \mathbb{R}, 0 < Y < \infty \)):

\[
\begin{cases}
\partial_T V_0 + V_0 \partial_X V_0 + W_1 \partial_Y V_0 + \partial_X (p_0|_{y=0}) = \partial^2_X V_0, \\
V_0|_{Y=0} = W_1|_{Y=0} = 0, \text{ (from (2.42))} \\
\lim_{Y \to \infty} V_0(T, X, Y) = v_0(t, x, 0), \\
\partial_X V_0 + \partial_Y W_1 = 0, \\
V_0(0, X, Y) = u_I(x, 0), Y > 0, \text{ if } u_I \text{ contains no boundary layer.}
\end{cases}
\]

This is a degenerate parabolic equation for \( V_0 \) (the term \( \partial^2_X V_0 \) is missing), wherein \( V_0 \) and \( W_1 \) are coupled by a linear equation of first order.

Combined approximation:

\[
\hat{v}(t, x, y) = V_0 \left( t, x, \frac{y}{\varepsilon^2} \right) + v_0(t, x, y) - v_0(t, x, 0),
\]
\[
\hat{w}(t, x, y) = w_0(t, x, y), \quad \text{(no correction of order } O(\varepsilon^0) \text{ because } W_0 \equiv 0),
\]
\[
\hat{p}(t, x, y) = p_0(t, x, y), \quad \text{(because pressure=const in } Y \text{ in boundary layer}).
\]

Result: in a boundary layer of vertical thickness \( O(\sqrt{\varepsilon}) \) the horizontal component \( v_0 \) of the velocity is corrected such that at \( y = 0 \) the no-slip condition \( u = 0 \) is satisfied. The vertical component of the velocity already satisfies \( w_0(t, x, 0) = 0 \) because it solves the Euler equations and hence does not need to be corrected.

References: [EGK] §6.6
3 Theory of elasticity

Aim: Model how a body deforms subject to external forces.

3.1 Notation

- \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \): Reference configuration = region occupied by the body when no forces are applied
- \( x \in \Omega \): particle
- \( \Phi : \Omega \to \mathbb{R}^d \): deformation field. The particle \( x \) is moved by the deformation to \( \Phi(x) \) (description in Lagrange coordinates; \( \Phi \) does not have to be volume preserving)
- \( \frac{\partial \Phi}{\partial x} \in \mathbb{R}^{d \times d} \): deformation gradient. We only consider orientation preserving deformations, i.e., such that \( \det \frac{\partial \Phi}{\partial x} > 0 \) (i.e. no reflections)
- \( u(x) := \Phi(x) - x \): displacement field

We now consider the relative change of length effected by \( \Phi \). Let \( \Delta x \in \mathbb{R}^d \) be a small displacement →

\[
\frac{\|\Phi(x + \Delta x) - \Phi(x)\|^2}{\|\Delta x\|^2} = \frac{\|\frac{\partial \Phi}{\partial x}(x) \cdot \Delta x + \mathcal{O}(\|\Delta x\|^2)\|^2}{\|\Delta x\|^2} = \frac{(\Delta x)^\top \cdot (\frac{\partial \Phi}{\partial x}(x))^\top \cdot \frac{\partial \Phi}{\partial x}(x) \cdot \Delta x}{\|\Delta x\|^2} + \mathcal{O}(\|\Delta x\|)
\]

Definition 3.1. The symmetric matrix

\[
C := \frac{\partial \Phi}{\partial x} \cdot \frac{\partial \Phi}{\partial x} = \left( \frac{\partial u}{\partial x} + I \right)^\top \cdot \left( \frac{\partial u}{\partial x} + I \right)
\]

(3.1)
3 Theory of elasticity

is called Cauchy-Green strain tensor and describes the local relative change of length in the body.

We have

\[ C = I \iff \exists Q \in O(d) ; \ b \in \mathbb{R}^d : \ \Phi(x) = Q \cdot x + b \]

(hence for rigid body movements there is no change of length).

**Definition 3.2.** The symmetric matrix

\[ E := \frac{1}{2}(C - I) = \frac{1}{2} \left( \frac{\partial u}{\partial x}^\top + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial x} \right) \]

is called Green strain tensor and vanishes for such rigid body movements (and is quadratic in \( u \)).

The matrix

\[ \epsilon := \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}^\top \right) \approx E \quad (\text{for small variations of the displacement}) \]

is called linear strain tensor.

### 3.2 Hyperelastic materials

A body is deformed by some force. The work done is saved as deformation energy. An elastic body completely returns this energy if the applied force is removed. A material is called hyperelastic if the deformation energy depends pointwise on the Cauchy-Green strain tensor \( C \):

\[ E_{\text{def}} = \int_{\Omega} W(C(x))dx, \]

with energy density \( W : \{ A \in \mathbb{R}^{d \times d} \mid A = A^\top \} \to \mathbb{R} \)

(resp. general \( W(x, C) \) for inhomogeneous materials).

**Ex.:** Rubber (isotropic; linear elasticity would be too inaccurate)

We only consider isotropic materials, i.e., material properties are the same in all directions \( \Rightarrow W \) is invariant under (rigid body) rotations \( \Rightarrow W(E) := W(I + 2E) \) depends only on \( \text{Sp} E, \text{Sp}(E^2) \) (for \( d = 2 \)) and for \( d = 3 \) additionally on \( \det E \). Derivation analogously to the proof of the form of the viscous stress tensor \( \sigma \left( \frac{\partial u}{\partial t} \right) \) in §2.2: \( E \) is symmetric, hence diagonalisable \( \Rightarrow W \) only depends on its eigenvalues.
3.3 Variational formulation

Lemma 3.3 (Hooke’s law). Let $E = 0$ be a local minimum of $\tilde{W}$ with (wlog) $\tilde{W}(0) = 0$. This in quadratic approximation,

$$\tilde{W}(E) \approx \frac{1}{2} \lambda (\text{Sp } E)^2 + \mu \text{Sp}(E^2),$$

with Lamé-constants $\lambda, \mu \in \mathbb{R}$ (cf. (2.6): $\sigma = \lambda (\text{div } u) I + 2\mu D$).

Proof (for $d = 3$).

Let $\tilde{W}(E) = \hat{W}(\text{Sp } E, \text{Sp}(E^2), \det E)$ with $\hat{W} : \mathbb{R}^3 \to \mathbb{R}$, cubic in $E$. If $E = 0$ is a local minimum of $\tilde{W}$, Taylor’s formula gives

$$\tilde{W}(E) = \hat{W}(0, 0, 0) + \partial_1 \hat{W}(0, 0, 0) \text{Sp } E + \frac{1}{2} \partial_2^2 \hat{W}(0, 0, 0) (\text{Sp } E)^2 + \partial_3 \hat{W}(0, 0, 0) \text{Sp}(E^2) + O(\|E\|^3).$$

Rem: Hooke’s law corresponds linear material law (cf. force-dilation relation in spring)

3.3 Variational formulation

Let $\partial \Omega = \Gamma_D \cup \Gamma_N$ (Dirichlet- resp. Neumann-boundary). Assume the body is fixed at $\Gamma_D$ and on $\Gamma_N$ an external surface force $b$ is acting. Moreover, assume that on $\Omega$ a volume force is acting, e.g, gravitation. The displacement $u$ caused by the forces implicates a total energy

$$E_{\text{tot}}(u) = \int_{\Omega} W(C(u(x))) \, dx - \int_{\Omega} f \cdot u \, dx - \int_{\Gamma_N} b \cdot u \, dS.$$  \hspace{1cm} (3.2)

Rem: region of integration $\Omega$ ... undeformed reference configuration

Aim: find equation for displacement $u$ — by minimizing $E_{\text{tot}}(u)$.

Admissible displacements satisfy $u|_{\Gamma_D} = 0$.

Let $u$ be the minimizing displacement and $v$ another admissible displacement, i.e., $v|_{\Gamma_D} = 0$.

$\Rightarrow \Psi : \mathbb{R} \to \mathbb{R}$, $\Psi(t) := E_{\text{tot}}(u + tv)$ has a minimum at $t = 0$ so

$$0 = \Psi'(0) = \int_{\Omega} \left[ \frac{dW}{dC}(C(u)) \right] : \left[ \frac{dC}{du} (v) \right] \, dx - \int_{\Omega} f \cdot v \, dx - \int_{\Gamma_N} b \cdot v \, dS.$$  \hspace{1cm} (3.3)
∀ admissible $v \Rightarrow$ gives minimality condition for $u$.

$\Psi'(0) = \delta E_{tot}(u,v)$ ... first variation of $E_{tot}$ at $u$ in direction $v$

**Notation:**

- $A : B := \sum_{i,j} A_{ij} B_{ij} = \operatorname{Sp}(A^\top \cdot B)$ ... Frobenius scalar product for (real) matrices.
- $(\frac{\partial v}{\partial x})_{ij} = \left( \frac{\partial v}{\partial x_i} \right)$
- $(\operatorname{div} A)_i = \sum_j \partial x_j A_{ij}$ ... divergence of a matrix function $A(x)$ is a vector field.

First variation of $E_2(u) := \int_\Omega f \cdot u \, dx$:

$$\frac{d\Psi_2}{dt} = \frac{d}{dt} \int_\Omega f \cdot (u + tv) \, dx = \int_\Omega f \cdot v \, dx \quad \checkmark$$

next aim: representation of $\frac{dw}{dc}(C) : \frac{dc}{du}(v)$.

- $\forall$ (small) symmetric matrices $\Delta \in \mathbb{R}^{d \times d}$:

$$W(C + \Delta) \overset{\text{Taylor}}{=} W(C) + \left[ \frac{dW}{dc}(C) \right] : \Delta + \mathcal{O}(\|\Delta\|^2)$$

The matrix $\Sigma := 2 \frac{dW}{dc}(C)$ is called second Piola-Kirchhoff stress tensor $\Sigma_{ij} = 2 \frac{\partial W}{\partial C_{ij}}$. Because $C$ is symmetric, $\Sigma$ is symmetric.

- $\forall$ (small) $t > 0$: $C(u + tv) = C(u) + \frac{dc}{du}(v)t + \mathcal{O}(t^2)$

Laut (3.1): $C(u + tv) = \left( \frac{\partial u}{\partial x} + t \frac{\partial v}{\partial x} + I \right)^\top \cdot \left( \frac{\partial u}{\partial x} + t \frac{\partial v}{\partial x} + I \right)

= C(u) + t \left[ \left( \frac{\partial u}{\partial x} + I \right)^\top \cdot \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}^\top \cdot \left( \frac{\partial u}{\partial x} + I \right) \right] + \mathcal{O}(t^2)$

$\Rightarrow \frac{dc}{du}(v) = \left( \frac{\partial u}{\partial x} + I \right)^\top \cdot \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}^\top \cdot \left( \frac{\partial u}{\partial x} + I \right)$ ... directional derivative of $C$ at $u$ in direction $v$
3.4 Linear elasticity

- From (3.3): 
  \[ \int_\Omega f \cdot v \, dx + \int_{\Gamma_N} b \cdot v \, dS = \int_\Omega \left[ \frac{dW}{dC} (C) \right] : \left[ \frac{dC}{du} (v) \right] \, dx \]

  \[ = \frac{1}{2} \int_\Omega \Sigma : \left[ \left( \frac{\partial u}{\partial x} + I \right) \cdot \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} \cdot \left( \frac{\partial u}{\partial x} + I \right) \right] \, dx \]

  \[ = \frac{1}{2} \int_\Omega \Sigma : \left[ \left( \frac{\partial u}{\partial x} + I \right) \cdot \frac{\partial v}{\partial x} \right] \, dx \]

\[ \forall \text{admissible } v \text{ and outer normal vector } n. \]

\[ \text{(\ast)} \quad \text{with} \quad A : (B \cdot C) = (B^T \cdot A) : C \]

\[ \Rightarrow \text{equation as well for integrand } \Rightarrow \]

- **Equations of elasticity theory** (for \( \frac{\partial u}{\partial x} \) and \( \Sigma = \Sigma \left( \frac{\partial u}{\partial x} \right) \)):

  \[ \begin{cases} 
  - \text{div} \left( \left( \frac{\partial u}{\partial x} + I \right) \cdot \Sigma \right) = f \text{ in } \Omega \\
  \left( \frac{\partial u}{\partial x} + I \right) \cdot \Sigma \cdot n = b \text{ on } \Gamma_N 
  \end{cases} \quad (3.4) \]

These are the **Euler-Lagrange equations** of (3.2).

From \( \frac{\partial u}{\partial x} \) and \( u|_{\Gamma_D} = 0 \) we obtain \( u(x), \forall x \in \Omega. \)

### 3.4 Linear elasticity

**Assumptions:**

- small displacements \( u \)
- small distortion, \( E \approx \epsilon \)
- Hooke’s law holds: \( W(C) = \frac{\lambda}{2} (\text{Sp } \epsilon)^2 + \mu \epsilon : \epsilon \)

\[ \Rightarrow \text{Minimization problem: find admissible displacement } u \text{ (i.e., satisfying } u|_{\Gamma_D} = 0), \text{ such that} \]

\[ E_{\text{ges}} (u) = \int_\Omega \frac{\lambda}{2} (\text{Sp } \epsilon)^2 + \mu \epsilon : \epsilon - f \cdot u \, dx - \int_{\Gamma_N} b \cdot u \, dS \to \min, \quad (3.5) \]

with \( \epsilon (u) := \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} ^T \right). \) Further assumption: \( \lambda, \mu > 0 \)
3 Theory of elasticity

Notation:

\[ a(u, v) := \int_{\Omega} \frac{\lambda}{2} (\text{Sp} \, \varepsilon(u)) (\text{Sp} \, \varepsilon(v)) + \mu \varepsilon(u) : \varepsilon(v) \, dx, \]
\[ l(u) := \int_{\Omega} f \cdot u \, dx + \int_{\Gamma_N} b \cdot u \, dS, \]

hence: \( J(u) := a(u, u) - l(u) \to \min \)

next aim: bilinear form \( a \) is coercive on “space of admissible displacements” \( H^1_D := \{ u \in (H^1(\Omega))^d : u|_{\partial \Omega} = 0 \} \).

Lemma 3.4 (Korn’s inequality). Let \( \Omega \) be a bounded region with piecewise smooth boundary and \( \mu_{d-1}(\Gamma_D) > 0 \). \( \Rightarrow \exists c > 0 \) with

\[ \int_{\Omega} \varepsilon(u) : \varepsilon(u) \, dx \geq c \sum_{i=1}^{d} \| u_i \|^2_{H^1(\Omega)} \quad \forall u \in H^1_D. \]

(3.6)

Proof. (here only for smooth \( u \) satisfying \( u|_{\partial \Omega} = 0 \))

We have the formula

\[ 2 \varepsilon(u) : \varepsilon(u) - \frac{\partial u}{\partial x} : \frac{\partial u}{\partial x} - (\text{div } u)^2 = \text{div} \left( \frac{\partial u}{\partial x} \cdot u - (\text{div } u) u \right). \]

\[ \Rightarrow \int_{\Omega} 2 \varepsilon(u) : \varepsilon(u) - \frac{\partial u}{\partial x} : \frac{\partial u}{\partial x} - (\text{div } u)^2 \, dx = \int_{\Omega} \text{div} \left( \frac{\partial u}{\partial x} \cdot u - (\text{div } u) u \right) \, dx \]

Gauß

\[ = \int_{\partial \Omega} \left( \frac{\partial u}{\partial x} \cdot u - (\text{div } u) u \right) \cdot n \, dS = 0, \quad \text{da } u|_{\partial \Omega} = 0. \]

(3.7)

From (3.7), Poincaré inequality for \( u_i \):

\[ 2 \int_{\Omega} \varepsilon(u) : \varepsilon(u) \, dx \geq \int_{\Omega} \frac{\partial u}{\partial x} : \frac{\partial u}{\partial x} \, dx = \sum_{i=1}^{d} \| \nabla u_i \|^2_{L^2} \geq c_p \sum_{i=1}^{d} \| u_i \|^2_{H^1}, \]

with constants \( c_p > 0 \).

for extension of proof: Poincaré inequality also holds for (smooth) \( u \) vanishing at only one boundary point.

\[ \blacksquare \]

Rem.: For \( d = 1 \) (3.6) corresponds to the Poincaré inequality. For \( d > 1 \) (3.6) is non-trivial because on the left hand side only the symmetric part of \( \frac{\partial u}{\partial x} \), that is \( \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}^T \right) \), occurs, but not all derivatives separately.
3.4 Linear elasticity

**Theorem 3.5.** Let $f \in L^2(\Omega)$, $b \in L^2(\Gamma_N)$. Under the conditions of Lemma 3.4, $E_{\text{tot}}$ in (3.5) has a unique minimizer $u \in H^1_D$.

**Proof.** The (symmetric) bilinear form $a$ is on $H^1_D$ continuous and coercive (due to Korn’s inequality). For a minimum the following (weak formulation) has to hold:

$$0 = \delta E_{\text{tot}}(u, v) = 2a(u, v) - l(v), \forall v \in H^1_D.$$

Claim follows with Lemma of Lax-Milgram. ■

Analogously to the derivation of (3.4) one obtains the linear equations of static elasticity as Euler-Lagrange equations of (3.5):

$$\begin{cases}
-\lambda \nabla (\text{div } u) - 2\mu \text{div } (\varepsilon(u)) &= f, \quad \Omega \\
(\lambda \text{div } u + 2\mu \frac{\partial u}{\partial x}) \cdot n &= b, \quad \Gamma_N
\end{cases} \quad (3.8)$$

[compare: the minimizer of $\frac{1}{2}\|\nabla u\|_{L^2}^2 - \int_\Omega fu \, dx$ satisfies $-\Delta u = f$.]

**References:** [EGK] §5.10, §6.1.9, [Schö] §1,§2

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3 Theory of elasticity
4 Diffusion filtering in image processing

Diffusion filters are

- optical lens attachment for photographic special effects → blur, softener;
- software-driven, digital image (post)processing, e.g. “Gaussian blur” in Photoshop.

Application/Aim: Smoothing of noisy images, blurring of too sharp/hard images, image sharpening, edge detection (e.g. for image segmentation)

We only consider greyscale images with scale \( f(x) \in [0,1], x \in \Omega \subset \mathbb{R}^2 \). Real-world application: \( f \) discrete (pixel) on a bounded region.

Here only \( \Omega = \mathbb{R}^2 \) to avoid problems with boundary conditions. Moreover, let \( f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \).

4.1 Linear diffusion filter

Simplest image smoothing by convolution with 2D Gauss Function

\[
K_\sigma(x) := \frac{1}{2\pi\sigma^2} \exp\left( -\frac{|x|^2}{2\sigma^2} \right)
\]

with standard deviation (“size”) \( \sigma > 0 \):

\[
(K_\sigma * f)(x) = \int_{\mathbb{R}^2} K_\sigma(x-y)f(y)dy
\]

(4.1)

Effects:

- Because \( K_\sigma \in C^\infty(\mathbb{R}^2) \Rightarrow K_\sigma * f \in C^\infty(\mathbb{R}^2) \), also for \( f \in L^1(\mathbb{R}^2) \).
- In frequency domain:

\[
\hat{K_\sigma * f}(\omega) = \hat{K_\sigma}(\omega) \cdot \hat{f}(\omega)
\]

(4.2)

with \( \hat{f}(\omega) = (\mathcal{F}f)(\omega) := \int_{\mathbb{R}^2} f(x)e^{-i\omega \cdot x}dx \)
4 Diffusion filtering in image processing

Because \( \widetilde{K}_\sigma(\omega) = 2\pi \exp\left(-\frac{|\omega|^2}{2\sigma^2}\right) \):

(4.1) is low pass filter, which (monotonously) dampens high (spacial) frequencies \( \Rightarrow \) edge smoothing, denoising.

