hp-Finite Elements with Local Exact Sequence Properties

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Outline

- Vector-valued function spaces and the de Rham Complex
- High-order finite elements with the local exact sequence property
- Robust preconditioning for problems with small parameters
- Applications and numerical results
- Next step: Tensor elements for elasticity

Function spaces and variational problems

We consider the vector-valued function spaces

$$H(\mathrm{curl}) = \left\{u \in \left[L_2(\Omega)\right]^3 \,:\, \mathrm{curl}\, u \in \left[L_2(\Omega)\right]^3\right\}$$

$$H(\mathrm{div}) = \left\{q \in \left[L_2(\Omega)\right]^3 \,:\, \mathrm{div}\, q \in L_2(\Omega)\right\}$$

Moreover, we focus on the parameter-dependent variational problems

$$\text{Find } u \in V: \qquad \int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\, v \, dx + \int_\Omega \kappa\, u \cdot v \, dx = \int f \cdot v \, dx \quad \forall v \in V$$

$$\text{Find } p \in Q: \qquad \int_\Omega \mathrm{div}\, p \; \mathrm{div}\, q + \int_\Omega \kappa\, p \cdot q \, dx = \int f \cdot q \, dx \quad \forall q \in Q$$

with $V = \{v \in H(\mathrm{curl}) : v_\tau = 0 \text{ on } \Gamma_D\}$ and $Q = \{q \in H(\mathrm{div}) : q_n = 0 \text{ on } \Gamma_D\}$.

Goals:

- Conforming hp-FE spaces for $H(\mathrm{div})$ and $H(\mathrm{curl})$ on unstructured hybrid meshes
- Parameter-robust preconditioners in $H(\mathrm{curl})$ and $H(\mathrm{div})$ for $0 < \kappa \ll 1$.

The de Rham Complex - exact sequences

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\nabla} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
W_h & \xrightarrow{\nabla} & V_h & \xrightarrow{\mathrm{curl}} & Q_h & \xrightarrow{\mathrm{div}} & S_h
\end{array}
$$

The exact sequence property

$$
\begin{aligned}
\mathrm{range}(\nabla) &= \ker(\mathrm{curl}) \\
\mathrm{range}(\mathrm{curl}) &= \ker(\mathrm{div})
\end{aligned}
$$

holds on the continuous and on the discrete level (contractable domains). [Bossavit],[Hiptmair]

Important for stability, convergence analysis, error estimates, ...

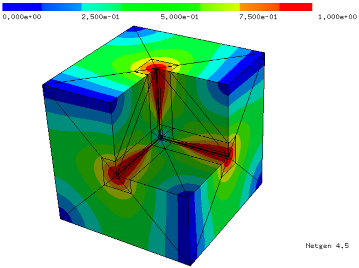
On the construction of high order finite elements for $H(\mathrm{curl})$ and $H(\mathrm{div})$

- [Webb] $H(\mathrm{curl})$-conforming shape functions with explicite gradients, convenient to implement up to order 3
- [Dubiner, Karniadakis+Sherwin] H^1-conforming shape functions in tensor product structure ($\rightarrow$ allows fast summation techniques)
- [Demkowicz, Rachowicz] Study of the de Rham diagram for *hp*-FE-spaces (global exact sequence property, minimal order condition), hp-adaptive code based on hexahedral meshes
- [Ainsworth, Coyle] Systematic construction of $H(\mathrm{curl})$- and $H(\mathrm{div})$-conforming elements of arbitrarily high order for tetrahedra
- [Nigan, Phillips] Pyramidal elements by transformation
- [JS, Zaglmayr] Based on local exact sequence property and using tensor-product structure we achieve a **systematic strategy** for the construction of $H(\mathrm{curl})$- and $H(\mathrm{div})$-conforming shape functions of **arbitrary** and **variable order for all types of element topologies** (hex, tet, prism, pyramid) [COMPEL 05, Thesis Zaglmayr 06]

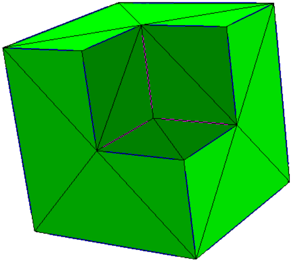
Hybrid meshes with various element topologies (1)

We assume unstructured, hybrid, conforming meshes in order to allow for

- *anisotropic 3D geometric h-refinement* in order to efficiently resolve singularities due to corners and edges



Smallest eigen-vector of Maxwell problem
(anisotropic order distribution $p = 3, \ldots, 6$)



Initial coarse tetrahedral mesh

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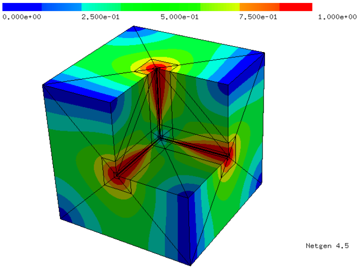

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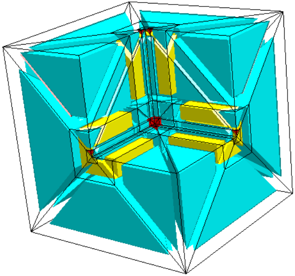
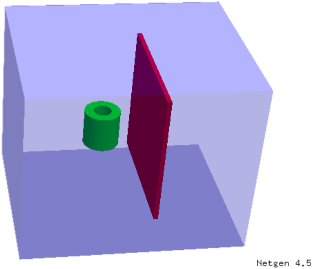


geometric h-refinement
$\rightarrow$ hexes (yellow), prisms (blue), tets (red).

