One-dimensional perturbations of unbounded selfadjoint operators with empty spectrum

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\section*{A B S T R A C T}
We study spectral properties of one-dimensional singular nonselfadjoint perturbations of an unbounded selfadjoint operator and give criteria for the possibility to remove the whole spectrum by a perturbation of this type. A counterpart of our results for the case of bounded operators provides a complete description of compact selfadjoint operators whose rank one perturbation is a Volterra operator.

\section*{1. Introduction}

1.1. \textit{Singular rank one perturbations}

We study \textit{singular} rank one perturbations of an unbounded selfadjoint operator. This paper is a continuation of [6], where the completeness of eigenvectors and the spectral synthesis property for these perturbations were considered.

Let $\mu$ be a \textit{singular} measure on $\mathbb{R}$ (i.e., singular with respect to the Lebesgue measure) and let $\mathcal{A}$ be the operator of multiplication by the independent variable $x$ in $L^2(\mu)$ (thus, $\mathcal{A}$ is a cyclic \textit{singular} selfadjoint operator). Moreover, we assume that $0 \notin \text{supp} \, \mu$, and so $\mathcal{A}^{-1}$ is a bounded operator in $L^2(\mu)$.
Now we define singular rank one perturbations of \( \mathcal{A} \). Let \( a(t), b(t) \) be functions such that
\[
\frac{a(t)}{t}, \frac{b(t)}{t} \in L^2(\mu),
\] (1.1)
however, possibly, \( a, b \notin L^2(\mu) \). Let \( \varkappa \in \mathbb{C} \) be a constant such that
\[
\varkappa \neq \int_{\mathbb{R}} t^{-1}a(t)b(t) d\mu(t) \quad \text{in the case when } a \in L^2(\mu).
\] (1.2)

We associate with any such data \( (a, b, \varkappa) \) a linear operator \( \mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa) \), defined as follows:
\[
\mathcal{D}(\mathcal{L}) \overset{\text{def}}{=} \{ y = y_0 + c \cdot \mathcal{A}^{-1}a : c \in \mathbb{C}, y_0 \in \mathcal{D}(\mathcal{A}), \varkappa c + \langle y_0, b \rangle = 0 \};
\]
\[
\mathcal{L}y \overset{\text{def}}{=} \mathcal{A}y_0, \quad y \in \mathcal{D}(\mathcal{L}).
\]
(1.3)

Condition (1.2) is equivalent to the uniqueness of the decomposition \( y = y_0 + c \cdot \mathcal{A}^{-1}a \) in the above formula for \( \mathcal{D}(\mathcal{L}) \), hence the operator \( \mathcal{L} \) is correctly defined. The operator \( \mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa) \) is said to be a singular rank one perturbation of \( \mathcal{A} \). In general, it is not selfadjoint.

Singular perturbations of selfadjoint operators have been studied for a long time, see, for instance, [29,2,36].

Essentially, singular rank one perturbations are unbounded algebraic inverses to bounded rank one perturbations of bounded selfadjoint operators. Namely, if the triple \( (a, b, \varkappa) \) satisfies (1.2) and \( \varkappa \neq 0 \), then the bounded operator \( \mathcal{A}^{-1} - \varkappa^{-1}\mathcal{A}^{-1}a(\mathcal{A}^{-1}b)^* \) has trivial kernel, and
\[
\mathcal{L}(\mathcal{A}, a, b, \varkappa) = (\mathcal{A}^{-1} - \varkappa^{-1}\mathcal{A}^{-1}a(\mathcal{A}^{-1}b)^*)^{-1}.
\] (1.4)

Here we denote by \( \mathcal{A}^{-1}a(\mathcal{A}^{-1}b)^* \) the bounded rank one operator \( \mathcal{A}^{-1}a(\mathcal{A}^{-1}b)^*f = (f, \mathcal{A}^{-1}b)\mathcal{A}^{-1}a, f \in L^2(\mu) \). Conversely, if \( \mathcal{A}_0 \) is a bounded selfadjoint operator with trivial kernel and \( \mathcal{L}_0 = \mathcal{A}_0 + a_0b_0^* \) is its rank one perturbation and \( \ker \mathcal{L}_0 = 0 \), then the algebraic inverse \( \mathcal{L}_0^{-1} \) is a singular rank one perturbation of \( \mathcal{A}_0^{-1} \). We refer to [6] for details and for similar statements for rank \( n \) singular perturbations.

During the last 20 years selfadjoint rank one perturbations of selfadjoint operators were extensively studied by Simon, del Rio, Makarov and many other authors in relation with the problem of stability of the point spectrum and the study of the singular continuous spectrum (see [11] and a survey [42]). Some recent developments can be found in [26,3]. In what follows, we consider only perturbations of compact selfadjoint operators (or selfadjoint operators with compact resolvent), but the perturbations are no longer selfadjoint. The spectral structure of this class becomes unexpectedly rich and complicated as soon as we leave the classes covered by classical theories (dissipative operators or weak perturbations in the sense of Macaev; see the books [12] and [13]).

1.2. Problems on removable spectra

In our preceding paper [6], we studied the completeness of eigenvectors of \( \mathcal{L} \) and \( \mathcal{L}^* \) as well as the possibility of the spectral synthesis for such perturbations. Our main tool in [6] was a functional model for rank one singular perturbations. This model realizes singular rank one perturbations as certain “shift” operators in a so-called model subspace of the Hardy space or in a de Branges space of entire functions.

In this paper, we address the following problem:

**Problem 1.** For which measures \( \mu \) does there exist a (nonselfadjoint) singular perturbation \( \mathcal{L} \) of \( \mathcal{A} \) of the above type whose spectrum is empty?
Clearly, if such a perturbation exists, then the resolvent of \( \mathcal{A} \) is compact, and so the measure \( \mu \) in question should necessarily be of the form \( \mu = \sum_n \mu_n \delta_{t_n} \), where \( t_n \in \mathbb{R} \) and \( |t_n| \to \infty, \ |n| \to \infty \). Here \( \{t_n\} \) may be either one-sided sequence (enumerated by \( n \in \mathbb{N} \)) or a two-sided sequence (enumerated by \( n \in \mathbb{Z} \)). Thus, the problem is to describe those spectra \( \{t_n\} \) for which the spectrum of the perturbation is empty. Such spectra will be said to be removable. It is clear that the property to be removable or nonremovable depends only on \( \{t_n\} \), but not on the choice of the masses \( \mu_n \).

The change of boundary conditions of an ordinary differential operator leads to a singular one-dimensional perturbation, see, for instance, [6]. This phenomenon of the disappearance of the spectrum if the boundary conditions are properly chosen is well-known, see [33,21,7].

As an example, consider the simplest first order selfadjoint operator \( \mathcal{A} f(t) = -if'(t) \) on \([0, 2\pi]\) with the boundary condition \( f(2\pi) = f(0) \), whose spectrum is \( \mathbb{Z} \). The operator \( \mathcal{L} f(t) = -if'(t) \) with the changed (nonselfadjoint) boundary condition \( f(0) = 0 \) satisfies \( \mathcal{A} = \mathcal{L} \) on \( D(\mathcal{A}) \cap D(\mathcal{L}) \); moreover, \( D(\mathcal{A}) \cap D(\mathcal{L}) \) has codimension one both in \( D(\mathcal{A}) \) and in \( D(\mathcal{L}) \). Therefore \( \mathcal{L} \) is a rank one singular perturbation of \( \mathcal{A} \) (see [6, Proposition 1.2]). Since the spectrum of \( \mathcal{L} \) is empty, the spectrum \( \sigma(\mathcal{A}) = \mathbb{Z} \) is removable.

In view of the relation between singular rank one perturbations and usual rank one perturbations of bounded selfadjoint operators, the problem is equivalent to the following:

**Problem 2.** Describe those compact selfadjoint operators that have a rank one perturbation which is a Volterra operator.

Recall that a compact operator is called a Volterra operator if its spectrum equals \( \{0\} \) (sometimes the assumption that the kernel is trivial is also included in the definition; all Volterra operators appearing in the present paper have this property).

A close problem was studied by Silva and Toloza [40] (see also their paper [41] for some more general results) who described entire (in the sense of Krein) operators in terms of spectra of two of their selfadjoint extensions. Their description is based on a theorem due to Woracek [44] characterizing de Branges spaces, which contain zero-free functions. Though our problem deals with only one spectrum, its solution is based on a functional model in a de Branges space and the problem essentially reduces to the existence of a zero-free function in it.

Finite rank perturbations of Volterra operators and their models in de Branges spaces also have been studied in several works by Gubreev and coauthors. These papers concern Riesz bases, completeness, generation of \( C_0 \) semigroups and the relation with the so-called quasi-exponentials, see [15,14,28] and references therein. The paper by Khromov [21] treats spectral properties of finite rank perturbations of Volterra operators from a different point of view; in particular, it contains results stated in terms of the asymptotics of the kernel \( M(x,t) \) of an integral Volterra operator near the diagonal.

### 1.3. Main results

In this paper, we solve Problems 1 and 2 and obtain a necessary and sufficient condition of removability in terms of entire functions of the so-called Krein class. We say that an entire function \( F \) (with \( F(0) \neq 0 \)) is in the *Krein class* \( \mathcal{K}_1 \) if it is real on \( \mathbb{R} \), has only real simple zeros \( t_n \) and may be represented as

\[
\frac{1}{F(z)} = q + \sum_n c_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right), \quad \sum_n t_n^2 |c_n| < \infty, \tag{1.5}
\]

where \( c_n = -1/F'(t_n) \) and \( q = 1/F(0) \) (see Section 3 and Lemma 5.1 for details).
Our main result reads as follows:

**Theorem 1.1.** Let $t_n \in \mathbb{R}$ and $|t_n| \to \infty$, $|n| \to \infty$. The following are equivalent:

(i) The spectrum $\{t_n\}$ is removable;

(ii) There exists a function $F \in \mathcal{K}_1$ such that the zero set of $F$ coincides with $\{t_n\}$.

An unexpected (and rather counterintuitive) consequence of Theorem 1.1 is that adding a finite number of points to the spectrum helps it to become removable, while deleting a finite number of points may make it nonremovable (see Corollaries 5.2 and 5.3 below).