Equivalence to linear diffusion filter:

\[
\begin{align*}
\frac{du}{dt} &= \Delta u, \quad x \in \mathbb{R}^2, t > 0 \\
u(x, 0) &= f(x), \quad f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)
\end{align*}
\]

has the unique solution (e.g. assuming Gaussian decay of \( u \) for \( |x| \to \infty \)):

\[
u(x, t) = T_t f = \begin{cases} 
  f(x) & t = 0 \\
  (K \sqrt{t} * f)(x) & t > 0.
\end{cases}
\]

\( \{T_t|t \geq 0\} \ldots \) evolution semigroup of the diffusion equation

Hence: time \( t \) corresponds to (spacial) size \( \sqrt{2t} \) of the Gauss function; smoothing of image structures up to order \( \sigma \) corresponds to stopping time \( T = \sigma^2/2 \) of diffusion process.

Maximum-minimum-principle:

\[
\inf_{\mathbb{R}^2} f \leq u(x, t) \leq \sup_{\mathbb{R}^2} f \text{ auf } \mathbb{R}^2 \times [0, \infty)
\]

- Images typically contain structures on a large bandwidth of scales (e.g. portrait with resolution of every single pore)
- Often it is a-priori unclear which scale represents the “desired information”. \( \Rightarrow \) It is desirable to have a representation of the image in different scales.
- Original image \( f \) is embedded in evolution process resp. scale of smoothed/simplified images \( \{u(x, t)|t \geq 0\} \).
- \( u(x, t) \xrightarrow{t \to \infty} 0 \) (uniformly on bounded regions)
  \( \Rightarrow \) More and more image structure gets lost. \( \Rightarrow \) Only “small” \( t \) is practically relevant.
- And image can only be seen as representative of an equivalence class which contains all images of the same object. The difference between two images of a class can be e.g. grey value adjustment, translation, rotation, . . .

Numerical aspects:

- Discrete version of convolution (4.1), multiplication (4.2) in frequency domain (via FFT) and discretization of diffusion equation are not equivalent.
- For this application mostly explicit finize difference schemes for (4.3).

Disadvantages of linear Gauss filtering:
Nonlinear diffusion filters

Figure 4.1: noisy original $f$, diffusion smoothing with mean curvature equation, Gaussian diffusion smoothing, diffusion smoothing with anisotropic diffusion orthogonal to the edges [Ma]

a) Isotropic diffusion smoothens noise but also image structures (e.g. edges).
   Local diffusion which is orthogonal to the edges is not desired.

b) Linear diffusion filters move edges when transitioning from fine to coarse image scale (i.e. for large $t$).

c) Topology of contour lines can change (in 2D), e.g. splitting in two contour lines when moving to a coarser scale.

d) Smoothing does not commute with (nonlinear, monotonic) mappings $F$ which change contrast or grey value: $T_t(F(f)) \neq F(T_t f)$

(a), (b) can be ameliorated with nonlinear diffusion filters; (c), (d) using morphological equations.

References: [We] §1.1, [Ma] §10

4.2 Nonlinear diffusion filters

Aim:
- Nonlinear PDEs as improved model of (4.3);
  * Image scale $\{T_t f \mid t \geq 0\}$ is still represented by an evolution semigroup $\{T_t \mid t \geq 0\}$.
- Use of scalar diffusivity which depends on local properties of the image.
- Extension to adaptive diffusion matrices for anisotropic diffusion filters.

4.2.1 The Perona-Malik model

Model:
4 Diffusion filtering in image processing

scalar diffusivity \( g(|\nabla u|^2) > 0 \) with

\[
g(s) \searrow ; \quad g(0) = 1, \quad g(s) \xrightarrow{s \to \infty} 0
\]

e.g.

\[
g(s^2) = \frac{1}{1 + s^2/\lambda^2} \quad \text{(with parameter } \lambda > 0) \quad (4.4)
\]

hence:

\[
\begin{aligned}
    u_t - \text{div}(g(|\nabla u|^2) \nabla u) &= 0 \quad ; x \in \mathbb{R}^2, t > 0 \\
    u(x, 0) &= f(x)
\end{aligned}
\quad (4.5)
\]

Motivation: little diffusion at edges, because there \(|\nabla u(x)|\) is large.

Edge sharpening:

1D-variant of (4.5) with flux function \( \Phi(s) := sg(s^2) \):

\[
u_t = \partial_x(\Phi(u_x)) = \Phi'(u_x)u_{xx}
\quad (4.6)
\]

For \( g \) of (4.4) we have:

\( \Phi'(u_x) \geq 0 \) for \(|u_x| \leq \lambda \) \( \Rightarrow \) (4.6) is forward parabolic,

\( \Phi'(u_x) < 0 \) for \(|u_x| > \lambda \) \( \Rightarrow \) (4.6) ist backwards parabolic

(i.e. sign for ill-posedness of (4.6)).

\( \lambda \) is a contrast parameter:

For \(|u_x| \leq \lambda \) (low contrast): smoothing;

for \(|u_x| > \lambda \) (high contrast): edge sharpening (for “small time”, then growing oscillations).

We now consider the local behaviour (in \( x \) and \( t \)) of edge sharpening:

\[
\begin{align*}
    u(\cdot, t) & \quad x
    \end{align*}
\]

Edge position for a “smoothened edge”
For a "smoothed edge" we define the edge position $x_0$ (at time $t$) as the inflection point of $u$, i.e. as maximum of $u_x^2$. Hence: $(u_xu_{xx})(x_0) = 0$ (with $u_x(x_0) \neq 0$) and $(u_xu_{xxx})(x_0) < 0$.

Calculate $\partial_t(u_x^2)(x_0, t)$:

$$\partial_t(u_x^2) = 2u_xu_x \phi''(u_x) u_{xx}^2 + 2\phi'(u_x) u_x u_{xxx} > 0 \text{ at } (x_0, t) \text{ exactly for } \phi'(u_x) < 0,$$

hence exactly for $|u_x| > \lambda$. Then we have temporal growth of $|u_x(x_0)|$, i.e., edge sharpening.

2D-Gleichung (4.5): Introduction of (local) coordinates $\xi, \eta$ tangential resp. orthogonal to contour lines of $u$ ⇒

$$u_t = g(|\nabla u|^2) \frac{\Delta u}{= u_{\xi\xi} + u_{\eta\eta}} + g'(|\nabla u|^2)2 \nabla^\top u \frac{\partial^2 u}{\partial \xi^2} \nabla u = g(|\nabla u|^2) u_{\xi\xi} + p(\xi) u_{\eta\eta},$$

$\phi'(s) = g(s^2) + 2s^2 g'(s^2)$, hence forwards diffusion along contour lines (e.g. parallel to the edges) and forwards/backwards diffusion (corresponding to sign of $\phi'$) in normal direction.

Results:

- Smoothing of small fluctuations (for $|\nabla u|$ small),
- Edge sharpening (normal to the edges) (for $|\nabla u|$ large);
- PM-filter work very well practically (i.e., numerically) (although tending to be ill posed, which is not proven yet though).

Reason: numerical schemes give "implicit" regularization/stabilization (disappearing for finer and finer meshes).

- Disadvantage: noise (with $|\nabla u|$ large) is misinterpreted as "edge" ⇒ is retained or even amplified.

systematic way out with following regularization ...

References: [We] §1.2, [Ma] §10, [TE]

### 4.2.2 Regularized Perona-Malik model

Replace diffusivity $g(|\nabla u|^2)$ in (4.5) by $g(|\nabla u_{\sigma}|^2)$ with $u_{\sigma} := K_{\sigma} \ast u ⇒$

$$\begin{cases}
  u_t = \text{div}(g(|\nabla u_{\sigma}|^2) \nabla u), t > 0, \\
  u(x, 0) = f(x).
\end{cases}$$

(4.7)
\( \sigma > 0 \) is another scale parameter: noise on length scale smaller than \( \sigma \) is being smoothened.

Consider (4.7) on \( \Omega := (0, a_1) \times (0, a_2) \) with “extension by reflection" of \( f|_\Omega \) on \( \mathbb{R}^2 \) (necessary for definition of \( u_\sigma \)).

**Theorem 4.1.** Let \( f \in L^\infty(\Omega) \). \( \Rightarrow \) (4.7) has a unique distributional solution \( u(x, t) \) with:

\[
\begin{align*}
    u &\in C([0, \infty); L^2(\Omega)) \cap L^2_{\text{loc}}(0, \infty; H^1(\Omega)) \cap C^\infty(\overline{\Omega} \times (0, \infty)), \\
    \partial_t u &\in L^2_{\text{loc}}(0, \infty; H^2(\Omega)).
\end{align*}
\]

For \( a \leq f \leq b \) \( u \) satisfies the Minimum/Maximum principle:

\[
a \leq u(x, t) \leq b \quad \forall x \in \Omega, t \geq 0.
\]

**Idea of proof.**

a) Existence by Schauder fixed point theorem for the mapping \( v \mapsto w =: U(v) \) in \( W(0, T) := \{ w, \frac{dw}{dt} \in L^2(0, T; H^1(\Omega)) \} \) for fixed \( T > 0 \). \( w \) solves the linear equation

\[
\begin{align*}
    w_t &= \text{div}(g(|\nabla u_\sigma|^2)\nabla w), t > 0 \\
    w(x, 0) &= f(x)
\end{align*}
\]

(4.8)

b) Regularity via “bootstrapping" argument; i.e. from \( u(t) \in H^1(\Omega) \) \( \forall t > 0 \) follows \( u(t) \in H^2(\Omega) \) \( \forall t > 0 \), and so on.

c) Uniqueness & continuous dependence on initial conditions via Gronwall Lemma for difference of two solutions.

d) Minimum/Maximum principle with truncation method.

Details: [CLMC], Th. 2.1 in [We]

**Remark 4.2.**

1) Iteration of (4.8) converges in \( C([0, T]; L^2(\Omega)) \) \( \forall T > 0 \) (see [CLMC]).

2) possible discretization of (4.7): finite differences; \( g(|\nabla u_\sigma|^2) \) explicitly, rest implicitly in time [CLMC].

regular grid \((ih, jh, n\Delta t)\),

\[
h = \frac{1}{N + 1}, 0 \leq i, j \leq N + 1, u_{i,j}^n \approx u(ih, jh, n\Delta t)
\]

Let \( \alpha_{i,j}^n \approx g(|\nabla K_\sigma * u|^2)(ih, jh, n\Delta t) \).

Discretization of \( \partial_{x_1}(\alpha(x)u_{x_1}) \) an \((ih, jh, n\Delta t)\):

\[
\frac{1}{2h^2} \left[ (\alpha_{i+1,j}^n + \alpha_{i,j}^n)(u_{i+1,j}^{n+1} - u_{i,j}^{n+1}) - (\alpha_{i,j}^n + \alpha_{i-1,j}^n)(u_{i,j}^{n+1} - u_{i-1,j}^{n+1}) \right],
\]

analogously for \( \partial_{x_2}(\alpha(x)u_{x_2}) \).
\[ u_{i,j}^{n+1} - u_{i,j}^n \frac{\Delta t}{2h^2} \left\{ (\alpha_{i,j}^n + \alpha_{i,j-1}^n)u_{i-1,j}^{n+1} + (\alpha_{i,j}^n + \alpha_{i,j+1}^n)u_{i+1,j}^{n+1} + (\alpha_{i,j}^n + \alpha_{i,j+1}^n)u_{i,j+1}^{n+1} + (\alpha_{i,j}^n + \alpha_{i,j-1}^n)u_{i,j-1}^{n+1} \right\} = 0, \]

IC: \[ u_{i,j}^0 = f(ih, jh), \quad 1 \leq i, j \leq N \]

Neumann-BC: \[ u_{i,0}^{n+1} = u_{i,1}^{n+1}, \quad u_{i,N}^{n+1} = u_{i,N+1}^{n+1}, \quad 0 \leq i \leq N+1, \]
\[ u_{0,j}^{n+1} = u_{1,j}^{n+1}, \quad u_{N,j}^{n+1} = u_{N+1,j}^{n+1}, \quad 0 \leq j \leq N+1 \]

total structure: \[ u_{i,j}^{n+1} - u_{i,j}^n \frac{\Delta t}{2h^2} + \frac{\Delta t}{h^2} (u_{i,j}^n) u_{i,j}^{n+1} = 0. \]

Hence one has to solve the following linear system:
\[ (I + \Delta t A_h(u^n)) u^{n+1} = u^n, \quad n \geq 0, \]
with \( A_h \) block-tridiagonal, positive definite \( \Rightarrow I + \Delta t A_h(u^n) \) invertible.

**Invariances:**

Let \( \{ T_t, t \geq 0 \} \) be the solution semigroup of (4.7).

a) Grey value shift:

Diffusivity \( g(|\nabla u|_\sigma^2) \) only depends on \( \nabla u \) but not on \( u \).
\[ T_t(0) = 0, \quad t \geq 0 \]
\[ T_t(f + C) = T_t(f) + C, \quad \forall t \geq 0; \forall C \in \mathbb{R} \]

One bounded regions one additionally needs homogeneous Neumann-BCs.

b) Contrast inversion:

\[ g(|-\nabla u|_\sigma^2) = g(|\nabla u|_\sigma^2) \]
\[ \Rightarrow T_t(-f) = -T_t(f), \quad \forall t \geq 0 \]

c) mean grey value:

\[ \mu := \frac{1}{|\Omega|} \int_\Omega f(x) dx = \frac{1}{|\Omega|} \int_\Omega T_t(f) dx \quad t > 0 \quad (4.9) \]

follows from divergence form of (4.7) and homogeneous Neumann-BC (compare extension by reflection).

d) Translation and rotation invariance for \( \Omega = \mathbb{R}^2 \).

**Reduction of information for** \( t > 0 \):

**Local** Extrema of \( u \) are not amplified in (4.7):
**Theorem 4.3.** Let $x_0 \in \Omega$ be a local extremum of $u(\cdot, t_0)$ for some $t_0 > 0$. \(\Rightarrow\)

- $u_t(x_0, t_0) \leq 0$ if $x_0$ local maximum,
- $u_t(x_0, t_0) \geq 0$ if $x_0$ local minimum.

**Proof.** Let $x_0$ be a local maximum, hence $\nabla u(x_0, t_0) = 0$, $\Delta u(x_0, t_0) \leq 0$.

At $(x_0, t_0)$ we have by (4.7):

$$u_t(x_0, t_0) = g(|\nabla u|_{\sigma}^2) \cdot \nabla u \leq 0.$$

\[\Box\]

Convergence of solution $u$ from (4.7) towards mean grey value $\mu$:

**Theorem 4.4.** Let $f \in L^\infty(\Omega)$, $\Omega = (0, a_1) \times (0, a_2)$.

$$\Rightarrow \|u(t) - \mu\|_{L^p(\Omega)} \leq C e^{-\lambda t}, 1 \leq p < \infty, t \geq 0,$$

with $C, \lambda$ depending on $\Omega, p, \|f\|_{\infty}$.

**Proof.** $e(x, t) := u(x, t) - \mu$ satisfies (4.7).

According to maximum principle in Theorem 4.1:

$$\|e(t)\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} + |\mu| \quad \forall t \geq 0. \quad (4.10)$$

$$\Rightarrow \nabla e_\sigma(t) = (\nabla K_\sigma) * e(t)$$ satisfies (with Young inequality for convolution):

$$\|\nabla e_\sigma(t)\|_{L^\infty(\Omega)} \leq \|\nabla K_\sigma\|_{L^1(\mathbb{R}^2)} \|e(t)\|_{L^\infty(\Omega)} \leq C_1 \quad \forall t \geq 0$$

$$\Rightarrow \exists \nu > 0 \text{ with } g(|\nabla e_\sigma(x, t)|^2) \geq \nu \quad \forall t > 0, \forall x \in \Omega.$$

First proof for $p = 2$: From (4.7) for $e(t)$ we infer, using $\nabla e \cdot n|_{\partial \Omega} = 0$:

$$\int_\Omega ee_t \, dx = \int_\Omega \text{div}(g(|\nabla e|_{\sigma}^2) \nabla e) \, dx = -\int_\Omega |\nabla e|^2 g(|\nabla e|_{\sigma}^2) \, dx,$$

hence

$$\frac{1}{2} \frac{d}{dt} \|e(t)\|^2_{L^2(\Omega)} \leq -\nu \|\nabla e(t)\|^2_{L^2(\Omega)} , \quad t > 0.$$

For $t > 0$ fixed: $e(t) \in C^\infty(\Omega)$, $\int_\Omega e(x, t) \, dx = 0$ (due to (4.9)) $\Rightarrow \exists x_0 \in \Omega \text{ with } e(x_0, t) = 0$.

According to Poincaré inequality with $C_2 = C'_2(\Omega) > 0$:

$$\|e(t)\|^2_{L^2(\Omega)} \leq C_2 \|\nabla e(t)\|^2_{L^2(\Omega)} \quad \forall t > 0,$$
4.2 Nonlinear diffusion filters

\[
\frac{d}{dt} \|e(t)\|_{L^2(\Omega)}^2 \leq -2\nu C_2^{-1} \|e(t)\|_{L^2(\Omega)}^2, \quad t \geq 0.
\]

\[
\Rightarrow \|e(t)\|_{L^2(\Omega)} \leq e^{-\nu C_2^{-1}t} \|f - \mu\|_{L^2(\Omega)}, \quad t \geq 0.
\]

(4.11)

Analogously for \(\|e(t)\|_{L^p(\Omega)}\) with \(1 \leq p < 2\) because \(L^2(\Omega) \subset L^p(\Omega)\). Result for \(2 < p < \infty\) follows from (4.10) and (4.11) by Interpolation (Hölder inequality).

\[\Box\]

References: [We] §1.2, 2.3-4

4.2.3 Anisotropic diffusion filter

- so far only scalar, i.e., isotropic diffusivity in \(u_t = \text{div}(\Phi(\nabla u))\); flux

\[
j = -\Phi(\nabla u) = -g(|\nabla u|^2) \nabla u \quad \text{always } \|zu\nabla u\|
\]

- compare to PM-model written in local coordinates (\(\xi\) tangential, \(\eta\) normal to level curves of \(u\)):

\[
u_t = g(|\nabla u|^2) u_{\xi\xi} + \Phi'(\nabla u) u_{\eta\eta}
\]

- an efficient anisotropic diffusion model: diffusion only tangential to level curves/contour lines (i.e. \(\|\) to edges)

Ex.: mean curvature filter:

linear diffusion filter in local coordinates:

\[u_t = u_{\xi\xi} + u_{\eta\eta}\]

anisotropic analogon (with diffusion only tangential to contour lines):

\[
\begin{cases}
u_t = u_{\xi\xi}, \quad t > 0 \\
u(x, 0) = f(x)
\end{cases}
\]

(4.12)

This is a nonlinear degenerate parabolic equation; in local coordinates the diffusion matrix reads \(A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\).

