Continuous - Hybrid Discontinuous Galerkin Methods for Vector-valued Applications

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Linear Elasticity

$\Omega \subset \mathbb{R}^d$. Find displacement $u \in [H^1]^d$ such that $u = u_D$ on Γ_D and

$$\int_\Omega D\varepsilon(u) : \varepsilon(v) = \int_\Omega f v \qquad \forall\, v \in V_0$$

with the linear strain operator $\varepsilon(\cdot) : [H^1]^d \to [L_2]^{d \times d, sym}$

$$\varepsilon(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big) = \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)_{i,j=1,..d}$$

and the isotropic material operator $D : [L_2]^{d \times d} \to [L_2]^{d \times d}$

$$D\varepsilon = 2\mu\varepsilon + \lambda\,\mathrm{tr}(\varepsilon) I$$

The stress tensor is

$$\sigma = D\varepsilon(u)$$

Continuous and elliptic in $[H^1]^d$

BUT: Constants depend on λ/μ, and on the domain (Korn's inequality)

Incompressible flows

Stokes Equation:

$\Omega \subset \mathbb{R}^d$. Find $u \in [H^1]^d$ such that $u = u_D$ on Γ_D, $p \in Q := L_2$ such that

$$\int_\Omega D\varepsilon(u) : \varepsilon(v) + \int_\Omega \operatorname{div} v \, p = \int_\Omega f v \qquad \forall \, v \in V_0$$

and incompressibility constraint

$$\int \operatorname{div} u \, q = 0 \qquad \forall \, q \in Q$$

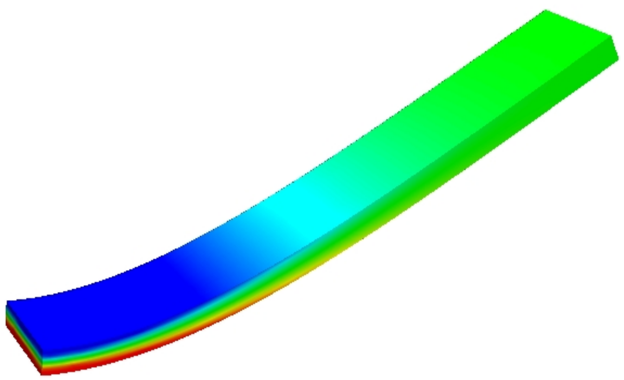
with Dirichlet b.c. (no slip and inflow), point-wise mixed b.c. (slip) and Neumann (outflow).

Difficulty: Incompressibility constraint

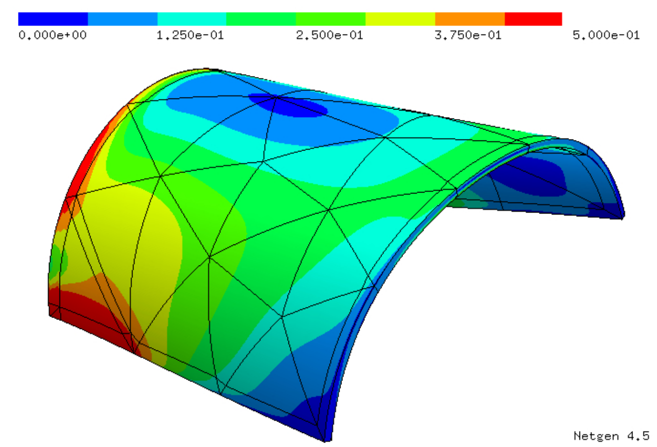
Mixed finite elements: continuous pressure ? discontinuous pressure ? stabilized methods ?

