A Non-Matching Domain Decomposition Method for Maxwell's Equations

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Eidesstattliche Erklärung

Ich versichere hiermit, die vorliegende Arbeit ohne fremde Hilfe verfasst und mich keiner anderen als der angegebenen Hilfsmittel und Quellen bedient zu haben.
Diese Diplomarbeit wurde bisher weder im In- noch im Ausland als Prüfungsarbeit vorgelegt.

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Chapter 1

Introduction

Domain decomposition expands the application range of finite element methods in different ways by splitting the solution domain into two or more subdomains. For example, the decomposition of a large problem into smaller ones might make its solution feasible for certain solvers, or it could facilitate computation on a parallel architecture. Some domain decomposition methods require the finite element discretization (i.e. the mesh) to match on different sides of the interface (i.e. the shared boundary of two subdomains). Others, such as the method described in this thesis, do not. One significant advantage of the latter type in particular arises when different parts of the domain move in relation to each other.

Consider, for example, a simulation of the magnetic field inside an electric motor. Typically for many types of motors, a current-carrying armature rotates in the static magnetic field of the field magnet or field coil. Thus, the configuration of the parts changes continually. In order to obtain a matching discretization of the space between rotor and stator for different stages of the simulation, the mesh would have to be recomputed at every time step. With non-matching domain decomposition, however, one could split off a cylindrical subdomain around the armature and rotate the subdomain. Because the submeshes remain the same, we will also see that only the coupling equations of the system matrix have to be reassembled.

One standard approach to non-matching domain decomposition is the mortar method, where the continuity of the field is enforced by Lagrange multipliers on the interface. This leads to an indefinite system of equations.

Unlike some mortar methods, the approach described in this thesis does not utilize a saddle-point problem for the variational formulation. Instead, it bears a relation to Nitsche's method of enforcing Dirichlet boundary conditions, where additional stabilization and symmetry terms are introduced to the variational equation. These fit the problem into the Lax-Milgram / Strang framework for convergence of non-conforming, coercive formulations. This technique was introduced for the Poisson problem in [1].

In order to arrive at a practicable method, one also has to consider the process of finite element system assembly. Specifically, element matrices at the interface need to be computed independently from the elements of any other subdomain. To that end, we modify the method and introduce degrees of freedom on the interface similar to those found in hybrid discontinuous Galerkin methods, as described in [2] for the Poisson problem. The convergence results for this hybrid method allow for great flexibility in the choice of the finite element space for the hybrid degrees of freedom on the interface. We propose a space spanned by B-spline basis functions, and analyze the hybrid method for the Poisson problem in some detail.

This technique is applied to Maxwell's equations for electromagnetic fields. As it turns out, a directly equivalent treatment of the boundary conditions that arise from Maxwell's equations leads to an over-penalization of certain components of the solution field. This causes the energy norm induced by the bilinear form to scale poorly, which has a negative impact on convergence. We address this problem by imposing weaker transmission conditions using a Helmholtz-type decomposition of the field. This yields a stronger norm for convergence at the cost of additional degrees of freedom on the interface.

The methods for the Poisson problem and Maxwell's equations were implemented in the open source software package NETGEN / NGSOLVE to demonstrate their feasibility. Numerical experiments were carried out to verify the theoretical results and examine the behavior of the methods in practical applications.

We begin this thesis by reiterating some of the theory of Sobolev function spaces which forms the basis for the analysis. Next, the general framework of finite element methods is introduced, including standard methods for Maxwell's equations and some results for non-conforming methods pertinent to our domain decomposition method. We go on to discuss spline spaces and some of their properties, before introducing the domain decomposition method for the Poisson problem in the following chapter. After some numerical results, the method is applied to Maxwell's equation. We close with the discussion of a more complex numerical simulation employing domain decomposition.

Chapter 2

A Short Introduction to function spaces

We begin with an introduction to function spaces, from Lebesgue spaces to general Sobolev spaces and vector valued function spaces, where only certain differential operators exist. This chapter adheres to [6], chapters 4 and 5.

2.1 The L_2 space

The Lebesgue function space $[L_2]^n$ is a superset to all function spaces utilized in this thesis, and all differential operators are derived from the properties of its Lebesgue integral. The system of subsets of $\mathbb{R}^n$ that integration will be defined on, must form a σ-algebra.

Definition 1 *A set Σ of subsets of $\mathbb{R}^n$, such that*

(a) $\mathbb{R}^n \in \Sigma$

(b) $A \in \Sigma \Rightarrow A^c \in \Sigma$

(c) $A_k \in \Sigma,\ k \in \mathbb{N}\ \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \Sigma$

*is called a σ-*algebra*.*

Integration on $\mathbb{R}^n$ induces a volume function $\mu\ :\ \mathbb{R}^n \to \mathbb{R}$ via $\mu(A) := \int_A 1$. Contrarily, we will proceed by prescribing certain properties of a volume function and using it to construct the Lebesgue integral.

Definition 2 *For a σ-algebra Σ, a* measure *is a function $\mu\ :\ \Sigma \to \mathbb{R}_0^+ \cup \{\infty\}$, such that*

(a) $\mu(A) \geq 0$ *for all* $A \in \Sigma$

(b) $\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$ *for* $A_k \in \Sigma$ *and* $A_k \cap A_l = \emptyset$ *for* $k \neq l$

Property (a) is called non-negativity*, (b) is called* σ-additivity*.*

We call a measure μ normed, if $\mu([0,1]^n) = 1$, and μ is translation-invariant, if $\mu(q + A) = \mu(A)$ for all $A \in \Sigma,\ q \in \mathbb{R}^n$. It can be shown that there exists no non-negative, sigma additive, normed, and translation invariant measure on $\mathcal{P}(\mathbb{R}^n)$. For a *box* E, i.e. an interval of the form $E = [a_1, b_1) \times [a_2, b_2) \times \ldots \times [a_n, b_n)$, let $\mu_0(E) := \prod_{i=1}^{m}(b_i - a_i)$. Then we can define a measure on the set Σ_0 of finite unions of discrete boxes via $\mu_0(\bigcup_{i=1}^{m} E_i) := \sum_{i=1}^{m} \mu_0(E_i)$. Note that although

Σ_0 is not a σ-algebra, we can construct a measure μ on a σ-algebra Σ, such that $\Sigma_0 \subset \Sigma$ and $\mu(A) = \mu_0(A)$ for $A \in \Sigma_0$ by applying an extension argument to μ_0. We call $N \subset \mathbb{R}^n$ a *null set*, iff $\mu(N) = 0$. See [7] for a proof of the following result:

Proposition 1 *There exists a σ-algebra Σ on $\mathbb{R}^n$ and a measure μ on Σ, called* Lebesgue measure*, such that*

(a) All open subsets of $\mathbb{R}^n$ are in Σ.

(b) For every $A \in \Sigma$ and $\epsilon > 0$, there exists an open set $B \supset A$, such that $\mu(B \backslash A) \leq \epsilon$.

(c) $A \subset B$ and $B \in \Sigma$, $\mu(B) = 0 \quad \Rightarrow \quad A \in \Sigma$ and $\mu(A) = 0$

(d) $A = \{x \in \mathbb{R}^n \ : \ a_k \leq x_k \leq b_k, \ k = 1, \ldots, n\} \in \Sigma$ and $\mu(A) = \prod_{k=1}^n (b_k - a_k)$

(e) $\mu(A) = \mu(x + a)$ for $A \in \Sigma, x \in \mathbb{R}^n$

Using the Lebesgue measure, we can identify a class of functions we will assign an integral to.

Definition 3 *Let $\Omega \subset \mathbb{R}^n$ be measurable and $u \ : \ \Omega \to \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. We call u* measurable*, iff $\{x \ : \ u(x) > a\}$ is measurable for all $a \in \mathbb{R}$.*

The integral is first introduced on the simple class of step functions.

Definition 4 *We call $s \ : \ \mathbb{R}^n \to \mathbb{R}$ a* step function*, iff $s(x) = \sum_{k=1}^m a_k \chi_{A_k}$ for some measurable sets $A_1, \ldots, A_m \subset \mathbb{R}^n$, and $a_1, \ldots, a_m \in \mathbb{R}$. Let $\Omega \subset \mathbb{R}^n$ be measurable and $s(x) = \sum_{k=1}^m a_k \chi_{A_k}$ a step function. Define*

$$\int_\Omega s(x) := \sum_{k=1}^m a_k \mu(A_k)$$

Proposition 2 *To every measurable function $u \geq 0$, there exists a sequence of monotonically increasing step functions (s_k), such that $s_k(x) \to u(x)$ for every $x \in \Omega$.*

Now we extend integration to all measurable functions:

Definition 5 *For a positive measurable function $\tilde{u}$, let*

$$\int_\Omega \tilde{u}(x) := \sup \left\{ \int_\Omega s(x) \ : \ s \in T(\Omega), \ 0 \leq s \leq \tilde{u} \right\}$$

For a measurable function u, let $\int_\Omega u_+(x) < \infty$ or $\int_\Omega -u_-(x) < \infty$. Then we define

$$\int_\Omega u(x) := \int_\Omega u_+(x) - \int_\Omega -u_-(x).$$

If both integrals are bounded, u is said to be integrable*.*

In order to construct a complete space from the Lebesgue-measurable functions, we must identify all functions that are identical up to a null set under the Lebesgue measure.

Definition 6 *Let $1 \leq p < \infty$. The set of equivalence classes u of measurable functions on Ω under the equivalence relation*

$$u \sim v \quad :\Leftrightarrow \quad u = v \quad a.e. \ on \ \Omega,$$

such that $|u|^p$ is integrable on Ω, is called the space $L^p(\Omega)$.

Proposition 3 *For $1 \le p \le \infty$, $L^p(\Omega)$ with the norm*

$$\|u\|_p := \left(\int_\Omega |u(x)|^p\right)^{1/p}, \quad \text{if} \quad p < \infty$$
$$\|u\|_\infty := \inf_{\mu(N)=0} \sup_{x \in \Omega \setminus N} |u(x)|$$

is a Banach space.

Note that $L^2(\Omega)$ together with the inner product $(u,v)_{L^2(\Omega)} = \int_\Omega u(x) \cdot v(x)$ is a Hilbert space.

2.2 The Weak Derivative

To approximate partial differential equations with finite element methods, we need appropriate function spaces on which the differential operators that appear in the PDE are well-defined. The construction of such spaces requires a more general definition for differentiation of Lebesgue-measurable functions than classic differentiation.

Definition 7 *The space of smooth functions with compact support is defined as*

$$\mathcal{D}(\Omega) := \{u \in C^\infty(\Omega) \mid u \text{ has compact support in } \Omega\}$$

The dual space $\mathcal{D}'(\Omega)$ of continuous functionals on $\mathcal{D}(\Omega)$ (i.e. continuous linear mappings from $\mathcal{D}(\Omega)$ to $\mathbb{R}$, see 15) is called the space of distributions.

Definition 8 *The space of all measurable functions that are integrable on every compact subset of Ω is called $L^1_{loc}(\Omega)$. In other words,*

$$L^1_{loc}(\Omega) = \{f \in L^1(\Omega_0) \; : \; \Omega_0 \subset \Omega, \; \Omega_0 \text{ compact}\}$$

Definition 9 *For $f \in L^1_{loc}$, define $\Lambda_f \in \mathcal{D}'$ via*

$$\Lambda_f(\Phi) = \int_\Omega \Phi f$$

for $\Phi \in \mathcal{D}$

If $f \in C^\infty(\Omega)$, then integration by parts holds:

$$\int D^\alpha f \cdot \Phi = (-1)^{|\alpha|} \int f \cdot D^\alpha \Phi$$

for all $\Phi \in \mathcal{D}(\Omega)$. Or, in the distributional formulation for f:

$$\Lambda_{D^\alpha f}(\Phi) = (-1)^{|\alpha|} \Lambda_f(D^\alpha \Phi)$$

We can use this relation to generalize differentiation to all distributions in $\mathcal{D}'$:

Definition 10 (Differentiation of distributions) *For a multi-index α and a distribution $\Lambda \in \mathcal{D}'(\Omega)$, the αth* distribution derivative *of Λ is the linear functional $D^\alpha \Lambda$ on $\mathcal{D}'(\Omega)$ with*

$$(D^\alpha \Lambda)(\Phi) = (-1)^{|\alpha|} \Lambda(D^\alpha \Phi)$$

for all $\Phi \in \mathcal{D}(\Omega)$.

Partial differentiation shows that $D^\alpha \Lambda_f = \Lambda_{D^\alpha f}$ for $f \in C^\infty$. The equality does not hold for all functions that are not continuously differentiable, however. A distributional derivative can be assigned to every locally integrable function, but not every derivative can be assigned a function. With this in mind, we arrive at a definition for differentiation that is more general than classical differentiation but more restrictive than distributional differentiation:

Definition 11 (Weak derivative) *If there exists $g \in L^1_{loc}$, such that $\Lambda_g = D^\alpha \Lambda_f$ for $f \in L^1_{loc}$, then $D^\alpha f := g$ is the α*th weak derivative *of f.*

In similar fashion, we can define weak differential operators for gradient, curl, and divergence, respectively:

Definition 12 *For $w \in L_2(\Omega)$, $g = \text{div } w \in L_2(\Omega)$ is called the* weak divergence *of w, if*

$$\int g \cdot v = -\int w \text{ div } v \quad \textit{for all} \quad v \in [\mathcal{D}(\Omega)]^3$$

For $u \in [L_2(\Omega)]^3$, $c = \text{curl } w$ is called the weak curl *of w, if*

$$\int c \cdot v = \int u \text{ curl } v \quad \textit{for all} \quad v \in [\mathcal{D}(\Omega)]^3$$

For $q \in L_2(\Omega)$, $d = \nabla w$ is called the weak gradient *of w, if*

$$\int d \cdot v = -\int q \ \nabla v \quad \textit{for all} \quad v \in \mathcal{D}(\Omega)$$

Like the weak differential, these operators coincide with their strong counterparts for classically differentiable functions.

2.3 Sobolev Spaces

Using the weak derivative, we can now construct Banach spaces for finite element methods where the differential operators are well-defined. We begin with the general Sobolev spaces $W^{m,p}$.

Definition 13 *For $m, p \in \mathbb{N}$ we define the spaces*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \textit{ for } 0 \le |\alpha| \le m\},$$

where $D^\alpha u$ is the distributional partial derivative of u. Because the case $p = 2$ is of special interest, we write

$$H^m(\Omega) := W^{m,2}(\Omega)$$

Proposition 4 *$W^{m,p}(\Omega)$ is a Banach space w.r.t. the norm:*

$$\|u\|_{m,p} = \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha u\|_p^p \right)^{1/p} \quad \textit{for} \quad 1 \le p < \infty,$$
$$\|u\|_{m,\infty} = \max_{0 \le |\alpha| \le m} \|D^\alpha u\|_\infty.$$

Proof Let $v_j, j \in \mathbb{N}$ be Cauchy in $W^{p,k}(\Omega)$. From the definition of $W^{p,k}(\Omega)$, it follows that $D^\alpha v_j$ is Cauchy in $L_p(\Omega)$, and converges to some $v^\alpha \in l^p(\Omega)$. Let K be the (compact) support of ϕ. Then,

$$\begin{aligned}\int_\Omega (D^\alpha v_j - v^\alpha)\phi &= \int_K (D^\alpha v_j - v^\alpha)\phi \\ &\leq \|D^\alpha v_j - v^\alpha\|_{L_1(K)} \|\phi\|_{L_\infty} \\ &\leq \|D^\alpha v_j - v^\alpha\|_{L_p(K)} \|\phi\|_{L_\infty} \to 0\end{aligned}$$

Now, we show that $D^\alpha v = v^\alpha$:

$$\begin{aligned}\int_\Omega v^\alpha \phi &= \lim_{j\to\infty} \int_\Omega D^\alpha v_j \phi \\ &= \lim_{j\to\infty} (-1)^{|\alpha|} \int_\Omega v_j D^\alpha \phi \\ &= (-1)^{|\alpha|} \int_\Omega v D^\alpha \phi\end{aligned}$$

This proves $W^{m.p}(\Omega)$ is complete, which makes it a Banach space.