We have an immediate counterpart of Theorem 1.1 for compact operators which have Volterra rank one perturbations.

**Theorem 1.2.** Let $s_n \in \mathbb{R}$, $s_n \neq 0$, and $|s_n| \to 0$, $|n| \to \infty$, and let $A_0$ be a compact selfadjoint operator with simple point spectrum $\{s_n\}$. Then the following are equivalent:

(i) There exists a rank one perturbation $L_0 = A_0 + a_0 b_0^*$ such that $L_0$ is a Volterra operator;

(ii) The points $t_n = s_n^{-1}$ form the zero set of some function $F \in \mathcal{K}_1$.

As a corollary of Theorem 1.2 we give in Section 5 a condition on the spectrum $\{s_n\}$ which is sufficient for existence of rank $N$ perturbation of $A_0$ which is a Volterra operator (see Corollary 5.4).

### 1.4. Relations between real and imaginary parts of Volterra operators

There exists a vast range of results (mainly due to Krein, Gohberg and Macaev) relating the Schatten class properties of the imaginary part of a Volterra operator with the corresponding property for its real part. See [12, Ch. IV] or [13, Ch. III] for these results and for their generalizations to more general symmetric norm ideals. We refer to [31,35,38,43] for further results in this direction; see also [16] and references therein. Also, in [12, Ch. IV] and in [13, Ch. III] some partial results are given about the spectra of Volterra operators with finite-dimensional imaginary part. Let us also mention a remarkable theorem, which essentially goes back to Livšic [27] (for an explicit statement see [13, Ch. I, Th. 8.1]): *any dissipative Volterra operator, which is a rank one perturbation of a selfadjoint operator, is unitary equivalent to the integration operator*. In Section 7 we discuss this theorem in more detail and deduce it from our model. Another group of results concerns the existence of bases of eigenvectors for rank one perturbations of Volterra integral operators [21,32,37].

We observe that here we are speaking about Volterra operators of the form $L_0 = A_0 + a_0 b_0^*$, where $A_0$ is a compact selfadjoint operator and the perturbation $a_0 b_0^*$ has rank one, but need not be skew selfadjoint, so that $A_0$ is not equal to the real part of $L_0$. In general, $L_0$ has *rank two* imaginary part. The problem of finding possible spectra of real parts of Volterra operators is different. In fact, there exists a selfadjoint compact operator $A_0$, which is the real part of a Volterra operator with rank two imaginary part, but none of the operators of the form $A_0 + a_0 b_0^*$ is Volterra. We comment on it in Example 7.3 below.

The paper is organized as follows. In Sections 2 and 3 we give some preliminaries on the functional model from [6] and on de Branges’ theory. The proofs of Theorems 1.1 and 1.2 are given in Sections 4 and 5. In Section 5, we also briefly discuss some variations of Problems 1 and 2 (rank $N$ perturbations and the removal of a part of the spectrum). Section 6 contains some examples of removable and nonremovable spectra, while in Section 7 we discuss a simple proof of Livšic’s theorem by our methods. In Section 8 we show that sometimes the Volterra property may be achieved by sufficiently “smooth” perturbations and compare this result with a classical completeness theorem due to Macaev and some results from [6].

### 2. Functional model

We use the notations $\mathbb{C}^\pm = \{z \in \mathbb{C} : \pm \text{Im} z > 0\}$ for the upper and the lower half-planes and set $H^2 = H^2(\mathbb{C}^+)$. Recall that a function $\Theta$ is said to be an *inner function* in $\mathbb{C}^+$ if it is a bounded analytic
Theorem 2.1. (See [6].) Let \( \mathcal{L} = \mathcal{L}(A, a, b, \infty) \) be a singular rank one perturbation of \( A \), and let \( b \) be a cyclic vector for \( A^{-1} \). Then there exist an inner function \( \Theta \), such that \( \Theta \) is analytic in a neighborhood of \( 0 \), \( 1 + \Theta \notin H^2 \), \( \Theta(0) \neq -1 \), and a function \( \varphi \) satisfying

\[
\varphi \notin H^2, \quad \frac{\varphi(z) - \varphi(i)}{z - i} \in K_\Theta, \tag{2.1}
\]

such that \( \mathcal{L} \) is unitary equivalent to the operator \( \mathcal{T} = T(\Theta, \varphi) \) which acts on the model space \( K_\Theta \overset{\text{def}}{=} H^2 \ominus \Theta H^2 \) by the formulas

\[
\mathcal{D}(\mathcal{T}) \overset{\text{def}}{=} \{ f = f(z) \in K_\Theta : \text{there exists } c = c(f) \in \mathbb{C} : zf - c\varphi \in K_\Theta \},
\]

\[
T f \overset{\text{def}}{=} zf - c\varphi, \quad f \in \mathcal{D}(\mathcal{T}).
\]

Conversely, any inner function \( \Theta \) which is analytic in a neighborhood of \( 0 \) and satisfies \( 1 + \Theta \notin H^2 \), \( \Theta(0) \neq -1 \), and any function \( \varphi \) satisfying (2.1) correspond to some singular rank one perturbation \( \mathcal{L} = \mathcal{L}(A, a, b, \infty) \) of the operator \( A \) of multiplication by the independent variable in \( L^2(\mu) \), where \( \mu \) is some singular measure on \( \mathbb{R} \) and \( x^{-1}a(x), x^{-1}b(x) \in L^2(\mu) \).

The functions \( \Theta \) and \( \varphi \) appearing in the model for \( \mathcal{L}(A, a, b, \infty) \) are given by the following formulas. Put

\[
\begin{align*}
\beta(z) &= \varphi + zb^*(A - z)^{-1}A^{-1}a \\
&= \varphi + \int \left( \frac{1}{t - z} - \frac{1}{i} \right) a(t)b(t) d\mu(t), \tag{2.2}
\end{align*}
\]

\[
\begin{align*}
\rho(z) &= \delta + zb^*(A - z)^{-1}A^{-1}b = \delta + \int \left( \frac{1}{t - z} - \frac{1}{i} \right) |b(t)|^2 d\mu(t), \tag{2.3}
\end{align*}
\]

where \( \delta \) is an arbitrary real constant. Then \( \Theta \) and \( \varphi \) are defined as

\[
\Theta(z) = \frac{i - \rho(z)}{i + \rho(z)}, \quad \varphi(z) = \frac{\beta(z)}{2} (1 + \Theta(z)). \tag{2.4}
\]

The above functional model uses essentially the properties of the so-called Clark measures introduced in [10]. Recall that the Clark measure \( \sigma_\zeta, |\zeta| = 1 \), is the measure from the Herglotz representation

\[
\frac{i\zeta + \Theta(z)}{\zeta - \Theta(z)} = p_\zeta z + q_\zeta + \frac{1}{\pi} \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\sigma_\zeta(t), \quad z \in \mathbb{C}^+,
\]

where \( p_\zeta \geq 0, q_\zeta \in \mathbb{R} \) and \( \int_{\mathbb{R}} (1 + t^2)^{-1} d\sigma_\zeta(t) < \infty \). Note that if \( \Theta \) is meromorphic, then any Clark measure \( \sigma_\zeta \) is discrete.
It follows from the results of Ahern and Clark [1] that
\[ \zeta - \Theta \in H^2 \iff p_{\zeta} > 0. \] (2.5)
Note that in our model
\[ \frac{1 - \Theta(z)}{1 + \Theta(z)} = \delta + \int \left( \frac{1}{t - z} - \frac{1}{t} \right) |b(t)|^2 d\mu(t). \] (2.6)
Thus, the measure \( \pi |b|^2 \mu \) is the Clark measure \( \sigma_{-1} \) for \( \Theta \).
Let us mention the following result from [6].

**Proposition 2.2.** (See [6, Proposition 2.2].) Let \( a, b \) be functions that satisfy (1.1) and let \( \kappa \in \mathbb{R} \). Let \( \Theta \) and \( \varphi \) be defined by (2.4). Then we have:
1. \( 1 + \Theta \notin H^2 \), \( \Theta(0) \neq -1 \), and \( \frac{\varphi}{z+1} \in H^2 \);
2. If \( a \notin L^2(\mu) \), then \( \varphi \notin H^2 \);
3. If \( a \in L^2(\mu) \), then \( \varphi \in H^2 \) if and only if \( \kappa = \sum_n a_n b_n t_n^{-1} \mu_n \).

Since we are interested in the case when \( \mu \) is a discrete measure (i.e., \( \mu = \sum_n \mu_n \delta_{t_n} \), where \( |t_n| \to \infty \), \( |n| \to \infty \)), the function \( \Theta \) is meromorphic in the whole complex plane and analytic on \( \mathbb{R} \); so is \( \varphi \) and any element of \( K_{\Theta} \). This situation reduces to the study of de Branges spaces of entire functions (see Section 3 below).

From now on we assume that \( \Theta \) and \( \varphi \) are meromorphic. By the well-known properties of the model spaces \( K_{\Theta} \), a function \( f \in H^2(\mathbb{C}^+) \) is in \( K_{\Theta} \) if and only if the function \( \hat{f}(z) = \Theta(z)\hat{f}(\overline{z}) \) also is in \( H^2(\mathbb{C}^+) \). Analogously, we put
\[ \hat{\varphi}(z) = \Theta(z)\overline{\varphi(z)}. \]
Denote by \( Z_{\varphi} \) the set of zeros of \( \varphi \) in \( \text{clo}_{\overline{\mathbb{C}}} \mathbb{C}^+ \) and put \( \mathbb{Z}_{\hat{\varphi}} = \{ z \in \text{clo}_{\overline{\mathbb{C}}} \mathbb{C}^- : z \in Z_{\hat{\varphi}} \} \). It follows from [6, Lemma 2.1] that the functions
\[ h_{\lambda}(z) = \frac{\varphi(z)}{z - \lambda}, \quad \lambda \in Z_{\varphi} \cup \mathbb{Z}_{\hat{\varphi}} \]
belong to \( K_{\Theta} \), and, moreover, all eigenfunctions of the model operator \( \mathcal{T} \) are of the form \( h_{\lambda}, \lambda \in Z_{\varphi} \cup \mathbb{Z}_{\hat{\varphi}} \).