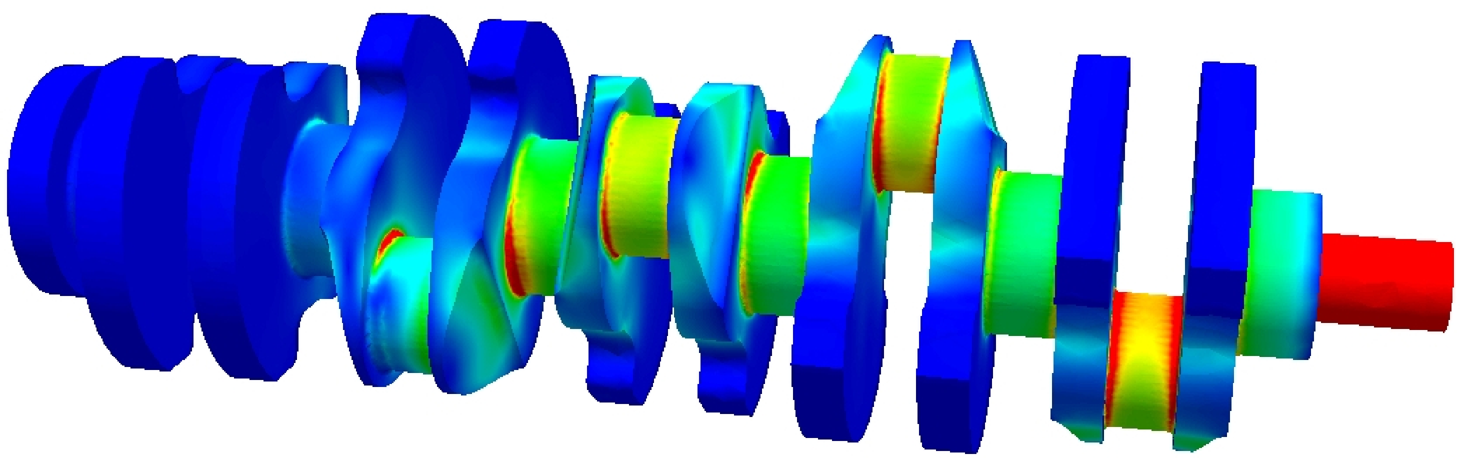
Elasticity examples: Visualization of stresses

Elastic beam:

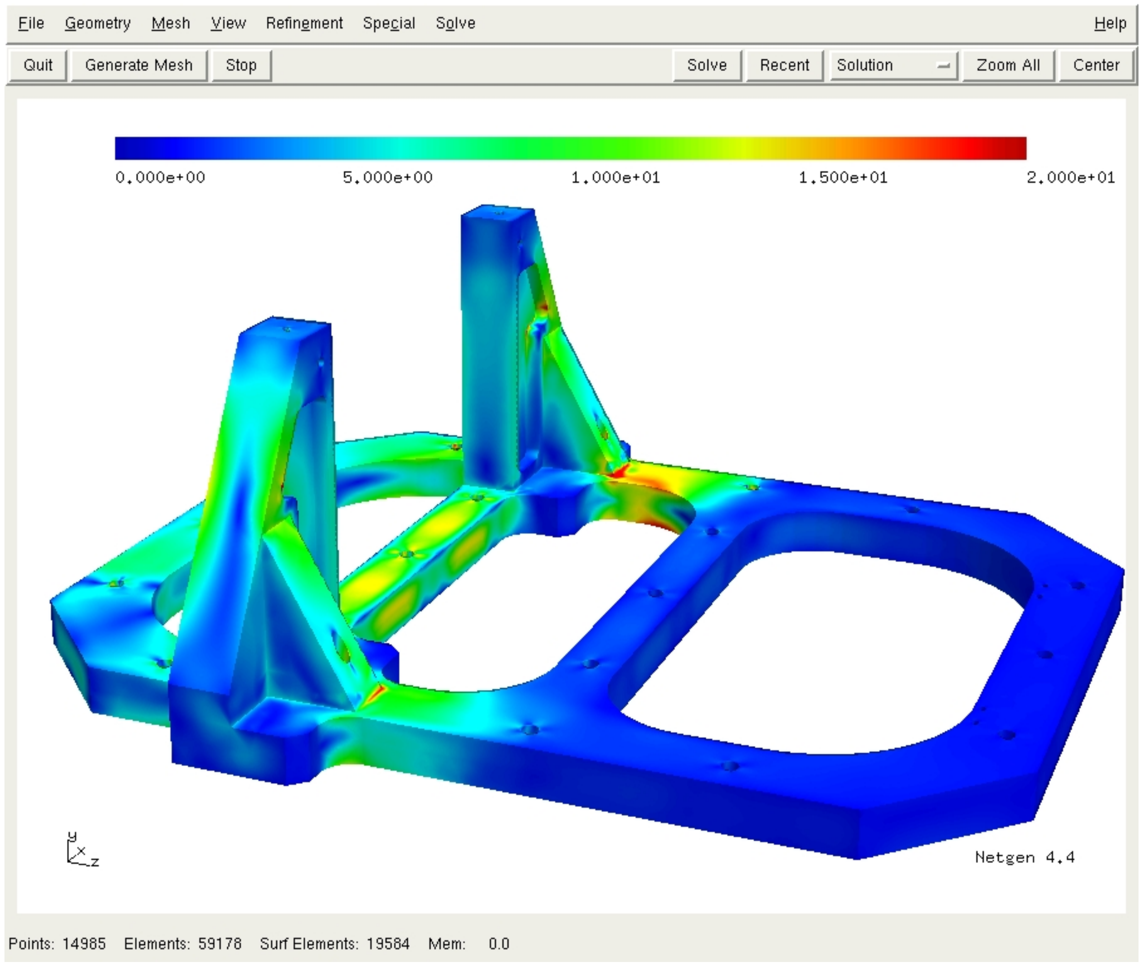


Shell structures:





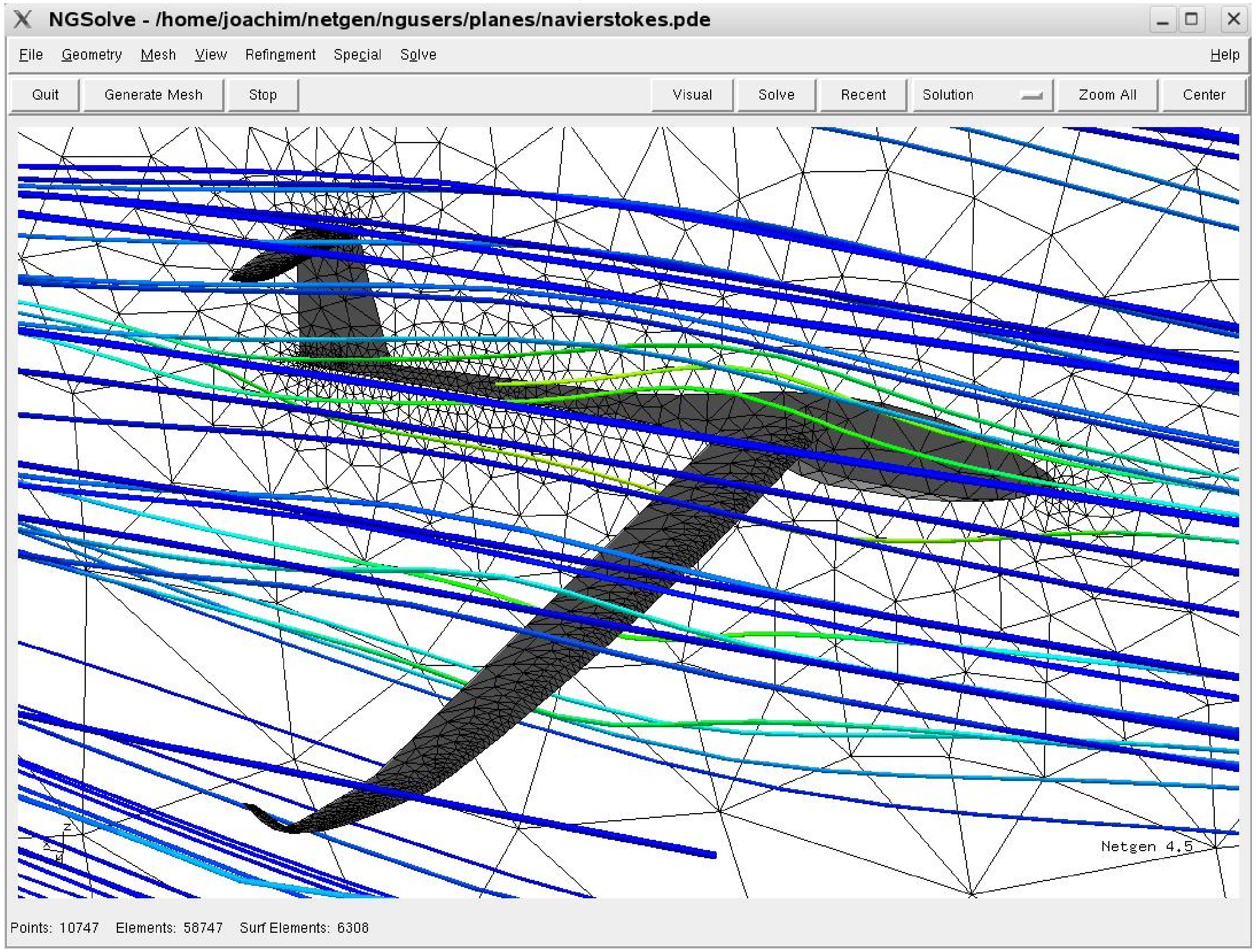
Von-Mises Stresses in a Machine Frame (linear elasticity)



Simulation with Netgen/NGSolve

45553 tets, p = 5, 3×1074201 unknowns, 5 min on 8 core 2.4 GHz 64-bit PC 6 GB RAM

Toy Example: Sailplane



Stokes Flow, 2^{nd}-order HDG elements, 59E3 elements, 1.65E6 dofs, 2GB RAM, 5 min (2-core 1.8GHz)

Function spaces $H(\text{curl})$ and $H(\text{div})$

$$
\begin{aligned}
H(\text{curl}) &= \{u \in [L_2]^d : \text{curl}\, u \in L_2^{d(d-1)/2}\} \\
H(\text{div}) &= \{u \in [L_2]^d : \text{div}\, u \in L_2\}
\end{aligned}
$$

Piece-wise smooth functions in

- $H(\text{curl})$ have continuous tangential components,
- $H(\text{div})$ have continuous normal components.