Note that $H^m(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{H^m(\Omega)} := \sum_{|\alpha|\leq m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u \cdot D^\alpha v.$$

The spaces $H^m(\Omega)$ are natural solution sets for PDE's where the differential operators manipulate the components of the field symmetrically, such as the Laplace, elasticity, or the bi-harmonic problem.

2.4 Vector-Valued Function Spaces

Some PDE's feature differential operators that couple only certain components of the solution field. The Maxwell equation, for example, that will be considered in this thesis, uses the *curl* operator.
The spaces

$$\begin{aligned}H^1 &= W^{1,2}(\Omega) \\ H(curl, \Omega) &= \{u \in [L_2(\Omega)]^3 \mid curl\ u \in [L_2(\Omega)]^3\} \\ H(div, \Omega) &= \{q \in [L_2(\Omega)]^3 \mid div\ q \in L_2(\Omega)\}\end{aligned}$$

together with the norms

$$\begin{aligned}\|w\|_{H^1} &= \|w\|_{1,2} = \|w\|_{L_2} + \|\nabla w\|_{L_2} \\ \|u\|_{H(curl,\Omega)} &= \|u\|_{L_2} + \|curl\ u\|_{L_2} \\ \|q\|_{H(div,\Omega)} &= \|q\|_{L_2} + \|div\ u\|_{L_2},\end{aligned}$$

are appropriate solution spaces to problems dominated by the respective differential operators.

2.5 Trace Theorems

In this section we examine the behavior of Sobolev space functions at the boundary of the domain Ω. In the following we assume Ω to satisfy a certain restriction to the shape of its boundary that makes it feasible for analysis:

Definition 14 *A domain Ω is* Lipschitz-bounded*, if there exist open sets O_i, such that $O_i \cap \partial\Omega$ is the graph of a Lipschitz function $\phi_i \ : \ \mathbb{R}^d - 1 \to \mathbb{R}$*

We will also need the following result for the analysis:

Proposition 5 *A domain Ω is Lipschitz bounded, iff there exist open sets $U_j \in \mathbb{R}^n$, $\partial\Omega \subset U_j$ together with diffeomorphisms $g_j \ : \ U_j \to B_1(0)$*

At a glance, extension of functions in $H^1(\Omega)$ to $\partial\Omega$ seems like a contradiction of terms, as Sobolev functions are only defined up to a null set, and the boundary of the domain is, in fact, a null set. It can be shown, for example, that $C_0^\infty(\Omega)$ is dense in $H^1(\Omega)$ for Lipschitz bounded domains. As it turns out, extension to the boundary is possible, however, in the sense that a completion for the extension operator on C^∞ exists:

Proposition 6 (Trace Theorem) *For a Lipschitz-bounded domain Ω, there exists a well-defined continuous operator*

$$\mathrm{tr} \ : \ H^1(\Omega) \to L_2(\partial\Omega)$$

with

$$u|_{\partial\Omega} = \mathrm{tr}\ u \quad \textit{for} \quad u \in C^1(\overline{\Omega})$$

In order to prove the trace theorem, it can be shown that the operator

$$\begin{aligned} \mathrm{tr} \ &: \ C^1(\overline{\Omega}) \to L_2(\partial\Omega) \\ &\qquad u \mapsto u|_{\partial\Omega} \end{aligned}$$

is bounded. Because $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, it follows that there exists a unique completion on the whole space. First, we map the norm of the continuous trace to the reference domain $(0,1)^2$ using the transformation from proposition 5:

$$\begin{aligned} \|\mathrm{tr}\ u\|^2_{L_2(\partial\Omega)} &= \sum_{i=1}^{M} \int_{\Gamma_i} u(x)^2 \\ &= \sum_{i=1}^{M} \int_{\Gamma_i} u(s_i(\zeta,0))^2 \left| \partial \frac{s_i}{\partial\zeta}(\zeta,0) \right| d\zeta \end{aligned}$$

The norm in the domain is transformed in the same way:

$$\begin{aligned} \|u\|^2_{H^2(\Omega)} &\geq \sum_{i=1}^{M} \int_{S_i} |\nabla_x u|^2 \\ &= \sum_{i=1}^{M} \int_{Q} |\nabla_x (u \circ s)|^2 \det(s') \\ &\geq c \sum_{i=1}^{M} \int_{Q} |\nabla_x (\tilde{u})|^2 \end{aligned}$$

The boundedness of the operator now boils down to the estimate on the reference domain. See [13] for the two-dimensional case, while [6] has the proof in greater generality.
The Range of the trace operator is a Banach space:

Proposition 7 *The space*

$$W = \{\text{tr } u \mid u \in H^1(\Omega)\}$$

with the norm

$$\|\text{tr } u\|_W = \inf\{\|v\| \mid v \in H^1(\Omega),\ \text{tr } u = \text{tr } v\}$$

is a Banach space.

It is possible to define spaces $H^s(\Omega)$ for non-integer values $s \in \mathbb{R}$ via the *broken Sobolev norm.* Let $s = m + \sigma$, $m \in \mathbb{N}_0$, $0 < \sigma < 1$. Define

$$|u|^2_{\sigma,\Omega} = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\sigma}} dx\ dy.$$

Then, the broken Sobolev norm of $u \in H^2(\Omega)$ is

$$\|u\|^2_{s,\Omega} = \|u\|^2_{m,\Omega} + \sum_{|\alpha|=m} |D^\alpha u|^2_{\sigma,\Omega}$$

Using the Lipschitz property, we can define broken Sobolev spaces on the boundary of our domain. As it turns out, the trace space W from proposition 7 is identical to the Sobolev space $H^{1/2}(\partial\Omega)$. In fact, there exist continuous trace operators from $H^s(\Omega)$ to $H^{s-1/2}(\delta\Omega)$ for all $s > \frac{1}{2}$. On the other hand, there also exist continuous extension operators $F\ :\ H^{s-1/2}(\partial\Omega) \to H^s(\Omega)$, such that $\text{tr } F = \text{Id}$.
The dual space of $H^s(\Omega)$ is referred to as $H^{-s}(\Omega)$. The following trace theorems for the vector-valued Sobolev spaces hold:

Proposition 8 *For a Lipschitz-bounded domain Ω, there exists a well-defined, continuous operator*

$$\text{tr}_n\ :\ H(\text{div}, \Omega) \to H^{-1/2}(\partial\Omega)$$

such that for $u \in [C(\overline{\Omega})]^3 \cap H(\text{div})$

$$\text{tr}_n\ u = [u(.) \cdot n(.)]|_{\partial\Omega}$$

The inverse normal trace theorem also holds for $H(div, \Omega)$.

Proposition 9 *For every $q_n \in H^{-1/2}(\partial\Omega)$, there exists $q \in H(div)$, such that*

$$\text{tr}_n u = q_n.$$

Proposition 10 *For a Lipschitz-bounded domain Ω, there exists a well-defined, continuous operator*

$$\text{tr}_\tau\ :\ H(\text{curl}, \Omega) \to H^{-1/2}(\partial\Omega)$$

such that for $u \in [C(\overline{\Omega})]^3 \cap H(\text{curl})$

$$\text{tr}_\tau\ u = [u(.) \times n(.)]|_{\partial\Omega}$$

Proposition 11 *For a domain decomposition $\Omega_1, \Omega - 2, \ldots, \Omega_m$, let $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. If $u_i \in H(\text{curl}, \Omega_i)$, $i = 1, \ldots, m$, such that $\text{tr}_{\tau,\Gamma_{ij}} u_i = \tau, \Gamma_{ij} u_i u_j$, then*

$$u \in H(\text{curl}, \Omega) \textit{ and } (\text{curl } u)|_{\Omega_i} = \text{curl } u_i,$$

where $u|_{\Omega_i} := u_i$.

This last result is useful because it allows us to construct finite dimensional subspaces of $H(\text{curl})$ on a partition of Ω. These will be the solution spaces for the finite element method.

Chapter 3

Finite Element Methods

3.1 Variational Equations

Finite element methods compute a numerical solution to a PDE in the weak form, i.e. they operate on a variational equation that is equivalent to the differential problem. Thus, the theory of variational equations provides the basic framework for the analysis of the method's approximation behavior. We begin with some elementary concepts from functional analysis.

Definition 15 *A* functional *is a linear mapping $l : v \to \mathbb{R}$, where V is a Vector Space. A linear mapping $A : V \times V \to \mathbb{R}$ is called a* bilinear form. A *is called symmetric, if $A(u,v) = A(v,u)$ for all $u, v \in V$.*

Every element x of a Hilbert space V can be mapped to a functional f_x on V via $f_x(y) := (y, x)$. Riesz's representation theorem demonstrates, that $j : V \to V'$, $x \mapsto f_x$ is a bijective mapping from V to the space of continuous functionals on V, called the dual space V'.

Proposition 12 (Riesz Representation Theorem) *For every continuous linear functional l on a Hilbert space V, there exists $u_l \in V$, such that*

$$l(v) = (u_l, v) \quad \text{for all} \quad v \in V,$$

and

$$\|l\|_{V^*} = \|u_l\|_V.$$

Definition 16 *A bilinear form A on a Hilbert space V is called* coercive*, if there exists $c_1 > 0$, such that*

$$A(u,u) \geq c_1 \|u\|^2 \quad \text{for all} \quad u \in V.$$

It is bounded*, if there exists $c_2 \in \mathbb{R}$, such that*

$$A(u,v) \leq c_2 \|u\|_V \|v\|_V \quad \text{for all} \quad u, v \in V.$$

A *variational equation* is an equation of the form

Proposition 13 (Lax-Milgram) *Let V be a Hilbert space, A a bilinear form, and f a functional on V. If A is coercive and bounded, then there exists a unique $u \in V$, such that*

$$A(u,v) = f(v) \quad \text{for all} \quad v \in V,$$

i.e. the variational equation is uniquely solvable.

Proof According to Riesz's representation theorem, there exits an isomorphism $J_V : V^* \longrightarrow V$, such that:

$$(J_V g, v)_V = g(v)$$

3.2 The Galerkin Approximation

Finite element methods methods solve a variational equation on a finite-dimensional subspace of V, the *finite element space*:

Definition 17 *For a closed subspace V_h of a Hilbert space V, the Galerkin approximation of the solution u to the variational problem*

$$A(u,v) = f(v) \quad \text{for all} \quad v \in V_h$$

with a coercive and bounded bilinear form A is the unique solution u_h of the variational problem

$$A(u_h, v_h) = f(v_h) \quad \text{for all} \quad v_h \in V_h$$

Proposition 14 (Cea's Lemma) *For u, u_h as above, it holds that*

$$\|u - u_h\|_V \leq \frac{c_2}{c_1} \inf_{v \in V_h} \|u - v_h\|.$$

Proof For the Galerkin approximation, the so-called Galerkin orthogonality holds:

$$(u - u_h, v_h) = (u, v_h) - (u_h, v_h) = f(v_h) - f(v_h) = 0$$

Now, let $v_h \in V_h$

$$\|u - u_h\|^2 \leq c_1^{-1}$$

Thus, a framework for basic finite element methods is almost complete. If a weak form of a PDE satisfies coercivity and boundedness, it is uniquely solvable, and the Galerkin approximation is bounded by the best approximation to the solution in the finite element space. What is left is to choose a finite element space, so that the best approximation is sufficiently close to the exact solution.

3.3 Finite Elements

A finite element method (FEM) consists of a Variational equation

$$A(u,v) = f(v),$$

where A and f are defined on some finite-dimensional Hilbert space V_h spanned by basis functions φ_i. The solution of the FEM is a function $u_h \in V_h$, such that

$$A(u_h, v_h) = f(v_h) \quad \text{for all} \quad v_h \in V_h.$$

If the exact solution u of the problem that is to be approximated satisfies

$$A(u,v) = f(v) \quad \text{for all} \quad v \in V. \tag{3.1}$$

for some Hilbert space V, and V_h is a subspace of V, then the FEM is called *conforming*. The basis functions are generally derived from finite elements, i.e. triples (T, V_T, Ψ_T), where

- T is a bounded set in Ω
- V_T is a function space on T of finite dimension N_T

- $\Psi_T = \{\psi_T^1, \ldots, \psi_T^{N_T}\}$ is a set of linearly independent functionals on V_T.

The element basis $\{\varphi_T^1, \ldots, \varphi_T^{N_T}\}$ of V_T is the dual basis to Ψ_T, i.e.

$$\psi_T^i(\varphi_T^j) = \delta_{i,j} \quad \text{for} \quad 1 \leq i, j \leq N_T. \tag{3.2}$$

For $v \in C(\overline{\Omega})$, the local nodal interpolation operator defined as

$$I_T v := \sum_{\alpha=1}^{N_T} \psi_T^\alpha(v)\phi_T^\alpha. \tag{3.3}$$

is a projection into V_T. A triangulation $\mathcal{T} = \{T_1, \ldots, T_M\}$ is a subdivision of Ω into bounded subsets T_i, such that $\bigcup T_i = \overline{\Omega}$, and $\mathring{T}_i \cap \mathring{T}_j = \emptyset$. We can define a global Interpolation Operator on all of Ω via

$$I_{\mathcal{T}} v|_T = I_T v|_T \quad \text{for all} \quad T \in \mathcal{T}. \tag{3.4}$$

Then the finite element space V_h is the range of the smooth functions $C^m(\overline{\Omega})$ under the projection Operator $I_{\mathcal{T}}$:

$$V_{\mathcal{T}} := \{v = I_{\mathcal{T}} w \mid w \in C^m(\overline{\Omega})\}$$

$V_{\mathcal{T}}$ is said to be of regularity r, if $V_{\mathcal{T}} \subset C^r$. The regularity of $V_{\mathcal{T}}$ depends on the choice of functionals ψ_T^α and the local function space V_T.

3.4 Nédelec Finite Elements

We can use 11 to construct $H(curl)$-conforming finite elements (T, V_T, Ψ_T), beginning with the lowest order *Nédelec finite element* in 2D. Here,

- T is a Triangle
- $V_T = \mathcal{N}_0$, where

$$\mathcal{N}_0 := \left\{ v : T \to \mathbb{R}^2 \ : \ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a_x \\ b_x \end{pmatrix} + b \begin{pmatrix} y \\ -x \end{pmatrix},\ a_x, a_y, b \in \mathbb{R} \right\}$$

- the functionals for each edge $E_{\alpha\beta}$ of the triangle are

$$\psi_{E_{\alpha\beta}} : v \mapsto \int_{E_{\alpha\beta}} v \cdot \tau$$

Note that $[P^0]^2 \subset \mathcal{N}_0 \subset [P^1]^2$. The local space and the functionals are chosen to fulfill certain properties that give Curl-conformity:

- The 2d curl ($curl\ v = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$) is constant for functions in $\mathcal{N}_0$:

$$curl\ \mathcal{N}_0 = P^0$$

- The tangential component of a function in $\mathcal{N}_0$ is constant along a line.