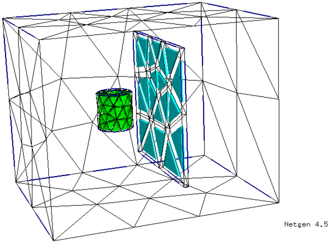
Hybrid meshes with various element topologies (2)

We are interested in unstructured, hybrid, conforming meshes in order to allow for

- Resolve boundary layers or local tensor-product meshes (e.g. for thin shields)



Geometry - Coil and Thin Shield



Coil(tets), Shield (prism-layer),

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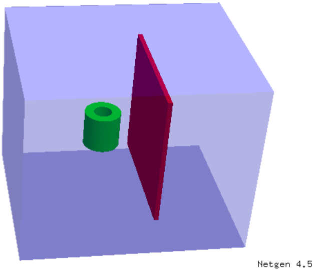

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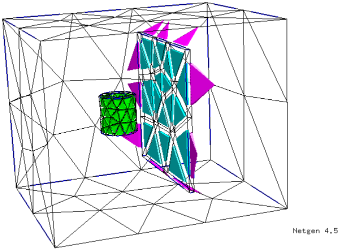
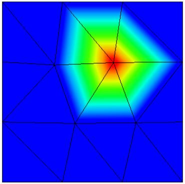


Coil(tets), Shield (prism-layer), Air (pyramids and tets)

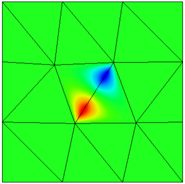
Hierarchical V-E-F-C basis for H^1-conforming Finite Elements

We use a **hierarchical Vertex-Edge-Face-Cell** basis for the high-order continuous FE-space $W_{h,p} \subset H^1(\Omega)$.

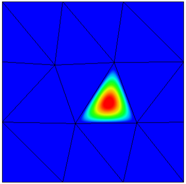
Vertex basis function



Edge basis function p=3



Cell basis function p=3



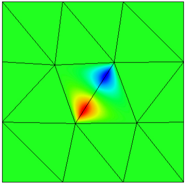
- allows variable polynomial order on each edge, face(3D) and cell.

A new strategy for high-order $H(\mathrm{curl})$-conforming elements

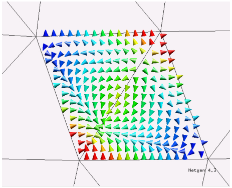
The deRham Complex states $\nabla H^1 \subset H(\mathrm{curl})$ and $\nabla W_{h,p+1} \subset V_{h,p}$.

- Take the lowest-order Nédélec element.
- Explicitely use the gradients of a H^1-conforming high-order basis functions.

Edge-basis functions:

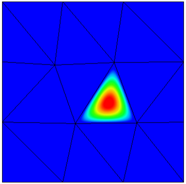


$\xrightarrow{\nabla}$

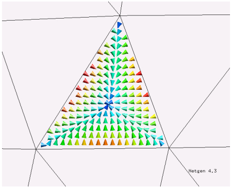


$$\boxed{\nabla W^E_{p_E+1} = V^E_{p_E}}$$

Face basis functions



$\xrightarrow{\nabla}$

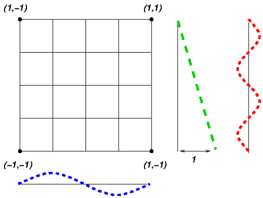


$$\boxed{\nabla W^F_{p_F+1} \subset V^F_{p_F}}$$

- Extend face/cell shape functions to a complete polynomial basis of $V_{h,p}$.

high-order H^1-conforming shape functions in tensor product structure

Exploit the tensor product structure of quadrilateral elements to build edge and face shapes



Family of orthogonal polynomials $(\, P_k^0[-1,1]\,)_{2\le k\le p}$ vanishing in ± 1.

$$\begin{aligned} \phi_i^{E_1}(x,y) &= P_i^0(x)\,\frac{1-y}{2}, \\ \phi_{i\,j}^{F}(x,y) &= P_i^0(x)\,P_j^0(y). \end{aligned}$$

.

high-order H^1-conforming shape functions in tensor product structure

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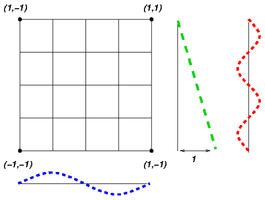

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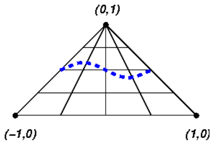
.

Degenerated tensor-product structure for triangle [Dubiner],[Karniadakis,Sherwin]:



Triangle as degenerated quadrilateral by Duffy transformation $x \to \frac{x}{1-y}$

$$\phi_i^{E_1}(x,y) = P_i^0\big(\tfrac{x}{1-y}\big)\,(1-y)^i$$
$$\phi_{ij}^{F}(x,y) = \underbrace{P_i^0\Big(\frac{x}{1-y}\Big)(1-y)^i}_{u_i(x,y)}\ \underbrace{P_j(2y-1)y}_{v_j(y)}$$

Remark: Implementation can be done division-free!