Transformation to \(x = (x_1, x_2)\)-coordinates gives mean curvature equation:

\[
u_t = \left(\nabla \nabla^\top u\right) \cdot \frac{\partial^2 u}{\partial x^2} \cdot \nabla^\top u = \left|\nabla u\right|^2 \frac{u_{x_2}^2 u_{x_1} - 2u_{x_1}u_{x_2}u_{x_1} + u_{x_1}^2 u_{x_2}}{u_{x_1}^2 + u_{x_2}^2}
\]

\[
= |\nabla u| \text{ div} \left(\frac{\nabla u}{|\nabla u|}\right). \quad (4.13)
\]
Diffusion filtering in image processing

\[ \kappa(x,t) := \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) \ldots \text{(mean curvature of contour line of } u(\cdot,t) \text{ through } x). \]

**Theorem 4.5.** Assume that \( f \) is bounded and uniformly continuous on \( \mathbb{R}^2 \). \( \Rightarrow \)

(4.13) hat a unique viscosity solution \( u(x,t) \) on \( \mathbb{R}^2 \times [0,\infty) \). It satisfies a max/min-principle:

\[ \inf_{\mathbb{R}^2} f \leq u(x,t) \leq \sup_{\mathbb{R}^2} f. \]

Solution is \( L^\infty \)-stable, i.e., for 2 solutions \( u_{1,2}(t) \) with ICs \( f_{1,2} \) we have:

\[ \| u_1(t) - u_2(t) \|_{L^\infty(\mathbb{R}^2)} \leq \| f_1 - f_2 \|_{L^\infty(\mathbb{R}^2)} \quad \forall \, t \geq 0 \]

Rem.: vague motivation of viscosity solution: because (4.12) is degenerate parabolic, consider \( u_t = u_{\xi \xi} + \varepsilon \Delta u, \varepsilon \to 0 \) (precise notions is very technical).

Reformulation of (4.13) as transport equation:

\[ u_t + \kappa(x,t)n(x,t) \cdot \nabla u = 0; \quad (4.14) \]

with

\[ n(x,t) := -\frac{\nabla u(x,t)}{|\nabla u(x,t)|} \ldots \text{unit normal vector on level curve of } u(\cdot,t) \]

(nonlinear, because \( \kappa, n \) depend on \( u \! \! \) )

Solution of (4.14) using method of characteristics:

\( u = \text{const along characteristics, given by } \dot{x} = \kappa(x,t)n(x,t) \).

Result:

- Velocity of level curves is proportional to local curvature; in direction of decreasing \( u \)
- Smoothing by alignment of curvature of each level curve:
  - Each level curve asymptotically tends to a circle and collapses to a point in finite time.
- (4.14) cannot amplify contrast:

References: [We] §1.2.3, 1.4-5, [Ma]§10

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4.3 Edge sharpening, shock filter

opposing processes:

- smoothing, blur
- sharpening, deblur

1D-situation:

\begin{align*}
\text{sharpening?} & \quad \text{smoothing} \\
1 & \quad 1 \\
-1 & \quad -1
\end{align*}

\textbf{Aim:} find a PDE for image sharpening as “time” evolution process

\textbf{Example 4.6.} Let \( f(x) = \cos(x) \).

\textbf{Conclusion:}

- Direction of movement of 1D-“level points” \( u(x, t) \) depends on \( \text{sign}[u_x(x,t)u_{xx}(x,t)] \).
- for \( u_x(x,t) = 0 \) or \( u_{xx}(x,t) = 0 \): no movement desired

\textbf{Proposed model} (in 1D): “shock filter” by Osher & Rudin:

\begin{align*}
\begin{cases}
    u_t = -\text{sign}(u_x u_{xx}) u_x = -|u_x| \text{sign}(u_{xx}), x \in \mathbb{R}, t > 0 \\
    u(x, 0) = f(x)
\end{cases}
\end{align*}

(4.15)
This is a transport equation with velocity $\pm 1$, e.g. in the region where $u_x(x,t) > 0$, $u_{xx}(x,t) > 0$: $u_t + u_x = 0$. But in total the equation is fully nonlinear.

Preliminary study of a simplified model:
In the above example the local convexity/concavity do not change.

$$
\begin{align*}
    \begin{cases}
        u_t = -|u_x| \text{sign}(f_{xx}), & x \in \mathbb{R}, t > 0 \\
        u(x,0) = f(x) := \cos(x)
    \end{cases}
\end{align*}
$$

1st case: consider (4.16) on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}^+$

$\rightarrow \text{sign}(f_{xx}) = -1 \Rightarrow u_t = |u_x|$ (a Hamilton-Jacobi equation.)

Solution by method of characteristics:

$$
    u(x,t) = \begin{cases}
        \cos(x + t), & -\frac{\pi}{2} < x < -t \\
        1, & t \geq |x| \\
        \cos(x - t), & t < x < \frac{\pi}{2}
    \end{cases}
$$

This is a rarefaction wave, analogously to §1.2; weak solution is only unique if we demand continuity.
2nd case: consider (4.16) on \( \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \times \mathbb{R}^+ \)

\[ \rightarrow \text{sign}(f_{xx}) = 1 \implies u_t = -|u_x| \]

analogous rarefaction wave:

\[ u(x, t) = \begin{cases} 
\cos(x + t) & , \quad \frac{\pi}{2} < x < \pi - t \\
-1 & , \quad t \geq |x - \pi| \\
\cos(x - t) & , \quad t + \pi < x < \frac{3\pi}{2} 
\end{cases} \]

Solution of (4.16) by periodic extension:

has shocks at \( x = (2k + 1)\frac{\pi}{2}, k \in \mathbb{Z} \):

for \( t \geq \frac{\pi}{2} \): \( u(x, t) = (-1)^k \) for \( (2k - 1)\frac{\pi}{2} < x < (2k + 1)\frac{\pi}{2} \). \( \Rightarrow \) (4.16) sharpens the curves up to perfect step functions (in finite time!) with jumps where \( f_{xx} = 0 \).

Information gain (because of sharpening) seemingly possible because of restriction to \( u \in \{-1, 1\} \).

Generalization of “shock filters” (4.15):

\[
\begin{align*}
\begin{cases}
u_t = -|u_x|F(u_{xx}), x \in \mathbb{R}, t > 0 \\
u(x, 0) = f(x)
\end{cases}
\end{align*}
\]

(4.17)

with \( F \in \text{Lip}(\mathbb{R}) \) and \( F(0) = 0; \text{sign}(s)F(s) > 0, \forall s \neq 0. \)

e.g. with \( F(s) = s \):

\[ u_t = -|u_x|u_{xx} = -(u_{xx} \text{sign}(u_x))u_x, \quad x \in \mathbb{R}, t > 0. \]
This is a transport equation with local propagation speed \( c(x,t) = \text{sign}(u_x)u_{xx} \).

Edge positions \( x_0 \) are defined as maxima of \( u^2 \Rightarrow u_{xx}(x_0) = 0, \) \( u_{xx} \) changes sign at \( x_0 \).

\[ \Rightarrow \text{sign change of } c(x) \text{ is “detector” for edges (and extrema of } u). \]

(4.18) is ill posed (backwards parabolic!), but works very well numerically (reason still unclear).

**Conjecture 4.7** (Osher-Rudin, 1990). Let \( f \in C(\mathbb{R}) \). \( \Rightarrow (4.17) \) has a unique solution with jumps (for \( t > 0 \)) only at inflection points of \( f(x) \). The total variation in \( x \) of \( u \) is constant in \( t \), the same holds for positions and values of local extrema.

**2D-generalization:**

\[ u_t = -|\nabla u|F(\Delta u), \quad x \in \mathbb{R}^2, \ t > 0; \]

\( \text{e.g. with } F(s) := \text{sign}(s). \)

**References:** [AK] §3.3.3, [Ma]§10
5 Pattern formation / reaction-diffusion equations

Examples for pattern formation processes:

- chemical reactions, e.g. spiral waves
- two-phase mixtures of liquids, e.g. “fingering” in oil-water flow in porous medium
- in biology: leaf structures, animal skin (“animal coat”), . . .

in biology: only the “recipe” for pattern formation processes is “stored” genetically, but not the pattern itself.

Aim: (nonlinear) mathematical models (e.g. parabolic PDEs) producing “such” patterns → as possible mechanism for pattern formation.

5.1 Reaction-diffusion equations

Derivation:

c(x, t) . . . (scalar) density function of a substance; x ∈ ℝ³
J(c, x, t) . . . flux function
f(c, x, t) . . . production rate of substance

Balance equation in region Ω ⊂ ℝ³:

$$\frac{d}{dt} \int_{\Omega} c(x, t) dx = - \int_{\partial\Omega} J \cdot \nu ds + \int_{\Omega} f(c, x, t) dx$$

$$\Omega \text{ arbitrary } \Rightarrow$$

$$c_t + \text{div} J = f(c, x, t) \quad (5.1)$$

classical diffusion: \( J = -D\nabla c \); here only \( D = \text{const} \).

Generalization on multiple interacting species or chemicals \( c_i(x, t); i = 1, \ldots, m. \)
Rate or production/reaction here only $f = f(c) \in \mathbb{R}^m$ (nonlinear!):

$$c_t = f(c) + D \Delta c.$$  \hspace{1cm} (5.2)

here: $0 \leq D =$ constant diagonal matrix; hence no cross-diffusion.

References: [Mu] §9.2

### 5.2 Turing mechanism

let $m = 2$; $c = (u, v)^\top$, after suitable scaling (spacial scale parameter $\gamma > 0, d > 0$):

$$\begin{align*}
  u_t &= \gamma f(u, v) + \Delta u \\
  v_t &= \gamma g(u, v) + d \Delta v
\end{align*}$$ \hspace{1cm} (5.3)

Turing mechanism:

1. Let $(u_0, v_0)^\top$ be a spacially homogeneous, stable stationary point of

$$u_t = \gamma f(u, v), \quad v_t = \gamma g(u, v).$$ \hspace{1cm} (5.4)

2. For suitable $f, g$ and $1 \neq d$ we have: (5.3) is linearly instable at $(u_0, v_0)^\top$, although diffusion “usually” stabilizes.

$\Rightarrow$ small disturbances of the homogeneous stationary state can produce spacially inhomogeneous patterns in the time evolution: “regular” patterns as stationary states $u_\infty(x) = \lim_{t \to \infty} u(x, t)$ resp. $v_\infty(x) = \lim_{t \to \infty} v(x, t)$. There are not unique!

Consider (5.3) on $\Omega \subset \mathbb{R}^2$ with BC:

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega,$$

i.e. 0-flux-BC to permit self-organizing patterns (without BC-effect!).

IC: $u(x, 0), v(x, 0)$ given.

Conditions for diffusion-driven instability:

**Definition 5.1** (linear stability). For an autonomous dynamical system $y' = F(y)$ a point $y_0 \in \mathbb{R}^m$ is called a linearly stable stationary point if $F(y_0) = 0$ and for all eigenvalues of $\frac{\partial F}{\partial y}(y_0)$ we have: $\text{Re}(\lambda_i) < 0$. If there is an eigenvalue satisfying $\text{Re}(\lambda_i) > 0$ then $y_0$ is called a linearly unstable stationary point.
5.2 Turing mechanism

The case $\text{Re}(\lambda_i) = 0$ is not covered here because it does not allow for a statement about the nonlinear system.

**Lemma 5.2.** $(u_0, v_0)^\top \in \mathbb{R}^2$ is a linearly stable stationary point of (5.4) $\iff$

$$f(u_0, v_0) = g(u_0, v_0) = 0,$$

$$f_u + g_v |_{u_0, v_0} < 0,$$

$$f_u g_v - f_v g_u |_{u_0, v_0} > 0.$$

**Proof.** Linearization of ODE (5.4):

$$w := \left( u - u_0 \atop v - v_0 \right);$$

for $|w|$ we have:

$$w_t \approx \gamma A w, \quad A = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} |_{u_0, v_0}.$$

$w = 0$ is linearly stable $\iff$ $\text{Re} \lambda_1, 2(A) < 0 \iff$

Conditions:

$$\text{tr} A = \lambda_1 + \lambda_2 = f_u + g_v |_{u_0, v_0} < 0,$$

$$\det A = \lambda_1 \lambda_2 = f_u g_v - f_v g_u |_{u_0, v_0} > 0.$$

\[\Box\]

**Theorem 5.3** (necessary condition for instability). Suppose (5.5) holds. Let $(u_0, v_0)^\top \in \mathbb{R}^2$ be a linearly unstable stationary point of (5.3) $\Rightarrow$

$$df_u + g_v |_{u_0, v_0} > 0,$$

$$(df_u + g_v)^2 - 4d(f_u g_v - f_v g_u) |_{u_0, v_0} > 0.$$  \hspace{1cm} (5.6)

(1st condition and (5.5) imply that $d \neq 1, f_u g_v < 0$)

**Proof.** Step 1: Solution formula for linearized RD-equations:

Linearization of (5.3) around stationary state $(w(x, t) \in \mathbb{R}^2)$:

$$\begin{cases}
    w_t = \gamma Aw + D\Delta w, & D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \\
    \text{BC: } \frac{\partial w_1, 2}{\partial \nu} = 0, & x \in \partial \Omega, \\
    \text{IC: } w(x, 0) \text{ is "small" disturbance of } (u_0, v_0)^\top. 
\end{cases}$$  \hspace{1cm} (5.7)

Consider first the scalar eigenvalue problem ($=\text{Helmholtz equation}$), $z(x) \in \mathbb{R}$:

$$\begin{cases}
    \Delta z + \mu^2 z = 0, & \Omega \ldots \text{bounded region} \\
    \frac{\partial z}{\partial \nu} = 0, & \partial \Omega 
\end{cases}$$  \hspace{1cm} (5.8)
\( \mu_n^2 \in \mathbb{R}_+^n \), \( n \in \mathbb{N}_0 \) ... discrete eigenvalues (increasing), \( \mu \) ... “wavenumbers”; \( \frac{1}{n} \) proportional to wave length 
\( z_n(x), n \in \mathbb{N}_0 \) ... (scalar) eigenfunctions; form ONB of \( L^2(\Omega) \).

In particular: \( \mu_0 = 0 \), \( z_0 \equiv |\Omega|^{-1/2} \). This \( x \)-homogeneous mode is stable by assumption (5.5).

Approach for system of two parabolic equations (5.7): based on eigenfunction expansion for (5.8):

\[
  w(x, t) = \sum_{n=0}^{\infty} c_n e^{\lambda_n t} z_n(x),
\]

(5.9)

Calculation of \( \lambda_n \in \mathbb{C}, c_n \in \mathbb{C}^2; n \in \mathbb{N}_0 \) by inserting in (5.7):

\[
  \lambda_n z_n c_n = \gamma z_n A c_n + \Delta z_n D c_n \overset{(5.8)}{=} z_n (\gamma A - \mu_n^2 D) c_n \quad \forall n \in \mathbb{N}_0
\]

This is a homogeneous linear system of equations for \( c_n \). Because \( z_n \neq 0 \):

\[
  0 = \det(\lambda_n I - \gamma A + \mu_n^2 D) = \lambda_n^2 + l(\mu_n^2) \lambda_n + h(\mu_n^2) = 0, \quad \text{(eq. for } \lambda_n^2) \quad \text{(5.10)}
\]

\[
  l(\mu) := \mu^2 (1 + d) - \gamma (f_u + g_v) \in \mathbb{R},
\]

\[
  h(\mu) := d \mu^4 - \gamma (d f_u + g_v) \mu^2 + \gamma^2 \det A \in \mathbb{R}.
\]

Let \( \lambda_j^l \in \mathbb{C}, j = 1, 2 \) be solutions of (5.10), i.e., eigenvalues of \( \gamma A - \mu_n^2 D \), and \( c_j^l \in \mathbb{C}^2 \) the corresponding eigenvectors. (Here we assume that \( \gamma A - \mu_n^2 D \) is diagonalizable.)

\[
  \Rightarrow c_n^1 e^{\lambda_1^l t} z_n(x) \text{ solves (5.7).}
\]

\( \lambda_n^{1,2} \) resp. \( c_n^{1,2} \) are conjugate complex or both real because \( \gamma A - \mu_n^2 D \) is real.

\[
  \Rightarrow w(x, t) = \sum_{n=0}^{\infty} \left[ \alpha_n c_n^{1} e^{\lambda_1^l t} + \beta_n^{1} e^{\lambda_2^l t} \right] z_n(x), \quad \text{(5.11)}
\]

and the coefficients \( \alpha_n, \beta_n \in \mathbb{C} \) are uniquely determined by the Fourier expansion of the ICw(., 0) \( \in L^2(\Omega; \mathbb{R}^2) \).

Step 2: proof of the two inequalities (5.6):

Homogeneous stationary state \( (u_0, v_0) \) of (5.3) is linearly stable \( \iff \) both solutions of (5.10) satisfy: \( \text{Re} \lambda_n^{1,2} < 0 \quad \forall n \in \mathbb{N}_0 \).

In any case we have

\[
  l(\mu^2) = \underbrace{\mu^2 (1 + d)}_{\geq 0} - \gamma (\underbrace{f_u + g_v}_{<0}) > 0 \quad \forall \mu.
\]

If \( \lambda_n \) is a double eigenvalue \( \Rightarrow \lambda_n = -l(\mu_n^2) / 2 < 0 \), i.e., stable mode.

Stationary state \( (u_0, v_0) \) is linearly unstable \( \iff \exists n \in \mathbb{N}, \exists j \in \{1, 2\} \) with \( \text{Re} \lambda_j^l > 0 \). (Rem.: \( n = 0 \) is stable mode.)
This happens exactly for \( h(\mu_n^2) < 0 \) in (5.10) for one \( n \in \mathbb{N} \); because (5.10) implies:

\[
2\lambda_1^2 = -l(\mu_n^2) \pm \sqrt{l^2(\mu_n^2) - 4h(\mu_n^2)},
\]

(5.12)

and \( \lambda_1^1 > 0 \iff h(\mu_n^2) < 0 \).

\[
h(\mu^2) = d\mu^4 - \gamma (d_f + g_v)\mu^2 + \gamma^2 \det A > 0
\]

from (5.5)

(5.13)

\( \Rightarrow h(\mu^2) < 0 \) only for \( d_f + g_v > 0 \) possible (= Condition 1).

As \( f_u + g_v < 0 \) (from (5.5)) \( \Rightarrow d \neq 1, f_ug_v < 0 \).

Minimum of \( h(\mu^2) \) as function of \( \mu^2 \):

\[
\begin{aligned}
 h_{\text{min}} &= \gamma^2 \left( \frac{\det A}{>0} - \frac{(d_f + g_v)^2}{4d >0} \right), \\
 \mu_{\text{min}}^2 &= \frac{d \gamma f_u + g_v}{2d} > 0 \quad \text{Cond.1}
\end{aligned}
\]

(5.14)

\( \Rightarrow \) condition for \( h(\mu^2) < 0 \) for one \( \mu \neq 0 \):

\[
\frac{(d \gamma f_u + g_v)^2}{4d >0} > \det A > 0, \quad (= \text{condition 2}).
\]

i.e. for \( 0 < d \ll 1 \) or \( d \gg 1 \).

\[\boxed{\textbf{Remark 5.4.}} \quad h(\mu^2) < 0 \iff \mu^2 < \mu_n^2 < \mu_c^2 \quad \text{possibly empty set, depending on } d, \gamma \]

with

\[
\mu_n^2, \mu_c^2 = \gamma \frac{d f_u + g_v \pm \sqrt{(d_f + g_v)^2 - 4d \det A}}{2d}
\]

(=zeros of (5.13)).