The resulting local basis functions are

$$\phi_{\alpha\beta} = \lambda_\alpha \nabla \lambda_\beta - \lambda_\beta \nabla \lambda_\alpha,$$

where λ_α and λ_β are the barycentric coordinates associated with the vertices V_α and V_β. (Here, E_α is the edge from V_α to V_β.) For the global finite element space, the local basis functions that belong to the same edge are identified. Because the tangential components of the local basis functions are constant along the shared edge, and the integral along that edge is the same as prescribed by the local functionals, the tangential components of the global basis functions are continuous across the edge. With 11, we have

Proposition 15 *The finite element space defined by the lowest order Nédelec elements is a subspace of $H(\text{curl})$.*

3.4.1 High Order 3D Nédelec elements on tetrahedra

For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we define the total degree by $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. Then we define the space of polynomials of *maximum* total degree k via

$$P_k = \left\{ p : \mathbb{R}^3 \to \mathbb{R} \; : \; p(x) = \sum_{|\alpha| \le k} a_\alpha x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \; a_\alpha \in \mathbb{R} \right\},$$

and the space of homogeneous polynomials of total degree *exactly* k via

$$\tilde{P}_k = \left\{ \tilde{p} : \mathbb{R}^3 \to \mathbb{R} \; : \; \tilde{p}(x) = \sum_{|\alpha| = k} a_\alpha x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \; a_\alpha \in \mathbb{R} \right\}.$$

Then we have the subspace

$$\mathcal{S}_k := \{ p \in [\tilde{P}^k]^3 \; : \; x \cdot p = 0 \}$$

of $[P_k]$, i.e. S_k is the kernel of the linear map $p \mapsto x \cdot p$ from $[\tilde{P}^3]$ to $\tilde{P}_{k+1}$, which is unto. Thus,

$$\dim(\mathcal{S}_k) = 3 \dim \tilde{P}_k - \dim \tilde{P}_{k+1} = k(k+2)$$

The following subspace of $[\tilde{P}_k]^3$ will be important for curl-conforming elements:

$$R_k := [P_{k-1}]^3 \oplus \mathcal{S}_k$$

For the dimension of R_k, we have

$$\dim(R_k) = 3 \dim(P_{k-1}) + \dim(\mathcal{S}_k) = \frac{1}{2}(k+3)(k+2)k$$

Because $R_k \cap \nabla \tilde{P}_{k+1} = \{0\}$, and $\dim([P_k]^3) = \dim(R_k) + \dim(\nabla \tilde{P}_{k+1})$, the space R_k is part of a Helmholtz-decomposition of $[P_k]^3$:

Proposition 16 *The following algebraic decomposition holds:*

$$[P_k]^3 = R_k \oplus \nabla \tilde{P}_{k+1}.$$

If curl $u = 0$ for $u \in [L_2(\Omega)]^3$, then $u = \nabla p$ for some $p \in H^1(\Omega)$. The polynomial spaces R_k and P_k reflect that property:

Proposition 17 *If $u \in R_k$ satisfies $\operatorname{curl} u = 0$, then $u = \nabla p$ for some $p \in P - k$.*

We will first define the Nédelec curl-conforming elements on the reference tetrahedron.

Definition 18 (Curl-conforming reference elements) *The Curl-conforming Nédelec reference elements consist of*

- *The reference tetrahedron $\hat{T}$*
- *The local finite element space R_k*
- *Three types of degrees of freedom:*
  (1) *Edge degrees of freedom:*

$$M_{\hat{e}}(\hat{u}) = \left\{ \int_{\hat{e}} \hat{u} \cdot \hat{\tau} \hat{q} \ : \ \hat{q} \in P_{k-1}(\hat{e}), \ \hat{e} \in E(\hat{T}) \right\}$$

Here, $E(\hat{T})$ is the set of edges of $\hat{T}$, and for $\hat{e} \in E(\hat{T})$, τ is a unit vector in the direction of $\hat{e}$.

  (2) *Face degrees of freedom:*

$$M_{\hat{f}}(\hat{u}) = \left\{ \frac{1}{\operatorname{area}(\hat{f})} \int_{\hat{f}} \hat{u} \cdot \hat{q} \ : \ \hat{f} \in F(\hat{T}), \ \hat{q} \in [P_{k-2}(\hat{f})]^3, \ \hat{q} \cdot \hat{\nu} = 0 \right\}$$

$F(\hat{T})$ is the set of faces of $\hat{T}$, and $\hat{\nu}$ is the unit normal of a face $\hat{f}$.

  (3) *Volume degrees of freedom:*

$$M_{\hat{T}}(\hat{u}) = \left\{ \int_{\hat{T}} \hat{u} \cdot \hat{q} \ : \ \hat{q} \in [P_{k-3}(\hat{T})]^3 \right\}$$

Let B_T be the Jacobi matrix of the transformation from the reference element $\hat{T}$ to a general element T. Then, the following transformation maps a $H(\text{curl})$-function from $\hat{T}$ to T while preserving its $H(\text{curl})$-property:

$$u \circ F_T = (B_T^T)^{-1} \hat{u} \tag{3.5}$$

The curl of u is related to the curl of $\hat{u}$ via

$$\text{curl } u = \frac{1}{\det(B_T)} B_T \operatorname{curl} u.$$

An edge tangent vector $\hat{\tau}$ is transformed as follows:

$$\tau = \frac{B_T \hat{\tau}}{|B_T \hat{\tau}|}$$

The next result demonstrates the advantage of choosing R_k as the local finite element space:

Proposition 18 *R_K is invariant under the transformation 3.5, i.e. if $\hat{u} \in R_k$, then $u \in R_k$, if $\hat{u}$ and u are related by 3.5.*

Now we define the Nédelec finite element on a general tetrahedron:

Definition 19 *The curl-conforming finite element consists of*

- *A tetrahedron T*
- *The local finite element space R_k*
- *Three types of degrees of freedom:*

  (1) Edge degrees of freedom:

$$M_e(u) = \left\{ \int_e u \cdot \tau q \ : \ q \in P_{k-1}(e), \ e \in E(T) \right\}$$

Here, $E(T)$ is the set of edges of T, and for $e \in E(T)$, τ is a unit vector in the direction of e.

  (2) Face degrees of freedom:

$$M_f(u) = \left\{ \frac{1}{\mathrm{area}(f)} \int_f u \cdot q \ : \ f \in F(T), \ q = B_T \hat{q} \ \hat{q} \in [P_{k-2}(\hat{f})]^3, \ \hat{q} \cdot \hat{\nu} = 0 \right\}$$

$F(T)$ is the set of faces of T, and $\hat{\nu}$ is the unit normal of a face $\hat{f}$, the face of the reference tetrahedron corresponding to f.

  (3) Volume degrees of freedom:

$$M_T(u) = \left\{ \int_T u \cdot q \ : \ q \circ F_T = (1/\det(B_T)) B_T \hat{q}, \ \hat{q} \in [P_{k-3}(\hat{T})]^3 \right\}$$

Because we use the transformation 3.5, the following relation holds:

Proposition 19 *If $\det(B_T) > 0$, then the sets of degrees of freedom for u on T equal those of $\hat{u}$ on $\hat{T}$.*

To see that the global finite element functions are curl-conforming, we will need the following proposition:

Proposition 20 *If for $u \in R_k$ the degrees of freedom of a face f and its edges vanish, then $u \times \nu = 0$ on f.*

Now, let V_h be the finite element space induced by the Nédelec elements on a triangulation $\mathcal{T}$ as described in section 3.3. If $u \in V_h$, then $u_1 := u|_{T_1} \in R_k$ and $u_2 := u|_{T_2} \in R_k$ for two neighboring tetrahedra T_1 and T_2 that share a face f. We can extend u_1 and u_2 to the full domain, and $u_1 - u_2 \in R_k$ holds as well. The degrees of freedom associated with the face f and its edges vanish for $u_1 - u_2$. Thus, according to the preceding proposition, $u_1 \times \nu = u_2 \times \nu$ on the face f, and it follows from 11 that $u \in H(\mathrm{curl})$.

3.5 Approximation Theorems

According to Cea's lemma, the approximation error $\|u - u_h\|$ for the Galerkin approximation u_h of an $H(\mathrm{curl})$-function u is linearly bounded by the best approximation error $\inf_{v \in V_h} \|u - v\|$ in the finite element space V_h. Thus, now that we have defined an appropriate curl-conforming finite element space, we must examine its approximation properties.

3.5.1 The H1-conforming space

To establish an error estimate in the Sobolev norm we will need the Bramble-Hilbert lemma:

Proposition 21 (Bramble-Hilbert Lemma) *Let the bound of $\Omega \in \mathbb{R}^n$ be Lipschitz continuous, $t \geq 2$, and L a bounded, linear Operator from $H^t(\Omega)$ into a normed space Y. If $\mathcal{P}_{t-1} \subset \ker L$, then there exists $c = c(\Omega)\|L\| \geq 0$, such that*

$$\|Lv\| \leq c|v|_t v \in H^t(\Omega) \tag{3.6}$$

To establish the relationship between the mesh size h of a triangulation $\mathcal{T}$ and the approximation error $\|u - I_h u\|_{m,\Omega}$ we will need the following transformation formula for affine transformations:

Proposition 22 *Let $\Phi : \widehat{T} \to T,\ x \mapsto x_0 + B_T x$ be bijective for a non-singular matrix B_T. For $v \in H^m(T)$ and $\widehat{v} := \Phi \circ v$, it holds that*

$$|\widehat{v}|_{m,\widehat{T}} \leq c \cdot \|B_T\|^m \cdot |\det B_T|^{-1/2} |v|_{m,T} \tag{3.7}$$

Now we can prove the following upper bound for the approximation error:

Proposition 23 *Let I_h be the interpolation operator induced by H1-conforming finite elements of piecewise polynomials of order $s-1$ on a Triangulation $\mathcal{T}_h$. Then, it holds that*

$$\|u - I_h u\|_{m,\Omega} \leq c \cdot h^{s-m} |u|_{s,\Omega}$$

Proof We decompose the approximation error into element-wise contributions and map to the reference element using the transformation formula 3.7. Then we apply the Bramble-Hilbert Lemma to $L := (id - I_h)$ and map back to the global element.

The following proposition follows from local scaling:

Proposition 24 *For $v_h \in \mathcal{V}_h$, it holds that*

$$h \left| \frac{\partial v_h}{\partial n} \right|^2_{\partial\Omega_i \cap \Gamma} \leq C_I \|\nabla v_h\|^2_{L^2(\Omega_i)}. \tag{3.8}$$

Proof For the edge E of an element T, transforming to the reference element gives:

$$|v|^2_E \leq h^{n-1} |\widehat{v}|^2_{\widehat{E}},$$

where $\widehat{E}$ is the edge of the reference element $\widehat{T}$ corresponding to E. This is straightforward to verify for the 2D case (i.e. $n = 2$):
Without loss of generality let $\widehat{E} = [0,1] \times \{0\}$. Then, for $p := \Phi((0,0))$, and $q := \Phi((1,0))$, we have $\Phi((s,0)) = p + s(q-p)$, and

$$|v|^2_E = \int_{[0,1]} (v \circ \Phi)^2((\cdot,0)) \cdot |\Phi'(\cdot,0)| = \int_{\widehat{E}} \widehat{v}^2 \cdot |p - q| \leq h |\widehat{v}|^2_{\widehat{E}}$$

Because $|.|_{\partial v_h \cap \Gamma}$ is a semi-norm on $\mathcal{V}_h$, and all norms are equivalent on finite dimensional spaces, it holds that

$$\left| \frac{\partial v_h}{\partial n} \right|^2_{\widehat{E}} \leq |\nabla v_h|^2_{\widehat{E}} \leq C \|\nabla v_h\|^2_{L_2(\widehat{T})}$$

Now, using the transformation formula to transform to the reference element and back, we have

$$\begin{aligned}
\left|\frac{\partial v_h}{\partial n}\right|^2_{\partial\Omega_i\cap\Gamma} &\leq |\nabla v_h|^2_{\partial\Omega_i\cap\Gamma} \\
&= \sum_{E\subset\Gamma} |\nabla v_h|^2_E \\
&\leq \sum_{E\subset\Gamma} h^{n-1}\left|\widehat{\nabla v_h}\right|^2_{\widehat{E}} \\
&\leq \sum_{T\in\mathcal{T}} h^{n-1}C\|\widehat{\nabla v_h}\|_{L_2(\widehat{T})} \\
&\leq \sum_{T\in\mathcal{T}} h^{n-1}C(\det B_T)^{-1}\|\nabla v_h\|_{L_2 T} \\
&\leq Ch^{-1}\|\nabla v\|^2_{L^2(\Omega_i)}
\end{aligned}$$

Here, we also used the Hadamard inequality, which implies $\det B_T \leq h^n$

3.5.2 The H(curl)-conforming space

Let I_h be the projector into V_h induced by the Nédelec high order finite elements on a regular triangulation $\mathcal{T}_h$ as described in section 3.3. Note that due to the edge functionals, I_h is not well-defined for all $H(\mathrm{curl})$-functions. The trace theorem only provides a tangential trace on the face of the tetrahedron. Thus, we need to impose additional regularity:

Proposition 25 *Let $u \in [H^1(T)]^3$, $\operatorname{curl} u \in [L^p(T)]^3$ with $p > 2$ for every $T \in \mathcal{T}_h$. Then $I_h u$ is well-defined and bounded.*

The following upper bound for approximation of certain $H(\mathrm{curl})$ functions by polynomials of degree k holds:

Proposition 26 *Let T be a Lipschitz domain, $k \in \mathbb{N}$. There exists a constant C, such that for $v \in [H^1(T)]^3$, $\operatorname{curl} v \in [H^1(T)]^3$*

$$\inf_{p\in\mathcal{P}_k} \|v-p\|_{H(\mathrm{curl},T)} \leq C\|v\|_{H(\mathrm{curl},T)} \tag{3.9}$$

Mapping the norm to the reference element and using estimates like the above gives the following result:

Proposition 27 *Let $\mathcal{T}_h$ be a regular Triangulation of Ω. If $u \in H^1(\Omega)$ and $\operatorname{curl} u \in H^1(\Omega)$, then*

$$\|u - I_h u\|_{H(\mathrm{curl},\Omega)} \leq Ch(\|u\|_{H^1(\Omega)} + \|\operatorname{curl} u\|_{H^1(\Omega)}) \tag{3.10}$$

Thus, we have established h-convergence of an elliptic and bounded problem on $H(\mathrm{curl})$ using Nédelec high order finite element.

3.6 Non-Conforming Methods

As it turns out, it is possible to drop or weaken certain assumptions we made on the finite element space and the variational equation without losing convergence. Strictly speaking, the term *non-conforming FEM* refers to methods that utilize finite element spaces that are not contained in the solution space of the PDE. We will also use it, however, to describe classes of finite element methods that violate one or more of a variety of such assumptions. These so-called *variational crimes* include, but are not limited to:

- As mentioned, the finite element space S_h does not necessarily have to be a subspace of the solution space V_h. For example, a function that is C^∞-smooth on the elements of a triangulation $\mathcal{T}$ is in $H^1(\Omega)$ if and only if it is continuous on the whole domain Ω. Thus, S_h is not a subset of V for discontinuous finite element basis functions.
- Instead of the exact variational equation $a(u,v) = f(v)$, one might solve an approximation $a_h(u,v) = f_h(v)$.

The following is a tweaked version of Strang's second lemma (see [5]), which we will use in the convergence analysis of our domain decomposition method. It serves as a generalization of Cea's lemma for a certain class of non-conforming methods:

Proposition 28 *Let a_h be a bilinear form on on a space $V + S_h$ and $u \in V$, such that*

$$a_h(u,v) = f(v) \quad \textit{for all} \quad v \in V. \tag{3.11}$$

If for a norm $\|.\|_h$ on $V + S_h$ it holds that

$$a_h(v_h, v_h) \geq \alpha \|v_h\|_h^2 \quad \textit{for all} \quad v_h \in S_h, \tag{3.12}$$
$$a_h(u + v_h, w_h) \leq C\|u + v_h\|_h \|w_h\|_h \quad \textit{for all} \quad v_h, w_h \in S_h \tag{3.13}$$

then the solution u_h of the Variational Problem

$$a_h(u_h, v_h) = f(v_h) \quad \textit{for all} \quad v_h \in S_h \tag{3.14}$$

satisfies

$$\|u - u_h\|_h \leq c \inf_{v_h \in S_h} \|u - v_h\|_h + \sup_{w_h \in S_h} \frac{|a_h(u, w_h) - a(u, w_h)|}{\|w_h\|_h} \tag{3.15}$$

Proof Let $v_h \in S_h$. Then it follows from boundedness and stability, that

$$\begin{aligned}
\alpha\|u_h - v_h\|_h^2 &\leq a_h(u_h - v_h, u_h - v_h) \\
&= a_h(u_h - u + u - v_h, u_h - v_h) \\
&= a_h(u - v_h, u_h - v_h) + a_h(u_h, u_h - v_h) - a_h(u, u_h - v_h) \\
&= a_h(u - v_h, u_h - v_h) + a(u, u_h - v_h) - a_h(u, u_h - v_h) \\
&\leq C\|u - v_h\|_h \cdot \|u_h - v_h\|_h + a(u, u_h - v_h) - a_h(u, u_h - v_h)
\end{aligned}$$

Dividing by $\|u_h - v_h\|$ gives

$$\|u_h - v_h\|_h \leq \frac{C}{\alpha}\|u - v_h\|_h + \frac{a(u, u_h - v_h) - a_h(u, u_h - v_h)}{\|u_h - v_h\|}$$

for all $v_h \in S_h$. Using the triangle inequality, we get

$$\begin{aligned}
\|u - u_h\|_h &= \|u - v_h + v_h - u_h\|_h \\
&\leq \|u - v_h\|_h + \|v_h - u_h\|_h \\
&\leq \left(1 + \frac{C}{\alpha}\right)\|u - v_h\|_h + \sup_{w_h \in S_h} \frac{|a_h(u, w_h) - a(u, w_h)|}{\|w_h\|_h}
\end{aligned}$$

for all $v_h \in S_h$.