**Lemma 2.3.** (See [6, Lemma 2.4].) Let meromorphic \( \Theta \) and \( \varphi \) correspond to a singular rank one perturbation of a cyclic selfadjoint operator \( \mathcal{A} \) with the compact resolvent. Then the following hold:
1. Operators \( \mathcal{L} \) and \( \mathcal{T} \) have compact resolvents;
2. \( \sigma(\mathcal{T}) = \sigma_p(\mathcal{T}) = Z_{\varphi} \cup \mathbb{Z}_{\hat{\varphi}} \);
3. The eigenspace of \( \mathcal{T} \) corresponding to an eigenvalue \( \lambda \in Z_{\varphi} \cup \mathbb{Z}_{\hat{\varphi}} \), is spanned by \( h_{\lambda} \).

3. Preliminaries on entire functions

An entire function \( E \) is said to be in the Hermite–Biehler class (which we denote by \( HB \)) if
\[ |E(z)| > |E(\overline{z})|, \quad z \in \mathbb{C}^+. \]
We also always assume that \( E \neq 0 \) on \( \mathbb{R} \). For a detailed study of the Hermite–Biehler class see [25, Chapter VII]. Put \( E^*(z) = \overline{E(\overline{z})} \). If \( E \in HB \), then \( \Theta = E^*/E \) is an inner function which is meromorphic in
the whole plane $\mathbb{C}$; moreover, any meromorphic inner function can be obtained in this way for some $E \in HB$ (see, e.g., [18, Lemma 2.1]).

Given $E \in HB$, we can always write it as $E = A - iB$, where

$$A = \frac{E + E^*}{2}, \quad B = \frac{E^* - E}{2i}.$$  

Then $A, B$ are real on the real axis and have simple real zeros. Moreover, if $\Theta = E^*/E$, then $2A = (1 + \Theta)E$.

Any function $E \in HB$ generates the de Branges space $\mathcal{H}(E)$ which consists of all entire functions $f$ such that $f/E$ and $f^*/E$ belong to the Hardy space $H^2$, and $\|f\|_E = \|f/E\|_{L^2(\mathbb{R})}$ (for the de Branges theory see [9]). It is easy to see that the mapping $f \mapsto f/E$ is a unitary operator from $\mathcal{H}(E)$ onto $K_\Theta$ with $\Theta = E^*/E$ (see, e.g., [18, Theorem 2.10]).

An entire function $F$ is said to be of Cartwright class if it is of finite exponential type and

$$\int_{\mathbb{R}} \frac{\log^+ |F(x)|}{1 + x^2} \, dx < \infty$$

(recall that $\log^+ t = \max(\log t, 0), t > 0$). For the theory of the Cartwright class we refer to [17,23]. It is well known that the zeros $z_n$ of a Cartwright class function $F$ have a certain symmetry: in particular,

$$F(z) = K z^m e^{icz} v.p. \prod_n \left(1 - \frac{z}{z_n}\right) \overset{\text{def}}{=} K z^m e^{icz} \lim_{R \to \infty} \prod_{\{z_n\} \leq R} \left(1 - \frac{z}{z_n}\right),$$  

(3.1)

where the infinite product converges in the “principal value” sense, $c \in \mathbb{R}$ and $K \in \mathbb{C}$ are some constants, $m \in \mathbb{Z}_+$. It follows from this representation that a Cartwright class function is determined uniquely by its zeros, up to a factor $Ke^{\gamma z}$, where $K \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ are constants.

A function $f$ analytic in $\mathbb{C}^+$ is said to be of bounded type if $f = g/h$ for some functions $g, h \in H^\infty(\mathbb{C}^+)$. If, moreover, $h$ can be taken to beouter, we say that $f$ is in the Smirnov class in $\mathbb{C}^+$. It is well known that if $f$ is analytic in $\mathbb{C}^+$ and $\text{Im} f > 0$ (such functions are said to be the Herglotz functions), then $f$ is in the Smirnov class [17, Part 2, Chapter 1, Section 5]. In particular, if $t_n \in \mathbb{R}$, $u_n > 0$ and $\sum_n u_n < \infty$, then the function $\sum_n \frac{u_n}{t_n - z}$ is in the Smirnov class in $\mathbb{C}^+$. Consequently, $\sum_n \frac{u_n}{t_n - z}$ is in the Smirnov class in $\mathbb{C}^+$ for any $\{v_n\} \in \ell^1$.

The following theorem due to M.G. Krein (see, e.g., [17, Part II, Chapter 1]) will be useful: If an entire function $F$ is of bounded type both in $\mathbb{C}^+$ and in $\mathbb{C}^-$, then $F$ is of Cartwright class. If, moreover, $F$ is in the Smirnov class both in $\mathbb{C}^+$ and in $\mathbb{C}^-$, then $F$ is a Cartwright class function of zero exponential type.

We also consider the class of entire functions introduced by M.G. Krein [24] (see also [25, Chapter 6]). Assume that $F$ is an entire function, which is real on $\mathbb{R}$, with simple real zeros $t_n \neq 0$ such that, for some integer $p \geq 0$, we have

$$\sum_n \frac{1}{|t_n|^{p+1}|F'(t_n)|} < \infty$$

and

$$\frac{1}{F(z)} = R(z) + \sum_n \frac{1}{F'(t_n)} \cdot \left(\frac{1}{z - t_n} + \frac{1}{t_n} + \frac{z}{t_n^2} + \cdots + \frac{z^{p-1}}{t_n^p}\right),$$  

(3.2)

where $R$ is some polynomial. The class of such functions $F$ will be denoted by $K_p$. If $F \in K_p$ for some $p$, then $F$ is of Cartwright class [25, Chapter 6].
4. First criterion of removability

Recall that the spectrum \( \{t_n\} \) with \( 0 \not\in \{t_n\} \) is said to be removable if there exists a singular perturbation \( L = L(A, a, b, \infty) \) of \( A \), whose spectrum is empty. Here \( A \) is the operator of multiplication by \( x \) in \( L^2(\mu) \), \( \mu = \sum \mu_n \delta_{t_n} \). In this case, given \( a, b \in L^2(\mu) \) we will write \( a_n \) and \( b_n \) in place of \( a(t_n) \) and \( b(t_n) \).

It is obvious that if the spectrum of \( L = L(A, a, b, \infty) \) is empty, then \( b \) must be a cyclic vector for \( \text{span} A \). Indeed, if \( b_n = 0 \) for some \( n \), then the vector \( e_n \) defined by \( e_n(t_k) = \delta_{nk} \) will be an eigenvector of \( L \) corresponding to the eigenvalue \( t_n \). Note that \( e_n \in \mathcal{D}(L) \) since \( (e_n, b) = 0 \) and we may take \( c = 0 \).

Since \( b \) is cyclic for \( \text{span} A \), we may apply the functional model from Section 2. Then, in view of Theorem 2.1 and Lemma 2.3, we have an immediate criterion of removability.

**Proposition 4.1.** The spectrum \( \{t_n\} \) is removable if and only if there exist a meromorphic inner function \( \Theta \) with \( \{t : \Theta(t) = -1\} \equiv \{t_n\} \), \( \Theta(0) \neq -1 \), \( 1 + \Theta \notin H^2 \), and a function \( \varphi \) which satisfies (2.1) such that both \( \varphi \) and \( \tilde{\varphi} \) have no zeros in \( \mathbb{C}^+ \cup \mathbb{R} \).

**Proof.** Indeed, if such pair \( (\Theta, \varphi) \) exists, then, by the converse statement in Theorem 2.1 there are a singular measure \( \mu \), functions \( a \) and \( b \) and a constant \( \infty \) such that \( L(A, a, b, \infty) \) is unitarily equivalent to the model operator \( T(\Theta, \varphi) \). Moreover, in this case \( \Theta \) and \( \varphi \) are related to the data \( (a, b, \infty) \) by formulas (2.2)–(2.4). Thus, \( |b|^2 \mu \) is the Clark measure \( \sigma_1 \) for \( \Theta \) whence \( \{t : \Theta(t) = -1\} \equiv \{t_n\} \). By Lemma 2.3 the spectrum of \( T(\Theta, \varphi) \) is empty if and only if \( \varphi \) and \( \tilde{\varphi} \) do not vanish in \( \mathbb{C}^+ \cup \mathbb{R} \). \( \Box \)

The following statement gives a more palpable description of removable spectra. In particular, we will see that the function \( \varphi \) may be chosen of the form \( 1/E \) for a function \( E \) in the Hermite–Biehler class.

**Theorem 4.2.** The spectrum \( \{t_n\} \) is removable if and only if the following two conditions hold:

1. The set \( \{t_n\} \) is the zero set of an entire function in the Cartwright class, and so the generating function of the set \( \{t_n\} \),

\[
A(z) = v.p. \prod \left(1 - \frac{z}{t_n}\right) = \lim_{R \to \infty} \prod_{n \leq R} \left(1 - \frac{z}{t_n}\right),
\]

is well-defined and belongs to the Cartwright class;

2. Moreover, there exists an entire function \( E \) of the Hermite–Biehler class such that \( E + E^* = 2A \), \( A \notin \mathcal{H}(E) \) and \( \frac{1}{(z+1)E} \in H^2 \).

If the spectrum is removable (and thus conditions 1 and 2 hold), then the pair \( (\Theta, \varphi) \) corresponding to a perturbation of \( A \) with empty spectrum and a function \( E \) in 2 may be chosen so that \( \Theta = E^*/E \) and \( \varphi = 1/E \).

**Proof.** Necessity of 1 and 2. If the spectrum \( \{t_n\} \) is removable, then there is a pair \( (\Theta, \varphi) \) satisfying all conditions of Proposition 4.1. Since \( \varphi \) is a function of bounded type (and even from Smirnov class) which does not vanish in \( \mathbb{C} \) and is analytic in a neighborhood of \( \mathbb{R} \), its inner–outer factorization (see, e.g., [22]) is of the form

\[
\varphi(z) = e^{ic_1 z} O(z),
\]

where \( c_1 \geq 0 \) and \( O \) is an outer function with \( |O| = |\varphi| \) on \( \mathbb{R} \). Since \( \tilde{\varphi} \) is also a function of Smirnov class, we have for \( t \in \mathbb{R} \),

\[
\tilde{\varphi}(t) = \Theta(t) \overline{O(t)} e^{-ic_1 t} = \zeta^2 O(t) e^{ic_2 t} = \zeta^2 \varphi(t) e^{i(c_2 - c_1) t}
\]
for some $c_2 \geq 0$ and some constant $\zeta \in \mathbb{C}$, $|\zeta| = 1$. We conclude that for $z \in \mathbb{R}$ and, hence, for any $z \in \mathbb{C}$,

$$\zeta^2 e^{2icz} = \frac{\varphi(z)}{\varphi(z)},$$

(4.2)

where $2c = c_1 - c_2$.