Important for constructing conforming finite elements such as Raviart Thomas, Brezzi-Douglas-Marini, and Nedelec elements.

Natural function space for Maxwell equations: Find $A \in H(\text{curl})$ such that

$$
\int_\Omega \mu^{-1}\,\text{curl}\, A \;\text{curl}\, v + \int_\Omega (i\sigma\omega - \varepsilon\omega^2) A v = \int j v \qquad \forall\, v \in H(\text{curl})
$$

Contents

- Introduction
- Hybrid Discontinuous Galerkin Method
- Finite Elements for $H(\mathrm{div})$ and $H(\mathrm{curl})$
- Tangential-continuous finite elements for elasticity
- Normal-continuous finite elements for Stokes

Hybrid Discontinuous Galerkin (HDG) Method

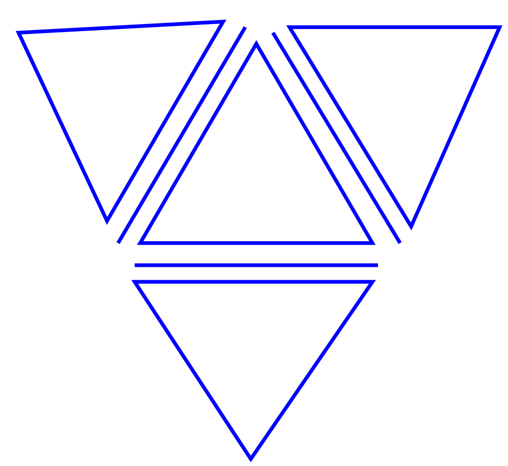
Model problem: $-\Delta u = f$

A mesh consisting of elements and facets (= edes in 2D and faces in 3D)

$$\mathcal{T} = \{T\} \qquad \mathcal{F} = \{F\}$$

Goal: Approximate u with piece-wise polynomials on elements and additional polynomials on facets:

$$u_N \in P^p(\cup T) \qquad \lambda_N \in P^p(\cup F)$$



HDG - Derivation

Exact solution u, traces on element boundaries: $\lambda := u|_{\cup F}$

Integrate against discontinuous test-functions $v \in H^1(\cup T)$, element-wise integration by parts:

$$\sum_T \Big\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} v \Big\} = \int_\Omega f v$$

Use continuity of $\frac{\partial u}{\partial n}$, and test with single-valued functions $\mu \in L_2(\cup F)$:

$$\sum_T \Big\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} (v - \mu) \Big\} = \int_\Omega f v$$

Use consistency $u = \lambda$ on ∂T to symmetrice, and stabilize ...

$$\sum_T \Big\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} (v - \mu) - \int_{\partial T} \frac{\partial v}{\partial n} (u - \lambda) + \frac{\alpha p^2}{h} \int_{\partial T} (u - \lambda)(v - \mu) \Big\} = \int_\Omega f v$$

Dirichlet b.c.: Imposed on λ, Neumann b.c.: $\int_{\Gamma_N} g \mu$

Inverse Inequality: For $u \in P^p(T)$ there holds

$$\int_{\partial T} \left| \frac{\partial u}{\partial n} \right|^2 \leq \frac{c_{inv} p^2}{h} \int_T |\nabla u|^2 \, dx$$

Proof for mapped quads: Numerical integration with Gauss-Lobatto integration rule ($x_0 = -1$, $x_n = 1$):

$$\int_{-1}^{1} f(x) dx \approx \sum_{i=0}^{n} \omega_i f(x_i)$$

Consider bottom edge:

$$\sum_i \omega_i |S_i| \, \left| n \cdot \nabla u\big(\Phi(x_i, -1)\big) \right|^2 \leq \max_i \left\{ \frac{|S_i|}{|V_{i,0}|} \right\} \frac{1}{\omega_0} \sum_{i,j} \omega_i \omega_j |V_{i,j}| \, \left| \nabla u\big(\Phi(x_i, y_j)\big) \right|^2$$

with surface measure and volume measures $|S_i|$ and $|V_{i,j}|$. There holds

$$\frac{1}{h_{op}} := \frac{|S|}{|V|} \qquad \text{and} \qquad \omega_0 \approx p^{-2}$$

For Gauss-Lobatto integration, a good constant c_{inv} is computable for free !