Chapter 4

Spline Spaces

The finite element space used for the domain decomposition is a product space $V_h \times \mathcal{M}_h$ of a standard finite element space V_h and a finite-dimensional subspace of $L_2(\Gamma)$ on the interface. For the latter, we will implement spline spaces, which possess certain properties that are beneficial to the accuracy and computational efficiency of the method. We follow the outline of [9].

4.1 The Spaces $\mathbb{P}_{k,\tau}$

Splines are piecewise polynomial functions of degree $k-1$ on a partition of an interval, that are $k-2$ times smoothly differentiable on the whole interval:

Definition 20 *For a sequence $\tau = \{\tau_0, \ldots, \tau_{l+1}\}$ satisfying $a = \tau_0 < \tau_1 < \ldots < \tau_{l+1} = b$, the* space of k-th order splines *is defined as*

$$\mathbb{P}_{k,\tau} = \left\{ f \in C^{k-2}([a,b]) \mid f|_{[\tau_i,\tau_{i+1})} \in P_{k-1},\ 0 \le i \le l \right\} \tag{4.1}$$

The following result is derived by enumerating the degrees of freedom in constructing a spline function:

Proposition 29 *It holds that*

$$\dim \mathbb{P}_{k,\tau} = k + l \tag{4.2}$$

Thus, $\mathbb{P}_{k,\tau}$ is a finite-dimensional vector space that the polynomials of degree $k-1$ are a subspace of.

4.2 B-Splines

Because of the piecewise structure of splines, it is possible to construct convenient bases for $\mathbb{P}_{k,\tau}$ that are non-zero on a limited set of subintervals $[\tau_i, \tau_{i+1})$. We define the *B-spline* basis functions via the de-Boor-iteration, which also allows for an efficient computation of function values:

Definition 21 *Let $\tau = \{\tau_0, \ldots, \tau_n\}$ be a sequence of knots as in definition 20. For $1 \le k < n$, let*

$$\begin{aligned} N_{j,1}(x) &:= \chi_{[t_j,t_{j+1})} \quad \textit{for} \quad j = 1, \ldots, n-1, \\ N_{j,k}(x) &:= \frac{x - t_j}{t_{j+k-1} - t_j} N_{j,k-1} + \frac{t_{j+k} - x}{t_{j+k} - t_{j+1}} N_{j+1,k-1}(x), \\ &\qquad \textit{for} \quad k = 2, \ldots, n-1, \quad \textit{and} \quad j = 1, \ldots, n-k \end{aligned}$$

The following observations on B-spline basis functions hold:

- We have

$$\text{supp } N_{j,k} \subset [t_j, t_{j+k}],$$

  i.e. $N_{j,k}$ vanishes outside of $[t_j, t_{j+k}]$. This will prove to be a significant advantage of spline spaces over polynomial spaces when used as a finite element space, because it reduces coupling in the linear equation system associated with the method.

- $N_{j,k}(x) > 0$ for $x \in (t_j, t_{j+k})$

- $(N_{j,k})|_{[t_i,t_{i+1})} \in P_{k-1}$

- $N_{j,k} \in C^{k-2}([t_1, t_n])$

The last two items imply that $N_{j,k} \in \mathbb{P}_{k,\tau}$. However, we have so far defined only $n-k$ different B-splines for a set of knots $\tau = \{\tau_0, \ldots, \tau_n\}$. To expand the B-spline functions to a basis of $\mathbb{P}_{k,\tau}$, we add arbitrary knots lower than τ_1 to τ as follows (producing a new set of knots T):

$$\begin{aligned} T = \{t_1, \ldots, t_n\} \quad &\text{where} \quad n := 2k + l, \\ t_1 < \ldots < t_k = \tau_0, & \\ t_k + j = \tau_j \quad &\text{for} \quad j = 1, \ldots, l, \\ \tau_{l+1} = t_{k+l+1} < \ldots < t_{2k+l}. & \end{aligned}$$

The B-splines associated with the expanded set of knots, restricted to the original interval $[a, b]$, are still in $\mathbb{P}_{k,\tau}$. Because there are $k + l$ functions in all, linear independence of the B-splines implies they are indeed a basis of the spline space (see e.g. [3]).

Proposition 30 *Let $N_{j,k}$ be the B-splines of order k associated with the expanded set of knots T for the interval $[a, b]$. It holds that*

$$\text{span } \{N_{j,k}|_{[a,b]} \;:\; 1 \leq j \leq k + l\} = \mathbb{P}_{k,\tau}$$

Thus, we have a convenient basis for the spline space that is non-zero on a limited set of subintervals of a partition of an interval

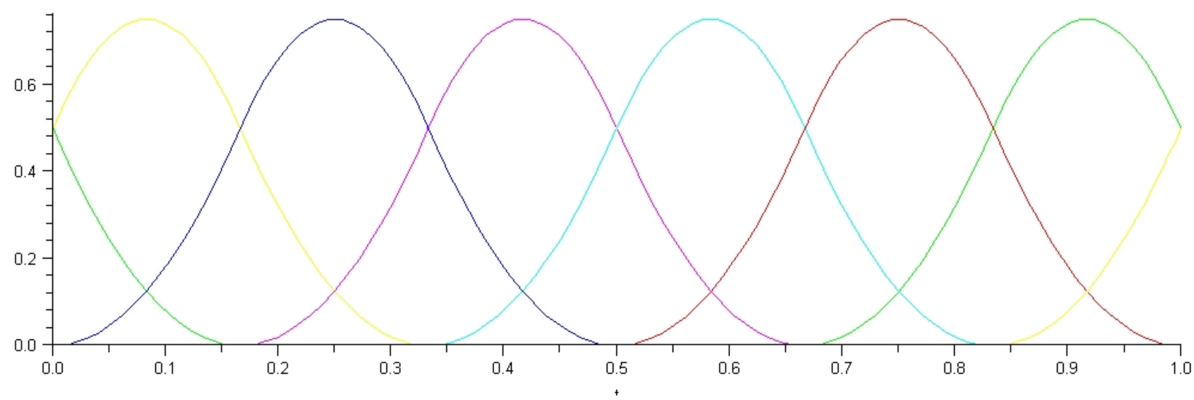


Figure 4.1: The spline basis functions of order three to a uniform set of knots.

4.3 Spline Approximation Properties

The next proposition gives an error bound for the best approximation of a function in the space of k-th order splines (see e.g. [3]):

Proposition 31 *For* $h_\Gamma = \max_{j=0,\dots,l}(\tau_{j+1} - \tau_j)$, *there exists* $c \leq \infty$ *such that for all* $f \in H^\sigma([a,b])$, *it holds that*

$$\min_{S_k \in \mathbb{P}} \|v - S_k\|_{H^r([a,b])} \leq c h_\Gamma^{\sigma - r} \|f\|_{H^\sigma([a,b])}$$

This last result allows us to bound the error of the finite element method by the knot spacing h, thus ensuring convergence for decreasing mesh size.

Chapter 5

Decomposing the Poisson Problem

5.1 The Poisson Transmission Problem

To introduce the concept of the Nitsche-type domain decomposition presented in this thesis, we will first apply the method to the model Poisson problem:

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{5.1}$$

where Ω is a bounded two- or three-dimensional domain and $f \in L_2(\Omega)$. For ease of notation, the L^2 scalar products will denoted by

$$\begin{aligned} (u, v)_\Omega &:= \int_\Omega uv\, dx, \\ \langle u, v\rangle_\Gamma &:= \int_\Gamma uv\, ds. \end{aligned}$$

Then the Poisson equation can be written in the weak form: Find $u \in H_0^1(\Omega)$, such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{5.2}$$

Let Ω be divided into two subdomains Ω_1 and Ω_2 with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$. Under the regularity assumption that $u \in H^s(\Omega)$ for $s > 3/2$, the Poisson equation is equivalent to the following transmission problem:

$$\begin{aligned} -\Delta u_i &= f \quad \text{in} \quad \Omega_i, \quad i = 1, 2, \\ u_i &= 0 \quad \text{on} \quad \partial\Omega, \quad i = 1, 2, \\ u_1 - u_2 &= 0 \quad \text{on} \quad \Gamma, \\ \frac{\partial u_1}{\partial n_1} - \frac{\partial u_2}{\partial n_2} &= 0 \quad \text{on} \quad \Gamma \end{aligned} \tag{5.3}$$

where $u_i = u|_{\Omega_i}$. The jump and mean value of a function u on the interface are defined as, respectively:

$$\begin{aligned} [u] &:= u_1 n_1 + u_2 n_2 \\ \{u\} &:= \frac{1}{2}u_1 + \frac{1}{2}u_2 \end{aligned}$$

Integrating the strong formulation of the Poisson equation over the subdomains separately and adding them up gives the weak form of the transmission problem.

$$\sum_i (\nabla u_i, \nabla v_i)_{\Omega_i} - \langle \{\nabla u_i\}, [v_i] \rangle_\Gamma = (f, v)_\Omega \qquad \forall v \in H_0^1(\Omega) \tag{5.4}$$

This variational formulation is not equivalent to the Poisson equation on the whole domain, however, as it is satisfied by any two solutions of the Poisson equations on the subdomains. To ensure uniqueness of the solution, we must take measures to enforce the continuity across the interface.

5.2 Nitsche-Type Method for Domain Decomposition of the Poisson Problem

The solution u being continuous across the interface according to 5.3, adding symmetry and stabilization terms to 5.4 retains consistency:

$$\sum_i (\nabla u_h, \nabla v_h)_{\Omega_i} - \langle \{\nabla u_h\}, [v_h] \rangle_\Gamma - \underbrace{\langle [u_h], \{\nabla v_h\} \rangle_\Gamma}_{=0} + \underbrace{\frac{\alpha}{h} \langle [u_h], [v_h] \rangle_\Gamma}_{=0} = (f, v_h)_\Omega \tag{5.5}$$

This formulation implies a first finite element method for the transmission problem as introduced in [1]:

Method 1 (Nitsche-Type Method) *Find $u_h \in \mathcal{V}_h$ such that*

$$\sum_i (\nabla u_h, \nabla v_h)_{\Omega_i} - \langle \{\nabla u_h\}, [v_h] \rangle_\Gamma - \langle [u_h], \{\nabla v_h\} \rangle_\Gamma + \frac{\alpha}{h} \langle [u_h], [v_h] \rangle_\Gamma = (f, v_h)_\Omega \tag{5.6}$$

for all $v_h \in \mathcal{V}_h$.

In the following, we will assume that $\mathcal{V}_h$ is a piecewise polynomial finite element space on a shape regular triangulation $\mathcal{T}_h$ with mesh-size h, such that $v_h|_{\Omega_i} \in H^1(\Omega_i)$ for $v_h \in \mathcal{V}_h$. Furthermore, let I_h denote the interpolation operator defined locally by $I_h v_h|_{\Omega_i} = I_{h,i} v_h$, where $I_{h,i}$ is the interpolation operator induced by the H1-conforming finite elements on Ω_i, respectively. Analysis will be performed using the following mesh-dependent energy norm:

$$|\!|\!| v |\!|\!|_{1,h} := \left(\sum_{i=1}^{2} \|\nabla v\|_{\Omega_i}^2 + \frac{1}{h} |v|_{\partial\Omega_i \cap \Gamma}^2 + h |\frac{\partial v}{\partial n}|_{\partial\Omega_i \cap \Gamma}^2 \right)^{1/2} \tag{5.7}$$

Note that the norm $|\!|\!| . |\!|\!|_{1,h}$ is well defined only for for functions that have a normal derivative in $L_2(\Gamma)$ on the interface. This is the case for the piecewise polynomial functions $v_h \in \mathbb{V}_h$ and, according to the trace theorem 6, for functions in $H^2(\Omega)$. Now we can prove the stability of the method:

Proposition 32 (Coercivity) *For α sufficiently large, there exists a positive constant C, such that*

$$B(u_h, u_h) \geq C |\!|\!| u |\!|\!|_{1,h}^2 \tag{5.8}$$

for all $u_h \in \mathcal{V}_h$.

Proof Inserting into B and applying the Cauchy-Schwarz inequality and Young's inequality gives:

$$
\begin{aligned}
& B\left(u_h, u_h\right) \\
= & \sum_i\left[\left(\nabla u_h, \nabla u_h\right)_{\Omega_i}-2\left\langle\frac{\partial u_h}{\partial n}, u_h\right\rangle_{\partial \Omega_i \cap \Gamma}+\frac{2 \alpha}{h}\left\langle u_h, u_h\right\rangle_{\partial \Omega_i \cap \Gamma}\right] \\
\geq & \sum_i\left[\|\nabla u_h\|_{L^2(\Omega_i)}^2-2 h\left|\frac{\partial u_h}{\partial n}\right|_{\partial \Omega_i \cap \Gamma} \frac{1}{h}\left|u_h\right|_{\partial \Omega_i \cap \Gamma}+\frac{2 \alpha}{h}\left|u_h\right|_{\partial \Omega_i \cap \Gamma}^2\right] \\
\geq & \sum_i\left[\|\nabla u_h\|_{L^2(\Omega_i)}^2-2\left(\frac{h}{4 C_I}\left|\frac{\partial u_h}{\partial n}\right|_{\partial \Omega_i \cap \Gamma}^2+\frac{C_I}{h}\left|u_h\right|_{\partial \Omega_i \cap \Gamma}^2\right)+\frac{2 \alpha}{h}\left|u_h\right|_{\partial \Omega_i \cap \Gamma}^2\right] \\
\geq & \sum_i\left[\|\nabla u_h\|_{L^2(\Omega_i)}^2-2\left(\frac{C_I}{4 C_I}\|\nabla u_h\|_{L^2(\Omega_i)}^2+\frac{C_I}{h}\left|u_h\right|_{\partial \Omega_i \cap \Gamma}^2\right)+\frac{2 \alpha}{h}\left|u_h-\right|_{\partial \Omega_i \cap \Gamma}^2\right] \\
= & \sum_i\left[\left(\frac{1}{2}\|\nabla u_h\|_{L^2(\Omega_i)}^2+\left(2 \alpha-C_I\right) \frac{1}{h}\left|u_h\right|_{\partial \Omega_i \cap \Gamma}^2\right)\right] \\
\geq & \frac{1}{2}\|u_h\|_{1, h}
\end{aligned}
$$

Proposition 33 (Boundedness) *There exists a positive constant C, such that*

$$
B\left(u, v_h\right) \leq C\|u\|_{1, h}\|v_h\|_{1, h} \tag{5.9}
$$

for all $u \in H^2(\Omega)$, $v_h \in \mathcal{V}_h$.