Tensor-product based high-order $H(\mathrm{curl})$-conforming elements with explicite high-order gradients:

- Lowest-order Nédélec (1st kind): $\varphi^{\mathcal{N}_0} = \lambda_i \nabla \lambda_j - \nabla \lambda_i \lambda_j$
- Edge-based shape functions (gradient fields):
  $$\varphi_i^E = \nabla \phi_i^{E,H^1} \qquad 2 \leq i \leq p_E + 1,$$
- Face-based shape functions (gradient fields and irrotationals)
  $$\begin{aligned} \varphi_{ij}^{F,1}(x,y) &= \nabla \phi_{ij}^{F,H^1}(x,y) = \nabla u_i\, v_j + v_j\, \nabla u_i \qquad 3 \leq i + j \leq p_F \\ \varphi_{ij}^{F,2a} &= \nabla u_i v_j - u_i \nabla v_j \\ \varphi_{ij}^{F,2b} &= \varphi^{\mathcal{N}_0}\, v_j \end{aligned}$$

→ Analogue principle for cell-based shape functions in 3D.

→ Thanks to tensor-product based construction this strategy extends systematically for all types of element topologies (quads, trigs, hexes, prisms, tets and pyramids).

Tensor product-based high-order $H(\text{div})$-conforming tetrahedral elements using explicite high-order curl fields

- Lowest-order Raviart-Thomas functions
  $$\varphi^{\mathcal{RT}_0} = \lambda_1 \nabla\lambda_2 \times \nabla\lambda_3 + \lambda_2 \nabla\lambda_3 \times \nabla\lambda_1 + \lambda_3 \nabla\lambda_1 \times \nabla\lambda_2$$
- Face-based shape functions (curl fields, solenoidal)
  $$\psi_{ij}^{F,k} = \nabla \times \varphi_{ij}^{\text{curl},F,k} \quad 3 \le i+j \le p_F+1, 1 \le k \le 3$$
- Cell-based shape functions (solenoidal and non-solenoidal fields)
  $$\begin{aligned} \psi_{i,j,k}^{C,1a} &= \nabla \times \varphi_{ijk}^{\text{curl},C,2a} = w_k \nabla u_i \times \nabla v_j \\ \psi_{j,k}^{C,1a} &= \nabla \times \varphi_{jk}^{\text{curl},C,2a} = \nabla \times \left(\varphi^{\mathcal{N}_0} v_j w_k\right) \end{aligned}$$
  $+$ set of lin.indep. non-solenoidal functions using factors u_i, v_j, w_k
  (corr. to H^1-conforming cell-based functions $\phi_{ijk}^{H^1,C} = u_i v_j w_k$.)

$\rightarrow$ Systemtatic strategy extends to all types of element topologies!

The local exact sequence property

Using explicite kernel functions leads to exact sequences in a local sense:

$$
\begin{array}{rcllllc}
W_{h,\mathbf{p}+1}(\mathcal{T}_h) & = & W^V_{h,1}(\mathcal{T}_h) & + \sum_E W^E_{p_E+1} & + \sum_F W^F_{p_F+1} & + \sum_C W^C_{p_C+1} & \subset H^1(\Omega) \\
 & & \downarrow\nabla & \quad\downarrow\nabla & \quad\downarrow\nabla & \quad\downarrow\nabla & \\
V_{h,\mathbf{p}}(\mathcal{T}_h) & = & V_h^{\mathcal{N}_0}(\mathcal{T}_h) & + \sum_E \nabla W^E_{p_E+1} & + \sum_F V^F_{p_F} & + \sum_C V^C_{p_C} & \subset H(\mathrm{curl},\Omega) \\
 & & \downarrow\nabla\times & & \quad\downarrow\nabla\times & \quad\downarrow\nabla\times & \\
Q_{h,\mathbf{p}-1}(\mathcal{T}_h) & = & Q_h^{RT_0}(\mathcal{T}_h) & & + \sum_F \nabla\times V^F_{p_F} & + \sum_C Q^C_{p_C-1} & \subset H(\mathrm{div},\Omega) \\
 & & \downarrow\nabla\cdot & & & \quad\downarrow\nabla\cdot & \\
S_{h,\mathbf{p}-2}(\mathcal{T}_h) & = & S_h^0(\mathcal{T}_h) & & & + \sum_C \nabla\cdot Q^C_{p_C-1} & \subset L_2(\Omega)
\end{array}
$$

- Local exact sequence property: Each sequence of local high-order spaces associated to a single edge, single face, or single cell is exact.

The local exact sequence property

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W_{h,\mathbf{p}+1}(\mathcal{T}_h) & = & W_{h,1}^{V}(\mathcal{T}_h) & + \sum_E W_{p_E+1}^{E} & + \sum_F W_{p_F+1}^{F} & + \sum_C W_{p_C+1}^{C} & \subset H^1(\Omega) \\
 & & \downarrow\nabla & \quad\downarrow\nabla & \downarrow\nabla & \downarrow\nabla & \\
V_{h,\mathbf{p}}(\mathcal{T}_h) & = & V_h^{\mathcal{N}_0}(\mathcal{T}_h) & + \sum_E \nabla W_{p_E+1}^{E} & + \sum_F V_{p_F}^{F} & + \sum_C V_{p_C}^{C} & \subset H(\mathrm{curl},\Omega) \\
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 & & \downarrow\nabla\cdot & & & \downarrow\nabla\cdot & \\
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\end{array}
$$

- Local exact sequence property: Each sequence of local high-order spaces associated to a single edge, single face, or single cell is exact.
- This implies the global exact sequence property for arbitrary and variable polynomial order on each single edge, face, and cell!
- Key to cheap, parameter-robust ASM-preconditioning.