HDG - Stability

HDG - norm:

$$\|(u,\lambda)\|_{1,HDG}^2 = \sum_T \left\{ \|\nabla u\|_{L_2(T)}^2 + \frac{p^2}{h} \|u - \lambda\|_{L_2(\partial T)}^2 \right\}$$

Lemma: Assume $\alpha > c_{inv}$. Then, for $(u,\lambda) \in P^p(\cup T) \times P^p(\cup F)$ there holds

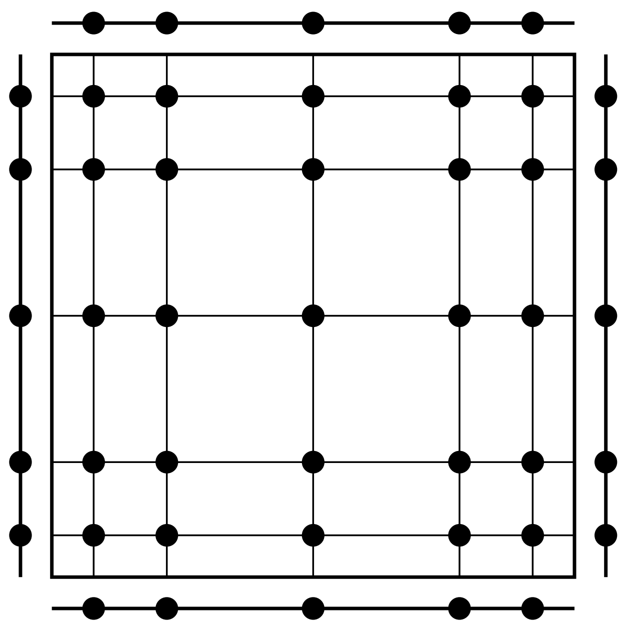
$$\|(u,\lambda)\|_{1,HDG}^2 \preceq A(u,\lambda;u,\lambda) \preceq \|(u,\lambda)\|_{1,HDG}$$

Proof of lower bound: Element by element:

$$\begin{aligned}
A^T(u,\lambda;u,\lambda) &= \int_T |\nabla u|^2 - 2 \int_{\partial T} \frac{\partial u}{\partial n}(u-\lambda) + \frac{\alpha p^2}{h} \int_{\partial T} (u-\lambda)^2 \\
&\geq \int_T |\nabla u|^2 - \frac{1}{\gamma} \int_{\partial T} \left| \frac{\partial u}{\partial n} \right|^2 - \gamma \int_{\partial T} (u-\lambda)^2 + \frac{\alpha p^2}{h} \int_{\partial T} (u-\lambda)^2 \\
&\geq \int_T |\nabla u|^2 - \frac{c_{inv} p^2}{\gamma h} \int_T |\nabla u|^2 - \gamma \int_{\partial T} (u-\lambda)^2 + \frac{\alpha p^2}{h} \int_{\partial T} (u-\lambda)^2
\end{aligned}$$

Choosing $\gamma = \sqrt{c_{inv}\alpha} p^2/h$ gives the result. Equivalence constants depend only on α/c_{inv}

Interpretation as low-order method on the collocation grid



$$
\begin{aligned}
\|u\|_{1,HDG}^2 \quad \approx \quad & \sum_{i,j} \delta_{x_i}\delta_{y_j} \left(\left(\frac{u(x_{i+1},y_j) - u(x_i,y_j)}{\delta_{x_i}} \right)^2 + \left(\frac{u(x_i,y_{j+1}) - u(x_i,y_j)}{\delta_{y_j}} \right)^2 \right) \\
+ \quad & \sum_{i} \delta_{x_i}\delta_{y_0} \left(\frac{u(x_i,y_0) - \lambda_{bot}(x_i)}{\delta_{y_0}} \right)^2 + E_{right} + E_{top} + E_{left}
\end{aligned}
$$

This holds since $\delta_{y_0} \approx \frac{h}{p^2}$.

Relation to standard Interior Penalty DG method

DG - space

$$V_N := P^p(\cup T)$$

Bilinearform

$$A^{DG}(u,v) = \sum_T \left\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n}[v] - \int_{\partial T} \frac{\partial v}{\partial n}[u] + \frac{\alpha p^2}{h} \int_{\partial T} [u][v] \right\}$$

Hybrid DG has

- even more unknowns, but less matrix entries
- allows element-wise assembling
- allows static condensation of element unknowns

Hybridization of standard DG methods [Cockburn+Gopalakrishnan+Lazarov]

Consistency + Stability $\Rightarrow$ Convergence

$$\|(u - u_N, \lambda - \lambda_N)\|_{1,HDG} \leq c\, \|(u - I_N^T u, u - I_N^F u)\|_{1,HDG}$$

c ... absolute constant

Relation to classical hybridization of mixed methods

First order system

$$A\sigma - \nabla u = 0 \qquad \text{and} \qquad \operatorname{div} \sigma = -f$$