The function $1/\varphi$ is a meromorphic function, which has no poles in $\mathbb{C}^+$ and on $\mathbb{R}$. Also, by (4.2),

$$\frac{1}{\varphi(z)} = \frac{\Theta(z)}{\varphi(z)} \zeta^2 e^{2icz}, \quad z \in \mathbb{C}^+,$$

and so $1/\varphi$ has no poles in $\mathbb{C}^-$. We conclude that $E = \zeta e^{icz}/\varphi$ is an entire function. The function $E$ is in $\text{HB}$, because $E^*/E = \Theta$. Also, $E$ is of bounded type both in the upper and the lower half-planes, and so is of Cartwright class by Krein’s theorem.

Now put $\varphi_1 = \zeta e^{-icz}\varphi = 1/E$. We assert that we can replace $\varphi$ with $\varphi_1$, that is, that $\varphi_1$ also satisfies the conditions of Proposition 4.1. Indeed, we know that

$$\varphi_1 = \frac{\varphi(z)}{z+i} \in H^2$$

(see (2.1)). We assert that $\frac{\varphi(z)}{z+i} = \zeta e^{-icz}\varphi_1$ is also in $H^2$. If $c \leq 0$, then it is obvious. On the other hand, since $E \in \text{HB}$ is of order at most one and does not vanish on $\mathbb{R}$, it admits the following factorization (see, e.g., [25, Chapter VII])

$$E(z) = Ke^{-iaz+bz} \prod_n \left(1 - \frac{z}{z_n}\right) e^{h_n z},$$

where $K \in \mathbb{C}$, $a \geq 0$, $b \in \mathbb{R}$, $\{z_n\}$ is a finite or infinite sequence of points in $\mathbb{C}^-$, satisfying the Blaschke condition, and $h_n = \text{Re} \frac{1}{z_n} \geq 0$. It follows that $|E(iy)| \to \infty$ when $y \to +\infty$. Hence, when $c > 0$, $\frac{\varphi(iy)}{z+i} = o(e^{-cy})$, and, thus, the function $\frac{\varphi(z)}{z+i}$ is divisible by $e^{icz}$ in $H^2$. Since $\varphi \notin H^2$, it follows that $\varphi \notin L^2(\mathbb{R})$, so that $\varphi_1 \notin H^2$. Next, since $\varphi_1 = \varphi_1$, it follows that $\Theta(\frac{\varphi_1(t)}{t+i}) |_{\mathbb{R}}$ is in $H^2$, which implies that $\frac{\varphi_1(z)-\varphi_1(i)}{z-i} \in K_{\Theta}$. Hence (2.1) holds.

We get that $E$ is both in the Hermite–Biehler and in the Cartwright class and satisfies $\frac{1}{(z+i)E} = \frac{\varphi(z)}{z+i} \in H^2$. Then $A = \frac{E+E^*}{2}$ is a Cartwright class function with zero set $\{t : \Theta(t) = -1\} = \{t_n\}$. Since $1+\Theta \notin H^2$, we conclude that $A = \frac{(1+\Theta^2)}{2} E \notin H(E)$.

Notice that this argument also proves the last statement of the theorem.

**Sufficiency of 1 and 2.** If $E$ and $A$ are given, we may put $\Theta = E^*/E$ and $\varphi = 1/E$. Then $\Theta$ and $\varphi$ satisfy all conditions of Proposition 4.1, except, may be, one: it may happen that $\varphi = 1/E \in H^2$.

However, the function $E$ such that $A = \frac{E+E^*}{2}$ is not unique. Namely, the function $E_1 = A - iB$ is in $\text{HB}$ if and only if $B_+ / A$ is a Herglotz function in $\mathbb{C}^+$, which holds if and only if there exist $\nu_n \geq 0$, $\sum_n t_n^2 \nu_n < \infty$, $p \geq 0$ and $r \in \mathbb{R}$ such that

$$\frac{B_+(z)}{A(z)} = pz + r + \sum_n \nu_n \left(\frac{1}{t_n^2 - z} - \frac{1}{t_n^2}\right).$$

(4.3)

In particular, there exist $\nu^0_n$ and $r_0$ such that for our initial $E = A - iB$ we have

$$\frac{B(z)}{A(z)} = r_0 + \sum_n \nu^0_n \left(\frac{1}{t_n^2 - z} - \frac{1}{t_n^2}\right).$$
Since $1 + \Theta \notin H^2$ and $\frac{\mathbf{B}}{A} = i \frac{1 + \Theta}{1 + \Theta}$, the corresponding summand $p_0 \varepsilon$ is absent by (2.5). Now consider the functions $B\varepsilon$ such that
\[ B\varepsilon(z) \over A(z) = r_0 + \sum_n \nu_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right), \]
where $\nu_n$ are free parameters; then
\[ E\varepsilon(z) = A(z) \left( 1 - ir_0 - i \sum_n \nu_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) \right). \]

Clearly, $E\varepsilon(t_n) = i \nu_n A'(t_n)$, and choosing $\{\nu_n\}$ rapidly decreasing we can achieve that $\frac{1}{E\varepsilon} \notin L^2(\mathbb{R})$. On the other hand, for the choice of $\nu_n = \nu_n^0$ we have $E\varepsilon = E$, and so $\frac{1}{(x + i)E} \in L^2(\mathbb{R})$. Since $E\varepsilon$ is a “continuous function of $\nu_n$”, it is not difficult to show that there exist data $\{\nu_n\}$ such that $\frac{1}{(x + i)E\varepsilon} \notin L^2(\mathbb{R})$, whereas $\frac{1}{E\varepsilon} \notin L^2(\mathbb{R})$.

For the convenience of the reader who might be not satisfied with the above “continuity” argument, we give a rigorous proof of the existence of such sequence $\{\nu_n\}$. It may be assumed that the sequence $\{t_n\}$ has infinitely many positive terms. We will choose a rapidly increasing subsequence $\{t_{n_k}\}_{k=1}^\infty$ of $\{t_n\}$ such that $t_{n_k} \to +\infty$. We will set
\[ E\varepsilon(z) = A(z) \eta\varepsilon(z), \quad \text{where } \eta\varepsilon(z) = 1 - ir_0 - i \sum_n \nu_{n, \varepsilon} \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right), \]
with
\[ \nu_{n, \varepsilon} = \begin{cases} \nu_n^0, & n \neq n_k, \\ \nu_n^\varepsilon, & n = n_k, \end{cases} \]
where $0 < \nu_n^\varepsilon \leq \nu_n^0$. We will also define auxiliary points $\tau_k$ such that $t_{n_k - 1} \leq \tau_{k - 1} \leq t_{n_k}$ for $k \geq 2$. The sequences $\{n_k\}$, $\{\tau_k\}$ and the weights $\{\nu_n^\varepsilon\}$ will be defined by induction. To do that, we first introduce some more notation. For $k \geq 0$, let
\[ \nu_{n, k} = \begin{cases} \nu_n^0, & n \neq n_\ell \text{ or } n = n_\ell, \ \ell > k, \\ \nu_n^\varepsilon, & n = n_\ell, \ 1 \leq \ell \leq k, \end{cases} \]
denote the weight, changed only in the points $t_{n_1}, \ldots, t_{n_k}$. Put
\[ E_k(z) = A(z) \eta_k(z), \quad \text{where } \eta_k(z) = 1 - ir_0 - i \sum_n \nu_{n, k} \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right), \]
(so that $E_0 = E$). It is easy to see that for any such choice, $\frac{1}{(x + i)E_k} \notin L^2(\mathbb{R})$ for all $k$.

The inductive definition is as follows. On the first step, choose any $t_{n_1} > 0$ and any $\tau_1 = \max(4, 2t_{n_1})$. Now suppose that the numbers $t_{n_\ell}, \nu_\ell$ and $\tau_\ell (\ell = 1, \ldots, k - 1)$ have been already chosen. On the $k$th step, $n_k, \nu_k^\varepsilon$ and $\tau_k$ will be defined. We will use the notation $J_\ell = [-\tau_\ell, \tau_\ell]$.

Choose $t_{n_k} > 2\tau_{k - 1}$ so that $\nu_{n_k} t_{n_k}^{-2} \leq 2^{-k - 1} \tau_k^{-1}$. It is possible because $\sum t_n^{-2} \nu_n < \infty$.

If $\frac{1}{(x + i)E_{k - 1}} \in L^2(\mathbb{R}\setminus J_{k - 1}) \geq 2\tau_{k - 1}^{-1}$, then we put $\nu_k^\varepsilon = \nu_{n_k}$ (so that $E_k = E_{k - 1}$). Otherwise, we choose $\nu_k^\varepsilon \in (0, \nu_{n_k})$ so that $\frac{1}{(x + i)E_k} \in L^2(\mathbb{R}\setminus J_k) = 2\tau_k^{-1}$. It is possible because this norm is continuous as a function of $\nu_k^\varepsilon$ and tends to infinity as $\nu_k^\varepsilon \to 0^+$. Next, in both cases choose $\tau_k > t_{n_k}$ such that $\frac{1}{(x + i)E_k} \in L^2(J_k \setminus J_{k - 1}) = \tau_k^{-1}$. Notice that $\tau_k > 2\tau_{k - 1}$, which gives that $\tau_k > 2^{k+1}$.

We claim that the following properties hold.
(i) $|\eta_\ell(x) - \eta_{\ell - 1}(x)| \leq 2^{-\ell}$ on $J_\ell$ for $\ell > k$;
(ii) $\left\| \frac{1}{(x+i)E_k} \right\|_{L^2(\mathbb{R})} \leq C$ for some constant $C$, independent of $k$;
(iii) The sequence of functions $\frac{1}{E_k}$ converges uniformly to $\frac{1}{E_0}$ on $J_k$ for any $k$.