Above conditions are necessary but not sufficient because \( \mu_{\text{min}}^2 \) isn’t an eigenvalue in general.
Remark 5.5 (sufficiency condition for instability). Exactly for the discrete eigenvalues \( \mu_n^2 \in (\bar{\mu}^2, \mu^2) \) (if they exist!) we have Re \( \lambda_n^1 > 0 \) (unstable modes! Follows from (5.12)).

The asymptotic behaviour of \( w \) (for large \( t \)) then is, following (5.11):

\[
w(x, t) \sim \sum_{\mu \leq \mu_n \leq \bar{\mu}} \alpha_n c_n^1 e^{\lambda_n^1 t} z_n(x)
\]

Sum only over discrete eigenvalues of (5.8) (possibly empty set) \( \Rightarrow \) only finitely many wavenumbers \( \mu_n \) (of the “pattern”) are unstable. Mode with maximal \( \lambda_n \) is dominant.

Idea: Linearly instable eigenfunctions are bounded by nonlinear effects \( \Rightarrow \) spatially inhomogeneous stationary states develop (proof exists only for special cases)

Java-Demo for Brusselator: http://crossgroup.caltech.edu/Patterns/Demo4_5.html (runs in Internet Explorer 11; not in Firefox)

Scale parameter \( \gamma \) \( (\sqrt{\gamma} \) proportional to typical length scale\) appears only in the interval boundaries (5.14) for instable \( \mu \)-interval: the larger \( \gamma \) is, the more instable (pattern) modes there are.

Remark 5.6. Let \( \Omega = \mathbb{R}^2 \Rightarrow \) Helmholtz equation (5.8) has continuous spectrum \( \mu^2 \geq 0 \).

For all modes with \( \mu^2 \in (\bar{\mu}^2, \mu^2) \) (5.9) is linearly instable.

\( \Rightarrow \) spatial pattern develops; with wavenumber \( \mu \) for maximal \( \lambda^1_{\mu} \).

References: [Mu] §14.2-3; [EGK] §16.2.12

5.3 Pattern formation in a sample system

Example for (5.3), first in 1D:

\[
\begin{align*}
    u_t &= \gamma f(u, v) + u_{xx} := \gamma (a - u + u^2 v) + u_{xx} \\
    v_t &= \gamma g(u, v) + d v_{xx} := \gamma (b - u^2 v) + dv_{xx} 
\end{align*}
\]

(5.15)

\( t > 0, x \in (0, p); a, b, d > 0 \)

Schnakenberg-System: Model for biochemical reaction between 2 substances with densities \( u(x, t), v(x, t) \) and 3-molecule reaction (e.g. additional enzyme reaction in system dynamics).

Pattern formation is independent from exact form of \( f, g \).

Homogeneous, positive stationary state:

\[ u_0 = a + b, v_0 = \frac{b}{(a + b)^2}, \quad b > 0, a + b > 0 \]
pattern formation in a sample system

\( f_u = \frac{b-a}{a+b}, \quad f_v = (a+b)^2 > 0, \quad g_u = \frac{-2b}{a+b}, \quad g_v = -(a+b)^2 < 0. \)

Consequence of (5.6): \( f_u g_v < 0 \Rightarrow b > a. \)

Conditions (5.5), (5.6) for linear ODE-stability resp. linear PDE-instability:

\[
\begin{align*}
\left\{ 
\begin{array}{l}
  f_u + g_v < 0 \quad \Rightarrow \quad 0 < b - a < (a+b)^3, \\
  \det A = f_u g_v - f_v g_u = (a+b)^2 > 0 \quad \checkmark \\
  d f_u + g_v > 0 \quad \Rightarrow \quad d(b-a) > (a+b)^3 \\
  \quad \left( d f_u + g_v \right)^2 - 4d(f_u g_v - f_v g_u) > 0 \quad \Rightarrow \quad \left[ d(b-a) - (a+b)^3 \right]^2 > 4d(a+b)^4 \\
\end{array}
\right. \\
\end{align*}
\]

These inequality for \((a, b, d)\) define area of instability (“Turing space”).

Eigenvalue problem (5.8) on \( \Omega = (0, p) \):

\[ z_{xx} + \mu^2 z = 0, \quad z_x(0) = z_x(p) = 0 \]

\( \Rightarrow \mu_n = \frac{n\pi}{p}, z_n(x) = \cos \frac{n\pi x}{p}, n \in \mathbb{N}_0 \)

Let \((a, b, d)\) be in the Turing space defined by (5.16).

\( \Rightarrow \) from (5.14): bad of instable wavenumbers = \( (\mu, \overline{\mu}) = (\sqrt{\gamma\sigma}, \sqrt{\gamma\overline{\sigma}}) \) with

\[ \sigma^2, \overline{\sigma}^2 := \frac{d(b-a) - (a+b)^3 \mp \sqrt{[d(b-a) - (a+b)^3]^2 - 4d(a+b)^4}}{2d(a+b)} \]

\( \Rightarrow \) all discrete modes with \( \mu_n = \frac{n\pi}{p} \in (\mu, \overline{\mu}) \) are linearly unstable.

Asymptotic behaviour (for large \( t \)) of \( w(x, t) \approx (u(x, t) - u_0, v(x, t) - v_0) \) from (5.15):

\[ w(x, t) \sim \sum_{n=0}^{\infty} \alpha_n c_n e^{\lambda_n t} \cos \frac{n\pi x}{p}, \]

\( \lambda_n \ldots \) positive solution of quadratic equation (5.10).

\( \mu, \overline{\mu} \) chosen such that the corresponding wavenumbers are in the band \( (\mu, \overline{\mu}) \).

Influence of scale parameter \( \gamma > 0 \):

typical length scale / system size \( \propto \sqrt{\gamma} \):

\[
\begin{array}{c|c|c|c|c|c}
\mu_0 & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_n \\
\hline
\gamma & \gamma & \mu = \sqrt{\gamma\sigma} & \overline{\mu} = \sqrt{\gamma\overline{\sigma}} & \mu
\end{array}
\]
Unstable interval \((\mu, \overline{\mu})\) translatable by \(\gamma\). Depending on \(\gamma\) there are 0, 1, ... linearly unstable modes:

- For \(\gamma < \gamma_c = \left(\frac{\mu}{\overline{\mu}}\right)^2\): all modes are linearly stable \(\Rightarrow (u_0, v_0)\) is stable.
- **Bifurcation** at \(\gamma = \gamma_c\) (critical value)
- For \(\gamma > \gamma_c\) with \(\mu < \mu_1 < \overline{\mu} < \mu_2\) \(\Rightarrow\) only mode 1 is linearly unstable:

\[
u(x, t) \sim u_0 + ce^{\lambda_1 t} \cos \frac{\pi x}{p}, \text{Re } \lambda_1 > 0.
\]

(valid in “linear region”)

Exponential growth of \(u\) is not restricted by nonlinear effects.

Hypothesis: \(u_\infty(x) \approx u_0 + \tilde{c} \cos \frac{\pi x}{p}\)

Expected 1D-pattern (for \(\tilde{c} > 0\)):

- If \(\mu_1 < \overline{\mu} < \mu_2 < \mu_3\) \(\Rightarrow\) only mode 2 is linearly unstable:

\[
u(x, t) \sim u_0 + ce^{\lambda_2 t} \cos \frac{2\pi x}{p}, \text{Re } \lambda_2 > 0.
\]

Expected 1D pattern:

Analogously for even larger systems. Also: system size and geometry (in 2D) are decisive for possible patterns.

2D-case:
5.4 Animal coat color patterns

Eigenvalue problem (5.8) on $\Omega = (0, p) \times (0, q)$:

$$\Delta z + \mu^2 z = 0 , \quad \frac{\partial z}{\partial \nu} = 0 \text{ on } \partial \Omega$$

$$\Rightarrow \mu^2_{n,m} = \pi^2 \left( \frac{n^2}{p^2} + \frac{m^2}{q^2} \right) ; z_{n,m}(x,y) = \cos \frac{n\pi x}{p} \cos \frac{m\pi y}{q} ; n, m \in \mathbb{N}_0$$

All discrete modes $z_{n,m}(x,y)$ with $\mu_{n,m} \in (\hat{\mu}, \overline{\mu})$ (from (5.17)) are unstable.

asymptotic behaviour:

$$w(x,y,t) \sim \sum_{n,m} \alpha_{n,m} c_{n,m} e^{\lambda_{n,m} t} \cos \frac{n\pi x}{p} \cos \frac{m\pi y}{q} \quad (\text{sum over unstable modes})$$

Expected 2D-pattern, mode $(1, 1)$:

<table>
<thead>
<tr>
<th>u∞ &gt; u0</th>
</tr>
</thead>
<tbody>
<tr>
<td>d∞ &gt; u0</td>
</tr>
<tr>
<td>-----------</td>
</tr>
<tr>
<td>d∞ &gt; u0</td>
</tr>
</tbody>
</table>

References: [Mu] §14.4

5.4 Animal coat color patterns

Explanation ansatz: coat color patterns correspond to a bio-chemical “prototypical pattern” which is formed during pregnancy.

“experimental” reaction-diffusion model:

$$\begin{cases}
u_t = \gamma f(u, v) + \Delta u \\
v_t = \gamma g(u, v) + d \Delta v
\end{cases} \quad (5.19)$$

$$f(u, v) := a - u - h(u, v), \quad g(u, v) := \alpha (b - v) - h(u, v),$$
Pattern formation / reaction-diffusion equations

\[ h(u,v) := \frac{\rho uv}{1 + u + Ku^2} \] (rather “invented” function)

Parameter \( a, b, \alpha, \rho, K > 0; d > 1 \)

Scale parameter \( \sqrt{\gamma} \) proportional to typical length scale.

Region \( \Omega \) for animal leg or tail: surface of cylinder (resp. trunacted pyramid)

Eigenvalue problem (5.8) on \( \Omega \) with \( 0 < x < s, 0 < \theta < 2\pi \) leads to (with periodic BCs in \( \theta \); \( r \) = radius):

\[ \mu_{n,m}^2 = \frac{n^2}{r^2} + \frac{m^2 \pi^2}{s^2}, z_{n,m}(\theta,x) = \cos n\theta \cos \frac{m\pi x}{s}; n, m \in \mathbb{N}_0 \]

and

\[ z_{-n,m}(\theta,x) = \sin n\theta \cos \frac{m\pi x}{s}; n \in \mathbb{N}, m \in \mathbb{N}_0 \]

All discrete modes \( z_{n,m} \) with \( \mu_{n,m} \in (\mu, \mu) \) are unstable.

Effects:

from numerical simulations with FEM; solution of (5.19) for \( t \to \infty \) (up to stationary state).

- long, thin cylinder \( (0 < r \ll 1) \): all circumferential modes \( n \geq 1 \) are outside the band of instability \( (\mu, \mu) \) ⇒ only horizontal stripes (with \( n = 0 \))
- the thicker the cylinder, the higher circumferential modes are possible

Conclusion:

- Effects are described qualitatively correctly.
- Whether model (5.19) describes their evolution correctly, is (still) unclear. The qualitative influence of the length scale on the possible patterns is “quite independent” of the equation.

References: [Mu] §15.1

5.5 Pattern formation in 2-component mixtures / Cahn-Hilliard equation

Application: Phase separation (under dominant diffusion) in binary fluid mixtures (e.g. (liquid) metallic alloys, emulsions: vinegar-oil, Ouzo-water microemulsion).

\[ 0 \leq c_{1,2}(x,t) \leq 1 \] ... local concentration of 2 components
5.5 Pattern formation in 2-component mixtures / Cahn-Hilliard equation

Derivation of Cahn-Hilliard equation:

\[ \partial_t c_i + \text{div} \, J_i = 0 ; \quad i = 1, 2 \]

Assumptions: system isotherm, isobar, incompressible

\[ \Rightarrow c_1 + c_2 = 1, \quad \partial_t (c_1 + c_2) = 0, \quad J_1 + J_2 = 0 \]

choose \( c := c_1 - c_2 \in [-1, 1], \quad J := J_1 - J_2 \)

\[ \Rightarrow c_t + \text{div} \, J = 0, \quad \Omega \subset \mathbb{R}^d. \quad (5.20) \]

**phenomenological Derivation** of flux \( J = -L \nabla \mu \):

- \( L \geq 0 \) \ldots (const.) mobility
- \( \mu \) \ldots chemical “potential” (e.g., \( \mu = c \) with diffusion);
  defined as derivative of a potential (resp. variational derivative of free energy); \( \nabla \mu \) is driving force for evolution

- free energy for mixture (= necessary energy for “generation” of a system with def. temperature \( T \) which is in balance with the environment.)

\[
E(c) := \int_\Omega \left[ f(c) + \frac{\gamma}{2} |\nabla c|^2 \right] \, dx \in \mathbb{R}, \quad \gamma > 0 \text{ const.}
\]

\[ \frac{\gamma}{2} |\nabla c|^2 \ldots \text{ energy of phase boundary between } c = \pm 1; \]
  “penalizes” phase transitions

\[ f : \mathbb{R} \rightarrow \mathbb{R}, \quad \text{given function, bistable (i.e. with 2 minima), e.g.} \]

\[ f(c) = \alpha (c^2 - a^2)^2; \quad \alpha, a > 0. \]

- system desires minimization of \( E(c) \)
- \( \mu \) ist variational derivative of (non-convex) functional \( E \) (cf. Gâteaux derivative):

\[
\delta E(c, v) := \lim_{\varepsilon \to 0} \frac{E(c + \varepsilon v) - E(c)}{\varepsilon}
\]

\[
= \lim_{\varepsilon \to 0} \int_\Omega \frac{f(c + \varepsilon v) - f(c)}{\varepsilon} + \frac{\gamma}{2} \frac{|\nabla (c + \varepsilon v)|^2 - |\nabla c|^2}{\varepsilon} \, dx
\]

int. by parts

\[
= \int_\Omega f'(c) v - \gamma \Delta c v \, dx
\]
(5.21) \[ \mu(c) = \delta E(c) = -\gamma \Delta c + f'(c) \quad \text{as lin. functional} \]

insert into (5.20) \( \Rightarrow \) Cahn-Hilliard equation:

\[ c_t = L \Delta (-\gamma \Delta c + f'(c)), \quad \Omega \quad \text{(semilinear, 4th order)} \quad (5.22) \]

• possible BCs:
  a) periodic BC
  b) \( \frac{\partial c}{\partial v} = 0, \quad J \cdot \nu = -L \frac{\partial}{\partial v} (-\gamma \Delta c + f'(c)) = 0, \quad \text{i.e. vanishing flux through boundary} \)

• Idea of evolution:
  const. solutions \( c \) with \( f''(c) < 0 \) can be unstable (because diffusion term \( L \div (f''(c) \nabla c) \) appears; is dominant for small variations) \( \rightarrow \) pattern formation (coarsening for \( t \uparrow 0 \); “grains” develop out of almost one substance)

**Theorem 5.7.** Let \( c \) be classical solution of the Cahn-Hilliard eq. in \( \Omega := (0,l)^d \) with periodic or 0-flux BCs. \( \Rightarrow \)

1. \( \frac{d}{dt} \int_{\Omega} c \, dx = 0 \)
   \( \Rightarrow \int_{\Omega} c_i \, dx = \text{const, because } \int c_1 + c_2 \, dx = \int 1 \, dx = \text{const} \)

2. \( \frac{d}{dt} E(c(t)) \leq 0 \quad \text{(free energy is Lyapunov-functional)} \)

**Proof.**

1. \[ \frac{d}{dt} \int_{\Omega} c \, dx = L \int_{\Omega} \Delta (-\gamma \Delta c + f'(c)) \, dx \]
   \( \text{div Thm } \frac{d}{dt} \int_{\partial \Omega} c \, \nu \cdot \nabla (-\gamma \Delta c + f'(c)) \, ds \text{ BC } = 0 \)

2. \[ \frac{d}{dt} \int_{\Omega} \left( \frac{\gamma}{2} |\nabla c|^2 + f(c) \right) \, dx = \int_{\Omega} \gamma \nabla c \cdot \nabla c_t + f'(c) c_t \, dx \]
   \( \text{int. by parts } \int_{\Omega} [-\gamma \Delta c + f'(c)] c_t \, dx \)
   \( \text{int. by parts } -L \int_{\Omega} |\nabla [-\gamma \Delta c + f'(c)]|^2 \, dx \leq 0 \)
5.5 Pattern formation in 2-component mixtures / Cahn-Hilliard equation

Remark: also holds for weak solution

Theorem 5.8 ([EF], Th. 2.1). Let $\Omega = (0, l)$, $f$ be double sink potential with $f(c) = \gamma_2 c^4 + \gamma_1 c^3 + \gamma_0 c^2$, $c_0 \in H^2_E(\Omega) := \{ y \in H^2(\Omega) \mid y_x(0) = y_x(l) = 0 \}$.

For the Cahn-Hilliard equation (5.22) with boundary condition (b) we have:

(i) $\forall T > 0 \exists!$ solution $c \in L^2((0, T); H^4(\Omega))$ with $c_t \in L^2((0, T); L^2(\Omega))$.

(ii) If $c_0 \in H^6(\Omega) \cap H^2_E(\Omega)$ and $\frac{\partial^2}{\partial x^2} c_0 \in H^2_E(\Omega)$ then the solution $c$ is classical.

linear instability:
All constants $c = c_m \in \mathbb{R}$ solve the Cahn-Hilliard equation (5.22) (homogeneous stationary solution).
Disturbance $c = c_m + u$, $u$ small with $\int_\Omega u \, dx = 0$ (conservation of mass); let e.g. $L = 1$.
Linearization at $c_m$:

$$u_t = c_t = \Delta [-\gamma \Delta u + f'(c) - f'(c_m)]$$
$$\approx \Delta [-\gamma \Delta u + f''(c_m)(c - c_m)]$$
$$= -\Delta [\gamma \Delta u - f''(c_m) u]$$

(5.23)

Eigenfunctions of operator $u \mapsto -\Delta(\gamma \Delta u - f''(c_m) u)$ on $\Omega = (0, l)^d$ with periodic BCs:

$$\varphi_k(x) = e^{ik \cdot x}, \quad k \in K := \frac{2\pi}{l} \mathbb{Z}^d \setminus \{0\} \quad \text{(because } \int u \, dx = 0)$$
$$\lambda_k = |k|^2 \left( -\gamma |k|^2 - f''(c_m) \right)$$
$$= -\gamma \left( |k|^2 + \frac{f''(c_m)}{2\gamma} \right)^2 + \frac{f''(c_m)^2}{4\gamma} \in \mathbb{R}$$

(5.24)

Remark: $\{\varphi_k\}_{k \in K}$ ... Basis of $\{L^2(\Omega) \mid \text{periodic BC}, \int f \, dx = 0\}$

$\Rightarrow$ solution of (5.23) as linear combination:

$$u(x, t) = \sum_{k \in K} \alpha_k e^{\lambda_k t} e^{ik \cdot x}$$

- $u \equiv 0$ is unstable if an eigenvalue $\lambda_k > 0$; only possible for $f''(c_m) < 0$.
  Let $f''(c_m) < 0$. 