Proof From the Cauchy-Schwarz inequality, it follows that

$$
\begin{aligned}
& B\left(u, v_h\right) \\
= & \sum_i\left\{\left(\nabla u, \nabla v_h\right)_{\Omega_i}-\left\langle\frac{\partial u}{\partial n}, v_h\right\rangle_{\partial \Omega_i \cap \Gamma}-\left\langle u, \frac{\partial v_h}{\partial n}\right\rangle_{\partial \Omega_i \cap \Gamma}+\frac{2 \alpha}{h}\left\langle u, v_h\right\rangle_{\partial \Omega_i \cap \Gamma}\right\} \\
\leq & \sum_i\left\{\|u\|_{L^2(\Omega_i)}\|v_h\|_{L^2(\Omega_i)}+\left|\frac{\partial u}{\partial n}\right|_{\partial \Omega_i \cap \Gamma}\left|v_h\right|_{\partial \Omega_i \cap \Gamma}\right. \\
& \left.+\left|\frac{\partial v}{\partial n}\right|_{\partial \Omega_i \cap \Gamma}|u|_{\partial \Omega_i \cap \Gamma}+\frac{2 \alpha}{h}|u|_{\partial \Omega_i \cap \Gamma}\left|v_h\right|_{\partial \Omega_i \cap \Gamma}\right\} \\
\leq & \sum_i\left\{\left(\|u\|_{L^2(\Omega_i)}+\left|\frac{\partial u}{\partial n}\right|_{\partial \Omega_i \cap \Gamma}+|u|_{\partial \Omega_i \cap \Gamma}+\frac{2 \alpha}{h}|u|_{\partial \Omega_i \cap \Gamma}\right)^{\frac{1}{2}} \cdot\right. \\
& \left.\left(\|v_h\|_{L^2(\Omega_i)}^2+\left|\frac{\partial v_h}{\partial n}\right|_{\partial \Omega_i \cap \Gamma}^2+\left|v_h\right|_{\partial \Omega_i \cap \Gamma}^2+\frac{2 \alpha}{h}\left|v_h\right|_{\partial \Omega_i \cap \Gamma}^2\right)^{\frac{1}{2}}\right\} \\
\leq & \left\{\sum_i\left(\|u\|_{L^2(\Omega_i)}+\left|\frac{\partial u}{\partial n}\right|_{\partial \Omega_i \cap \Gamma}+|u|_{\partial \Omega_i \cap \Gamma}+\frac{2 \alpha}{h}|u|_{\partial \Omega_i \cap \Gamma}\right)\right\}^{\frac{1}{2}} \cdot \\
& \left\{\sum_i\left(\|v_h\|_{L^2(\Omega_i)}^2+\left|\frac{\partial v_h}{\partial n}\right|_{\partial \Omega_i \cap \Gamma}^2+\left|v_h\right|_{\partial \Omega_i \cap \Gamma}^2+\frac{2 \alpha}{h}\left|v_h\right|_{\partial \Omega_i \cap \Gamma}^2\right)\right\}^{\frac{1}{2}}
\end{aligned}
$$

The following proposition now gives an error bound for the method:

Proposition 34 *For $u \in H^s(\Omega)$ with $3/2 < s \leq p+1$, it holds that*

$$\inf_{v \in V^h} \|\!|u - v|\!\|_{1,h} \leq C h^{s-1} \|u\|_{H^s(\Omega)} \tag{5.10}$$

Proof Follows separately for the components of the energy norm from approximation formula 23. For an element $T \in \mathcal{T}_h$, it holds that:

$$\|\nabla u - I_h \nabla u\|^2_{L_2(T)} \leq \|u - I_h u\|^2_{1,T} \leq \left(C h^{s-1} |u|_{s,T}\right)^2$$

Let $E \subset \Gamma$ for an edge E of T. Mapping to the reference element and back as in the proof of 24 gives

$$\frac{1}{h} |u - I_h u|^2_{L_2(E)} \leq C \frac{1}{h^2} \|u - I_h u\|^2_{0,T} \leq \left(C h^{s-1} |u|_{s,T}\right)^2$$

Because the restriction of $u - u_h$ to T is in $H^2(T)$, it follows from proposition 24, that

$$h |\frac{\partial}{\partial n}(u - I_h u)|^2_E \leq \|\nabla u - I_h \nabla u\|^2_T$$

Decomposing the approximation error into element-wise contribution thus gives the desired result.

This completes the *a priori* error estimate:

Proposition 35 *For α sufficiently large, and $u \in H^s(\Omega)$ for $3/2 < s \leq p+1$, it holds that*

$$\|\!|u - u_h|\!\|_{1,h} \leq C h^{s-1} \|u - v\|_{H^s(\Omega)}. \tag{5.11}$$

Proof Follows from coercivity, boundedness with Strang's second lemma and 34.

Thus, the method converges to the correct solution of the transmission problem. There is, however, a difficulty in implementing the matrix assembling for this method introduced by the interface integrals in the bilinear form:

$$\ldots - \langle \{\nabla u_h\}, [v_h] \rangle_\Gamma - \langle [u_h], \{\nabla v_h\} \rangle_\Gamma + \frac{\alpha}{h} \langle [u_h], [v_h] \rangle_\Gamma = \ldots \tag{5.12}$$

These terms introduce coupling in the corresponding system of linear equations between finite element basis functions defined on elements from different sides of the interface. On the other hand, matrix assembly for finite elements is done by integrating on the elements sequentially, so that function values for the basis functions are computed from the local barycentric coordinates only. In order to calculate the contributions from the interface integrals, one would have to compute the overlaps of surface elements from different sides of the interface.

5.3 Hybrid Glue Method

In this section we will introduce hybridization to decouple element basis functions across the interface. Starting from the formulation 1, we introduce the mean values $\lambda := \{u\}, \mu := \{v\}$ of the solution and the test function as hybrid variables. Then it holds that

$$[u] = u_1 n_1 + u_2 n_2 = 2 (u_1 - \lambda) n_1 = 2 (u_2 - \lambda) n_2$$
$$[v] = v_1 n_1 + v_2 n_2 = 2 (v_1 - \mu) n_1 = 2 (v_2 - \mu) n_2$$

Now some choice of a finite element space $\mathcal{M}_h \subset L^2(\Gamma)$ on the interface gives the hybrid method:

Method 2 (Hybrid Glue Method) *Find* $(u_h, \lambda_h) \in \mathcal{V}_h \times \mathcal{M}_h$ *such that*

$$B(u_h, \lambda_h; v_h, \mu_h) = F(v_h) \quad \textit{for all} \quad (v_h, \mu_h) \in \mathcal{V}_h \times \mathcal{M}_h \tag{5.13}$$

where

$$B(u_h, \lambda_h; v_h, \mu_h) := \sum_i \Bigg\{ (\nabla u_h, \nabla v_h)_{\Omega_i} - \left\langle \frac{\partial u_h}{\partial n}, v_h - \mu_h \right\rangle_{\Gamma_i} \tag{5.14}$$

$$- \left\langle u_h - \lambda_h, \frac{\partial v_h}{\partial n} \right\rangle_{\Gamma_i} + \frac{2\alpha}{h} \langle u_h - \lambda_h, v_h - \mu_h \rangle_{\Gamma_i} \Bigg\} \tag{5.15}$$

Proposition 36 *Method 2 is consistent.*

Proof Follows from the consistency of method 1 and

$$\begin{aligned}
&\sum_i (\nabla u, \nabla v)_{\Omega_i} - \langle \{\nabla u\}, [v] \rangle_\Gamma - \langle [u], \{\nabla v\} \rangle_\Gamma + \frac{\alpha}{h} \langle [u], [v] \rangle_\Gamma \\
=&\sum_i (\nabla u, \nabla v)_{\Omega_i} - \left\langle \frac{1}{2} \nabla u_1, [v] \right\rangle_\Gamma - \left\langle \frac{1}{2} \nabla u_2, [v] \right\rangle_\Gamma \\
&- \left\langle [u], \frac{1}{2} \nabla v_1 \right\rangle_\Gamma - \left\langle [u], \frac{1}{2} \nabla v_2 \right\rangle_\Gamma + \frac{\alpha}{2h} \langle [u], [v] \rangle_\Gamma + \frac{\alpha}{2h} \langle [u], [v] \rangle_\Gamma \\
=&\sum_i (\nabla u, \nabla v)_{\Omega_i} - \left\langle \frac{1}{2} \nabla u_1, 2(v_1 - \mu) n_1 \right\rangle_\Gamma - \left\langle \frac{1}{2} \nabla u_2, 2(v_2 - \mu) n_2 \right\rangle_\Gamma \qquad (5.16) \\
&- \left\langle 2(u_1 - \lambda) n_1, \frac{1}{2} \nabla v_1 \right\rangle_\Gamma - \left\langle 2(u_2 - \lambda) n_2, \frac{1}{2} \nabla v_2 \right\rangle_\Gamma \\
&+ \frac{\alpha}{2h} \langle 2(u_1 - \lambda) n_1, 2(v_1 - \mu) n_1 \rangle_\Gamma + \frac{\alpha}{2h} \langle 2(u_2 - \lambda) n_2, 2(v_2 - \mu) n_2 \rangle_\Gamma \\
=&\sum_i \left\{ (\nabla u, \nabla v)_{\Omega_i} - \left\langle \frac{\partial u}{\partial n}, v - \mu \right\rangle_{\Gamma_i} - \left\langle u - \lambda, \frac{\partial v}{\partial n} \right\rangle_{\Gamma_i} + \frac{2\alpha}{h} \langle u - \lambda, v - \mu \rangle_{\Gamma_i} \right\}
\end{aligned}$$

Next, we prove the ellipticity and boundedness of method 2 in the following norm on $(H^2(\Omega) + \mathcal{V}_h) \times L_2(\Gamma)$:

$$|||(v, \mu)|||_{1,h} := \left(\sum_i \|\nabla v\|^2_{L_2(\Omega_i)} + \frac{1}{h} |v - \mu|^2_{L_2(\partial\Omega_i \cap \Gamma)} + h \left| \frac{\partial v}{\partial n} \right|^2_{L_2(\partial\Omega_i \cap \Gamma)} \right)^{1/2}$$

Proposition 37 (Coercivity) *For α sufficiently large, there exists a positive constant C, such that*

$$B(u_h, \lambda_h; u_h, \lambda_h) \geq C ||| (u_h, \lambda_h) |||^2_{1,h} \tag{5.17}$$

for all $(u_h, \lambda_h) \in \mathcal{V}_h$.

Proof Coercivity follows as it did for method 1 from the Cauchy-Schwarz inequality and Young's inequality:

$$
\begin{aligned}
&B\left(u_h, \lambda_h; u_h, \lambda_h\right) \\
&= \sum_i \left[(\nabla u_h, \nabla u_h)_{\Omega_i} - 2 \left\langle \frac{\partial u_h}{\partial n}, u_h - \lambda_h \right\rangle_{\partial\Omega_i \cap \Gamma} + \frac{2\alpha}{h} \left\langle u_h - \lambda_h, u_h - \lambda_h \right\rangle_{\partial\Omega_i \cap \Gamma} \right] \\
&\geq \sum_i \left[\|\nabla u_h\|^2_{L^2(\Omega_i)} - 2h \left| \frac{\partial u_h}{\partial n} \right|_{\partial\Omega_i \cap \Gamma} \frac{1}{h} \left| u_h - \lambda_h \right|_{\partial\Omega_i \cap \Gamma} + \frac{2\alpha}{h} \left| u_h - \lambda_h \right|^2_{\partial\Omega_i \cap \Gamma} \right] \\
&\geq \sum_i \left[\|\nabla u_h\|^2_{L^2(\Omega_i)} - 2 \left(\frac{h}{4C_I} \left| \frac{\partial u_h}{\partial n} \right|^2_{\partial\Omega_i \cap \Gamma} + \frac{C_I}{h} \left| u_h - \lambda_h \right|^2_{\partial\Omega_i \cap \Gamma} \right) + \frac{2\alpha}{h} \left| u_h - \lambda_h \right|^2_{\partial\Omega_i \cap \Gamma} \right] \\
&\geq \sum_i \left[\|\nabla u_h\|^2_{L^2(\Omega_i)} - 2 \left(\frac{C_I}{4C_I} \|\nabla u_h\|^2_{L^2(\Omega_i)} + \frac{C_I}{h} \left| u_h - \lambda_h \right|^2_{\partial\Omega_i \cap \Gamma} \right) + \frac{2\alpha}{h} \left| u_h - \lambda_h \right|^2_{\partial\Omega_i \cap \Gamma} \right] \\
&= \sum_i \left[\left(\frac{1}{2} \|\nabla u_h\|^2_{L^2(\Omega_i)} + (2\alpha - C_I) \frac{1}{h} \left| u_h - \lambda_h \right|^2_{\partial\Omega_i \cap \Gamma} \right) \right] \geq \frac{1}{2} \| (u_h, \lambda_h) \|_{1,h}
\end{aligned}
$$

Note that the coercivity does not depend on the choice of the finite element space $\mathcal{M}_h$ on the interface.

Proposition 38 (Boundedness)

$$
B\left(u - u_h, \lambda - \lambda_h; v_h, \mu_h\right) \leq C \| (u - u_h, \lambda - \lambda_h) \| \| v_h, \mu_h) \| \tag{5.18}
$$

Proof Follows similarly as in the proof of 33 by applying the Cauchy-Schwarz inequality repeatedly:

$$
\begin{aligned}
&B\left(u - u_h, v_h\right) \\
&= \sum_i \Bigg\{ (\nabla(u - u_h), \nabla v_h)_{\Omega_i} - \langle \nabla(u - u_h), v_h \rangle_{\partial\Omega_i \cap \Gamma} \\
&\quad - \left\langle u - u_h, \frac{\partial v_h}{\partial n} \right\rangle_{\partial\Omega_i \cap \Gamma} + \frac{2\alpha}{h} \langle u - u_h, v_h \rangle_{\partial\Omega_i \cap \Gamma} \Bigg\} \\
&\leq \sum_i \Bigg\{ \|u - u_h\|_{L^2(\Omega_i)} \|v_h\|_{L^2(\Omega_i)} + \left| \frac{\partial(u - u_h)}{\partial n} \right|_{\partial\Omega_i \cap \Gamma} |v_h|_{\partial\Omega_i \cap \Gamma} \\
&\quad + \left| \frac{\partial v}{\partial n} \right|_{\partial\Omega_i \cap \Gamma} |u - u_h)|_{\partial\Omega_i \cap \Gamma} + \frac{2\alpha}{h} |u - u_h|_{\partial\Omega_i \cap \Gamma} |v_h|_{\partial\Omega_i \cap \Gamma} \Bigg\} \\
&\leq \sum_i \Bigg\{ \left(\|u - u_h\|_{L^2(\Omega_i)} + \left| \frac{\partial(u - u_h)}{\partial n} \right|_{\partial\Omega_i \cap \Gamma} + |u - u_h|_{\partial\Omega_i \cap \Gamma} + \frac{2\alpha}{h} |u - u_h|_{\partial\Omega_i \cap \Gamma} \right)^{\frac{1}{2}} \cdot \\
&\quad \left(\|v_h\|^2_{L^2(\Omega_i)} + \left| \frac{\partial v_h}{\partial n} \right|^2_{\partial\Omega_i \cap \Gamma} + |v_h|^2_{\partial\Omega_i \cap \Gamma} + \frac{2\alpha}{h} |v_h|^2_{\partial\Omega_i \cap \Gamma} \right)^{\frac{1}{2}} \Bigg\} \\
&\leq \Bigg\{ \sum_i \left(\|u - u_h\|_{L^2(\Omega_i)} + \left| \frac{\partial(u - h_h)}{\partial n} \right|_{\partial\Omega_i \cap \Gamma} + |u - u_h|_{\partial\Omega_i \cap \Gamma} + \frac{2\alpha}{h} |u - u_h|_{\partial\Omega_i \cap \Gamma} \right) \Bigg\}^{\frac{1}{2}} \cdot \\
&\quad \Bigg\{ \sum_i \left(\|v_h\|^2_{L^2(\Omega_i)} + \left| \frac{\partial v_h}{\partial n} \right|^2_{\partial\Omega_i \cap \Gamma} + |v_h|^2_{\partial\Omega_i \cap \Gamma} + \frac{2\alpha}{h} |v_h|^2_{\partial\Omega_i \cap \Gamma} \right) \Bigg\}^{\frac{1}{2}}
\end{aligned}
$$

Now it remains to choose a finite element interface space $\mathcal{M}_h$ with favorable properties in terms of approximation quality and efficient computation. One could, for example, use a basis of global Lagrange polynomials mapped to the interface. This would introduce a lot of coupling in the corresponding finite element stiffness matrix, however, because every interface basis function would couple with every volume basis function from elements at the interface.

5.4 A B-Spline Interface Space

The analysis from the previous chapter shows that the finite element space on the interface can be chosen with great flexibility. In the following, we will focus on the application of a space spanned by B-spline basis functions (see chapter 4). Now, let $\tau := \{\frac{i}{n} \mid i = 0, \dots, n\}$ be a uniform set of knots on $[0,1]$. To construct a spline interface space, we will assume there exists a parametrization $\gamma : \Gamma \to [0,1] \times [0,\epsilon]$ of the interface, such that γ is a diffeomorphism. Then, let $\mathcal{M}_h = \mathbb{P}_\Gamma := \{s \circ \gamma \mid s \in \mathbb{P}_{k,\tau}\}$. Let $\widehat{Q}$ be the operator from $L_2([0,1])$ into $\mathbb{P}_{k,\tau}$. Then Q with $Qu = \widehat{Q}(u \circ \gamma^{-1}) \circ \gamma$ is an operator from $L_2(\Gamma)$ into $\mathbb{P}_\Gamma$.