Mixed method: Find $\sigma \in H(\operatorname{div})$ and $u \in L_2$ such that

$$\begin{array}{lclcll} \int A\sigma\tau & - & \int \operatorname{div} \tau \, u & = & 0 & \forall \tau \in H(\operatorname{div}) \\ \int \operatorname{div} \sigma \, v & & & = & -\int f v & \forall v \in L_2 \end{array}$$

Breaking normal-continuity of σ_n, and reinforcing it by another Lagrange parameter

Find $\sigma \in H(\operatorname{div})$, $u \in L_2$, and $\lambda \in L_2(\cup F)$ such that

$$\begin{array}{lclcll} \int A\sigma\tau & + & \sum_T \int_T \operatorname{div} \tau \, u + \sum_F \int_F [\tau_n] \lambda & = & 0 & \forall \tau \in H(\operatorname{div}) \\ \sum_T \int_T \operatorname{div} \sigma \, v & & & = & -\int f v & \forall v \in L_2 \\ \sum_F \int_F [\sigma_n] \mu & & & = & 0 & \forall \mu \in L_2(\cup F) \end{array}$$

Allows to eliminate σ (and also u) leading to a coercive system in u and λ (or, only λ).

Comparison to mixed hybrid system

HDG method needs facet variable of one order higher ???

Solutions

- Project

$$
\begin{aligned}
A^{HDG}(u,\lambda;v,\mu) \quad = \quad & \sum_T \Big\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n}(v-\mu) \\
& - \int_{\partial T} \frac{\partial v}{\partial n}(u-\lambda) + \frac{\alpha p^2}{h} \int_{\partial T} P^{p-1}(u-\lambda)\, P^{p-1}(v-\mu) \Big\}
\end{aligned}
$$

- Choose orthogonal basis for facet element, leave highest order discontinuous. Estimate non-conformity error by Strang lemma.

Mixed Continuous / Hybrid Discontinuous Galerkin method

Vector-valued spaces with partial continuity and partial components

$$
\begin{array}{rclcrcl}
V_{\mathcal{T},n} & = & \{v \in [P^p(\cup T)]^d : [v_n] = 0\} & \qquad & V_{\mathcal{T},\tau} & = & \{v \in [P^p(\cup T)]^d : [v_\tau] = 0\} \\
V_{\mathcal{F},n} & = & \{v \in [P^p(\cup F)]^d : v_\tau = 0\} & \qquad & V_{\mathcal{F},\tau} & = & \{v \in [P^p(\cup F)]^d : v_n = 0\}
\end{array}
$$

$H(\mathrm{curl})$ - based formulation: Find $u \in V_{\mathcal{T},\tau}$ and $\lambda \in V_{\mathcal{F},n}$ such that

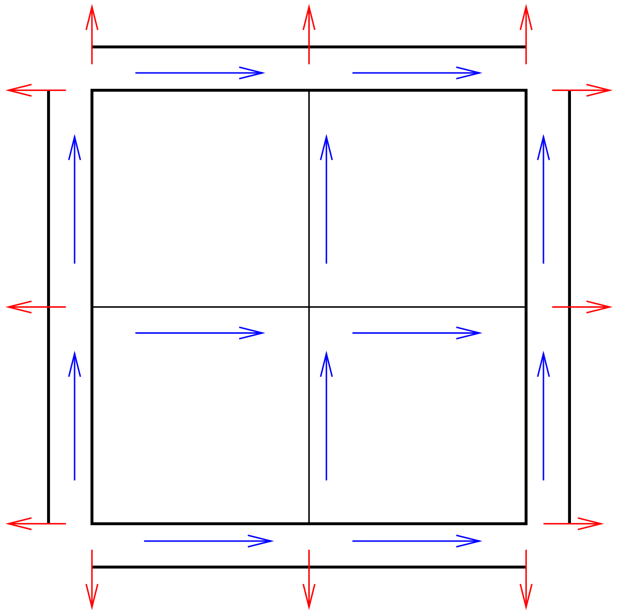
$$
A^\tau(u, \lambda; v, \mu) = \int f v \qquad \forall\, v \in V_{\mathcal{T},\tau}\ \forall \mu \in V_{\mathcal{F},\nu}
$$

$$
\begin{aligned}
A^\tau(u, \lambda; v, \mu) \quad = \quad & \sum_T \Big\{ \int_T D\varepsilon(u) : \varepsilon(v) - \int_{\partial T} (D\varepsilon(u))_{nn} (v - \mu)_n \\
& - \int_{\partial T} (D\varepsilon(v))_{nn} (u - \lambda)_n + \frac{\alpha p^2}{h} \int_{\partial T} (u - \lambda)_n (v - \mu)_n \Big\}
\end{aligned}
$$