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• from (5.24): most unstable mode for largest eigenvalue, hence
\[
\left| k \right|^2 + \frac{f''(c_m)}{2\gamma} \rightarrow \min
\]

Let the solution be \( k_0 \).

\( \rightarrow \) most unstable wave length:
\[
l_0 := \frac{2\pi}{|k_0|} \approx 2\pi \sqrt{-\frac{2\gamma}{f''(c_m)}} \quad \text{(because } k \text{ discrete)}.
\]

\[\lambda_k\]
\[|k|^2\]

unstable wave numbers

- wave numbers \( |k|^2 > -\frac{f''(c_m)}{\gamma} \) are (linearly) stable

\( \rightarrow \) Region with \( l = \frac{2\pi}{|k|} < 2\pi \sqrt{-\frac{\gamma}{f''(c_m)}} \) does not allow for instability, i.e., no pattern formation.

Long term behaviour:

**Theorem 5.9** ([EF], Th. 2.1). Assumption of Thm 5.8: let \( \frac{1}{L} \int c_0 \, dx =: M \), and \( c \) be the unique solution of the Cahn-Hilliard Eq. with BC (b) \( \Rightarrow \)

\((1)\) \( c(t) \xrightarrow{t \to \infty} c_\infty \) in \( L^2(\Omega) \) with \( c_\infty \) is **one** solution of the stationary problem:
\[
\begin{aligned}
\gamma c''_\infty &= f'(c_\infty) - \alpha, & 0 < x < l, \\
0 &= c''_\infty(l), \\
\int c_\infty \, dx &= \int c_0 \, dx,
\end{aligned}
\]

and integration constant \( \alpha \in \mathbb{R} \) to be determined.

\((2)\) Solution of (5.25) is equivalent to finding critical points of \( E(c) \) in \( H^1(\Omega) \cap L^4(\Omega) \) under constraint \( G(c) := \int_\Omega c \, dx = M \). (by calculus of variations then \( c_\infty \) satisfies:
\[
\delta E(c) + \lambda \delta G(c) \stackrel{(5.21)}{=} -\gamma \Delta c + f'(c) + \lambda = 0, \quad \text{with } \delta G(c) = 1 \text{ and Lagrange multiplier } \lambda \in \mathbb{R}.)
\]

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5.5 Pattern formation in 2-component mixtures / Cahn-Hilliard equation

(3) $c(t) \xrightarrow{t \to \infty} M \ (= \text{const}) \ in \ L^2(\Omega) \ (\text{hence no phase separation}), \ if \ one \ of \ the \ 3 \ following \ conditions \ holds:

a) $\gamma > \frac{l^2}{\pi^2}$ \ and $\|c_0\|_2$ small enough;

b) $|M|$ large (because then solution of (5.25) is unique);

c) $\int (f(c_0(x)) - f_m) \, dx + \frac{\gamma}{2} \|c'_0\|_{L^2}^2$ small enough and $f(c_0(x)) > f_m \ \forall x \in (0,l)$, where $f_m := f(c_m)$ is a local minimum of $f$ and $|c_m - M|$ is small enough.

Remark 5.10.

(1) Solution of (5.25) in general not unique; $c_\infty \equiv M$ is always a solution.

(2) Stationary problem of Cahn-Hilliard Eq. (5.22):

$$(-\gamma c_{xx} + f'(c))_{xx} = 0, \ 0 < x < l \ \text{with} \ c_x(0) = c_x(l) = 0$$

$$(-\gamma c_{xx} + f'(c))_x \big|_{x=0,l} = 0$$

integrating twice gives (5.25).

(3) ad stationary problem (5.25):

For $M = 0$ and $f(c) := \frac{c^4}{4} - \frac{c^2}{2}$ (5.25) has exactly $2N_0 + 1$ solutions, where $N_0 = \left\lfloor \frac{4}{\pi l \sqrt{\gamma}} \right\rfloor \ldots$ Gauss bracket. One solution is $c_\infty \equiv 0$. If $c(x)$ is solution $\Rightarrow -c(x)$ is solution.

Proof. of Theorem 5.9 (3c):

from $E(c(t)) \searrow$:

$$E(c) = \int_0^l f(c(x)) \, dx + \frac{\gamma}{2} \|c'_0\|_{L^2}^2 \leq E(c_0)$$

Sobolev embedding + Poincaré inequality (for $c - M \in H^1(\Omega)$, $\int_0^l (c - M) \, dx = 0$) $\Rightarrow$

$$\int_0^l f(c) \, dx + \frac{C\gamma}{2} \|c - M\|_{L^\infty}^2 \leq E(c) \leq \int_0^l f(c_0) \, dx + \frac{\gamma}{2} \|c'_0\|_{L^2}^2 \bigg| - f_ml$$

$\Rightarrow$ with $\frac{1}{2}\|c - c_m\|_{L^\infty}^2 \leq \|c - M\|_{L^\infty}^2 + |c_m - M|^2$:

$$\int_0^l (f(c) - f_m) \, dx + \frac{C\gamma}{4} \|c - c_m\|_{L^\infty}^2 \leq \int_0^l (f(c_0) - f_m) \, dx + \frac{C\gamma}{2} |c_m - M|^2 + \frac{\gamma}{2} \|c'_0\|_{L^2}^2 =: \varepsilon$$

(5.26)
by assumption $\varepsilon$ "small enough".

Now let $c_0$ such that $\varepsilon < \frac{C_\gamma}{8} (c_m - c_b)^2$ and $\int_0^l f(c_0) - f_m \, dx \geq 0$

Claim:

$$\|c(t) - c_m\|_{L^\infty} < c_m - c_b \quad \forall t \geq 0.$$  \hspace{1cm} (5.27)

Proof: From (5.26) for $t = 0$, hence $c = c_0$:

$$\|c_0 - c_m\|_{L^\infty}^2 \leq \frac{4}{C_\gamma} \varepsilon < \frac{1}{2} (c_m - c_b)^2 < (c_m - c_b)^2;$$

$c$ continuous in $t$ $\Rightarrow$ (5.27) holds on maximal interval $[0, t^*)$. Let $t^* < \infty$ and

$$\|c(t^*) - c_m\|_{L^\infty} \geq c_m - c_b.$$  \hspace{1cm} (5.28)

From (5.27): for $t \in [0, t^*)$: $c(x, t) \in (c_b, 2c_m - c_b)$; $f$ is convex

$$\Rightarrow \quad f(c(x, t)) \geq f_m \quad \forall x \in (0, l), \quad t \in [0, t^*)$$

$$\Rightarrow \quad \int_0^l f(c(t)) - f_m \, dx \geq 0 \quad \text{on} \quad [0, t^*)$$

$$\Rightarrow \quad (\text{from (5.26)}) \quad \frac{C_\gamma}{4} \|c(t) - c_m\|_{L^\infty}^2 \leq \varepsilon < \frac{C_\gamma}{8} (c_m - c_b)^2$$

$$\Rightarrow \quad \|c(t) - c_m\|_{L^\infty} < \frac{1}{\sqrt{2}} (c_m - c_b) \quad \text{on} \quad [0, t^*) \ldots \text{contradiction to (5.28).}$$

Hence (5.27) $\forall t \geq 0$.  \hspace{1cm} \square
5.5 Pattern formation in 2-component mixtures / Cahn-Hilliard equation

From (5.27): \( f''(c(x,t)) \geq 0 \quad \forall x \in (0,l), \quad t \geq 0 \)

\[
c_t = (-\gamma c_{xx} + f'(c))_{xx} \quad \cdot (c - M), \quad \int_0^l dx
\]

\[
\Rightarrow \quad \frac{1}{2} \frac{d}{dt} \|c - M\|_{L^2}^2 + \gamma \|c_{xx}\|_{L^2}^2 \quad \text{int. by parts} \quad \Rightarrow \quad \int_0^l f''(c) (c_x)^2 \, dx \leq 0.
\]

With 2x Poincaré inequality (due to \( \int_0^l (c - M) \, dx = 0 \)) and with \( c_x(0) = 0 \) we obtain:

\[
\|c - M\|_{L^2} \leq C_p \|c_x\|_{L^2} \leq \frac{C_p l}{\sqrt{2}} \|c_{xx}\|_{L^2}.
\]

\[
\Rightarrow \quad \frac{d}{dt} \|c - M\|_{L^2}^2 \leq -\frac{4\gamma}{C_p^2 l^2} \|c - M\|_{L^2}^2
\]

\[
\Rightarrow \quad \|c(t) - M\|_{L^2} \leq e^{-\frac{2\gamma}{C_p^2 l^2} t} \|c_0 - M\|_{L^2}, \quad t \geq 0
\]

\[\blacksquare\]

Remark: In Theorem 5.9 (3) \( f'' \geq 0 \) is essential, while for linear instability \( f''(c_m) < 0 \)
was necessary.

References: [EGK] §6.2.13, [EF], [TE]
5 Pattern formation / reaction-diffusion equations
6 Problems with free boundary / thin-film equation

Examples:

- Flow in porous medium \( u_t = \Delta u^\alpha, \alpha > 1 \); \( \partial(\text{supp } u(t)) \) is free boundary: dependent on time and solution

- Phenomena of melting and solidifying ("Stefan-Problem"): boundary layer between liquid and solid phase is free boundary

- Obstacle problem for elastic membrane → course "calculus of variations"

- Evolution (resp. flow) of thin (wetting) liquid films on flat surface; free boundary = \( \partial(\text{supp } h(t)) \)

6.1 Deduction from Navier-Stokes equation

NS-equation for homogeneous incompressible flow:

\[
\begin{align*}
\varrho_0 \left[ u_t + (u \cdot \nabla)u \right] + \nabla p &= \mu \Delta u \\
\text{div } u &= 0
\end{align*}
\]

im region

\[
\Omega(t) = \left\{ (x', x_3) = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x' \in \Omega', \ 0 < x_3 < h(x', t) \right\}; \ \Omega' \subset \mathbb{R}^2 \ldots \text{bounded region smooth, pos.}
\]
6 Problems with free boundary / thin-film equation

- on fixed boundary \((x', x_3)\) with \(x' \in \partial \Omega'\) or \(x_3 = 0\): no-slip boundary condition \(u = 0\).
- wanted: BC on free surface \(\Gamma(t) = \{(x', h(x', t)) \mid x' \in \Omega'\}\).

particle trajectory: \((x'(t), x_3(t))\) with tangential vector \(u(x'(t), x_3(t))\).

Idea: free boundary moves along with fluid:

\[
\frac{d}{dt} x'(t) = (u_1, u_2)(x'(t), h(x'(t), t), t) \quad \text{... projected trajectory,}
\]

\[
\frac{d}{dt} h(x'(t), t) = u_3(x'(t), h(x'(t), t), t)
\]

\[\Rightarrow \text{kinematic BC on } \Gamma(t):\]

\[
u_3 = \partial_t h + u_1 \partial_{x_1} h + u_2 \partial_{x_2} h
\]

(6.2)

- Balance of force on surface between tension and capillary forces:

\[
T\nu = \gamma \kappa \nu \quad \text{... surface tension (acts in normal direction)}
\]

Hence: tangential components of \(T\nu\) vanish:

\[
(T\nu)_{\text{tang}} = 0; \quad (T\nu)_{\text{norm}} = \gamma \kappa
\]

(6.4)

Stress tensor \(T = 2\mu D - pI\) (as \(\text{div} \, u = 0\))

Deformation tensor \(2D = \nabla \otimes u + (\nabla \otimes u)^T\)

\[
\gamma \quad \text{... const}
\]

\[
\kappa = \text{div}_{x'} \left( \frac{\nabla_{x'} h}{\sqrt{1 + |\nabla_{x'} h|^2}} \right) \quad \text{... mean curvature}
\]
Scaling:

\[ L \quad \text{typical length scale (horizontal)} \]
\[ H \quad \text{typical height of film} \]
\[ V \quad \text{typical velocity scale (horizontal)} \]

\[ x_i = L \hat{x}_i; \quad i = 1, 2; \quad x_3 = H \hat{h}_3; \quad h = H \hat{h} \quad \text{with} \quad \varepsilon := \frac{H}{L} \ll 1 \]
\[ u_i = V \hat{u}_i; \quad i = 1, 2; \quad u_3 = \varepsilon V \hat{u}_3; \quad t = \frac{L}{V} \hat{t}; \quad p = \frac{\varepsilon \gamma}{L} \hat{p}; \quad V := \frac{\varepsilon^3 \gamma}{\mu} \]
\[ Re := \frac{\rho_0 LV}{\mu} \quad \text{... Reynolds number} \]

The scalings of \( u_i, \ t, \ p \) arise naturally; the choice of \( V \) (later on) gives the “correct” balance between pressure term and viscosity.

Scaled NS-equation (notation ‘\( \hat{\cdot} \)’ for scaled variable is omitted from now on):

\[ \varepsilon^2 Re \left[ \partial_t u_i + (u \cdot \nabla)u_i \right] + \partial_{x_i} p = \left( \varepsilon^2 \partial^2_{x_1} + \varepsilon^2 \partial^2_{x_2} + \partial^2_{x_3} \right) u_i; \quad i = 1, 2 \]  
\[ \varepsilon^2 Re \left[ \partial_t u_3 + (u \cdot \nabla)u_3 \right] + \varepsilon^{-2} \partial_{x_3} p = \left( \varepsilon^2 \partial^2_{x_1} + \varepsilon^2 \partial^2_{x_2} + \partial^2_{x_3} \right) u_3 \]
\[ \text{div} \ u = 0 \quad \text{(6.5)} \]

\[ \varepsilon^2 Re \left[ \partial_t u_3 + (u \cdot \nabla)u_3 \right] + \varepsilon^{-2} \partial_{x_3} p = \left( \varepsilon^2 \partial^2_{x_1} + \varepsilon^2 \partial^2_{x_2} + \partial^2_{x_3} \right) u_3 \]
\[ \text{div} \ u = 0 \quad \text{(6.6)} \]

Assumptions: \( \varepsilon^2 Re \ll 1, \quad \varepsilon \ll 1 \)

\[ \Rightarrow \quad \text{dominant } \varepsilon\text{-order in (6.5), (6.6) (} \rightarrow \text{“lubrication-approximation”:\quad)} \]

\[ \partial^2_{x_3} u_i = \partial_{x_i} p; \quad i = 1, 2 \]
\[ \partial_{x_3} p = 0 \quad \text{(also } p = p(x', t)) \]

\[ \text{div} \ u = 0 \quad \text{(6.7)} \]

Solutions of (6.7) with BCs \( u_i(x_3=0)=0, \partial_{x_3} u_i(x_3=h)=0 \) (see (6.9) below); \( i = 1, 2 \):

\[ u_i(x, t) = \partial_{x_i} p(x', t) \left[ \frac{x_3^2}{2} - h(x', t) x_3 \right]; \quad i = 1, 2 \]
\[ \text{(cf. Poiseuille-flow)} \]

On free boundary \( x_3 = h(x', t) \) (with \( \partial_{x_3} h = O(\varepsilon) \)):

\[ \nu = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + O(\varepsilon), \quad \kappa = \varepsilon \Delta h + O(\varepsilon^2). \]
Consider (6.3) (unscaled !) for particular $\nu$:

$$T = \mu \left( \begin{array}{c}
\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \\
\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \\
2 \frac{\partial u_3}{\partial x_3}
\end{array} \right) - \left( \begin{array}{c}
0 \\
0 \\
p
\end{array} \right)$$

Magnitude of above terms after scaling: $O(\varepsilon^2), O(\varepsilon^4), O(\varepsilon^3), O(\varepsilon)$

Dominant $\varepsilon$-order of tangential component $(x_1, x_2)$ in balance (6.4) at $x_3 = h(x')$ is $O(\varepsilon^2)$:

$$\partial_{x_3} u_i = 0; \quad i = 1, 2.$$  \hfill (6.9)

Dominant $\varepsilon$-order of normal component $(x_3)$ in (6.4) is $O(\varepsilon)$:

$$-p = \Delta h \quad \text{(in scaled variables)}$$  \hfill (6.10)

Integrate $\text{div} \ u = 0$ in $x_3$:

$$0 = \int_0^{h(x_1,x_2,t)} (\partial_{x_1} u_1 + \partial_{x_2} u_2) \ dx_3 + u_3(x', h(x', t), t) - u_3(x', 0, t);$$

from kinematic BCs (6.2) at $x_3 = h(x')$:

$$\partial_t h = u_3 - u_1 \partial_{x_1} h - u_2 \partial_{x_2} h$$

$$= u_3 - \int_0^{h(x_1,x_2,t)} (\partial_{x_1} u_1 + \partial_{x_2} u_2) \ dx_3 - u_1 \partial_{x_1} h - u_2 \partial_{x_2} h$$

$$= - \text{div}_{x'} \left( \int_0^{h(x_1,x_2,t)} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \ dx_3 \right) \quad \text{(6.8)}$$

$$= - \text{div}_{x'} \left( -\nabla_{x'} p(x', t) \frac{h^3}{3} \right)$$

Flux function

With (6.10):

$$h_t = - \text{div} \left( \frac{h^3}{3} \nabla \Delta h \right) \ldots \text{ thin film equation for } h(x_1,x_2,t),$$

(quasilinear, 4th order)

- Evolution driven by surface tension, slowed down by viscosity
- Applications: movement of drop of water, (oil) lubrication, (paint) coating processes

References: [EGK] §7.10-11, [My]
6.2 Boundary conditions

more general thin film equations:

\begin{align*}
\begin{cases}
  h_t &= -\text{div}(h^n \nabla \Delta h), \quad x \in \mathbb{R}^d; \quad 0 < n \leq 3 \\
  h(\cdot, 0) &= h_0 \geq 0
\end{cases}
\end{align*}

(6.11) holds on \{h > 0\}.

wanted: BCs on free boundary \(\partial\{h > 0\}\).

Caution: in §6.1 the surface of the liquid was the free boundary, now it is the boundary of the liquid film.

(6.11) is parabolic eq. of 4th order with free boundary \(\rightarrow\) 3 BCs at every \(x \in \partial\{h > 0\}\) needed:

1) \(h = 0\) on \(\partial\{h > 0\}\)

2) contact angle \(\theta\) of the liquid at the intersection between fluid, der Flüssigkeit am Schnittpunkt zwischen Flüssigkeit, support, air \(\rightarrow\) results from three surface tensions between two materials each (Young-Dupré law)

a) \(\theta \neq 0\) (e.g. water drops on plastic)

b) \(\theta = 0\) (e.g. water drops on very clean glass, netting), \(h_x = 0\) on \(\partial\{h > 0\}\)

3) Speed of propagation of contact line:

First special case \(n = 1, d = 1\) with BC 2b); hence

\(h_t + (hh_{xxx})_x = 0\).

Formally \(V := h_{xxx}\) on \(\partial\{h > 0\}\) is the speed of propagation of the free boundary (compare linear transport equation \(\rightarrow\) hyperbolic). Movement of contact line back and forth is possible.