Proposition 39 *For $u \in H^2(\Omega)$, it holds that*

$$|||(u - \mathcal{I}_k u, u - Qu)|||_{1,h} \le Ch^s \|u\|_{s+1,h} \tag{5.19}$$

Proof From the definition of the norm, it follows that

$$|||(u - \mathcal{I}_k u, u - Qu \circ \gamma)|||_{1,h} \le \\ \left(\sum \|\nabla(u - \mathcal{I}_k u)\|^2_{\Omega_i} + \frac{1}{h}|u - \mathcal{I}_k u|^2_{\partial\Omega_i \cap \Gamma} + \frac{1}{h}|u - Qu|^2_{\partial\Omega_i\cap\Gamma} + h\left|\frac{\partial v}{\partial n}\right|^2_{L_2(\partial\Omega_i\cap\Gamma)}\right)^{1/2}$$

For the spline component we have:

$$\begin{aligned}
\frac{1}{h}|u - Q(u\circ\gamma^{-1})\circ\gamma|^2_{\partial\Omega_i\cap\Gamma} &= \frac{1}{h}\int_\Gamma \{u - Q(u\circ\gamma^{-1})\circ\gamma\}^2 \\
&= \frac{1}{h}\int_{[0,1]} \left\{\left[u\circ\gamma^{-1} - Q(u\circ\gamma^{-1})\right]\cdot|\det D\gamma^{-1}|\right\}^2 \\
&\le \frac{d^2}{h}\int_{[0,1]} \left[u\circ\gamma^{-1} - Q(u\circ\gamma^{-1})\right]^2 \\
&= \frac{d^2}{h}\|u\circ\gamma^{-1} - Q(u\circ\gamma^{-1})\|_{L_2([0,1])} \\
&\le \frac{ch_\Gamma^s}{h}\|u\circ\gamma^{-1}\|_{H^s([0,1])} \\
&\le \frac{ch_\Gamma^s}{h}\|u\circ\gamma^{-1}\|_{H^s([0,1]\times[0,\epsilon])} \\
&\le \frac{ch_\Gamma^s}{h}\|u\circ\gamma^{-1}\|_{s,h}
\end{aligned}$$

The other components of the approximation error are identical to those of the non-hybrid method.

This completes the *a priori* error estimate:

Proposition 40 *For α sufficiently large, and $u \in H^s(\Omega)$ for $3/2 < s \le p+1$, it holds that*

$$|||u - u_h|||_{1,h} \le Ch^{s-1}\|u\|_{H^s(\Omega)}. \tag{5.20}$$

Proof Follows from coercivity, boundedness with Strang's second lemma and 40.

As a basis for the spline interface space, *B-splines* can be computed efficiently using DeBoor iteration:

$$N_{i,r}\left(x\right) = \frac{x - t_i}{t_{i+r-1} - t_i} n_{i,r-1}\left(x\right) + \frac{t_{i+r} - x}{t_{i+r} - t_{i+1}} N_{i+1,r-1}\left(x\right),$$

where $N_{i,r}$, $r = 2, \dots n-1$, $j = 1, \dots n-r$ is the i-th B-spline of order r. B-splines minimize coupling in the stiffness matrix in the sense that the have minimal support supp $N_{i,r} \subset [\tau_i, \tau_{i+r}]$, i.e. they are non-zero only in the interval $[\tau_i, \tau_{i+r}]$.

5.5 On Numerical Integration

It is important to note that in the practical implementation of the hybrid glue method, the surface integrals on the interface will not be computed exactly, but numerically using Gauss quadrature with integration points derived from the finite elements along the interface. Let the integration points $x_k \in \Gamma$ and corresponding weights w_k be chosen such that numerical integration on the interface is exact for finite element functions in the subdomains, i.e.

$$\sum_k w_k u_h(x_k) = \int_\Gamma u_h \tag{5.21}$$

for all $u_h \in \mathcal{V}_h$. Now, the approximation of the bilinear form by Gauss quadrature is

$$\begin{aligned} B_{NI}\left(u_h, \lambda_h; v_h, \mu_h\right) = \sum_i \Bigg\{ & \left(\nabla u_h, \nabla v_h\right)_{\Omega_i} - \sum_k w_k \frac{\partial u_h}{\partial n}(x_k)[v_h(x_k) - \mu_h(x_k)] \\ & - \sum_k w_k \frac{\partial v_h}{\partial n}(x_k)[u_h(x_k) - \lambda_h(x_k)] \\ & + \sum_k w_k [u_h(x_k) - \lambda_h(x_k)][u_h(x_k) - \lambda_h(x_k)] \Bigg\}. \end{aligned}$$

Thus, stability has to be proven for the approximation of the bilinear form.

Proposition 41 *If*

$$\sum_k w_k \mu(x_k)^2 \geq c \int \mu^2 \tag{5.22}$$

for all $\mu \in \mathbb{P}_{k,\tau}$*, then it holds that*

$$B_{NI}(u_h, \lambda_h; u_h, \lambda_h) \geq \|(u_h, \lambda_h)\|^2 \tag{5.23}$$

Proof Using that numerical integration is exact for functions in $\mathcal{V}_h$, it holds that

$$
\begin{aligned}
&\sum_i \left\{ \|\nabla u_h\| - 2\sum_k \left[w_k \frac{\partial u_h}{\partial n}(x_k)\left(u_h(x_k) - \lambda_h(x_k)\right)\right] + \frac{2\alpha}{h}\left[\sum_k w_k\left(u_h(x_k)-\lambda_h(x_k)\right)\right]^2 \right\} \\
\geq &\sum_i \left\{ \|\nabla u_h\| - 2\left(\sum_k \left[\sqrt{w_k}\frac{\partial u_h}{\partial n}(x_k)\right]^2\right)^{1/2} \left(\sum_k \left[\sqrt{w_k}\left(u_h(x_k)-\lambda_h(x_k)\right)\right]^2\right)^{1/2} \right. \\
&\left. + \frac{2\alpha}{h}\sum_k \left[w_k\left(u_h(x_k)-\lambda_h(x_k)\right)^2\right] \right\} \\
\geq &\sum_i \left\{ \|\nabla u_h\| - \frac{h}{2C_I}\sum_k \left[\sqrt{w_k}\frac{\partial u_h}{\partial n}(x_k)\right]^2 - \frac{2}{h}\sum_k \left[\sqrt{w_k}\left(u_h(x_k)-\lambda_h(x_k)\right)\right]^2 \right. \\
&\left. + \frac{2\alpha}{h}\sum_k \left[w_k\left(u_h(x_k)-\lambda_h(x_k)\right)^2\right] \right\} \\
= &\sum_i \left\{ \|\nabla u_h\| - h\left|\frac{\partial u_h}{\partial n}\right|^2_{\partial\Omega_i\cap\Gamma} + \frac{1}{h}(2\alpha-1)\sum_k \left[w_k\left(u_h(x_k)-\lambda_h(x_k)\right)^2\right] \right\}
\end{aligned}
$$

Furthermore, using the spline interpolation operator Q, we have

$$
\begin{aligned}
\int_\Gamma [u_h-\lambda_h]^2 &= \int_\Gamma [u_h - Qu_h + Qu_h - \lambda_h]^2 \\
&\leq \int_\Gamma [u_h - Qu_h]^2 + \int_\Gamma [Qu_h-\lambda_h]^2 \\
&\leq \int_\Gamma [u_h - Qu_h]^2 + \sum_k w_k[Qu_h(x_k)-\lambda_h(x_k)]^2 \\
&\leq \int_\Gamma [u_h - Qu_h]^2 + \sum_k w_k[u_h(x_k)-Qu_h(x_k)]^2 + \sum_k w_k[u_h(x_k)-\lambda_h(x_k)]^2
\end{aligned}
$$

Now, $\int_\Gamma [u_h - Qu_h]^2 \leq C\|u_h\|_{H^1(\Omega_i)}$ according to 31, and $\sum_k w_k[u_h(x_k)-Qu_h(x_k)]^2 \leq \|u_h\|_{H^1(\Omega_i)}$ because

$$
\sum_{x_k\in\delta T\cap\Gamma} w_k\left|u(x_k)-Qu(x_k)\right|^2 \prec \max_{x\in\delta T\cap\Gamma}\left|u(x)-Qu(x)\right|^2 \leq \|u-Qu\|^2_{L_\infty(\delta T\cap\Gamma)} \leq \|u\|^2_{H^1(\Omega_i)} \quad (5.24)
$$

Thus, it follows that

$$
\sum_k w_k[u_h(x_k)-\lambda_h(x_k)]^2 + c\|u_h\|_{H^1(\Omega_i)} \geq \int_\Gamma [u_h-\lambda_h]^2 \quad (5.25)
$$

5.6 Numerical tests and examples

5.6.1 Tests in 2D

We consider the Laplace model problem on two adjacent squares where the right side of the equation is given by $f = 1$ on the left domain and $f = 0$ on the right. The boundary condition

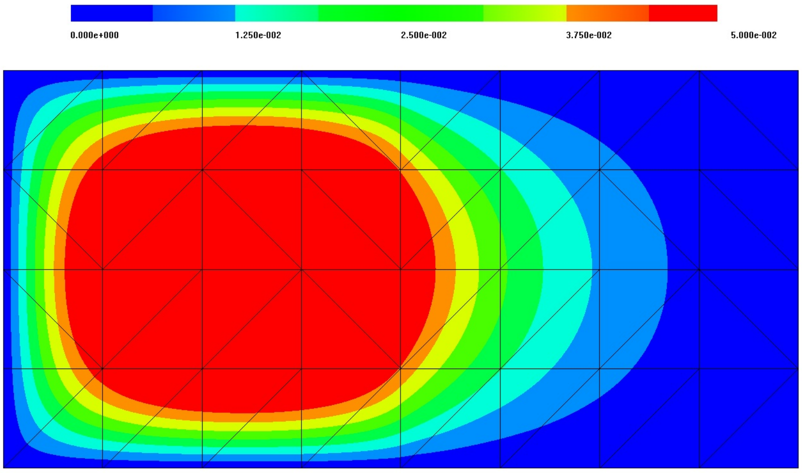


Figure 5.1: A simple model problem without domain decomposition.

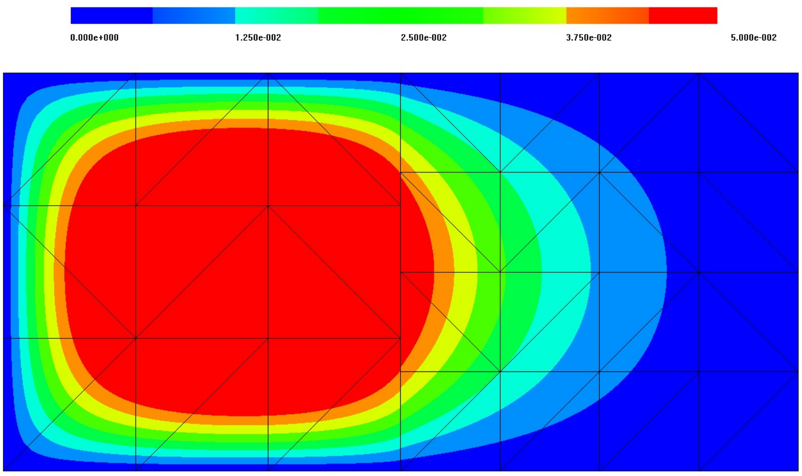


Figure 5.2: The same problem using domain decomposition. Note that the mesh is non-matching across the interface. The Nitsche-terms enforce continuity.

$u = 0$ is set via a Robin penalty term. Inside the domains, the solution is approximated by H^1-conforming Polynomial Finite elements of varying order.

To measure the effect of the domain decomposition on the approximation quality, we model the following problem:

- The domain $\Omega = [-1, 1] \times [-1, 1]$ is divided into the subdomains $\Omega_1 = [-1, 0.1] \times [-1, 1]$ and $\Omega_2 = [0.1, 1] \times [-1, 1]$. The interface is off-center to obtain a solution that is non-symmetrical across the interface.

- The mesh is non-matching across the interface.

- We choose the right side f of the Poisson problem $-\Delta u = f$, such that the solution u satisfies
$$u((x, y)) = \cos(5 \cdot \frac{\pi}{2} \cdot x) \cdot \cos(5 \cdot \frac{\pi}{2} \cdot y)$$
i.e. the solution consists of 25 sinus bubbles on the domain (see 5.3).

- The boundary condition $u(x) = 0$ is enforced by a large Robin penalty term.
- High-order H1-conforming triangular finite elements are used inside the subdomains. The interface space is spanned by B-spline basis functions on $0 \times [-1, 1]$.

We compute the approximation error $\|u - u_h\|_{L_2(\Omega)}$ numerically using Gauss quadrature integration points on the elements. The results for different configurations of polynomial degrees for the volume elements and on the interface space are plotted as a function of the mesh size.

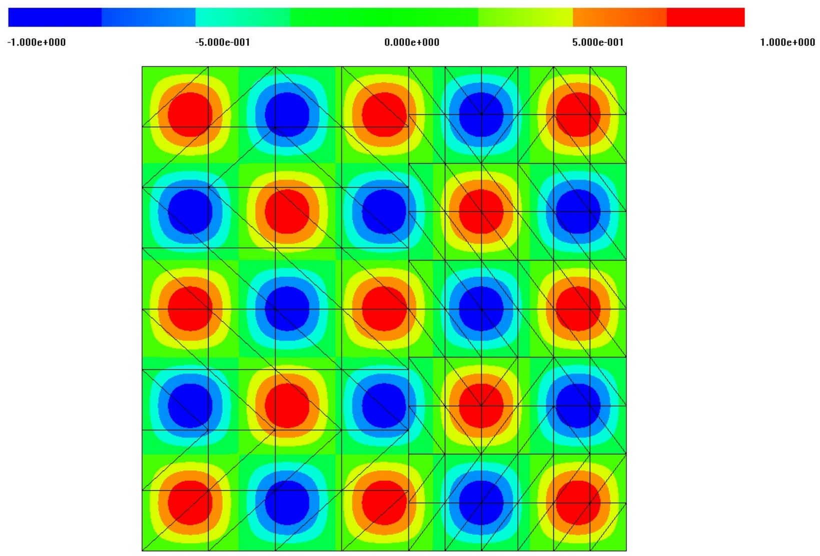


Figure 5.3: The solution using a coarse mesh. The polynomial order of the finite elements is already sufficient to resolve the analytical solution relatively well. There is little distortion from the interface approximation visible.

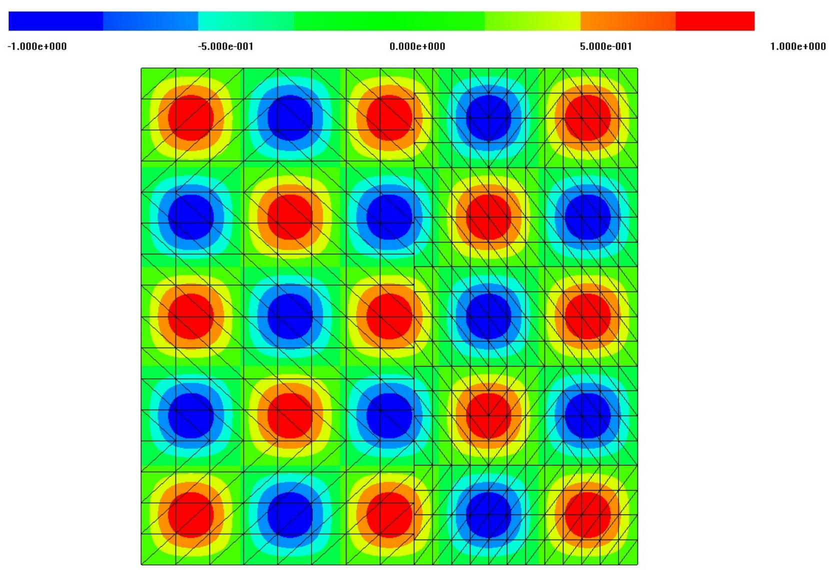


Figure 5.4: An example of a finer triangulation of the domain.