Or, vice versa ... [idea from notes by Cockburn, Gopalakrishnan, Lazarov]

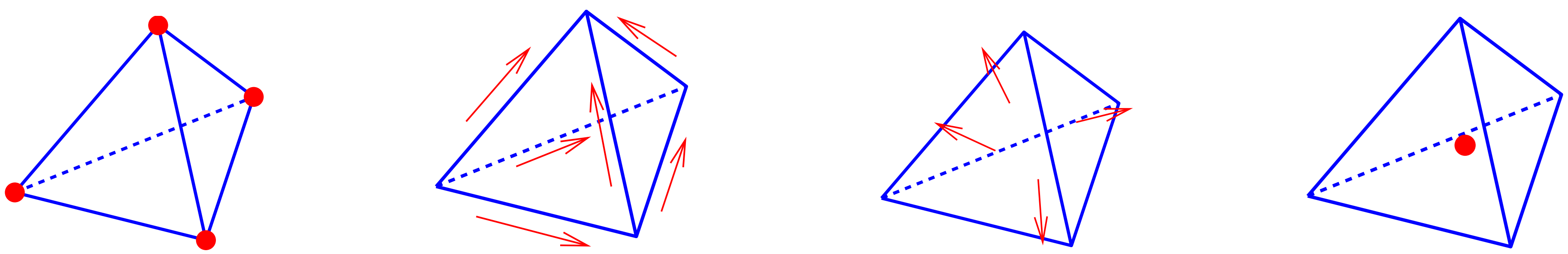
Collocation elements

$$u_T \in \mathcal{N}_1 = P^{p,p+1} \times P^{p+1,p}, \qquad \lambda_F \in P^{p+1}$$



The de Rham Complex

$$\begin{array}{ccccccc}
H^1 & \xrightarrow{\nabla} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
W_h & \xrightarrow{\nabla} & V_h & \xrightarrow{\mathrm{curl}} & Q_h & \xrightarrow{\mathrm{div}} & S_h
\end{array}$$



satisfies the **complete sequence property**

$$\begin{aligned}
\mathrm{range}(\nabla) &= \ker(\mathrm{curl}) \\
\mathrm{range}(\mathrm{curl}) &= \ker(\mathrm{div})
\end{aligned}$$

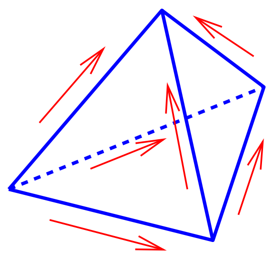
on the continuous and the discrete level.

Important for stability, error estimates, preconditioning, ...

Low-order $H(\mathrm{curl})$ finite elements

First order Nédélec I elements:

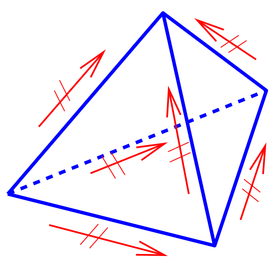
$$V_h = \{v \in H(\mathrm{curl}) : v|_T = a_T + b_T \times x\}$$



first order approximation for A-field and B-field

First order Nédélec II elements:

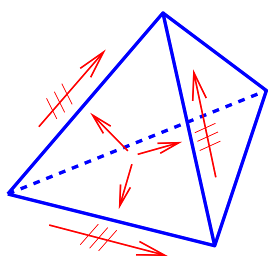
$$V_h = \{v \in H(\mathrm{curl}) : v|_T \in [P^1]^3\}$$



second order for A-field, first order for B-field

Second order Nédélec II elements:

$$V_h = \{v \in H(\mathrm{curl}) : v|_T \in [P^2]^3\}$$



third order for A-field, second order for B-field

On the construction of high order $H(\mathrm{curl})$ finite elements

- [Dubiner, Karniadakis+Sherwin] H^1-conforming shape functions in tensor product structure
  $\longrightarrow$ allows fast summation techniques

- [Webb] $H(\mathrm{curl})$ hierarchical shape functions with local complete sequence property
  convenient to implement up to order 4

- [Demkowicz+Monk] Based on global complete sequence property construction of Nédélec elements of variable order (with constraints on order distribution) for hexahedra