Formulation as free boundary value problem:

\begin{align*}
\begin{cases}
  h_t + (hh_{xxx})_x &= 0, \quad \text{in} \quad \{h > 0\} \\
  h &= h_x = 0 \quad \text{on} \quad \partial\{h > 0\} \\
  V &= h_{xxx} \quad \text{on} \quad \partial\{h > 0\} \\
  h(\cdot, 0) &= h_0
\end{cases}
\end{align*}

This is a coupled evolution system for \(h(x, t)|_{\{h>0\}}, a(t), b(t)\).
6 Problems with free boundary / thin-film equation

![Diagram](https://via.placeholder.com/150)

\( h(x, t) \)

\( V_a \leftarrow \frac{\partial h}{\partial x} V_b \rightarrow \)

\( a(t) \quad b(t) \)

Deduction of \( V = h_{xxx} \) for smooth solutions:

wlog let the (only) free boundary at \( t = 0 \) be at \( x = 0 \).

coordinate transformation

\[
y := x - \int_0^t V(\tau) \, d\tau \quad \Rightarrow \quad \text{problem with fixed boundary for } \tilde{h}(y, t) := h(x, t)
\]

\[
\begin{align*}
\tilde{h}_t - \tilde{h}_y V(t) + (\tilde{h} \tilde{h}_{yyy})_y &= 0, \quad \text{in } (0, \infty)^2 \\
\tilde{h} &= \tilde{h}_y = 0, \quad y = 0 \quad t > 0 \\
\tilde{h}(\cdot, 0) &= h_0
\end{align*}
\]

\( \partial_y \) in (6.12):

\[
\Rightarrow \quad 0 = \tilde{h}_{yy} - \tilde{h}_y V + (\tilde{h} \tilde{h}_{yyy})_y
\]

\[
= \tilde{h}_{yy} - \tilde{h}_y V + \tilde{h}_{yy} \tilde{h}_{yyy} + 2 \tilde{h}_y \partial_y^2 \tilde{h} + \tilde{h} \partial_y^2 \tilde{h}
\]

At \( y = 0 \) we have with (6.13):

\( \tilde{h}_{yy}(0, t) \left[ V(t) - \tilde{h}_{yyy}(0, t) \right] = 0 \)

If \( \tilde{h}_{yy}(0, t) \neq 0 \), then \( V = h_{xxx} \).

Generalisation to \( d \in \mathbb{N}, n > 0 \) (Proof: [GR] §9):

\[
V(x_0) = \lim_{x \to x_0} \frac{h^{n-1}}{x \in \text{supp}(h(x,t))} \frac{\partial}{\partial \nu} \Delta h(x, t), \quad x_0 \in \partial \{ h > 0 \}
\]

References: [Kn] §1.1, §2.12

6.3 Positivity of the solution

Parabolic equations of 4th order in general have no maximum principle (→ Exercises), but degenerateness of (6.11) “prevents” \( h < 0 \).

- technical aid: integral estimates
6.3 Positivity of the solution

• multiply (6.11) by $\Delta h$; integration over $\mathbb{R}^d \times (0, T)$ formally gives

$$-\frac{1}{2} \int_0^T \partial_t \|\nabla h\|^2 \, dt = \int_0^T h^n |\nabla \Delta h|^2 \, dx \, dt$$

and hence the energy estimate:

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla h|^2 (T) \, dx + \int_0^T \int_{\mathbb{R}^d} h^n |\nabla \Delta h|^2 \, dx \, dt = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla h|^2 (0) \, dx$$  \hspace{1cm} (6.14)

⇒ energy $\searrow$ (if $\|\nabla h(0)\|_{L^2} < \infty$)

• “entropy” $\int_{\mathbb{R}^d} G(h) \, dx$ defined using

$$G(s) := \int_A^s g(r) \, dr, \quad g(s) := \int_A^s |r|^{-n} \, dr, \quad (A > 0; \text{large enough})$$

Entropy $\geq 0$ (see (6.19)).

• multiply (6.11) by $G'(h) = g(h)$; integration over $\mathbb{R}^d \times (0, T)$ formally gives

$$\int_{\mathbb{R}^d} \int_0^T h_t G'(h) \, dt \, dx = \int_0^T \int_{\mathbb{R}^d} (h^n \nabla \Delta h) \cdot \nabla g(h) \, dx \, dt$$

and hence the entropy estimate:

$$\int_{\mathbb{R}^d} G(h(T)) \, dx + \int_0^T \int_{\mathbb{R}^d} (\Delta h)^2 \, dx \, dt = \int_{\mathbb{R}^d} G(h(0)) \, dx.$$  \hspace{1cm} (6.15)

⇒ entropy $\searrow$ (if $\int G(h(0)) \, dx < \infty$)

Problem: the above calculations are only valid for “smooth solutions”!

For the following rigorous result consider with $1 < n < 4$:

$$\begin{cases}
  h_t = -(h^n h_{xxx})_x, & x \in \Omega = (-a, a), \quad t > 0 \\
  h_x = h_{xxx} = 0, & x = \pm a \\
  h(., 0) = h_0 \in H^1(-a, a)
\end{cases}$$  \hspace{1cm} (6.16)
Theorem 6.1.

a) \( \exists \) “weak solution” \( h \in C([-a,a] \times [0,\infty)) \) (details in [BF] §3);  
(Rem: in general no uniqueness because weak formulation has “not enough” BCs. Subject largely unsettled.)

b) Additionally suppose \( n \geq 2, h_0 \geq 0 \) and \( \int_\Omega |\ln h_0| \, dx < \infty \) (if \( n = 2 \)) resp. \( \int_\Omega h_0^{2-n} \, dx < \infty \) (if \( 2 < n < 4 \)) \( \Rightarrow \int_\Omega G(h_0) \, dx < \infty \).  
\( \Rightarrow \) solution from (a) satisfies \( h(x,t) \geq 0 \).

Idea of proof:

a) non-degenerate approximation problems:

\[
\begin{aligned}
\partial_t h_\varepsilon &= - ( |h_\varepsilon|^n + \varepsilon ) \partial_3^2 h_\varepsilon , \quad \Omega \times (0,\infty) \\
\partial_x h_\varepsilon &= \partial_3^2 h_\varepsilon = 0 , \quad x = \pm a \\
h_\varepsilon(\cdot,0) &= h_{0\varepsilon} \in C^{4,\alpha}(\Omega) \quad \text{(Hölder continuous)}
\end{aligned}
\]  
(6.17)

with \( h_{0\varepsilon} \geq h_0, \ h_{0\varepsilon} \xrightarrow{\varepsilon \to 0} h_0 \) in \( H^1(\Omega) \), \( \partial_x h_{0\varepsilon} = \partial_3^2 h_{0\varepsilon} = 0 \) on \( x = \pm a \).

\( \Rightarrow \) (6.17) has unique classical solution \( h_\varepsilon \); subsequence satisfies \( h_{\varepsilon} \to h \) uniformly in \( [-a,a] \times [0,T] \) \( \forall T > 0 \) (via a-priori estimates, compactness; details in [BF] §2-3). Sign of \( h_\varepsilon \) can change!

b) Step 1: deduction of 2 integral estimates for \( h_\varepsilon \) is rigorous.

Analogously to (6.14):

\[
\frac{1}{2} \int_\Omega |\partial_x h_\varepsilon|^2 (T) \, dx + \int_0^T \int_\Omega \left( |h_\varepsilon|^n + \varepsilon \right) |\partial_3^2 h_\varepsilon|^2 \, dx \, dt = \frac{1}{2} \int_\Omega |\partial_x h_\varepsilon|^2 (0) \, dx
\]

\( \Rightarrow \int_\Omega |h_{\varepsilon,x}|^2 (T) \, dx \leq \int_\Omega |h_{0\varepsilon,x}|^2 \, dx \leq 2 \int_\Omega |h_{0,x}|^2 \, dx \quad \forall \varepsilon \leq \varepsilon_1 \) (from \( H^1 \)-convergence)

(6.18)

(6.17) is in divergence form \( \Rightarrow \int_\Omega h_\varepsilon(T) \, dx = \int_\Omega h_{0\varepsilon} \, dx \)

\( \Rightarrow \) with Sobolev embedding, Poincaré, (6.18):

\[
|h_\varepsilon(x,t)| \leq C \|h_\varepsilon(t)\|_{H^1} \leq C + C \|\partial_x h_\varepsilon(t)\|_{L^2} \leq: A \quad \forall x \in \Omega, \quad \forall t > 0, \quad \forall \varepsilon \leq \varepsilon_1.
\]

Analogously zu (6.15):

\[
\text{with } g_\varepsilon(s) := - \int_s^A \frac{dr}{|r|^n + \varepsilon} \leq 0 , \quad G_\varepsilon(s) := - \int_s^A g_\varepsilon(r) \, dr \geq 0 \quad \text{(für } s \leq A) \quad (6.19)
\]
6.3 Positivity of the solution

\[\int_{\Omega} G_\varepsilon(h_\varepsilon(T)) \, dx + \int_0^T \int_{\Omega} |\partial_x^2 h_\varepsilon|^2 \, dx \, dt = \int_{\Omega} G_\varepsilon(h_{0\varepsilon}) \, dx \]

\[\leq \int_{\Omega} G(h_{0\varepsilon}) \, dx \leq \int_{\Omega} G(h_0) \, dx < \infty \quad (6.20)\]

Step 2: to show: \(h(x, t) \geq 0\).

Assumption: let \(h(x_0, t_0) < 0\)

\[\Rightarrow \text{ (due to uniform convergence } h_\varepsilon) \begin{array}{c} \exists \delta > 0, \varepsilon_0 > 0 \text{ with } \\
\end{array} \]

\[h_\varepsilon(x, t_0) < -\delta \quad \text{for} \quad |x - x_0| < \delta, \ x \in \Omega, \ \varepsilon < \varepsilon_0.\]

For these \(x\) we have:

\[G_\varepsilon(h_\varepsilon(x, t_0)) = \left[ - \int_{h_\varepsilon(x, t_0)}^A g_\varepsilon(r) \, dr \right]_{\leq 0}^{\leq 0} - \int_{-\delta}^{0} g_\varepsilon(r) \, dr \underset{\varepsilon \to 0}{\longrightarrow} - \int_{-\delta}^{0} g(r) \, dr \overset{n \geq 2}{\longrightarrow} +\infty\]

\[\Rightarrow \lim_{\varepsilon \to 0} \int_{\Omega} G_\varepsilon(h_\varepsilon(t_0)) \, dx = \infty \quad \text{(contradiction to (6.20))}\]

Rem:

1) Discrete analoga of energy and entropy estimates are important for numerical schemes
\[\Rightarrow \text{num. solution } \geq 0, \text{ (probably) uniqueness (subject still unsettled).}\]

2) Film rupture (i.e. \(h(x_0, t_0) = 0\)) for \(n < \frac{1}{2}\) possible (rigorously proven) \(\rightarrow\) no max-principle!

3) \(h > 0\) (i.e. prevention of film rupture) is of technological importance: oil lubrication, continuous coverage of paint.

References: [Be] §3, [BG] §2, [BF] §3, 4
6 Problems with free boundary / thin-film equation
7 Collective behaviour - kinetic equations

Applications: Many self-moving objects of similar size and shape (insects, fish, birds, pedestrians, many robots) often show complex global behaviour – despite simple individual rules of interaction.

The models described here are based on detailed observations of individual interactions (much more well-founded as with most applications of turing instabilities).

For the interactions there often are 3 typical distances around a central object:

7.1 microscopic ODE-models

Model 1 (2006)
Collective behaviour - kinetic equations

\[ x_i \in \mathbb{R}^d; \quad i = 1, \ldots, N \quad \text{positions of } N \text{ objects} \]
\[ v_i \in \mathbb{R}^d \quad \text{their velocities} \]

Evolution in Newtonian Form:
\[
\frac{dx_i}{dt} = v_i \\
\frac{dv_i}{dt} = (\alpha - \beta |v_i|^2)v_i - \frac{1}{N} \sum_{j \neq i} \nabla U(|x_i - x_j|) \\
\] (7.1)

→ asymptotic speed = \(\sqrt{\alpha/\beta}\)

typical pair potentials (cf. Morse-, Lennard-Jones potentials in atom physics):
\[ U(r) = -C_A e^{-r/l_A} + C_R e^{-r/l_R}, \]
with \(C_R > C_A > 0, \quad l_A > l_R > 0, \quad \frac{C_A}{l_A^2} > \frac{C_R}{l_R^2}\)

Possible long-term effects in model (7.1): cluster formation (rotation); flock formation
(translation \(\forall i: \quad v_i = \hat{v} \in \mathbb{R}^d, \quad |\hat{v}| = \sqrt{\alpha/\beta}\))

Cucker-Smale model (2007)
\[
\frac{dx_i}{dt} = v_i \\
\frac{dv_i}{dt} = \sum_{j=1}^{N} a(|x_i - x_j|) (v_j - v_i) \\
\] (7.2)

with \(a(r) = \frac{1}{(1+r^2)^\gamma}; \quad \gamma \geq 0 \quad \text{... rate of communication} \)

Possible long-term effects in model(7.2): alignment of velocity, flock formation for \(\gamma < \frac{1}{2}\):
Theorem 7.1 (Flock formation; [CS]). Let \( d = 3, \gamma < \frac{1}{2} \). \( \exists \hat{X} \in \mathbb{R}^{3N \times 3N} \):
\[
\mathbb{R}^{3N \times 3N} \ni X(t) := (x_i(t) - x_j(t))_{1 \leq i, j \leq N} \xrightarrow{t \to \infty} \hat{X} \quad \text{(convergence of pair distances)};
\]
\exists \hat{v} \in \mathbb{R}^3: v_i(t) \xrightarrow{t \to \infty} \hat{v} \quad \forall i.

Model improvement: (e.g. for birds): alignment of velocities only in field of sight:
replace sum in (7.2) by
\[
\sum_{j \in \sigma_i(t)} \ldots \text{field of sight around own velocity vector, with}
\]
\[
\sigma_i(t) := \left\{ l \neq i \mid \frac{(x_l - x_i) \cdot v_i}{|x_l - x_i| |v_i|} \geq \cos \phi \right\} \quad \text{for some } \phi \in (0, \pi).
\]

7.2 mesoscopic PDE-models

For \( N \gg 1 \) it is often more practicable not to consider each individual “point” but only averaged models.

For \( x, v \in \mathbb{R}^d \) consider the \( x - v \)-phase space with probability density \( f(x, v, t) \); hence \( f \geq 0, \int \int f(x, v, t) \, dx \, dv = 1 \forall t \). \( f(x, v) \) should decay “sufficiently” fast for \( |x|, |v| \to \infty \).

Evolution of \( f \) according to kinetic equation:
\[
f_t + v \cdot \nabla_x f + \text{div}_v[(\alpha - \beta |v|^2)vf] - \text{div}_v[(\nabla_x U(|x|) *_x \rho)f] = 0, \quad t \geq 0, \quad (7.3)
f(x, v, 0) = f^0(x, v) \geq 0,
\]
with \( \rho(x, t) := \int_{\mathbb{R}^d} f(x, v, t) \, dv \geq 0 \ldots \text{location density (this is a boundary density and} \int \rho \, dx = 1). \)

This is a quadratic nonlinear Fokker-Planck-like equation (cf. plasma physics: for ion dynamics under electrostatic force).

Charakteristics for the second and third term of (7.3): \( \dot{X} = V, \dot{V} = (\alpha - \beta |V|^2)V, \) vgl. (7.1)

\( v \)-integration of (7.3) leads to continuity equation:
\[
\rho_t + \text{div}_x j = 0, \quad (7.4)
\]
with flux \( j(x,t) := \int v f(x,t) \, dv \).

Total energy:
\[
\mathcal{E}(t) := \frac{1}{2} \int \int f(x,v,t)|v|^2 \, dx \, dv + \frac{1}{2} \int \int U(|x-y|) \rho(x,t) \rho(y,t) \, dy =: E_{\text{kin}} + E_{\text{pot}} .
\]

**Lemma 7.2.**
\[
\mathcal{E}(t) \leq \max\{ \mathcal{E}(0), C + \frac{\alpha}{2\beta} \},
\]
with \( C := \frac{1}{2} \sup |U| \). (This implies \( E_{\text{pot}} \leq C \), as \( \int \rho \, dx = 1 \).)

**Proof.** For the kinetic energy of the second term of (7.3) we have:
\[
-\frac{1}{2} \int \int v \cdot \nabla_x f |v|^2 \, dx \, dv = -\frac{1}{2} \int \int \text{div}_x (v |v|^2 f) \, dx \, dv = 0 .
\]

For the kinetic energy of the 4th term of (7.3) we have with 2x integration by parts and (7.4):
\[
\frac{1}{2} \int \int |v|^2 \text{div}_v [(\nabla_x U(|x|) * \rho) f] \, dx \, dv = -\int \int v \cdot (\nabla_x U(|x|) * \rho) f \, dx \, dv
\]
\[
= \int (U(|x|) * \rho) \text{div}_x \left( \int v f \, dv \right) \, dx = -\int (U(|x|) * \rho) \rho_t \, dx
\]

The last tem cancels with the time derivative of the potential energy:
\[
\frac{dE_{\text{pot}}}{dt} = \frac{1}{2} \int \int U(|x-y|)[\rho_t(x) \rho(y) + \rho(x) \rho_t(y)] \, dy = \int \int U(|x-y|) \rho(y) \rho_t(x) \, dy \, dx
\]

With \( \int \int f \, dx \, dv = 1 \) we conclude:
\[
\frac{d\mathcal{E}}{dt} = \int \int f[\alpha - \beta |v|^2] |v|^2 \, dx \, dv \overset{\text{Hölder}}{\leq} \alpha \int \int f |v|^2 \, dx \, dv - \beta \left( \int \int f |v|^2 \, dx \, dv \right)^2 \leq 0 ;
\]

where the last inequality holds for \( \int \int f |v|^2 \, dx \, dv \geq \frac{\alpha}{\beta} \).

Hence: \( \frac{d\mathcal{E}}{dt} \leq 0 \) for \( \mathcal{E} \geq C + \frac{\alpha}{2\beta} \), as then \( E_{\text{kin}} = \mathcal{E} - E_{\text{pot}} \geq C + \frac{\alpha}{2\beta} - C = \frac{\alpha}{2\beta} \). \( \blacksquare \)

**Relationship to ODE-Model (7.1):**
(7.3) can be rigorously derived as “self-consistent” limit of (7.1) (cf. discrete vortex models). Conversely, (7.1) can be considered als numeric method (particle method) for (7.3); is also in use.

**Definition 7.3.** \( \mathcal{M}(\mathbb{R}) \) ... signed Radon measures with finite mass (can also be negative; inner regular and locally finite); can be identified with \( C_0(\mathbb{R})' \) (\( C_0 \) ... continuous functions with compact support).

\( \mathcal{P}^1(\mathbb{R}) \subset \mathcal{M}(\mathbb{R}) \) ... the subset of probability measures (which means \( \mu \geq 0, \int d\mu = 1 \)).
Let \((x_i^0, v_i^0)\) be the IC of (7.1) and
\[
f_N^0 := \sum_{j=1}^{N} m_j \delta_{(x_j^0, v_j^0)} \in \mathcal{P}^1(\mathbb{R}^{2d}) ,
\]
(7.5)
with \(m_j = \frac{1}{N}\) be the corresponding empirical measure in \(x-v\)-phase space.