Local preconditioners for $H(\mathrm{curl})$

In various formulations time-harmonic, quasi-static, but also in non-linear, time-stepping, eigenvalue iterations for Maxwell we face the parameter-dependent system

$$A_\kappa(u, v) = \int \mathrm{curl}\, u \cdot \mathrm{curl}\, v \,+\, \kappa\, u \cdot v \, dx$$

with non-trivial kernel $\ker(\mathrm{curl}) = \nabla H^1(\Omega)$.

- *Problem:* Classical preconditioners (Jacobi, symmetric GS, standard multigrid) fail on above parameter-dependent problems for $0 \leq \kappa \ll 1$..

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with non-trivial kernel $\ker(\mathrm{curl}) = \nabla H^1(\Omega)$.

- *Problem:* Classical preconditioners (Jacobi, symmetric GS, standard multigrid) fail on above parameter-dependent problems for $0 \leq \kappa \ll 1$..
- A two scale-problem of solenoidal and gradient fields
  A additive Schwarz preconditioner is defined by the splitting $V_h = \sum_i V_i$. For kernel functions $v = \sum_i v_i \in \nabla W_h$ we obtain

$$A(\nabla w, \nabla w) = \kappa \|\nabla w\|_0^2,$$

$$C(\nabla w, \nabla w) = \inf_{v_i \text{ s.t. } \sum v_i = \nabla w} \sum \|\,\mathrm{curl}\, v_i\|_0^2 + \kappa \|v_i\|_0^2.$$

$\rightarrow$ For general splittings : $\mathrm{cond}(C^{-1}A) = \mathcal{O}(\kappa^{-1})$.

Robust additive Schwarz methods for parameter-dependent problems

The general situation:

$$A^{\kappa}(u,v) = (\Lambda u, \Lambda v) + \kappa(u,v) \qquad u,v \in V$$

with an operator Λ with non-trivial kernel $V_0 := \ker(\Lambda)$

<u>**Theorem:**</u>

If the splitting is kernel-preserving

$$V_{h,p} = \sum V_i \quad \text{and} \quad V_0 = \sum (V_i \cap V_0),$$

then the AS-preconditioner C with

$$C(v,v) = \inf_{v_i \text{s.t. } v = \sum v_i} \sum A(v_i, v_i)$$

is robust in the sense of

$$\text{cond}(C^{-1}A) \qquad \text{is bounded uniformelty for } \kappa \to 0$$

JS 96,98,99: Nearly incompressible elasticity, Reissner Mindlin Plates
Arnold-Falk-Winther, Hiptmair: 98,2000: $H(\text{curl})$ and $H(\text{div})$, Xu: 06

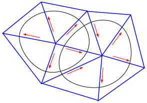
Two classical realizations of sub-spaces: *h*-version

Lowest-order case for $H(\mathrm{curl})$

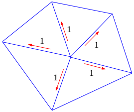
$$V_0 = \sum \nabla W_i \subset V_{h,p} \qquad \text{with } W_h = \mathrm{span}\{\phi_i : i \in \mathcal{V}\} \subset H^1$$

can be realized by the subspace splittings

Arnold-Falk-Winther: Large kernel-preserving blocks:



Hiptmair blocks: single-edge blocks plus kernel functions



$$V = \sum_{i \in \mathcal{V}} V_i \text{ with } \nabla \phi_i \in V_i \quad ,\text{or} \qquad V = \sum_{j \in \mathcal{E}} V_j + \sum_{i \in \mathcal{V}} \mathrm{span}(\nabla \phi_i)$$

Two classical realizations of sub-spaces: h-version

Lowest-order case for $H(\mathrm{curl})$

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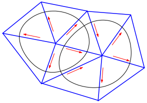

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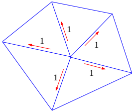

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Requirements in general (h.o.): for $H(\mathrm{curl})$ or $H(\mathrm{div})$

- AFW-smoother: overlapping block-Jacobi preconditioner (according to vertex patches or edge patches(3d))
- Hiptmair-smoother: discrete differential operators (B_∇,$B_{\nabla\times}$) and Jacobi-preconditioner for Poisson or curl-curl matrix.