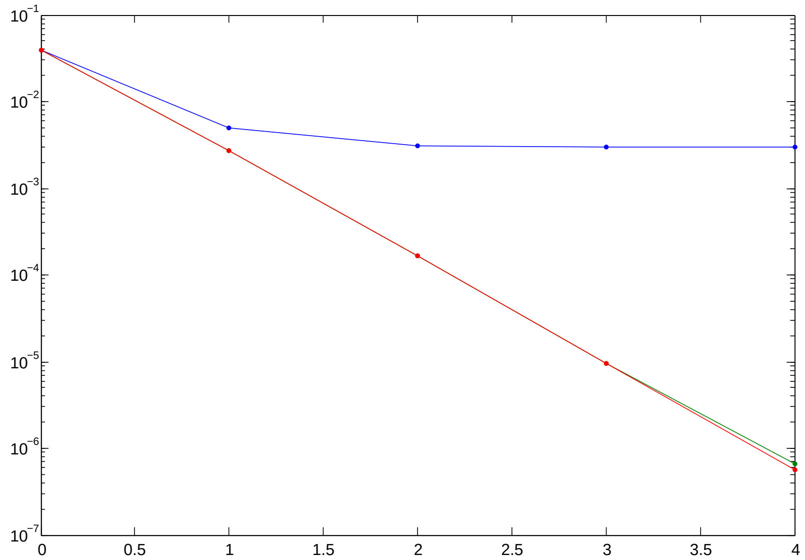


Figure 5.5: The approximation error plotted logarithmically against the decreasing mesh-size for three levels of density for the interface space: 10 degrees of freedom (blue), 100 degrees of freedom (green), and 1000 degrees of freedom (red). The polynomial order of the volume elements as well as the spline order on the interface is three.

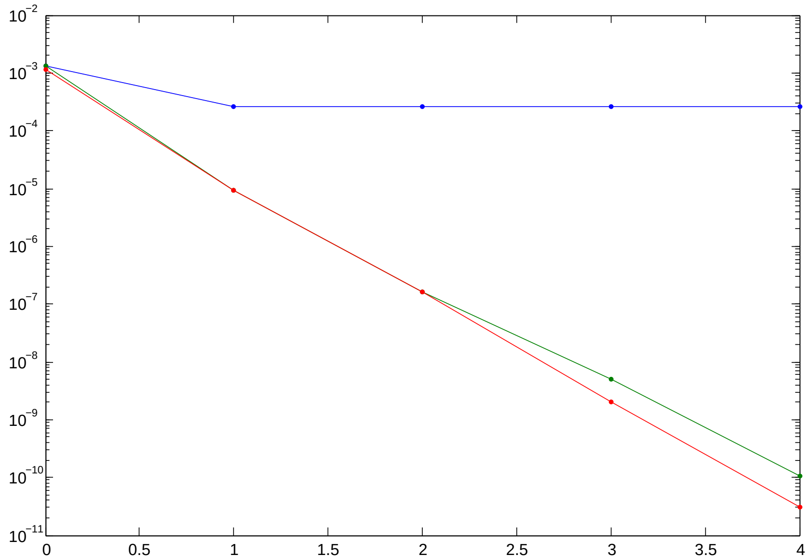


Figure 5.6: The same set-up with volume and interface basis functions of order five. Note how the approximation error for 100 and 1000 interface knots splits for dense triangulations.

We see that for a decreasing mesh-size of the volume elements, the approximation error with the sparse interface space levels out as the method converges to an inexact solution.

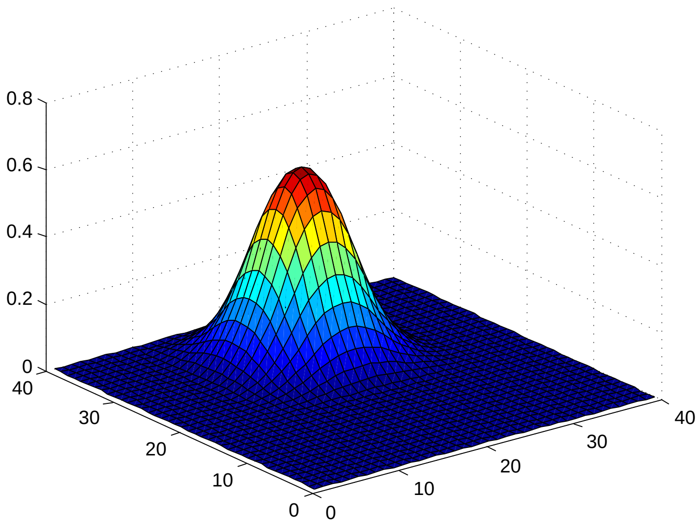


Figure 5.7: A two-dimensional B-spline patch in the reference domain. Mapped to the interface, these patches were used as basis functions for the two-dimensional interface space.

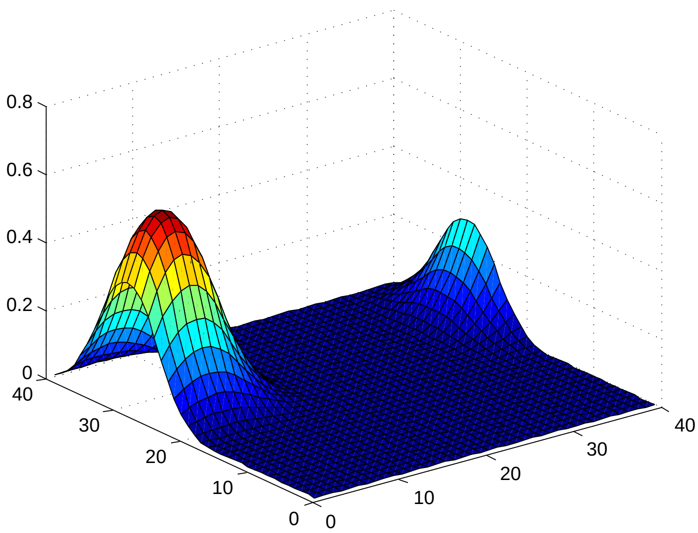


Figure 5.8: B-spline basis patch that ensures continuity and smoothness for a cylindrical interface.

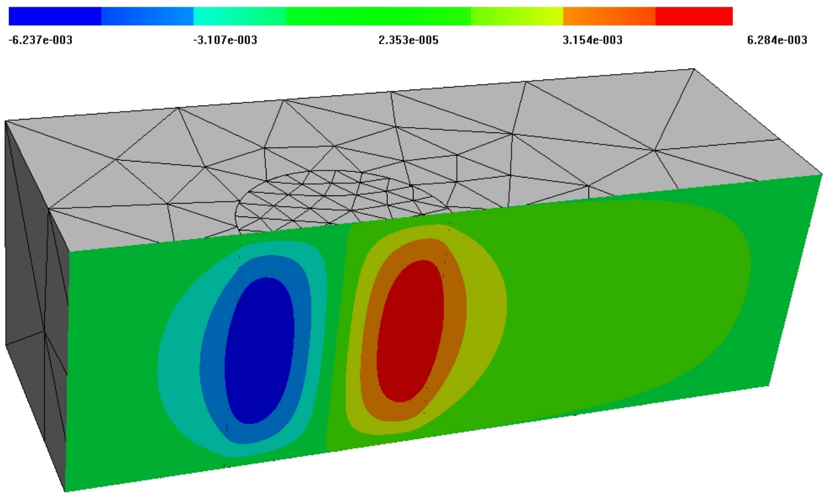


Figure 5.9: A simple problem in three dimensions using domain decomposition with a cylindrical interface.

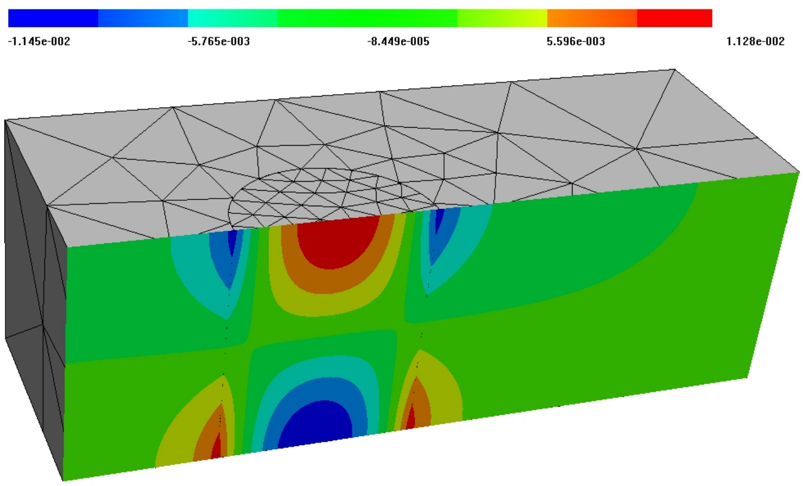


Figure 5.10: One component of the gradient of the solution. Note that the gradient is still continuous across the interface.

Chapter 6

A Method for Maxwell's Equations

6.1 Hybrid Method

The time harmonic formulation of Maxwell's equations is

$$\operatorname{curl} \mu^{-1} \operatorname{curl} u + \kappa u = j \quad \text{in } \Omega$$

with $\kappa = i\omega\sigma - \omega^2\epsilon$, and

$$E = -i\omega u, \quad H = \mu^{-1} \operatorname{curl} u.$$

As we did for the Poisson problem, we decompose the Maxwell equation into a problem on two subdomains Ω_1 and Ω_2. Now, the transmission conditions are as follows:

$$\begin{aligned} u_1 \times n_1 &= -u_2 \times n_2, \\ \mu_1^{-1} \operatorname{curl} u_1 \times n_1 &= -\mu_2^{-1} \operatorname{curl} u_2 \times n_2. \end{aligned}$$

Proceeding from the weak form as with the Poisson equation

$$\int_{\Omega_i} \{\mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} v + \kappa u \cdot v\} + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u \cdot (v \times n) = \int_{\Omega_i} j \cdot v$$

we add symmetry and penalty terms, arriving at the hybrid version:

Method 3 *Find* (u, λ) *such that*

$$\begin{aligned} \sum_{i=1}^{2} \Bigg\{ \int_{\Omega_i} \mu^{-1}\{\operatorname{curl} u \cdot \operatorname{curl} v + \kappa u \cdot v\} + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u \cdot [(v - \mu) \times n] \\ + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} v \cdot [(u - \lambda) \times n] + \frac{\alpha p^2}{\mu h} \int_{\partial\Omega_i} [(u - \lambda) \times n] \cdot [(v - \mu) \times n] \Bigg\} = \int_{\Omega} j \cdot v, \end{aligned}$$

where $u, v \in H(\operatorname{curl}, \Omega_1) \times H(\operatorname{curl}, \Omega_2)$, and λ, μ are tangential vector valued fields on the interface. The method is coercive with respect to the norm

$$|||(u, \lambda)|||^2 = \sum_{i=1}^{2} \left\{ \mu^{-1} \|\operatorname{curl} u\|_{\Omega_i}^2 + \kappa \|u\|_{\Omega_i}^2 + \frac{p^2}{\mu h} \|(u - \lambda) \times n\|_{\partial\Omega_i}^2 \right\}$$

Convergence in this norm is less than ideal for reasons we will address in the next chapter. The Nitsche terms in the formulation require vector valued interface DOF's. Note that only the tangential component of the interface space comes into play. Thus, we choose a basis of two-dimensional, tensorized B-spline patches mapped tangentially to the interface (see fig. 6.1).

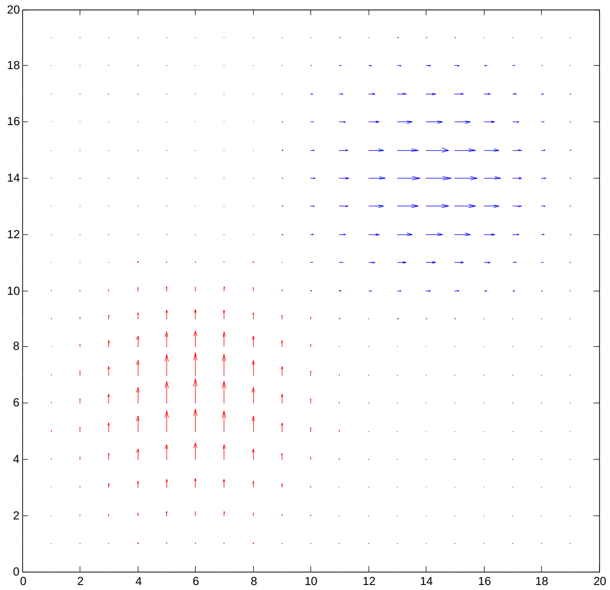


Figure 6.1: Two separate basis functions (blue and red) for the vector-valued interface space mapped together in the reference domain. They consist of tensorized B-spline patches.

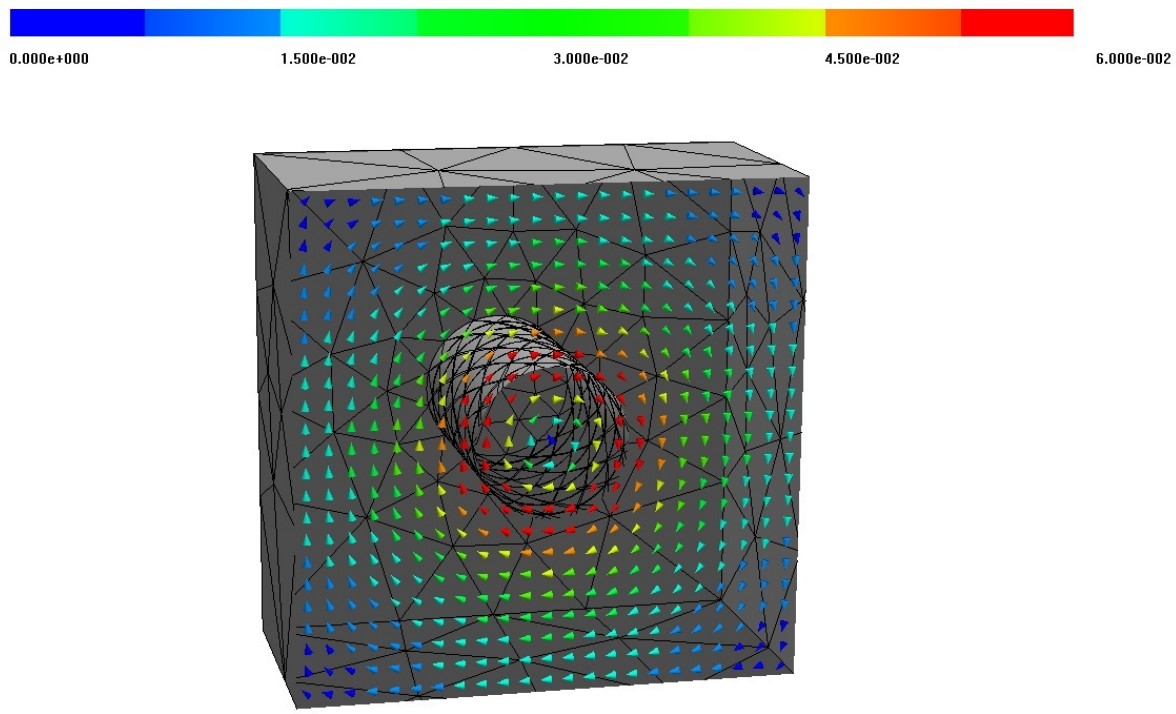
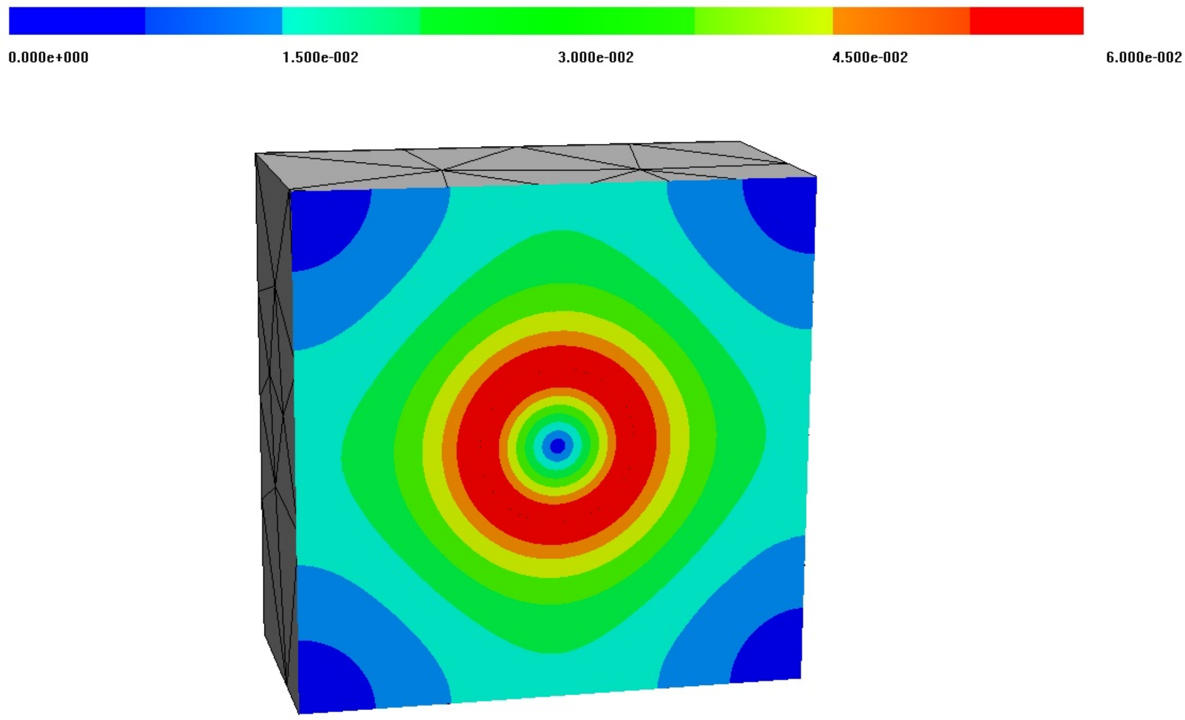


Figure 6.2: Numerical result for a simple model problem.