Idea:
\[
f_N^0 \xrightarrow{N \to \infty} f^0 \quad \text{(weak * as measure, predual is } C_0(\mathbb{R}^{2d})).
\]
(7.6)

\textbf{Theorem 7.4} ("self-consistent" limit; cf. [BH, Ne, Do] for Vlasov equation). Let \(U \in C^2_0(\mathbb{R}_0^+)\) with \(U'(0) = 0\).

\(a\) Let \((x_i, v_i) \in C([0,T); \mathbb{R}^{2d}); i = 1, ..., N\) be solution of particle system (7.1).

⇒ The probability measure
\[
f_N(t) := \sum_{j=1}^{N} m_j \delta_{(x_j(t), v_j(t))} \in \mathcal{P}^1(\mathbb{R}^{2d}) ,
\]
with \(\sum_{j=1}^{N} m_j = 1\) (z.B. \(m_j = \frac{1}{N}\)) satisfies \(f_N \in C([0,T); \mathcal{P}^1(\mathbb{R}^{2d}))\) (weak *) and solves (7.3) with IC (7.5).

\(b\) Let \(f^0 \geq 0\) with \(|\mathcal{E}[f^0]| < \infty\). Assume that an approximative sequence (of empirical measures) of the IC satisfies (7.6) (weak * as measure), and that \(\mathcal{E}[f_N^0]\) is uniformly bounded.

⇒ \(\forall T > 0: f_N \xrightarrow{N \to \infty} f \in C([0,T]; \mathcal{P}^1(\mathbb{R}^{2d}))\) (weak *), where \(f\) is the unique solution of (7.3).

\textit{Idea of proof.} (only part a)

\textbf{Step 1:}

Let the “force field” \(E(x,t) := -\nabla_x U * \rho\) be given.

Assumptions: let \(E \in C(\mathbb{R}^d \times [0,T])\) be locally Lipschitz in \(x\) (uniformly in \(t \in [0,T]\)).

\[
f_t + v \cdot \nabla_x f + \text{div}_v[(\alpha - \beta |v|^2)v f] + E(x,t) \cdot \nabla_v f = 0 , \quad t \geq 0
\]
(7.7)
is a linear hyperbolic equation; corresponding characteristic equations:
\[
\begin{align*}
\frac{dX}{dt} &= V \\
\frac{dV}{dt} &= E(X,t) + (\alpha - \beta |V|^2)V
\end{align*}
\]
(7.8)
The measure transported by the flux of (7.8) solves (7.7).

**Step 2:**

For

\[
\rho_N(t) := \sum_{j=1}^N m_j \delta_{x_j(t)} \in \mathcal{P}^1(\mathbb{R}^d)
\]

we have

\[
\left( \nabla_x U(|.|) * \rho_N \right)(x) = \sum_{j=1}^N m_j \nabla U(|x - x_j|) \in C^1_b(\mathbb{R}^d).
\]

Hence the nonlinear term \((\nabla_x U(|.|) * \rho_N)f\) of (7.3) is also well-defined for empirical measures \(f_N\), and the coefficient function \(\nabla_x U(|.|) * \rho_N\) satisfies the assumptions of Step 1.

**References:** [CS], [BH, Ne, Do]
8 Nonlinear waves – Solitons

(only up to WS 2011/12)

• 1D wave equation: \( u_{tt} - c^2 u_{xx} = 0, \ x \in \mathbb{R}, \ t \in \mathbb{R} \)
  Solution: travelling waves \( u(x, t) = f(x - ct) + g(x + ct) \) with const. velocity, not changing profile
  linear equation \(\to\) superposition principle

• transport equation: \( u_t + cu_x = 0 \)
  \(\rightarrow\) wave propagation in only one direction

• dispersive wave equation: \( u_t + u_x + u_{xxx} = 0 \)
  harmonic wave solutions: \( u(x, t) = e^{i(kx - \omega t)} \)
  \(\rightarrow\) dispersion relation: \( \omega(k) = k - k^3 \)
  \( \omega, k \) (angular)frequency
  \( c = \frac{\omega}{k} = 1 - k^2 \) ... speed of propagation (phase velocity)
  \(\Rightarrow\) waves with different wave number are travel with different speeds \(\rightarrow\) wave “disperses”; profile of wave is not preserved.

  \[
  \text{Superposition: } u(x, t) = \int_{\mathbb{R}} A(k) e^{i(kx - \omega(k)t)} dk
  \]

• inviscid Burgers’ equation: \( u_t + uu_x = 0 \) develops shocks discontinuities (“shocks” \(\rightarrow\) large wave numbers \( k \) in solution) in finite time.
  nonlinear equation \(\rightarrow\) no superposition

• Korteweg - de Vries (KdV) equation: \( u_t + uu_x + u_{xxx} = 0 \)
  Change of variables \( u \mapsto \alpha u, \ t \mapsto \beta t, \ x \mapsto \gamma x \) \( (\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}) \) gives general form of KdV:
  \[
  u_t + \frac{\alpha \beta}{\gamma} uu_x + \frac{\beta}{\gamma^3} u_{xxx} = 0
  \]
  Standard choice of parameters:
  \[
  u_t - 6uu_x + u_{xxx} = 0 \quad (8.1)
  \]
Smooth solution exists for $t \in \mathbb{R}$; “dispersive regularization” of Burgers’ equation, i.e., wave components with large $|k|$ “travel away” more quickly. Dispersive term dampens large slopes; balance with nonlinearity.

(8.1) is invariant under the following group of transformations:

$$G_t, l \in \mathbb{R} \setminus \{0\} : X = lx, T = l^3 t, U = l^{-2} u$$

→ suggests the existence of similarity solutions

References: [DJ] §1

### 8.1 Applications of KdV

Long waves in a shallow canal can (seldom) have the form of solitons, i.e., do not change their shape:

$$u(x, t) = a \text{sech}^2[b(x - ct)], \quad (8.2)$$

$$b^{-2} = 4h^2(h + a)/3a, c^2 = g(h + a)$$

Assumption for “shallow water waves”: wave length $\gg$ water depth

$u$ ... wave height over level at rest

$a > 0$ ... amplitude

$h$ ... water depth

$c$ ... speed of propagation (depending on amplitude!)

$g$ ... gravitation constant

sech = $1/\cosh$ ... secans hyperbolikus

$u(x, t) = a \text{sech}^2[b(x - ct)]$,

$\lambda^2 = 4h^2(h + a)/3a, c^2 = g(h + a)$
8.1 Applications of KdV

Observed 1834 by J.S. Russel in Scotland (Fig. 8.1); is gavitational wave with constant mass transport in $x$-direction.

(8.2) satisfies KdV (with $\alpha\beta/\gamma = c/4b^2$, $\beta/\gamma^3 = 3bc/a$).

KdV can be derived for $\frac{a}{b} \ll 1$ from 2D incompressible, rotation-free, inviscid fluid equations (over horizontal plane with free surface) ([DJ] §1.2, [De] §9.3), or from 2D Euler equation ([Jo] §3.2.1).

(8.2) is gravitational wave, i.e. transport of mass.

Further applications: (simple) tsunami-model.

Superimposition of solitons:

fast, high soliton “overtakes” slow, low soliton: short “interaction” (with phase shift) but no change of form (Fig. 8.2).

→ almost a superposition principle, altough nonlinear equation.

Further completely integrable systems with soliton solutions:
Figure 8.2: 2 interacting solitons as function of $x$, $t$: the interaction effects a local displacement of both solitons.

- kubic nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} \pm |\psi|^2 \psi = 0, \quad x \in \mathbb{R}, \quad t > 0$$

Applications: nonlinear optics (dispersion-free message transmission in fiber optic cables), Bose-Einstein condensate

- Sinus-Gordon equation

$$\frac{1}{c^2} \psi_{tt} - \psi_{xx} + \sin \psi = 0$$

Applications: differential geometry (for surfaces with constant negative Gauss curvature), displacements in a crystal with periodicity $\sin \psi$

References: [DJ] §1.2-4, §8.2, [TE]
8.2 Schrödinger scattering problems for KdV

Aim: Solution (resp. construction of solution) of IVP

\[
\begin{aligned}
    u_t - 6uu_x + u_{xxx} &= 0, \quad x \in \mathbb{R}, \ t > 0 \\
    u(x, 0) &= u_0(x), \quad x \in \mathbb{R}
\end{aligned}
\quad (8.3)
\]

Approach: transformation of (8.3) in family of linear eigenvalue problems (with parameter \( t \geq 0 \); \( \psi \in \mathbb{C} \):

\[
\left[ -\frac{\partial^2}{\partial x^2} + u(x; t) \right] \psi(x; t) = \lambda(t) \psi(x; t).
\]

Gives stationäry Schrödinger equation for (real) potential \( u \).

“Miura-transformation”

\[
u = v^2 + v_x
\quad (8.4)
\]
gives from (8.3):

\[
(2v + \frac{\partial}{\partial x})(v_t - 6v^2v_x + v_{xxx}) = 0.
\quad \text{modified KdV (mKdV)}
\]

Hence: if \( v \) solves mKdV then \( u \) solves KdV.

Solution of the Riccati equation (8.4) (for \( t \) fixed) with substitution

\[
v = \psi_x/\psi, \quad \psi(x; t) \neq 0
\quad (8.5)
\]

\[\Rightarrow \psi_{xx} - u\psi = 0\]

KdV is Galilei invariant, i.e., invariant under transformations \( \tilde{x} = x + 6\lambda t, \tilde{u} = u - \lambda \) for \( \lambda \in \mathbb{R} \). Inserting into (8.4), (8.5) gives (\( t \) is only parameter!)

\[
\psi_{xx} + (\lambda - u)\psi = 0
\quad (8.6)
\]

Idea: 1) Solution of linear EVP (8.6) for \( \psi(x; t), t \geq 0 \).
2) (8.4), (8.5) then gives \( u(x, t) \).

At first this sounds “weird” because \( u \) is given coefficient in (8.6), but we need the scattering data \( S \) (i.e. eigenvalues \( \lambda(t) \), (generalized) eigenfunctions \( \psi(x; t) \)) only for \( t = 0 \), i.e. \( u_0(x) \):
8 Nonlinear waves – Solitons

\[ u_0(x) \quad \text{scattering data} \quad S(0) \]

KdV

\[ u(x, t) \quad \text{inverse scattering problem} \quad S(t) \]

Spectral theory of \( L = -\frac{\partial^2}{\partial x^2} + u \):

let \( u = u(x; t) \) be bounded, smooth; rapidly decays for \( |x| \to \infty \), because solution of KdV. \( t \geq 0 \) . . . parameter in operator \( L \).

a) finitely many eigenvalues:

\[ \lambda_n = -\kappa_n^2 < 0, \ \kappa_n > 0; \quad n = 1, 2, \ldots, N \]

asymptotic behaviour of real eigenfunctions (“bounded states”):

\[ \psi_n(x; t) \sim c_n(t)e^{-\kappa_n x}, \ x \to \infty, \quad (8.7) \]

\( c_n(t) \) from normalization \( \|\psi_n\|_{L^2(\mathbb{R})} = 1 \), \( \psi_n(x; t) \) also decays exponentially for \( x \to -\infty \).

b) continuous spectrum:

\( \lambda = k^2 > 0 \). Discussion here for \( k > 0 \); for \( k < 0 \) analogously:

generalized eigenfunctions (“scattering states”; \( \not\in L^2 \)) oscillate for \( |x| \to \infty \):

\[ \psi(x; t) \sim \begin{cases} e^{-ikx} + b(k; t)e^{ikx}, & x \to \infty \\ a(k; t)e^{-ikx}, & x \to -\infty \end{cases} \quad (8.8) \]

\( a \in \mathbb{C} \ldots \) transmission coefficient

\( b \in \mathbb{C} \ldots \) reflection coefficient

We have: \( |a|^2 + |b|^2 = 1 \) (conservation of momentum resp. flow in scattering process)
Remark: (8.6) even has \( \forall k = \sqrt{\lambda} \in \mathbb{C} \) solutions of the form (8.8), except in the upper half-plane for \( k_n = i\kappa_n; \ n = 1, ..., N \).

If \( u = u(x, t) \) solves KdV then also the scattering data of (8.6) have a simple \( t \)-dependence:

**Theorem 8.1.** Let \( u = u(x, t) \) be solution of (8.3). \( \Rightarrow \) The “bounded states” satisfy (for \( n = 1, \ldots, N; t \geq 0 \)):

\[
N = \text{const in } t; \\
\lambda_n(t) = \lambda_n(0); \\
c_n(t) = c_n(0)e^{4\kappa_n^3}.
\] (8.9)

**Proof.** Step 1: Differentiating (8.6) with respect to \( x \) resp. \( t \):

\[
\psi_{xxx} - u_x \psi + (\lambda - u) \psi = 0 \tag{8.10}
\]

\[
\psi_{xxt} + (\lambda_t - u_t) \psi + (\lambda - u) \psi_t = 0 \tag{8.11}
\]

Define

\[
R(x, t) := \psi_t + u_x \psi - 2(\lambda + 4\kappa_n^3)\psi_x
\]

\[
\Rightarrow \frac{\partial}{\partial x}(\psi_x R - \psi R_x) = \ldots = \psi_{xxx}(\psi_t + u_x \psi - 2u\psi_x - 4\lambda\psi_x)
\]

\[
- \psi(\psi_{xxt} + u_{xxx} \psi - 3u_x \psi_{xx} - 2u\psi_{xxx} - 4\lambda\psi_{xxxx})
\]

[\( \psi_{xxx} \) and \( \psi_{xxt} \) with (8.10), (8.11) eliminieren]

\[
\psi_{xxx}(\psi_t - 2u\psi_x - 4\lambda\psi_x) - \psi(\psi_{xxx} \psi - 4u\psi_{xx})
\]

\[
- \psi(u\psi_t - \lambda \psi_t - \lambda_t \psi + u_t \psi) + \psi(2u + 4\lambda)(u_x \psi - \lambda \psi_x + u\psi_x)
\]

\[
\equiv \psi^2(\lambda_t - u_t + 6uu_x - u_{xxx}) = \lambda_t \psi^2 \tag{8.12}
\]

Remark: (8.12) also holds for continuous spectrum \( \lambda > 0 \).
Let now \( \lambda = \lambda_n = -\kappa_n^2 < 0 \), \( \psi = \psi_n \), \( R = R[u, \psi_n] =: R_n \).

\( \psi_n \), \( R_n \) decay exponentially for \( |x| \to \infty \).

\[ 0 = \psi_{n,x} R_n - \psi_n R_{n,x} \bigg|_{-\infty}^{\infty} = \lambda_n t \int_{\mathbb{R}} \frac{\psi_n^2}{\psi_n} \, dx = \lambda_n t \quad \checkmark \]

Step 2:

\( \Rightarrow \) indefinite \( x \)-integral of (8.12) (i.e. \(-\partial_x(\psi_{n,x} R_n - \psi_n R_{n,x}) = 0\), because \( \lambda_{n,t} = 0 \)) gives:

\[ \psi_n R_{n,x} - \psi_{n,x} R_n = g_n(t), \quad g_n(t) \ldots \text{arbitrary integration constant} \quad (8.13) \]

\( \psi_n \), \( R_n \) decay for \( |x| \to \infty \) \( \Rightarrow \) \( g_n = 0 \) \( \forall t \geq 0 \).

indefinite \( x \)-integral of (8.13) (i.e. \( \frac{\psi_n R_{n,x} - \psi_{n,x} R_n}{\psi_n^2} = \frac{\partial_x R_n}{\psi_n} = 0 \)) :

\[ \frac{R_n}{\psi_n} = h_n(t), \quad h_n(t) \ldots \text{arbitrary integration constant} \quad (8.14) \]

Multiply by \( \psi_n^2 \), use (8.6):

\[ R_n \psi_n = [\psi_t + u_x \psi - 2(u + 2\lambda) \psi_x] \psi = \frac{1}{2} (\psi_n^2)_t + (u \psi_n^2 - 2 \psi_{n,x}^2 - 4 \lambda \psi_n^2)_x = h_n(t) \psi_n^2 \]

\( \int_{\mathbb{R}} \, dx \)-integration:

\[ 0 = \frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}} \psi_n^2 \, dx \right) = h_n(t) \int_{\mathbb{R}} \psi_n^2 \, dx \]

\( \Rightarrow \) \( h_n(t) = 0 \), \( \forall t \geq 0 \)

(8.14), d.h. \( R_n = 0 \) gives evolution of \( \psi_n(x; t) \):

\[ \psi_{n,t} = -u_x \psi_n + 2(u + 2\lambda_n) \psi_{n,x} \]

use \( u \to 0 \) \( \Rightarrow \) \( \psi_n \) —asymptotics (8.7):

\[ \Rightarrow c'_n(t) - 4\kappa_n^3 c_n(t) = 0 \]

\[ \Rightarrow c_n(t) = c_n(0) e^{4\kappa_n^3 t} \quad \checkmark \]
8.2 Schrödinger scattering problems for KdV

**Theorem 8.2.** Let \( u = u(x, t) \) be solution of (8.3). ⇒ The “scattering states” satisfy (\( \forall k > 0; t \geq 0 \)):

\[
a(k; t) = a(k; 0), \quad b(k; t) = b(k; 0)e^{8ik^3t}.
\]  

(8.15)

*Proof.* Let \( \lambda = k^2 > 0 \) be fixed (i.e. const in \( t \), because continuous spectrum \((0, \infty)\) is \( t \)-indep.); \( \psi \) the corresponding generalized eigenfunction; \( R = R[u, \psi] \).

Integrate (8.12) with respect to \( x \) (with \( \lambda t = 0 \)):

\[
\psi_x R - \psi R_x = g(t; k) \ldots \text{arbitrary integration constant}
\]  

(8.16)

According to (8.8):

\[
\psi(x; t, k) \sim a(k; t)e^{-ikx}, \quad x \to -\infty
\]

⇒ \( R(x, t; k) \sim \psi_t - 4\lambda \psi_x \sim \left( \frac{da}{dt} + 4ik^3a \right) e^{-ikx}, \quad x \to -\infty \)

⇒ \( \psi_x R - \psi R_x \xrightarrow{x \to -\infty} 0 \quad \Rightarrow \quad g(t; k) = 0 \quad \forall t \geq 0 \)

\( x \)-integration of (8.16):

\[
\frac{R}{\psi} = h(t; k) \ldots \text{beliebig}; \quad R = h\psi
\]  

(8.17)

\( x \to \infty \)—asymptotics of \( \psi, R \) leads to:

\[
\frac{da}{dt} + 4ik^3a = ha
\]  

(8.18)

analogous behaviour for \( x \to \infty \):

\[
R(x, t; k) \sim \frac{db}{dt}e^{ikx} + 4ik^3(e^{-ikx} - be^{ikx}) \quad \overset{(8.17),(8.8)}{=} \quad h(e^{-ikx} + be^{ikx}) \sim h\psi
\]

Because \( e^{\pm ikx} \) is linearly independent (comparing coefficients):

\[
\frac{db}{dt} - 4ik^3b = hb, \quad h(t; k) = 4ik^3
\]

⇒ \( b(k; t) = b(k; 0)e^{8ik^3t}, \)

\[
a(k; t) = a(k; 0) \quad \text{(from (8.18))}
\]

**Remark 8.3.** In formulas (8.9), (8.15) the exact form of \( u(x, t) \) does *not* enter. They give a lot of a-priori information for the KdV-evolution (“similar” to conserved quantity of evolution).