These properties imply our statement. Indeed, (i) gives that for $\ell > m \geq k$,

$$
1 - \frac{\eta_m(x)}{\eta_\ell(x)} \leq |\eta_\ell(x) - \eta_m(x)| \leq \sum_{j=m}^{\ell-1} 2^{-j-1} \leq 2^{-m} \quad \text{for } x \in J_k.
$$

(4.4)

Next, (ii) and (iii) imply that $\frac{1}{(x+i)E_\ast}$ is in $L^2(\mathbb{R})$ (because the integrals of $\frac{1}{|z+iE_k|}$ over $J_k$ are bounded by a constant independent of $k$). Fix some $k$. For any $\ell > k$, $|\eta_\ell/\eta_\ell| \geq \frac{1}{2}$ on $J_k$, and therefore

$$
\left\| \frac{1}{E_\ell} \right\|_{L^2(J_\ell \setminus J_{\ell-1})}^2 = \left\| \frac{\eta_{\ell - 1}}{\eta_\ell} \cdot \frac{1}{E_k} \right\|_{L^2(J_\ell \setminus J_{\ell-1})}^2 \geq \frac{\tau_k - 1}{2} \left\| \frac{1}{(x+i)E_k} \right\|_{L^2(J_\ell \setminus J_{\ell-1})}^2 \geq \frac{1}{2},
$$

which by (iii) implies that $\left\| \frac{1}{E_\ast} \right\|_{L^2(J_\ell \setminus J_{\ell-1})} \geq \frac{1}{2}$ for any $k$. Therefore $\frac{1}{E_\ast} \notin L^2(\mathbb{R})$.

So it remains to check (i)-(iii).

**Proof of (i):** Let $\ell > k$, and let $x \in J_k \subset J_{\ell-1}$. Then

$$
|\eta_\ell(x) - \eta_{\ell - 1}(x)| = (\nu_{n_\ell} - \nu'_{n_\ell}) \frac{|x|}{t_{n_\ell}^2 - x} \leq \frac{\nu_{n_\ell} \cdot 2^{\tau_k - 1}}{t_{n_\ell}^2} \leq 2^{-\ell}.
$$

In particular, (4.4) holds.

**Proof of (ii):** For any index $k$ such that $E_k \neq E_{k-1}$, one has

$$
\left\| \frac{1}{(x+i)E_k} \right\|_{L^2(\mathbb{R})}^2 = \left\| \frac{\eta_{k - 1}}{\eta_k} \cdot \frac{1}{(x+i)E_{k-1}} \right\|_{L^2(J_k \setminus J_{k-1})}^2 + \left\| \frac{1}{(x+i)E_k} \right\|_{L^2(\mathbb{R} \setminus J_{k - 1})}^2
\leq (1 + 2^{-k})^2 \left\| \frac{1}{(x+i)E_{k-1}} \right\|_{L^2(\mathbb{R})}^2 + 4\tau_k^{-2}.
$$

Since $4\tau_k^{-2} < 2^{-2k}$, one gets that

$$
1 + \left\| \frac{1}{(x+i)E_k} \right\|_{L^2(\mathbb{R})}^2 \leq (1 + 2^{-k})^2 \left( 1 + \left\| \frac{1}{(x+i)E_{k-1}} \right\|_{L^2(\mathbb{R})}^2 \right).
$$

This inequality also holds if $E_k = E_{k-1}$. Since $\prod_{k \geq 1} (1 + 2^{-k})^2$ converges and $\frac{1}{(x+i)E_0}$ is in $L^2(\mathbb{R})$, property (ii) follows.

**Proof of (iii):** It follows from (4.4) that there are constants $C_k$ such that $\left\| \frac{1}{E_\ell} \right\|_{L^2(J_k)} \leq C_k$ for all $\ell$. Now it is easy to get from the formulas $E_\ell = A\eta_\ell$ and (4.4) that for any fixed interval $J_k$, $\{ \frac{1}{E_\ell} \}$ is a Cauchy sequence in $C(J_k)$. Since $\frac{1}{E_\ell}$ tend pointwise to $\frac{1}{E_\ast}$ on $\mathbb{R}$, (iii) follows.  

**Example 4.3.** 1. Let $t_n = n + \delta$, $n \in \mathbb{Z}$, $\delta \in (0, 1)$. This spectrum may be annihilated by a one-dimensional perturbation, since we can take $E(z) = ie^{-\pi(z-\delta)}$ or $E(z) = \sin \pi(z - \delta) + 2i \cos \pi(z - \delta)$.

2. The spectrum $\{t_n\} = \mathbb{N}$ is not removable, because $\mathbb{N}$ is not a zero set of a Cartwright class function.
Remark 4.4. If \( \frac{1}{(z+t)^{1/2}} \in H^2 \) for a Hermite–Biehler function \( E \), then, by Theorem 4.2, the spectrum \( \{t_n\} \) (the zero set of \( A = (E + E^*)/2 \)) is removable. A number of conditions in terms of the zeros of \( E \) ensuring this inclusion (which is equivalent to the fact that \( 1 \) is a function associated to \( H(E) \) in the terminology of [9]) have been obtained in [19,4], while a criterion in terms of zeros of \( A \) and \( B \) was given in [44]. A slightly stronger property \( 1 \in H(E) \) is closely related to the existence of positive minimal majorants for \( H(E) \) [18].

5. Conditions in terms of the generating function \( F \). Proofs of Theorems 1.1 and 1.2

We will use the following simple lemma about the Krein class \( \mathcal{K}_1 \).

**Lemma 5.1.** Let \( A(z) = v.p. \prod_n (1 - \frac{z}{t_n}) \) be a Cartwright class function with simple real zeros \( t_n \). Then \( A \in \mathcal{K}_1 \) if and only if

\[
\sum_n \frac{1}{t_n^2 |A'(t_n)|} < +\infty. \tag{5.1}
\]

Moreover, in this case in (3.2), \( R(z) \equiv R \equiv \text{const.} \)

**Proof.** We need to show only that (5.1) implies representation (3.2) with \( R \equiv \text{const.} \) Put

\[
R(z) = \frac{1}{A(z)} - \sum_n \frac{1}{A'(t_n)} \left( \frac{1}{z-t_n} + \frac{1}{t_n} \right).
\]

Obviously, \( R \) is an entire function. Moreover, since \( A \) is in the Cartwright class and \( |A(iy)| \to \infty \) as \( y \to \infty \), we conclude that \( 1/A \) is in the Smirnov class in the upper and in the lower half-planes. The function

\[
\sum_n \frac{1}{A'(t_n)} \left( \frac{1}{z-t_n} + \frac{1}{t_n} \right) = z^2 \sum_n \frac{1}{t_n^2 A'(t_n)} \cdot \frac{1}{z-t_n} - z \sum_n \frac{1}{t_n^2 A'(t_n)}
\]

is also in the Smirnov class (see Section 2.3). Thus \( R \) is of zero exponential type by Krein’s theorem. Finally, note that \( |R(iy)| = o(|y|), |y| \to \infty \), and so \( R \) is a constant. \( \square \)

**Proof of Theorem 1.1.** Assume that the spectrum \( \{t_n\} \) is removable. Then there exist \( E = A - iB \) and \( \Theta = E^*/E \) as in Theorem 4.2, so that \( 1 + \Theta \notin H^2 \). By (2.4), \( \varphi = 1/E \) is of the form

\[
\varphi(z) = \frac{1 + \Theta(z)}{2} \left( z + \sum_n c_n \left( \frac{1}{t_n-z} - \frac{1}{t_n} \right) \right),
\]

where \( c_n = a_n \bar{b}_n \mu_n \), and so \( \sum_n t_n^{-2} |c_n| < \infty \). On the other hand,

\[
\varphi = \frac{1}{E} = \frac{1 + \Theta}{2A},
\]

and we conclude that \( 1/A \) has a representation of the form (1.5).

Conversely, let \( A \) be an entire function in \( \mathcal{K}_1 \) and let \( \{t_n\} \) be its zero set. Then

\[
\frac{1}{A(z)} = q + \sum_n c_n \left( \frac{1}{t_n-z} - \frac{1}{t_n} \right), \quad \sum_n |c_n| < \infty;
\]

where \( q = \sum_n \frac{1}{t_n^2} \).
therefore $A$ is of Cartwright class. Now for any masses $\mu_n > 0$ we may choose $a_n$ and $b_n$ so that $c_n = a_n b_n \mu_n$ and
\[
\sum_n |a_n|^2 t_n^{-2} \mu_n < \infty, \quad \sum_n |b_n|^2 t_n^{-2} \mu_n < \infty.
\]
Indeed, note that $c_n = -1/A'(t_n) \neq 0$, and take $a_n = |c_n|^{1/2} \mu_n^{-1/2}$ and $b_n = \overline{c_n} |c_n|^{-1/2} \mu_n^{-1/2}$. Define $\Theta$ by formula (2.6) (with an arbitrary real constant $\delta$). By construction, $\Theta$ is inner and $1 + \Theta \notin H^2$ (see the equivalence (2.5)). Then put
\[
E = \frac{2A}{1 + \Theta}.
\]
Clearly, $E$ is an entire function (the zeros sets of $1 + \Theta$ and of $A$ coincide) and
\[
\frac{E^*(z)}{E(z)} = \frac{1 + \Theta(z)}{1 + \Theta(z)} = \Theta(z)
\]
(5.2) since $\overline{\Theta(z)} = (\Theta(z))^{-1}$. Thus, $E$ is a Hermite–Biehler function and $A \notin \mathcal{H}(E)$ since $1 + \Theta \notin H^2$. Finally, for the function $\varphi = 1/E$ we have
\[
\varphi(z) = \frac{1 + \Theta(z)}{2A(z)} = \frac{1 + \Theta(z)}{2} \left( q + \sum_n c_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) \right).
\]
We see that $\varphi$ is of the form (2.2), by Proposition 2.2, $\frac{1}{(z+i)E} \in H^2$. So the spectrum $\{t_n\}$ is removable by Theorem 4.2. \ □

Proof of Theorem 1.2. $(ii) \implies (i)$. Assume that the spectrum of $\mathcal{A} = \mathcal{A}_0^{-1}$ is the zero set of a function in $\mathcal{K}_1$. The above proof of Theorem 1.1 gives us a construction of a singular perturbation $\mathcal{L} = \mathcal{L}(\mathcal{A}, \kappa, a, b)$ of $\mathcal{A}$ with empty spectrum. In this construction, $\beta = 2\varphi/(1 + \Theta) = 1/A$. Therefore $\kappa = \beta(0) = 1/A(0) \neq 0$. Formula (1.4) gives that $\mathcal{L}^{-1}$ is a rank one Volterra perturbation of $\mathcal{A}_0$.