6.2 Helmholtz-Type Decomposition

Let the solution u to 6.1 be a gradient field, i.e. $u = \nabla \phi$ for some $\phi \in H^1(\Omega)$. Then, the energy norm of u should scale as

$$\|u\|^2 = \mu^{-1} \|\operatorname{curl} u\|^2 + \kappa \|u\|^2 = \kappa \|\nabla \phi\|^2 = O(\kappa)$$



However, $\|(u-\lambda)\times n\|$ scales like $O(1)$ We have $u = \nabla \underbrace{\phi}_{\in H^1} + \underbrace{z}_{\in [H^1]^3}$, Introduce scalar field variable ϕ to be evaluated only at the interface Introduce hybrid variable ϕ_Γ to glue ϕ across the interface Impose continuity for $(u - \nabla\phi) \times n$ and ϕ

$$\sum_{i=1}^{2} \Bigg\{ \int_{\Omega_i} \{\mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} v + \kappa u v\} + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u \, [(v-\mu)\times n] + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} v \, [(u - \nabla\phi - \lambda)\times n] + \frac{\alpha p^2}{\mu h} \int_{\partial\Omega_i} [(u-\nabla\phi-\lambda)\times n][(v-\nabla\psi-\mu)\times n] + \frac{\alpha p^2}{h} \int_{\partial\Omega_i} \kappa(\phi-\phi_\Gamma)(\psi-\psi_\Gamma) \Bigg\} = \int_\Omega j v$$

Set $v = \nabla\psi$ in the weak formulation:

$$\int_{\Omega_i} \kappa u \cdot \nabla\psi + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u \cdot (\nabla\psi \times n) = \int_\Omega j \cdot \nabla\psi$$

Apply div operator to Maxwell's equation:

$$\int_{\Omega_i} \operatorname{div}(\kappa u)\,\psi = \int_{\Omega_i} \operatorname{div} j\,\psi$$

$$-\int_{\Omega_i} \kappa u \cdot \nabla\psi + \int_{\partial\Omega_i} \kappa u_n \psi = -\int_{\Omega_i} j \cdot \nabla\psi + \int_{\partial\Omega_i} j_n \psi$$

Add:

$$\sum_{i=1}^{2}\left\{\int_{\partial\Omega_i}\mu^{-1}\operatorname{curl}u\cdot(\nabla\psi\times n)+\int_{\partial\Omega_i}\kappa u_n\psi\right\}=\sum_{i=1}^{2}\int_{\partial\Omega_i}j_n\psi$$

$$\sum_{i=1}^{2}\Bigg\{\int_{\Omega_i}\{\mu^{-1}\operatorname{curl}u\cdot\operatorname{curl}v+\kappa uv\}+\int_{\partial\Omega_i}\mu^{-1}\operatorname{curl}u\,[(v-\nabla\psi-\mu)\times n]$$
$$+\int_{\partial\Omega_i}\mu^{-1}\operatorname{curl}v\,[(u-\nabla\phi-\lambda)\times n]+\frac{\alpha p^2}{\mu h}\int_{\partial\Omega_i}[(u-\nabla\phi-\lambda)\times n][(v-\nabla\psi-\mu)\times n]$$
$$-\int_{\partial\Omega_i}\kappa u_n(\psi-\psi_\Gamma)-\int_{\partial\Omega_i}\kappa v_n(\phi-\phi_\Gamma)+\frac{\alpha p^2}{h}\int_{\partial\Omega_i}\kappa(\phi-\phi_\Gamma)(\psi-\psi_\Gamma)\Bigg\}=$$
$$\sum_{i=1}^{2}\left\{\int_{\Omega_i}jv-\int_{\partial\Omega_i}j_n\psi\right\}$$

$u, v\ldots$ $H(\operatorname{curl})$ conforming element basis functions on Ω_i $\phi,\psi\ldots$ H^1 conforming element basis functions on $\Omega_i\cap\Gamma$ $\lambda,\mu\ldots$ tangential vector valued spline functions on Γ $\phi_\Gamma,\psi_\Gamma\ldots$ scalar spline functions on Γ

6.3 Numerical Example

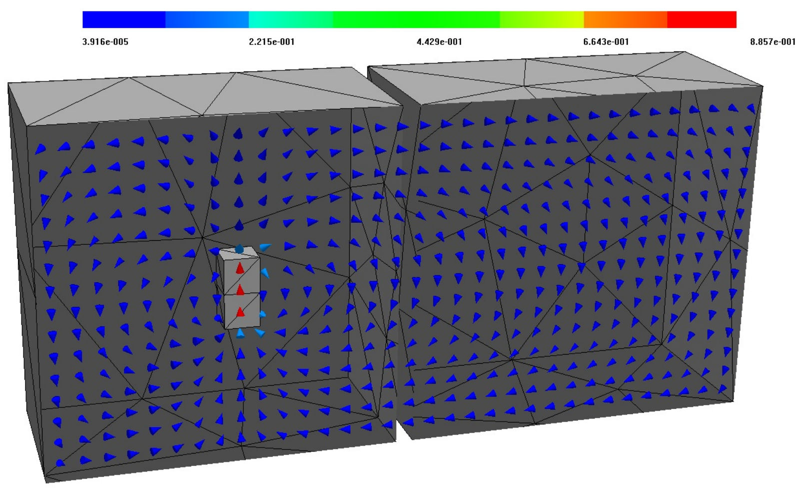


Figure 6.3: Simulation of the magnetic field of a permanent magnet using domain decomposition.

6.3.1 Simulation of an LWD

We analyze the results from the domain decomposition in a real-world application, the simulation of a logging-while-drilling (LWD) tool. Here, a sensor composed of two conducting loops, the sending and receiving antenna, respectively, is lowered into a bore hole. See figure 6.5 for a schematic of the sensor configuration. We make use of domain decomposition by splitting off a cylindrical domain around the sensor at the boundary where the wall of the rock formation meets the mud inside the bore hole.

The following parameters were set for the simulation:

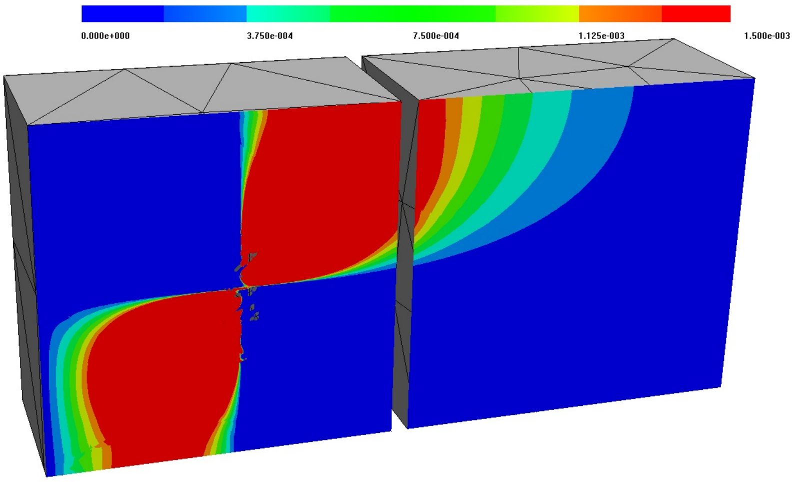


Figure 6.4: A cross section of one component of the field

frequency [kHz]	standard, rec volt [nV] 185 810 dofs	Nitsche, rec volt [nV] 195 383 dofs
20	25.44 - i18.38	25.43 - i18.37
100	71.68 - i197.5	71.65 - i197.3
400	124.9 - i963.0	124.9 - i962.3
2000	-635.9 - i5295	-634.8 - i5255

Table 6.1: Numerical results for the induced voltage in the receiver using first order volume elements

- Electric conductivity of the bore tool: $1.0 \cdot 10^6$ S/m. The tool body is modeled using a surface impedance boundary condition.
- Electric conductivity of the mud inside the bore hole: 1.0 S/m
- Electric conductivity of the mud inside the bore hole: 0.01 S/m
- Relative electric permittivity: $\epsilon_r = 26.67$ for a frequency of 2 MHz, and $\epsilon_r = 38.63$ for all other frequencies.
- Relative permeability: $\mu_r = 1.0$
- Current in the transmitter: 1.0 A

In order to quantify the effect of the domain decomposition, the experiment was conducted using standard finite elements on a matching mesh and on a comparable non-matching mesh using domain decomposition. For the latter, an interface consisting of 5 by 150 B-spline basis functions in azimuthal and axial direction of order 5 was used. In both cases, the induced voltage in the receiver resulting from the finite element solution was computed.

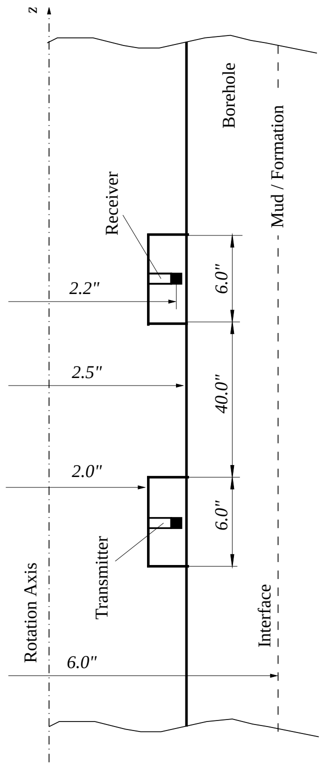


Figure 6.5: Sketch of the LWD tool with dimensions in inches.

frequency [kHz]	standard, rec volt [nV] 733 881 dofs	Nitsche, rec volt [nV] 736 939 dofs
20	24.99 - i18.47	24.98 - i18.47
100	70.25 - i196.7	70.23 - i196.7
400	121.7 - i957.9	121.7 - i957.7
2000	-648.1 - i5256	-647.9 - i5255

Table 6.2: Numerical results for the induced voltage in the receiver using second order volume elements

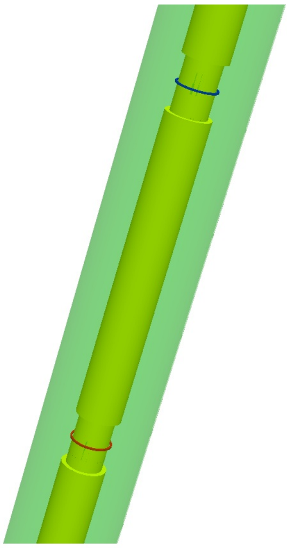


Figure 6.6: The geometry of the sensor domain with the sending and receiving antennae.

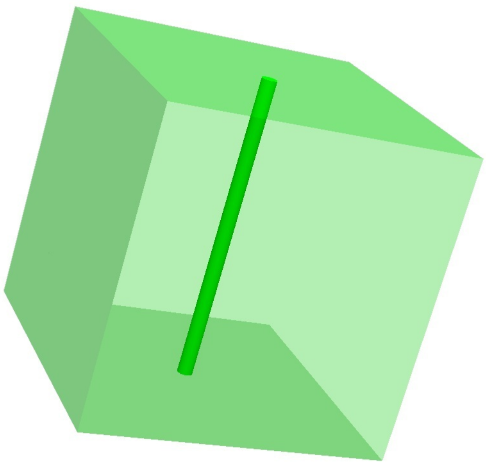


Figure 6.7: The geometry of the Two subdomains: The cylindrical bore hole and a cubic section of the formation surrounding it.

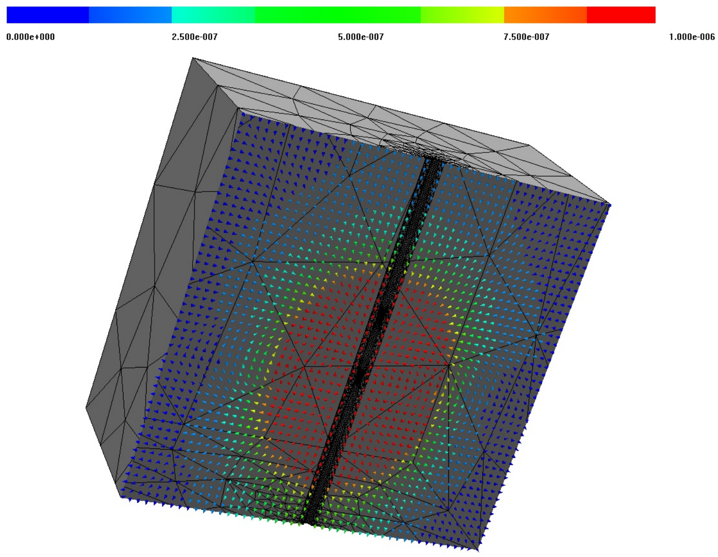


Figure 6.8: A cross section of a finite element solution vector field.

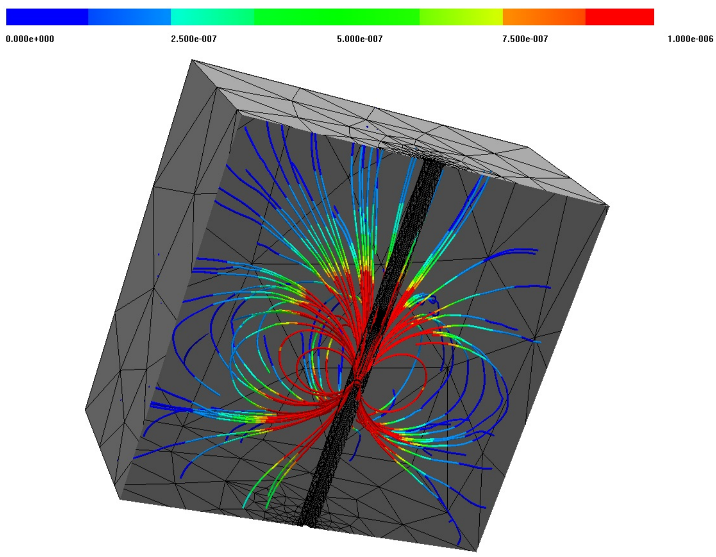


Figure 6.9: Field lines corresponding to the solution.

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