References: [De] §9.7, [DJ] §3.1-2,4.1-3, [Wh] §17.3
8.3 inverse scattering problem

Aim: Solution of nonlinear IVP

\[
\begin{aligned}
&\begin{cases}
u_t - 6u u_x + u_{xxx} = 0, & x \in \mathbb{R}, t > 0 \\
u(x, 0) = u_0(x), & x \in \mathbb{R}
\end{cases}
\end{aligned}
\]

in 3 steps:

1) linear eigenvalue problem \( \psi_{xx} + (\lambda - u_0(x))\psi = 0 \), \( x \in \mathbb{R} \) \( \rightarrow \) scattering data \( S(0) \)

2) explicit evolution of scattering data \( S(t), t \geq 0 \) (according to Thm 8.1, 8.2)

3) inverse scattering problem: reconstruction of \( u(x, t) \) from \( S(t) \) with linear integral equation

inverse scattering problem for \( t \) fixed:

\[
\psi_{xx} + (k^2 - u(x))\psi = 0 \quad , \quad x \in \mathbb{R}
\]

given: scattering data of (8.19) \( S = S(t) := \{-\kappa_1^2, \ldots, -\kappa_N^2; c_1, \ldots, c_N; b(k), k \in \mathbb{R}\} \) (e.g. obtained using Thm. 8.1, 8.2 from \( S(0) \))

wanted: potential \( u(x) = u(x; t) \)

Define for suitable decaying reflection coefficient \( b(k) \):

\[
F(\xi) := \sum_{n=1}^{N} c_n^2 e^{-\kappa_n \xi} + \frac{1}{2\pi} \int_{\mathbb{R}} b(k) e^{ik\xi} dk \quad , \quad \xi \in \mathbb{R}
\]

inverse Fourier trans.

Theorem 8.4 (inverse scattering theorem). Let \( F \) be rapidly decaying. \( \Rightarrow \)

\[
u(x) = -2 \frac{d}{dx} K(x, x),
\]

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with: $K(x, z)$ is the unique function on $\mathbb{R}^2$ such that $K(x, z) = 0$ for $z < x$ and satisfying the linear Fredholm integral equation:

$$K(x, z) + F(x + z) + \int_{x}^{\infty} K(x, y) F(y + z) dy = 0, \quad -\infty < x < z$$

("Gelfand-Levitan-Marchenko" (GLM)-equation).

**Idea of proof.** First discussion of direct scattering problem (8.19); Deduction of GLM-equation:

**Case 1:**

$L := -\frac{\partial^2}{\partial x^2} + u(x)$ has only continuous spectrum (e.g. for $u \geq 0$).

We are looking for solutions (for $k \in \mathbb{R}$ fixed) of the form ("Jost solutions")

$$\Phi_k(x) = e^{ikx} + \int_{x}^{\infty} K(x, z) e^{ikz} dz,$$

$$\Phi_{-k}(x) = e^{-ikx} + \int_{x}^{\infty} K(x, z) e^{-ikz} dz.$$  \hspace{1cm} (8.21)

If $K$ decays (suitably), then

$$\lim_{x \to \infty} \Phi_{\pm k}(x) = e^{\pm ikx}.$$  \hspace{1cm} (8.22)

**Aim:** Find equation for $K$ by inserting $\Phi_{\pm k}$ in (8.19):

$$\text{aus (8.21): } \Phi_{kxx} = e^{ikx} \left[ -k^2 - \frac{d}{dx} K(x, x) \right] + \int_{x}^{\infty} K_{xx} e^{ikz} dz$$

2x integration by parts in (8.21):

$$\Phi_k = e^{ikx} \left[ 1 + \frac{iK(x, x)}{k} - \frac{K_z(x, x)}{k^2} \right] - \frac{1}{k^2} \int_{x}^{\infty} K_{zz} e^{ikz} dz,$$

if $K(x, z), K_z(x, z) \to 0$ (such that the integrals exist):

$$\Rightarrow 0 \overset{(8.19)}{=} \Phi_{kxx} + (k^2 - u) \Phi_k =$$

$$= -e^{ikx} \left[ u + 2 \frac{d}{dx} K(x, x) \right] + \int_{x}^{\infty} (K_{xx} - K_{zz} - u(x) K) e^{ikz} dz$$
This holds if
\[ K_{xx} - K_{zz} - u(x)K = 0 \text{, } z > x, \text{ and } u(x) = -2\frac{d}{dx}K(x, x) = -2[K_x(x, x) + K_z(x, x)]. \] (8.23)

next aim: equation for \( K \) which only contains scattering data (but not \( u \)).

- \( \Phi_{\pm k}(x) \) linearly independent \( \Rightarrow \) are fundamental solutions of (8.19)
- generalized eigenfunctions according to (8.8):
  \[ \psi(x; t) \sim \begin{cases} 
  e^{-ikx} + b(k; t)e^{ikx}, & x \to \infty \\
  a(k; t)e^{-ikx}, & x \to -\infty 
  \end{cases} \]

\( \Rightarrow \) The particular solution with
  \[ \psi_k(x) \sim e^{-ikx} \text{ for } x \to -\infty, \text{ hence } \psi_k(x) = \frac{1}{a_k}\psi(x) \]

is:
\[ \psi_k(x) = \frac{1}{a(k)} \Phi_{-k}(x) + \frac{b(k)}{a(k)} \Phi_k(x) \] (8.24)

\( \Rightarrow a(k)\psi_k(x) = e^{-ikx} + \int_x^\infty K(x, z)e^{-ikz}dz \]
  \[ + b(k) \left[ e^{ikx} + \int_x^\infty K(x, z)e^{ikz}dz \right] \quad \forall x \in \mathbb{R}; \forall k \in \mathbb{R} \text{ fixed.} \]

inverse Fourier-transformation \( (k \to y) \) gives for \( y > x \):
\[ \frac{1}{2\pi} \int_{\mathbb{R}} a(k)\psi_k(x)e^{iky}dk \] (8.25)
\[ = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ik(y-x)}dk + \int_x^\infty K(x, z) \left[ \frac{1}{2\pi} \int_{\mathbb{R}} e^{ik(y-z)}dk \right]dz \]
  \[ = \delta(y-x) = 0 \text{ as } y > x \]
\[ + \frac{1}{2\pi} \int_{\mathbb{R}} b(k)e^{ik(x+y)}dk + \int_x^\infty K(x, z) \left[ \frac{1}{2\pi} \int_{\mathbb{R}} b(k)e^{ik(y+z)}dk \right]dz \]
  \[ = F(x+y) \text{ according to } (8.20) \]
\[ = K(x, y) + F(x + y) + \int_x^\infty K(x, z)F(y + z)dz, \]
8.3 inverse scattering problem

because \( L \) has no discrete spectrum (by assumption).

Calculation of the integral (8.25) with residue theorem and complex contour integral:

\[
\int_{\mathbb{R}} a(k)\psi_k(x)e^{iky}dk = 0, \quad \forall x, y \text{ fixed}
\]

because \( a(k), b(k), \psi_k \) are analytic in upper half-plane (details: [DJ], §3.3)

\( \Rightarrow K \) satisfies (with \( y \leftrightarrow z \)):

\[
K(x, z) + F(x + z) + \int_x^\infty K(x, y)F(y + z)dy = 0, \quad -\infty < x < z. \tag{8.26}
\]

inverse scattering problem:

\( F \) given by scattering data \( \Rightarrow K(x, z) \) can be calculated from integral equation (8.26) \( \Rightarrow u \) from (8.23).

case 2: \( L \) has \( N \geq 1 \) eigenvalues \( \lambda_1, \ldots, \lambda_N \).

We have: \( a(k), b(k) \) are meromorphic in the upper half-plane with \( N \) simple poles at \( k = i\kappa_n \) \( (\kappa_n > 0, \lambda_n = -\kappa_n^2) \)

Calculation of the integral (8.25):

With

\[
\psi_{i\kappa_n}(x) = c_{\kappa_n} \Phi_{i\kappa_n}(x) \quad \text{(cf. (8.24))}
\]

\[
\overset{(8.21)}{=} c_{\kappa_n}(e^{-\kappa_n x} + \int_x^\infty K(x, z)e^{-\kappa_n z}dz)
\]

one can show (details [DJ] §3.2-3):

\[
\frac{1}{2\pi} \int_{\mathbb{R}} a(k)\psi_k(x)e^{iky}dk = -\sum_{n=1}^N c_{\kappa_n} \psi_{i\kappa_n}(x)e^{-\kappa_n y}
\]

\[
= -\sum_{n=1}^N \kappa_n^2 \left[ e^{-\kappa_n (x+y)} + \int_x^\infty K(x, z)e^{-\kappa_n (y+z)}dz \right]
\]

Inserting into (8.25) again gives (8.26).

Rem: (8.26) implies (as desired) \( K_{xx} - K_{zz} - u(x)K = 0, \ z > x \) for \( u(x) := -2 \frac{d}{dx} K(x, x) \) (see Exercises).
Remark 8.5. 1) The Fredholm integral equation (8.26) can be written as fixed point iteration for $K \in C(\mathbb{R}^2)$ (or $\in C^\infty(\mathbb{R}^2)$):

$$K \mapsto K^*(x, z) := -F(x + z) - \int_x^\infty K(x, y)F(y + z)dy.$$  \hspace{1cm} (8.27)

Mapping (8.27) is Lipschitz with constant $\|F\|_{L^1(\mathbb{R})}$.

Let $\|F\|_{L^1} < 1 \Rightarrow$ GLM-equation has unique solution.

2) Special case: Let $F$ be separable; i.e.,

$$F(x + z) = \sum_{n=1}^N X_n(x)Z_n(z), \quad N \in \mathbb{N} \text{ with } Z_n \text{ l.u.}$$

(e.g. for $b \equiv 0$, which means a reflection-free potential). $\Rightarrow$ GLM-equation becomes

$$K(x, z) + \sum_{n=1}^N X_n(x)Z_n(z) + \sum_{n=1}^N Z_n(z) \int_x^\infty K(x, y)X_n(y)dy = 0$$

$\Rightarrow$ Ansatz for solution: $K(x, z) = \sum_{n=1}^N L_n(x)Z_n(z)$

$$\Rightarrow \quad L_n(x) + X_n(x) + \sum_{m=1}^N L_m(x) \int_x^\infty Z_m(y)X_n(y)dy = 0; \quad n = 1, \ldots, N$$

hence: $N$ linear algebraic equations for $N$ unknowns $L_n(x)$

3) We have: number of bounded states of operator $L$ ($= N$) = number of solitons a solution develops for $t \to \infty$.

Example 8.6 (Reflection coefficient with $N = 1$ pole). Scattering data are given as

1) $b(k) = \frac{-\beta}{\beta + ik}$ (for some $0 < \beta = \text{const}$), hence pole at $k = i\beta$, which means one eigenvalue $\lambda_1 = -\kappa_1^2 = -\beta^2$ of $L$.

2) $\psi_1(x) \sim \sqrt{\beta} e^{-\beta x}$ for $x \to \infty$; d.h. $c_1 = \sqrt{\beta}$

Aim: calculate corresponding potential $u$.

$$\rightarrow F(\xi) = \beta e^{-\beta \xi} - \frac{\beta}{2\pi} \int_R \frac{e^{ik\xi}}{\beta + ik}dk = \ldots = \beta e^{-\beta \xi} H(-\xi)$$

(with residue theorem; $H \ldots$ Heaviside function)
From GLM-equation (8.26): $K(x, z) = 0$ for $x + z > 0$.

GLM for $x + z < 0$ (as $F(y + z) \neq 0$ only for $y + z < 0$):

$$K(x, z) + \beta e^{-\beta(x+z)} + \beta \int_x^{-z} K(x, y)e^{-\beta(y+z)}\,dy = 0, \quad x < \min(z, -z).$$

(unique) solution: $K = -\beta$, hence

$$K(x, z) = \begin{cases} 0, & x + z > 0 \\ -\beta, & x + z < 0 \end{cases} = -\beta H(-x - z)$$

$$K(x, x) = -\beta H(-2x) = -\beta H(-x)$$

$\Rightarrow u(x) = -2\frac{d}{dx}K(x, x) = -2\beta \delta(x)$.

Furthermore: initial profile $u_0 = -2\beta \delta$ splits in one soliton

$$u(x, t) \sim 2\beta^2 \text{sech}^2 \left[ \beta(x - 4\beta^2 t + \ln \frac{2}{2 \beta}) \right]$$

and a dispersive wave.

Figure 8.3: initial condition $u_0 = -2\beta \delta$ (see (a)) splits in one soliton and a dispersive wave (siehe (b)) [DJ].

Rem: inverse (scattering) problems in many applications: e.g. computed tomography scan, acoustic exploration of soil geology

References: [De] §9.7, [DJ] §3.3, 4.4, [Wh] §17.3-5
8 Nonlinear waves – Solitons
A Bibliography


B Slides
Traffic flow diagram: velocity as multivalued function of density

Characteristics for traffic light example

BC: $u = -1$, red light

Figure B.1: Traffic light: red phase, 1st part of green phase
Numerical methods for linear advection equation (smooth solutions 1)

\[ u_t + u_x = 0, \quad x \in \mathbb{R}, \quad t > 0 \]
\[ u_0(x) = \sin(2\pi x) \]

num. solution on \([0, 1]\) with periodic boundary conditions.

The central scheme is unstable.

left: Lax-Friedrichs ist conditionally stable (for \(\gamma := \frac{|a|k}{h} \leq 1\));
right: the downwind scheme is unstable.
Numerical methods for linear advection equation (smooth solution 2)

\[ u_t + u_x = 0, \quad x \in \mathbb{R}, \quad t > 0 \]

\[ u_0(x) = \sin(2\pi x) \]

num. solution on \([0, 1]\) with periodic boundary conditions.

The upwind scheme is conditionally stable (for \(0 \leq \gamma := \frac{ak}{h} \leq 1\)); here \(\gamma = 0.75\).

For \(\gamma > 1\) the upwind scheme is unstable; here \(\gamma = 1.18\).
Numerical methods for discontinuous solutions (1)

\[ u_t + u_x = 0, \quad x \in \mathbb{R}, t > 0 \]

\[ u_0(x) = \begin{cases} 
1, & x < 0 \\
0, & x > 0
\end{cases} \]

exact solution (---) at \( t = 0.5 \) and numerical solution (\( \cdots \)) with \( h = 0.01, k/h = 0.5 \) (from [LV])
Numerical methods for discontinuous solutions (2)

\[ u_t + u_x = 0, \quad x \in \mathbb{R}, \ t > 0 \]

\[ u_0(x) = \begin{cases} 
1, & x < 0 \\
0, & x > 0 
\end{cases} \]

exact solution \((-\)) at \(t = 0.5\) and numerical solution \((\cdots)\) with \(h = 0.0025,\ k/h = 0.5\). Order of convergence: \(1/2\) resp. \(2/3\) [LV]
Riemann-Problem for Burgers’ equation

\[ u_t + u u_x = 0, \quad x \in \mathbb{R}, t > 0 \]

\[ u_l = 1.2, \quad u_r = 0.4, \quad \rightarrow \text{shock speed} \ s = 0.8 \]

exact solution (—) at \( t = 1 \) and numerical solution (···) with non-conservative scheme:

\[ u_j^{n+1} = u_j^n - \frac{k}{h} u_j^n (u_j^n - u_{j-1}^n) \]

num. solution with conservative upwind scheme (from [LV]):

\[ u_j^{n+1} = u_j^n - \frac{k}{h} \left( \frac{1}{2}(u_j^n)^2 - \frac{1}{2}(u_{j-1}^n)^2 \right) \]
Linear Gaussian diffusion filter

original image $f$; diffusion filters $K_\sigma \ast f$ with growing “thickness” $\sigma$ in (4.1) (created with *Photoshop*)
Diffusion filter (triangle and rectangle)

noisy input image

filtered with linear diffusion (+ automatic stopping time)

filtered with isotropic nonlinear diffusion [Perona-Malik equation] (+ automatic stopping time)

filtered with anisotropic nonlinear diffusion (+ automatic stopping time)

from [Pavel Mrazek, Dissertation, Prag, 2001]
(mean) curvature equation

\[ u_t = |\nabla u| \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) \]

Evolution of curves under (mean) evolution equation, [AK]. All closed curves asymptotically become circles and collapse in finite time.
Shock filter

\[ u_t = -|\nabla u| \text{sign}(\Delta u), \quad x \in \mathbb{R}^2, \quad t > 0 \]

Initial condition is Gauss-smoothened original image. Image reconstruction: convergence (in finite time) towards a step function (i.e. perfectly sharp image), [AK].
Brusselator (reaction-diffusion equation)

\[ u_t = a - (b + 1)u + u^2v + d_1 \Delta u \]
\[ v_t = bu - u^2v + d_2 \Delta v \] (B.1)

Model for autocatalytic, oscillating chemical reaction (i.e. a reaction product is also a reaction partner);
equation for 2 substances with densities \( u(x, t) \), \( v(x, t) \)

homogeneous stationary state \((u_0, v_0) = (a, b/a)\);
turing instability for \( b > (1 + a \sqrt{\frac{d_1}{d_2}})^2 \) \( (= 2\text{nd necessary condition}) \)

2 numerical examples for spacially inhomogeneous stationary states \( u_\infty(x) = \lim_{t \to \infty} u(x, t) \) (with same parameters \( a, b \)); are not unique!
stable, but not asymptotically stable.
(B.1) is invariant under translations and rotations (modulo BC)
pattern formation in chemical process (experiment)

evolution of concentration waves in chemical reaction (Belousov-Zhabotinsky reaction)
Pattern formation with reaction-diffusion equations (1)
Pattern formation with reaction-diffusion equations (2)
a) numerical simulation (Monte Carlo) of the Cahn-Hilliard Gleichung; \( t = 0, 20, 100, 400, 1000, 3000, 5000 \)

b) Magnification of \( t = 20, 400, 1000, 3000 \) shows scale invariance.

c) Movie of experiment (fat bubbles) [T. Ursell, 2007]:
http://www.youtube.com/watch?v=kDsFP67_ZSE&NR=1
Cahn-Hilliard: simulation for $t \to \infty$

$$f(c) := c^4/12 - c^2/2; \quad \text{local minima at } c_m = \pm \sqrt{3}$$

$\gamma = 0.03$, $L = 6$

FEM–simulation of Cahn-Hilliard equation [EF]:

- solution converges towards $c_\infty$
- $c_\infty$ almost piecewise constant (values at $\pm \sqrt{3}$)
- still unclear, whether $c_\infty$ stationary state or only metastable
thin films: “fingering”–instability of front

[Huppert, Nature, 1982]
FEM–simulation of (extended) thin film equation [BG]:
- IC: homogeneous film with small disturbance
- film ruptures
- evolution to few large droplets
data transmission with solitons Datenübertragung mit Solitonen

- practical comparison for impulse transmission at 4 Gbit/s: soliton vs. linear
- at 300 km: signal cannot be recognized with linear transmission
- with soliton-transmission almost unchanged
- practical applications still in preparation