$(i) \implies (ii)$. This implication also follows from (1.4) and Theorem 1.1. \ □

A somewhat unexpected consequence of Theorem 1.1 is that the addition of finite many points to the spectrum helps it to become removable, while the deletion of a finite number of points may make it non-removable.

Corollary 5.2. (i) If $\{t_n\}$ is removable, then for any finite set $\{\tilde{t}_m\}_{m=1}^M$ the spectrum $\{t_n\} \cup \{\tilde{t}_m\}_{m=1}^M$ is removable.

(ii) If the spectrum $\{t_n\}$ is removable, then by deleting a finite number of elements of this sequence and adding the same number of other elements we will always obtain a removable spectrum.

Proof. The statements follow immediately from Theorem 1.1, because the multiplication by a polynomial maps the Krein class $\mathcal{K}_1$ into itself. \ □

Corollary 5.3. There exists a removable spectrum $\{t_n\}$, such that $\{t_n\}_{n \neq m}$ is nonremovable for any $m$.

Proof. Clearly, the spectrum $\{t_n\}_{n \in \mathbb{Z}}$, where $t_n = n$ for $n \in \mathbb{Z} \setminus \{0\}$, and $t_0$ is any real noninteger number is removable (take $A(z) = \frac{\sin \pi z}{z}$ and $|F'(n)| \propto |n|^{-1}$. Hence the series $\sum_{n \neq 0} \frac{1}{n^2 |A(t_n)|}$ diverges. Thus, $A \notin \mathcal{K}_1$ and so the spectrum is nonremovable. \ □
A natural problem is to give a similar description of spectra of compact selfadjoint operators which can be turned to a Volterra operator by a rank \( N \) perturbation.

**Problem 3.** Given \( N \in \mathbb{N} \), describe those possible spectra of a compact selfadjoint operator \( A_0 \) such \( V = A_0 + B_0 \) is Volterra for a rank \( p \) operator \( B_0 \).

The following sufficient condition is an immediate corollary of Theorem 1.2.

**Corollary 5.4.** Let \( A_0 \) be a compact selfadjoint operator with simple point spectrum \( \{ s_n \} \), where \( s_n \in \mathbb{R} \), \( s_n \neq 0 \), and \( |s_n| \to 0 \), \( n \to \infty \). Let \( N \in \mathbb{N} \). If the points \( t_n = s_n^{-1} \) form the zero set of some function \( F \) which is a product of at most \( N \) functions from the class \( \mathcal{K}_1 \), then there exists a rank \( N \) perturbation \( V \) of \( A_0 \), which is a Volterra operator.

**Proof.** Let \( \{ s_n \} = \bigcup_{j=1}^{N} S_j \), where \( S_j = \{ s_j^{(j)} \} \) and \( \{ (s_j^{(j)})^{-1} \} \) is the zero set of some function from the class \( \mathcal{K}_1 \). Clearly, \( A_0 \) is an orthogonal direct sum of operators \( A_j \) of multiplication by the independent variable in \( L^2(\mu_j) \), where \( \mu_j = \mu|_{S_j} \), \( j = 1, \ldots, N \). By Theorem 1.2, each of the operators \( A_j \) has a rank one perturbation which is a Volterra operator. Thus, \( A_0 \) has a rank \( N \) perturbation which is a Volterra operator. \( \square \)

Gohberg, Krein and Makacov have several necessary conditions on the asymptotics of spectrum of \( A_0 \) for the case when \( A_0 = \text{Re} V \), see Theorems 8.2, 9.1, 10.1 and Corollary 11.1 of Chapter IV of the book [12]. These necessary conditions remain true in a general case of a finite rank perturbation \( B_0 \). Indeed, since \( A_0 - \text{Re} V \) is selfadjoint and of finite rank, the spectral asymptotics of the compact selfadjoint operators \( A_0 \) and \( \text{Re} V \) are the same.

We do not know whether the condition of Corollary 5.4 is necessary for the existence of rank \( N \) perturbation, which is a Volterra operator. See also Example 7.3 below.

We conclude this section with a discussion of one more problem, slightly more general than Problem 1. Let \( T = \{ t_n \} \) be, as before, an increasing sequence of real numbers such that \( |t_n| \to \infty \), \( n \to \infty \), and let \( A \) be a selfadjoint operator with simple spectrum \( T \).

**Problem 4.** For which partitions \( T = T_1 \cup T_2 \), \( T_1 \cap T_2 = \emptyset \), does there exist a singular perturbation \( L \) of \( A \) with simple spectrum \( T_1 \)?

In other words, we look for the description of those parts \( T_2 \) of the spectrum \( \{ t_n \} \) which can be removed by a singular rank one perturbation. The answer to this question may be obtained by an application of precisely the same methods as those used in the solution of Problem 1. Indeed, by the functional model the existence of such a perturbation is equivalent to the existence of a pair \(( \Theta, \varphi )\) as in Theorem 2.1 such that for some Hermite–Biehler function \( E \) with \( \Theta = E^*/E \) the zero set of the function \( G = E \varphi \) coincides with \( T_1 \). Repeating the arguments from Sections 4 and 5 we get the following theorem.

**Theorem 5.5.** In the conditions of Problem 4, a singular rank one perturbation with simple spectrum \( T_1 \) exists if and only if \( T_2 \) is the zero set of some function in the Krein class \( \mathcal{K}_1 \).

Note that the multiplication by a polynomial (with simple real zeros) preserves the Krein class \( \mathcal{K}_1 \). Thus, if we can remove all but finite part of the spectrum by a singular rank one perturbation, then the whole spectrum is removable.
6. Examples of removable and nonremovable spectra

In this subsection, we give some examples of removable and nonremovable spectra with power growth (one-sided and two-sided). To analyze the behavior of $|A'(t_n)|$ for the power growth of zeros we will use the Levin–Pfluger theory of functions of completely regular growth [25, Chapter 2]. Assume that $t_n$ are all situated on the ray $\mathbb{R}_+$ and the counting function $n(r) = #\{n : t_n \in [0, r]\}$ satisfies for some $\rho \in (0, 1)$,

$$\lim_{r \to \infty} \frac{n(r)}{r^\rho} = D \in (0, \infty).$$ (6.1)

Assume also that, for some $d > 0$,

$$t_{n+1} - t_n \geq dt_n^{1-\rho}. \quad (6.2)$$

Consider the discs $B_n = \{z : |z - t_n| < dt_n^{1-\rho}/2\}$. Then we have

$$\lim_{t \to +\infty, t \notin \cup_n B_n} \log \frac{|A(t)|}{t^\rho} = \pi D \cot \pi \rho \quad (6.3)$$

and

$$\lim_{t \to -\infty} \log \frac{|A(t)|}{|t|^\rho} = \frac{\pi D}{\sin \pi \rho}. \quad (6.4)$$

Moreover, it follows that

$$\frac{\log |A'(t_n)|}{t_n^\rho} \to \pi D \cot \pi \rho, \quad n \to +\infty. \quad (6.5)$$

**Example 6.1.** 1. *Two-sided symmetric power growth.* Assume that for some $\rho \in (0, 1)$, the spectrum $\{t_n\}$ satisfies

$$\lim_{r \to \infty} \frac{\#\{t_n \in (-r, 0)\}}{r^\rho} = \lim_{r \to \infty} \frac{\#\{t_n \in (0, r)\}}{r^\rho} = D \in (0, \infty),$$

and $t_{n+1} - t_n \geq d|t_n|^{1-\rho}$, $d > 0$. Then $t_n \asymp C|n|^{1/\rho}$ as $|n| \to \infty$. It follows from (6.3)–(6.5) that

$$\log |A'(t_n)| \sim \pi D|t_n|^\rho \cot \frac{\pi \rho}{2} \asymp |n|, \quad |n| \to +\infty,$$

and so, by Lemma 5.1, $A \in \mathcal{K}_1$ and the spectrum $t_n$ is removable.

In particular, the spectrum $t_n = |n|^\gamma \text{sign} n$, $n \in \mathbb{Z} \setminus \{0\}$, where $t_0$ is any nonzero number in $(0, 1)$, is removable for any $\gamma > 1$. In the case $\gamma = 1$ the spectrum $\{t_n\}$ is nonremovable (Corollary 5.3). Finally, if $\gamma < 1$, then the spectrum $\{t_n\}$ is not a zero set of a function of exponential type and, hence, is nonremovable.

2. *One-sided power growth.* Now let $t_n \in \mathbb{R}_+$ satisfy conditions (6.1)–(6.2). It follows from (6.5) and Lemma 5.1 that $\log |A'(t_n)| \asymp |t_n|^\rho$ and the spectrum $\{t_n\}$ is removable when $\rho < 1/2$, while for $\rho \in (1/2, 1)$ we have $\log |A'(t_n)| \asymp -|t_n|^\rho$ and the spectrum $\{t_n\}$ is nonremovable. In particular, the power spectra $t_n = n^\gamma$, $n \in \mathbb{N}$, are removable for $\gamma > 2$ and nonremovable for $\gamma < 2$.

3. *The limit case: square growth.* For one-sided power distributed zeros the limit case is the growth $t_n = n^2$, $n \in \mathbb{N}$. This situation is more subtle. In this case

$$A(z) = \prod_{n \in \mathbb{N}} \left(1 - \frac{z}{n^2}\right) = \frac{\sin(\pi \sqrt{z})}{\pi \sqrt{z}},$$

one can ask whether $A$ is entire or not.
and so $|A'(t_n)| = (2t_n)^{-1} = (2n^2)^{-1}$. Then the series (5.1) converges and, by Lemma 5.1, the spectrum is removable. However, if we consider the spectrum \( \{n^2\}_{n \geq 2} \), then the corresponding generating function \( A_1 \) satisfies \( |A'_1(t_n)| \asymp |t_n|^{-2} \), and the spectrum is nonremovable.