Robust preconditioning in the case of local exact sequences

- A κ-robust additive Schwarz preconditioner has to fulfill

$$V_{h,p} = \sum_{i=1}^{m} V_i \quad \text{and} \quad \nabla W_h = \sum_{i=1}^{m} \left(V_i \cap \nabla W_h\right) \qquad \text{for } H(\text{curl}),$$

$$Q_{h,p} = \sum_{i=1}^{m} Q_i \quad \text{and} \quad \text{curl } V_h = \sum_{i=1}^{m} \left(Q_i \cap \text{curl } V_h\right) \qquad \text{for } H(\text{div}).$$

- Due to the *local exact sequence property*

$$\begin{array}{rcllll}
W_{h,\mathbf{p}+1}(\mathcal{T}_h) & = & W^V_{h,1}(\mathcal{T}_h) & + \sum_E W^E_{p_E+1} & + \sum_F W^F_{p_F+1} & + \sum_C W^C_{p_C+1} \\
 & & \downarrow\nabla & \quad\downarrow\nabla & \quad\downarrow\nabla & \quad\downarrow\nabla \\
V_{h,\mathbf{p}}(\mathcal{T}_h) & = & \mathcal{N}_0(\mathcal{T}_h) & + \sum_E V^E_{p_E} & + \sum_F V^F_{p_F} & + \sum_C V^C_{p_C} \\
 & & \downarrow\nabla\times & & \quad\downarrow\nabla\times & \quad\downarrow\nabla\times \\
Q_{h,\mathbf{p}}(\mathcal{T}_h) & = & \mathcal{RT}_0(\mathcal{T}_h) & & + \sum_F \nabla\times V^F_{p_F} & + \sum_C Q^C_{p_C}
\end{array}$$

paramter-robustness is guaranteed for simple $\mathcal{N}_0$-E-F-C as well as $\mathcal{RT}_0$-F-C splitting.

In practise, this means ....

Simple Block-Preconditioning in H(curl)

The global stiffness matrix is split into the according unknowns:

$$A_h = \begin{pmatrix} A_{\mathcal{N}_0\mathcal{N}_0} & A_{\mathcal{N}_0 E} & A_{\mathcal{N}_0 F} & A_{\mathcal{N}_0 C} \\ A_{E\mathcal{N}_0} & A_{EE} & A_{EF} & A_{EC} \\ A_{F\mathcal{N}_0} & A_{FE} & A_{FF} & A_{FC} \\ A_{C\mathcal{N}_0} & A_{CE} & A_{CF} & A_{CC} \end{pmatrix}.$$

The cheap preconditioner is the $\mathcal{N}_0$-E-F-C block Jacobi-preconditioner

$$C_h = \begin{pmatrix} A_{\mathcal{N}_0\mathcal{N}_0} & 0 & 0 & 0 \\ 0 & \operatorname{diag}(A_{EE}) & 0 & 0 \\ 0 & 0 & \operatorname{diag}(A_{FF}) & 0 \\ 0 & 0 & 0 & A_{CC} \end{pmatrix}.$$

yields a parameter-robust method !

In fact, we apply a two-level concept:

- The lowest-order space (coarse level) is solved exactly, or by Hiptmair or AFW multigrid, or Reitzinger-Schöberl AMG.
- Local smoothing for the high-order unknowns

Application: Reduced Basis Gauging for magnetostatic problems

We consider the magnetostatic problem:
Find the vector potential $A \in H(\text{curl})$ s.t.

$$\int_\Omega \text{curl } A \, \text{curl}\, v \, dx = \int_\Omega jv \, dx, \quad \forall\, v \in V.$$

The solution A is determined up to gradients.

Application: Reduced Basis Gauging for magnetostatic problems

We consider the magnetostatic problem:
Find the vector potential $A \in H(\text{curl})$ s.t.

$$\int_\Omega \text{curl } A \text{ curl } v \, dx + \kappa \int_\Omega u \, v \, dx = \int_\Omega jv \, dx, \quad \forall \, v \in V.$$

The solution A is determined up to gradients.

$\rightarrow$ Gauging by adding a small regularization term with $0 < \kappa \ll 1$.
$\rightarrow$ Suitable, since numerical methods are robust in κ.

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Furthermore, we introduce the reduced basis gauging where

- the explicite high-order gradient basis functions are locally skipped,
- gauging is only needed for the lowest-order subspace.

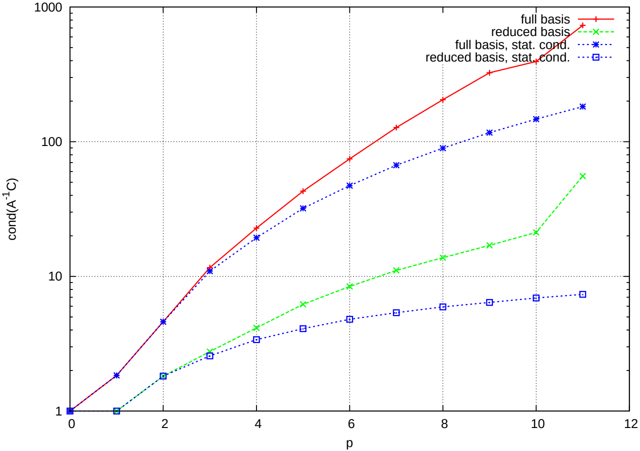
Advantages:

- The reduced system has ≈ 60 % of unknowns of the full system
- The reduced problem is better conditioned.