4. Two-sided nonsymmetric growth. More generally, suppose that

$$\lim_{r \to +\infty} \frac{\#\{\pm t_n \in (0, r)\}}{r^{\rho_\pm}} = D_{\pm} \in (0, \infty),$$

and \( t_{n+1} - t_n \geq d|t_n|^{1-\rho_\pm} \) for \( \pm n > 0 \), where \( \rho_\pm \in (0, 1) \) and \( d > 0 \). Define \( u_-, u_+ \) by

$$u_\pm = D_\pm \cot \pi \rho_\pm + \frac{D_{\mp}}{\sin \pi \rho_{\mp}}.$$

Then the same arguments as above imply that the spectrum is removable if both \( u_- \) and \( u_+ \) are positive and is not removable if at least one of these numbers is negative. In particular, if \( \rho_-, \rho_+ < 1/2 \), then the spectrum is removable.

**Remark 6.2.** 1. The special role of the exponent 2 in the power distributed spectra is well known. See, for example, [12, Theorem IV.10.1]. This theorem gives some restrictions on the possible spectra of real parts of Volterra operators with finite rank imaginary part. The special role of the exponent 2 can also be seen, e.g., in the problems of weighted polynomial approximation on discrete subsets of \( \mathbb{R} \) [8] or in Beurling–Malliavin type theorems on admissible majorants [5].

2. In the study of power growth, the regularity of the sequence is important. It is easy to see that for any \( \gamma > 2 \) there exists a subset of \( \{n^\gamma\}_{n \in \mathbb{N}} \), which is nonremovable (take the set \( n^\gamma, n \in [m_k, m_k + l_k] \) for appropriately chosen \( m_k, l_k \to \infty \)).

**Example 6.3.** Let \( a > 0 \) and consider two shifted progressions:

$$t_n = \begin{cases} n + a, & n \geq 0, \ n \in \mathbb{Z}, \\ n + 1 - a, & n < 0, \ n \in \mathbb{Z}, \end{cases}$$

that is, \( \{t_n\} = \{\ldots, -a - 1, -a\} \cup \{a, a + 1, \ldots\} \). Then

$$A(z) = \prod_{n=0}^{\infty} \left( 1 - \frac{z^2}{(n+a)^2} \right) = \frac{\Gamma(a)^2}{\Gamma(a+z)\Gamma(a-z)} = \frac{\Gamma(a)^2}{\pi \Gamma(a+z)} \sin \pi(a-z) \Gamma(1-a-z).$$

Therefore for positive \( k \in \mathbb{Z} \),

$$|A'(k + a)| = \frac{\Gamma(a)^2}{\Gamma(k+2a)} \Gamma(k+1) \asymp \Gamma(a)^2 k^{1-2a} \quad \text{as} \quad k \to +\infty.$$

Since \( A \) is an even function, the series \( \sum_{k>0} \frac{1}{k^2|A'(t_k)|} \) converges (and the spectrum is removable) if and only if \( a < 1 \).

**Example 6.4.** One more class of examples with a nonremovable spectrum can be obtained if we take a sequence of pairs of close points. In this case the spectrum is “almost multiple” and thus nonremovable. Let \( \{t_n\} \) be a separated sequence (i.e., \( \inf_n (t_{n+1} - t_n) > 0 \)) and consider the set \( \{t_n\} \cup \{t_n + \delta_n\} \), where \( \delta_n \to 0 \). If \( \delta_n \) are sufficiently small, then we can achieve that \( |A'(t_n)| \) be small and, thus, (5.1) is not satisfied.
7. Livšic’s theorem on dissipative Volterra operators with one-dimensional imaginary part

In this section we use our model to prove the above-mentioned theorem of Livšic; it says that any dissipative Volterra operator, which differs from a selfadjoint operator in an operator of the form $iB_0$, where $B_0$ is rank one and selfadjoint, is unitary equivalent to the integration operator ([27] or [13, Ch. I, Th. 8.1]).\footnote{We express our gratitude to N. Nikolski who attracted our attention to Livšic’s theorem and suggested to deduce it using our methods.} Namely, we show the following:

**Theorem 7.1.** (See Livšic, [27].) Let $\mathcal{L}_0 = \mathcal{A}_0 + i\mathcal{B}_0$ be a dissipative Volterra operator (in some Hilbert space $H$) such that both $\mathcal{A}_0$ and $\mathcal{B}_0$ are selfadjoint and $\mathcal{B}_0$ is of rank one. Then the spectrum of $\mathcal{A}_0$ is given by $s_n = c(n + 1/2)^{-1}$, $n \in \mathbb{Z}$, for some $c \in \mathbb{R}$, $c \neq 0$.

From this, one may deduce that $\mathcal{A}_0$ is unitary equivalent to the selfadjoint integral operator (having the same spectrum)

$$\langle \tilde{A}f \rangle(x) = i \int_0^{2\pi} f(t) \text{sign}(x - t) \, dt, \quad f \in L^2(0, 2\pi),$$

while $\mathcal{L}_0$ is unitary equivalent to the integration operator $\langle \tilde{L}_c f \rangle(x) = 2i \int_0^x f(t) \, dt$ on $L^2(0, 2\pi)$.

Since $\mathcal{B}_0 \geq 0$, we have $\mathcal{B}_0 x = (x, b_0) b_0$ for some $b_0 \in H$. By passing to the unbounded inverses, we obtain (after an obvious unitary equivalence) a singular rank one perturbation $\mathcal{L} = \mathcal{L}(A, a, b, \kappa)$ of the operator $A$ of multiplication by the independent variable in some space $L^2(\mu)$, where $\mu = \sum_n \mu_n \delta_{t_n}$, $t_n = s_n^{-1}$. Moreover, in the case of the positive imaginary part, we may assume that $\kappa = -1$ and $a = ib$.

Applying the functional model from Section 2, we construct a pair $(\Theta, \varphi)$ as in Theorem 2.1. Let $E = A - iB \in HB$ be such that $\Theta = E^*/E$ and let $g = \varphi E$. Then by (2.2), (2.3), we have

$$B(z) \over A(z) = \delta + \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) |b_n|^2 \mu_n,$$

$$g(z) \over A(z) = -1 + i \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) |b_n|^2 \mu_n,$$

whence $g = -A + i(B - \delta A)$.

Since $\mathcal{L}$ (and, thus, the model operator $T$) is the inverse to a Volterra operator, the spectrum of $T$ is the point at infinity. By Lemma 2.3, $g$ has no zeros in $\mathbb{C}$. Also, by Theorem 4.2, the function $E$ is of Carthwright class, and the same is true for $g$. We conclude that $g(z) = \exp(i\pi cz)$ for some real $c$. Thus,

$$e^{i\pi cz} = -A(z) + i(B(z) - \delta A(z)).$$

The functions $A$ and $B$ are real on the real axis. Taking the real parts, we get $A(z) = -\cos \pi cz$, and so $t_n = c^{-1}(n + 1/2)$, $n \in \mathbb{Z}$, as required. □

**Remark 7.2.** We observe that in general, whenever $\mathcal{L}_0 = \mathcal{A}_0 + i\mathcal{B}_0$ is a dissipative Volterra operator and $\mathcal{B}_0 \geq 0$ is of trace class, the spectrum $\{s_n\}$ of $\mathcal{A}_0 = \text{Re} \mathcal{L}_0$ has the symmetric linear asymptotics $s_n \sim cn^{-1}$, $n \to \pm \infty$, where $c = 1/2 \text{trace}(\mathcal{B}_0) > 0$ (see [12, Theorem IV.7.2]). One can compare this fact with the situation of Theorem 1.2, where other asymptotics of growth of $\mathcal{A}_0$ is possible. Notice that the perturbation $\mathcal{L}_0 = \mathcal{A}_0 + a_0 b_0^*$ is dissipative if and only if $b_0 = \rho a_0$, where $\rho \in \mathbb{C}$, $\text{Im} \rho \leq 0$. 
At the end of this section, let us use Livšić’s operators to deduce the following fact (already mentioned in the Introduction).

**Example 7.3.** There exists a selfadjoint compact operator $A_0$, which is the real part of a Volterra operator $V$ with rank two imaginary part, such that none of the operators of the form $A_0 + ab^*$ is Volterra.

To see it, put $V = \mathcal{L}_{c_1} \oplus \mathcal{L}_{c_2}$ on $L^2(0, 2\pi c_1) \oplus L^2(0, 2\pi c_2)$, where $\mathcal{L}_{c_i}$ are the integration operators, defined at the beginning of this section. Put $A_0 = \text{Re } V$, $B_0 = \text{Im } V$, so that $A_0$, $B_0$ are compact selfadjoint, $V = A_0 + iB_0$ and $B_0$ has rank two. The spectrum $\{t_n\}$ of the unbounded operator $A_0^{-1}$ is the union $\{t_1^1\} \cup \{t_2^1\}$ of two arithmetic progressions $t_n = c_j^{-1}(n + 1/2)$, $n \in \mathbb{Z}$. Suppose that $\alpha = c_2/c_1 > 0$ is well approximated by rational numbers: $|\alpha - p_k/q_k| \leq 1/q_k^2$, where $\{p_k\}$, $\{q_k\}$ are sequences of odd numbers that tend to $\infty$. (Any such $\alpha$ is transcendental, by the Thue–Siegel–Roth theorem.) There are infinitely many numbers $\alpha$ with this property, as it is easy to see by applying the usual construction of Liouville numbers with an odd base.

The generating function, defined by (4.1), equals $A(z) = \cos(\pi c_1 z) \cdot \cos(\pi c_2 z)$. Put $r_k = (q_k - 1)/2$. Since

$$|A'(t_{rk})| \leq C/(t_{rk})^2,$$

it follows by Lemma 5.1 that $A$ does not belong to $K_1$.

One shows in the same way, by taking $\alpha$ to be a Liouville number, that none of the conditions $A \in K_p$ is necessary for $A$ to be a generating function of the spectrum of an operator $A_0^{-1}$, where $A_0 = \text{Re } V$, $V$ is Volterra and $\text{Im } V$ has rank two.