A simple model problem: Condition numbers in full vs. reduced basis

... compared on the unit cube covered with 6 tetrahedra with/without static condensation (for $\kappa = 1e-6$):

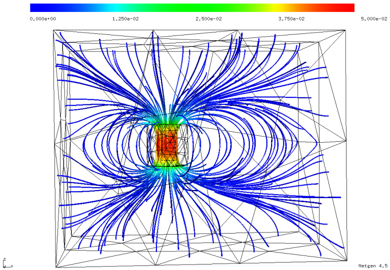
$$A_\kappa(u, v) = \int \operatorname{curl} u \operatorname{curl} v \, dx + \kappa \int u \, v \, dx.$$



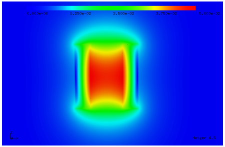
Polynomial order vs. Condition number ($\kappa(C^{-1}A)$)

Magnetostatic boundary value problem - Numerical Results

Simulation of the magnetic field induced by a coil with prescribed currents:



Magnetic field induced by a coil, p=6.



Absolute value of $|B| = |\,\mathrm{curl}\, A|$.

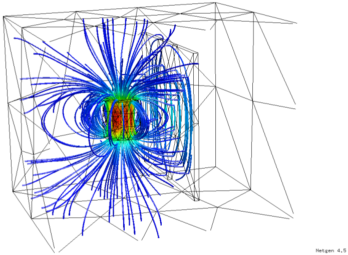
Comparison of simulation with full and with reduced basis:

p	dofs	grads	$\kappa(C^{-1}A)$	iter	solvertime
4	104350	yes	79.86	62	20.2 s
4	61744	no	21.01	39	3.2 s
6	303009	yes	207.02	91	120.3 s
6	186052	no	33.33	48	13.5 s
8	664380	yes	398.03	114	430.1 s
8	416064	no	43.38	54	41.7 s

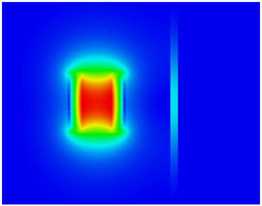
Note, the computed $B = \mathrm{curl}\, A$ is equal in both versions.

Magnetostatic BVP - The shielding problem

Simulation of the magnetic field induced by a coil with prescribed currents:



Electromagnetic shielding problem: magnetic field, p=5



Absolute value of magnetic flux, p=5

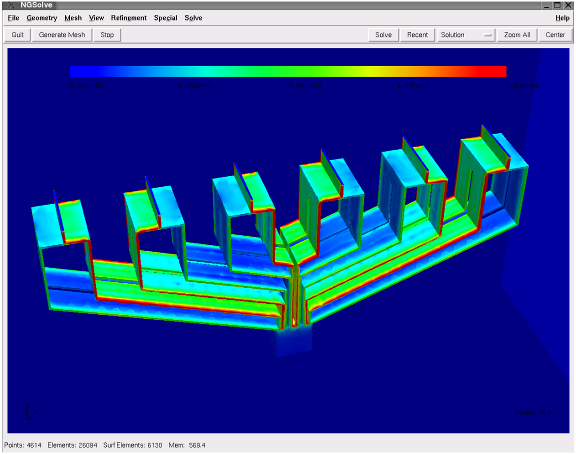
... prism layer in shield, unstructured mesh (tets, pyramids) in air/coil.

Comparision of simulation with full and with reduced basis

p	dofs	grads	$\kappa(C^{-1}A)$	iter	solvertime
4	96870	yes	34.31	37	24.9 s
4	57602	no	31.14	36	6.6 s
7	425976	yes	140.74	63	241.7 s
7	265221	no	72.63	51	87.6 s

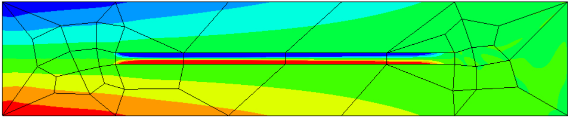
Application: Simulation of eddy-currents in bus bars

... gradients can be skipped in non-conducting domains (air).

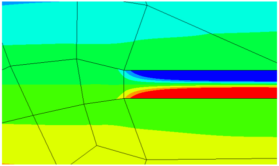


Full basis for $p = 3$ in conductor, reduced basis for $p = 3$ in air
$n \approx 450k$, 20 min on P4 Centrino, 1600MHz

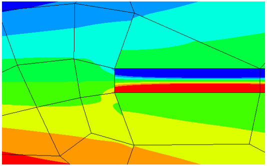
Elasticity Problem: A beam in a beam



Reenforcement with $E = 50$ in medium with $E = 1$.



New mixed FEM, $p = 2$



Primal FEM, $p = 3$

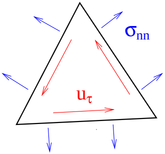
joint work with Astrid Sinwel, Start-project “hp-FEM”, RICAM Linz [Tech Report 07]

Degrees of freedom for TD-NNS elements

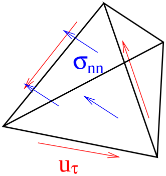
Mixed elements for approximating displacements and stresses.

- tangential components of displacement vector
- normal-normal component of stress tensor

Triangular Finite Element:



Tetrahedral Finite Element:



Hellinger Reissner mixed methods for elasticity

Primal mixed method:

Find $\sigma \in L_2^{sym}$ and $u \in [H^1]^2$ such that

$$\begin{array}{rcrcll} \int A\sigma : \tau & - & \int \tau : \varepsilon(u) & = & 0 & \forall\, \tau \\ -\int \sigma : \varepsilon(v) & & & = & -\int f \cdot v & \forall\, v \end{array}$$

Dual mixed method:

Find $\sigma \in H(\mathrm{div})^{sym}$ and $u \in [L_2]^2$ such that

$$\begin{array}{rcrcll} \int A\sigma : \tau & + & \int \operatorname{div} \tau \cdot u & = & 0 & \forall\, \tau \\ \int \operatorname{div} \sigma \cdot v & & & = & -\int f \cdot v & \forall\, v \end{array}$$

[Arnold+Falk+Winther]

Reduced Symmetry mixed methods

Decompose

$$\varepsilon(u) = \nabla u + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \operatorname{curl} u = \nabla u + \omega$$

Impose symmetry of the strain tensor by an additional Lagrange parameter:

Find $\sigma \in [H(\mathrm{div})]^2$, $u \in [L_2]^2$, and $\omega \in L_2^{skew}$ such that

$$\begin{array}{lclcll} \int A\sigma : \tau & + & \int u \operatorname{div} \tau + \displaystyle\int \tau : \omega & = & 0 & \forall\, \tau \\ \int v \operatorname{div} \sigma & & & = & -\int fv & \forall\, v \\ \int \sigma : \gamma & & & = & 0 & \forall\, \gamma \end{array}$$

The solution satisfies $u \in L_2$ and $\omega = \operatorname{curl} u \in L_2$, i.e.,

$$u \in H(\mathrm{curl})$$

Arnold+Brezzi, Stenberg,... 80s

Choices of spaces

$\int \operatorname{div}\sigma \cdot u$ understood as

$\langle \operatorname{div}\sigma, u\rangle_{H^{-1}\times H^1} = -(\varepsilon(u), \sigma)_{L_2}$	$\langle \operatorname{div}\sigma, u\rangle_{H(\mathrm{curl})^* \times H(\mathrm{curl})}$	$(\operatorname{div}\sigma, u)_{L_2}$

Displacement

$u \in [H^1]^2$ continuous f.e.	$u \in H(\mathrm{curl})$ tangential-continuous f.e.	$u \in [L_2]^2$ non-continuous f.e.

Stress

$\sigma \in L_2^{sym}$ non-continuous f.e.	$\sigma \in L_2^{sym}, \operatorname{div}\operatorname{div}\sigma \in H^{-1}$ normal-normal cont (σ_{nn}) f.e.	$\sigma \in H(\mathrm{div})^{sym}$ normal-cont (σ_n) f.e.

The mixed system is well posed for all of these pairs.

Continuity properties of the space $H(\operatorname{div}\operatorname{div})$

Lemma: Let σ be a piece-wise smooth tensor field on the mesh $\mathcal{T} = \{T\}$ such that $\sigma_{nt} \in H^{1/2}(\partial T)$. Assume that $\sigma_{nn} = n^T \sigma n$ is continuous across element interfaces. Then there holds $\operatorname{div} \sigma \in H(\operatorname{curl})^*$.

Proof: Let v be a smooth test function.

$$\begin{aligned}
\langle \operatorname{div} \sigma, v \rangle &:= -\int \sigma : \nabla v = \sum_T \Big\{ \int_T \operatorname{div} \sigma \cdot v - \int_{\partial T} \sigma_n \cdot v \Big\} \\
&= \sum_T \Big\{ \int_T \operatorname{div} \sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \Big\} + \sum_E \int_E \underbrace{[\sigma_{nn}]}_{=0} v_n \\
&\leq \sum_T \| \operatorname{div} \sigma \|_{L_2(T)} \| v \|_{L_2(T)} + \| \sigma_{n\tau} \|_{H^{1/2}(\partial T)} \| v_\tau \|_{H^{-1/2}(\partial T)} \\
&\preceq C(\sigma) \, \| v \|_{H(\operatorname{curl})}
\end{aligned}$$

By density, the continuous functional can be extended to the whole $H(\operatorname{curl})$:

$$\langle \operatorname{div} \sigma, v \rangle = \sum_T \Big\{ \int_T \operatorname{div} \sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \Big\}$$

The TD-NNS-continuous mixed method

Assuming piece-wise smooth solutions, the elasticity problem is equivalent to the following mixed problem: Find $\sigma \in H(\operatorname{div}\operatorname{div})$ and $u \in H(\operatorname{curl})$ such that

$$
\begin{array}{lclcl}
\int A\sigma : \tau & + & \sum_T \left\{ \int_T \operatorname{div}\tau \cdot u - \int_{\partial T} \tau_{n\tau} u_\tau \right\} & = & 0 \\
\sum_T \left\{ \int_T \operatorname{div}\sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \right\} & & & = & -\int f \cdot v
\end{array}
$$

Proof: The second line is equilibrium, plus tangential continuity of the normal stress vector:

$$\sum_T \int_T (\operatorname{div}\sigma + f) v + \sum_E \int_E [\sigma_{n\tau}] v_\tau = 0 \qquad \forall\, v$$

Since the space requires continuity of σ_{nn}, the normal stress vector is continuous.
Element-wise integration by parts in the first line gives

$$\sum_T \int_T (A\sigma - \varepsilon(u)) : \tau + \sum_E \int_E \tau_{nn} [u_n] = 0 \qquad \forall\, \tau$$

This is the constitutive relation, plus normal-continuity of the displacement.
Tangential continuity of the displacement is implied by the space $H(\operatorname{curl})$.