This example shows that the question of describing possible spectra of $A_0 = A_0^*$ such that $V = A_0 + iB_0$ is a Volterra operator for some rank two selfadjoint operator $B_0$ is different from Problem 2.

8. **Volterra rank one perturbations generated by “smooth” vectors**

Let $A_0$ be a compact selfadjoint operator with the simple point spectrum $\{s_n\}$, that is, the operator of multiplication by $x$ in $L^2(\nu)$, where $\nu = \sum_n \nu_n \delta_{s_n}$. In this section we show that, for a rank one perturbation $L_0 = A_0 + ab^*$, the property of being a Volterra operator is compatible with a certain smoothness of the vectors $a$ and $b$. On the other hand, recall that the classical completeness theorem of Macaev [30] (see also [12, Chapter V]) implies that in the case when $a$ or $b$ is in the range of $A_0$ (i.e., $a \in xL^2(\nu)$ or $b \in xL^2(\nu)$) and $\ker L_0 = 0$, the perturbed operator $L_0$ has a complete set of eigenvectors (or root vectors in case of multiple spectrum):

**Theorem.** (See Macaev (1961) [30].) If $L_0 = A_0(I + S)$, where $A_0$, $S$ are compact operators on a Hilbert space, $A_0$ is selfadjoint and $S$ is in the Macaev ideal $\mathcal{G}_\omega$ (i.e., its singular numbers $s_k$ satisfy $\sum_{k \geq 1} \frac{s_k}{\omega_k} < \infty$) and $\ker A_0 = \ker (I + S) = 0$, then $L_0$ and $L_0^*$ have complete sets of eigenvectors.

The following theorem shows that any weaker smoothness of $a$ and $b$ can be achieved for Volterra rank one perturbations. A special case of this result was given in [6, Theorem 0.6].

**Theorem 8.1.** Let $s_n \to 0$, $s_n \neq 0$, and assume that $\{t_n\}$, where $t_n = s_n^{-1}$, is a removable spectrum. Let $A$ be the corresponding function in the Krein class $K_1$, given by (4.1). Assume that for some $\gamma \in (0, 2)$ we have

$$\sum_n \left| t_n \right|^\gamma |A'(t_n)| < \infty.$$

Let $A_0$ be a selfadjoint operator with the point spectrum $\{s_n\}$ and trivial kernel (i.e., $A_0$ is the operator of multiplication by $x$ in $L^2(\mu^0)$ where $\mu^0 = \sum_n \mu_n \delta_{s_n}$). Then for any $\alpha_1, \alpha_2 \in [0, 1)$ with $\alpha_1 + \alpha_2 \leq 2 - \gamma$
there exist $a \in |x|^{\alpha_1} L^2(\mu^0)$ and $b \in |x|^{\alpha_2} L^2(\mu^0)$ such that $a, b \notin |x| L^2(\mu^0)$ and the spectrum of the perturbed operator $L_0 = A_0 + ab^*$ equals $\{0\}$.

**Proof.** Consider the unbounded operator $A$ on $L^2(\mu)$, where $\mu = \sum_n \mu_n \delta_{t_n}$; it is unitary equivalent to $A_0^{-1}$. We pass to a reformulation concerning a singular perturbation of $A$. By [6, Proposition 1.5], whenever $a, b$ satisfy the above requirements and $\varkappa \in \mathbb{R}$, $\varkappa \neq 0$, $A_0 - \varkappa^{-1} ab^*$ has a (unbounded) inverse, which is unitarily equivalent to $L(A, a', b', \varkappa)$, where $a' = A^{-1} a$, $b' = A^{-1} b$, $a', b' \in x L^2(\mu)$, so that $a'_n = t_n a_n$, $b'_n = t_n b_n$. Thus, for any $\alpha_1$ and $\alpha_2$ as above, we need to find a nonzero $\varkappa \in \mathbb{R}$ and $a', b' \notin L^2(\mu)$ such that

$$
\sum_n |a'_n|^2 |t_n|^{2\alpha_1 - 2} \mu_n < \infty, \quad \sum_n |b'_n|^2 |t_n|^{2\alpha_2 - 2} \mu_n < \infty
$$

(8.1)

and the function

$$
\varphi(z) = \frac{1 + \Theta(z)}{2} \cdot \left( \varkappa + \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) a'_n b'_n \mu_n \right)
$$

(8.2)

has no zeros in $\mathbb{C}$.

Since $A \in K_1$, we have

$$
\frac{1}{A(z)} = q + \sum_n c_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right),
$$

(8.3)

where $q = 1/A(0)$, $c_n = -1/A'(t_n)$ and $\sum |t_n|^{-\gamma} |c_n| < \infty$. We will represent $c_n$ as $c_n = a'_n b'_n \mu_n$, where $a'_n$ and $b'_n$ have the required properties. Once such $a'$ and $b'$ have been constructed, we define the function $\Theta$ by the formulas (2.2) and (2.4) with $b'_n$ in place of $b_n$, that is, we put

$$
i \frac{1 - \Theta(z)}{1 + \Theta(z)} = \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) |b'_n|^2 \mu_n.
$$

Next, define $\varphi$ by (8.2), with $a'_n b'_n \mu_n = c_n$ and with $q$ in place of $\varkappa$. If we now put $E = \frac{2A}{1 + \Theta}$, then, clearly, $E$ is an entire function, $\varphi = 1/E$ by (8.3), and $\Theta = E^*/E$ (see (5.2)) whence $E \in H B$. Thus, the function $\varphi$ has no zeros.

Define sequences $a', b'$ as follows:

$$
a'_n = \frac{|c_n|^{|1/2| t_n|^{|2 - 2\alpha_1 - \gamma|/2}}}{\mu_n^{|1/2|}}, \quad b'_n = \frac{|c_n|^{|1/2| t_n|^{|2\alpha_1 + \gamma - 2|/2}}}{\mu_n^{|1/2|}}.
$$

Then

$$
\sum_n |a'_n|^2 |t_n|^{2\alpha_1 - 2} \mu_n = \sum_n \frac{|c_n|^2}{|t_n|^?} < \infty,
$$

$$
\sum_n |b'_n|^2 |t_n|^{2\alpha_2 - 2} \mu_n = \sum_n |c_n||t_n|^{2\alpha_1 + 2\alpha_2 + \gamma - 4} < \infty,
$$

since $\alpha_1 + \alpha_2 \leq 2 - \gamma$.

If $a'$ and $b'$ are not in $L^2(\mu)$, then the theorem is already proved. In the general case, choose a sequence $p_{2n+1} \geq 1$ such that

$$
\sum_n p_{2n+1}^2 |a'_{2n+1}|^2 |t_{2n+1}|^{2\alpha_1 - 2} \mu_{2n+1} < \infty, \quad \sum_n p_{2n+1}^2 |a'_{2n+1}|^2 \mu_{2n+1} = \infty
$$

$$
\sum_n p_{2n+1}^2 |a_{2n}|^2 |t_{2n}|^{2\alpha_1 - 2} \mu_{2n} < \infty, \quad \sum_n p_{2n+1}^2 |a_{2n}|^2 \mu_{2n} = \infty
$$

(5.2)
(this is, obviously, possible, because \(|t_n| \to \infty\) and \(\alpha_1 < 1\)). Analogously, we choose \(p_{2n} \leq 1\) so that
\[
\sum_n p_{2n}^{-2} |b_n'|^2 |t_{2n}|^{2 \alpha_2 - 2} \mu_{2n} < \infty, \quad \sum_n p_{2n}^{-2} |b_n'|^2 \mu_{2n} = \infty.
\]

Then, clearly, \(\tilde{a}_n = p_n a_n'\) and \(\tilde{b}_n = p_n^{-1} b_n'\) are not in \(L^2(\mu)\), \(\tilde{a} \in |x|^{\alpha_1 - 1} L^2(\mu)\), \(\tilde{b} \in |x|^{\alpha_2 - 1} L^2(\mu)\) and
\[
\tilde{a}_n \tilde{b}_n \mu_n = c_n. \quad \Box
\]

**Remark 8.2.** One can compare Theorem 8.1 with Macaev’s theorem mentioned above as well as with the following result from [6] (Theorem 3.3, statement (2)): Let \(\mathcal{A}\) be an unbounded cyclic selfadjoint operator with discrete spectrum \(\{t_n\}\), \(t_n \neq 0\), and let the data \((a', b', \varkappa)\) satisfy
\[
\sum_n \frac{|a_n'| |b_n'| \mu_n}{|t_n|} < \infty, \tag{8.4}
\]
\[
\sum_n \frac{a_n' \overline{b_n} \mu_n}{t_n} \neq \varkappa. \tag{8.5}
\]

Then the singular rank one perturbation \(\mathcal{L} = L(\mathcal{A}, a, b, \varkappa)\) and its adjoint \(\mathcal{L}^*\) have complete sets of eigenvectors.

Note that for \(a', b'\) constructed as in the above proof, (8.4) will be satisfied if \(1 \leq \alpha_1 + \alpha_2\). However, there is no contradiction with Theorem 8.1. Indeed, looking at the asymptotics when \(y \to \infty\) in
\[
\frac{1}{A(iy)} = \varkappa - \sum_n \frac{c_n}{t_n} + \sum_n \frac{c_n}{t_n - iy},
\]
where \(c_n = a_n' \overline{b_n} \mu_n\), we see that (8.5) is not satisfied for the perturbation constructed in Theorem 8.1.

Thus, in contrast to Macaev’s theorem (which applies to the so-called weak perturbations of the form \(\mathcal{A}_0(I + S)\) or \((I + S)\mathcal{A}_0\)), we have the following corollary of Theorem 8.1:

**Corollary 8.3.** For any \(\alpha_1, \alpha_2 \in (0, 1)\) and any selfadjoint compact operator \(\mathcal{A}_0\) as in Theorem 8.1 there exists a rank one perturbation \(\mathcal{L}_0\) of \(\mathcal{A}_0\) of the form
\[
\mathcal{L}_0 = \mathcal{A}_0 + \mathcal{A}_0^{\alpha_1} S \mathcal{A}_0^{\alpha_2},
\]
where \(S\) is a rank one operator and \(\text{Ker} \mathcal{L}_0 = \text{Ker} \mathcal{L}_0^* = 0\), such that \(\mathcal{L}_0\) is a Volterra operator.

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**References**