The 3-step 'exact sequence'

$$H^1 \cap H^2(\mathcal{T}) \quad \xrightarrow{\nabla} \quad H(\mathrm{curl}) \cap [H^1(\mathcal{T})]^2 \quad \xrightarrow{\sigma_{\mathcal{T}}(\cdot)} \quad H(\mathrm{div\,div}) \quad \xrightarrow{\mathrm{div}} \quad H^{-1}(\mathrm{div}) \quad \xrightarrow{\mathrm{div}} \quad H^{-1}$$

with the stress operator

$$\sigma(v) = \begin{pmatrix} \frac{\partial v_y}{\partial y} & -\frac{1}{2}\left\{\frac{\partial v_y}{\partial x} + \frac{\partial v_y}{\partial x}\right\} \\ sym & \frac{\partial v_x}{\partial x} \end{pmatrix}.$$

The composite operators are

$$\begin{aligned} \mathrm{airy}(w) = \sigma(\nabla w) &= \begin{pmatrix} \frac{\partial^2 w}{\partial y^2} & -\frac{\partial w}{\partial x \partial y} \\ sym & \frac{\partial w}{\partial x^2} \end{pmatrix} \\ \mathrm{div}\,\sigma(v) &= \frac{1}{2}\,\mathrm{Curl\,curl}\,v \end{aligned}$$

There holds

$$\begin{aligned} \mathrm{range}(\sigma(\nabla\cdot)) &= \ker(\mathrm{div}) \\ \mathrm{range}(\mathrm{div}\,\sigma(\cdot) &= \ker(\mathrm{div}) \end{aligned}$$

Finite elements for $H(\operatorname{div}\operatorname{div})$

Start with C^0-continuous finite elements for $H^1 \cap H^2(\mathcal{T})$

Finite elements for $H(\operatorname{div}\operatorname{div})$ can be built with

edge basis functions: $\sigma(\nabla\varphi^E)$

ad hoc internal basis functions: $\operatorname{Sym}[\nabla\lambda_\alpha^\perp \otimes \nabla\lambda_\beta^\perp]\, \lambda_\gamma P^{k-1}$

Alternative: Take airy functions of internal C^0-continuous f.e., plus some more.
Potential to save dofs for subdomains with $\operatorname{div}\sigma = 0$.

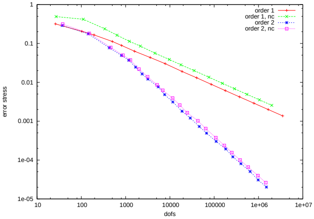
Unit square, left side fixed, vertical load, adaptive refinement

Proven to be robust with respect to volume locking ($\nu \to 0.5$)

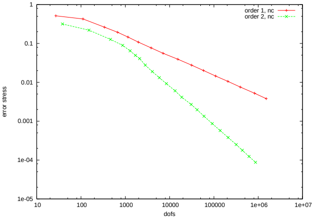
$\sigma \in P^1$
2 dof σ_{nn} per edge

	conforming	non-conforming
$u \in P^1$	2 dof u_τ	1 dof u_τ
$u \in P^2$	3 dof u_τ	2 dof u_τ

$\nu = 0.3$:

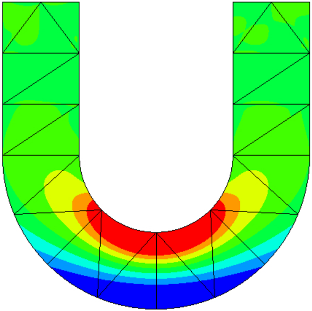


$\nu = 0.4999$:



Curved elements

fixed left top, pull right top
Elements of order 5

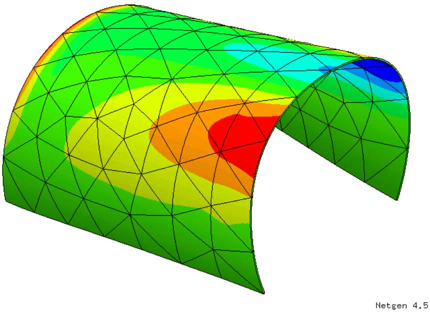


σ_{xx}

Shell structure

Proven to be robust with respect to shear locking (flat anisotropic elements).

R $=$ 0.5, t $=$ 0.005
$\sigma \in P^2$, $u \in P^3$



stress component σ_{yy}

Conclusions:

- A new systematic strategy for the construction of $H(\mathrm{curl})$ and $H(\mathrm{div})$-conforming Finite Elements using explicitely high-order kernel functions. This introduces the local exact sequence property and its advantages
  - variable and arbitrary polynomial degree on each edge, face, and cell preserving the global exact sequence property,
  - simple block ASM-preconditioners for curl-curl and div-div systems are parameter-robust,
  - reduced basis gauging,
  - trivial discrete differential operators B_∇, B_{curl}, B_{div}.
- Application to Maxwell Source Problems and Eigenvalue Problems [Thesis S. Zaglmayr, 06]
- Tensor-valued elements for elasticity [with A. Sinwel]
- These elements are available in the open source package Netgen/NgSolve.