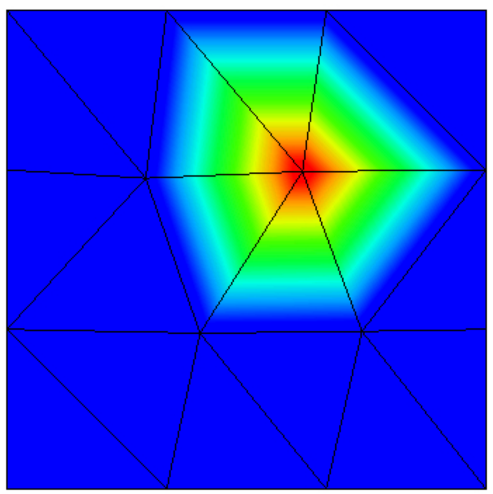
- [Ainsworth+Coyle] Systematic construction of $H(\mathrm{curl})$-conforming elements of arbitrarily high order for tetrahedra

- [JS+Zaglmayr] Based on **local complete sequence property** and by using **tensor-product structure** we achieve a **systematic strategy** for the construction of $H(\mathrm{curl})$-conforming hierarchical shape functions of **arbitrary** and **variable order for common element geometries** (segments, quadrilaterals, triangles, hexahedra, tetrahedra, prisms, pyramids).
  [COMPEL, 2005], PhD-Thesis Zaglmayr 2006

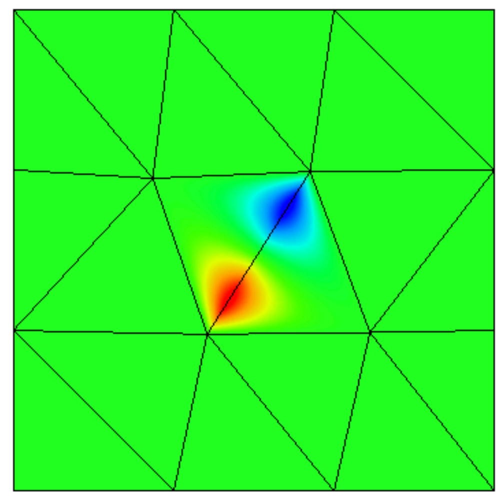
Hierarchical $VEFC$ basis for H^1-conforming Finite Elements

The high order elements have basis functions connected with the vertices, edges, (faces,) and cell of the mesh:

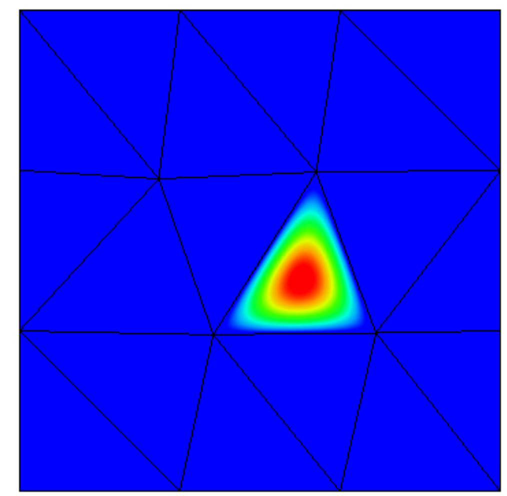
Vertex basis function



Edge basis function p=3



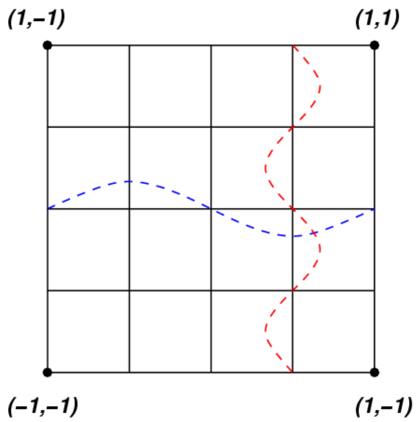
Inner basis function p=3



This allows an individual polynomial order for each edge, face, and cell..

High-order H^1-conforming shape functions in tensor product structure

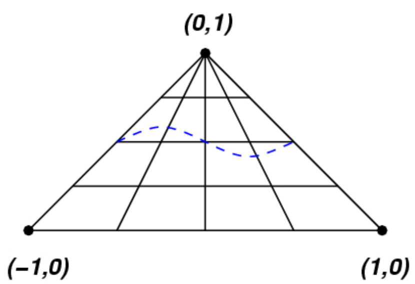
Exploit the tensor product structure of quadrilateral elements to build edge and face shapes



Family of orthogonal polynomials $(\,P_k^0[-1,1]\,)_{2 \leq k \leq p}$ vanishing in ± 1.

$$
\begin{aligned}
\varphi^F_{i\,j}(x,y) &= P_i^0(x)\,P_j^0(y), \\
\varphi^{E_1}_i(x,y) &= P_i^0(x)\,\tfrac{1-y}{2}.
\end{aligned}
$$

Tensor-product structure for triangle [Dubiner, Karniadakis+Sherwin]:



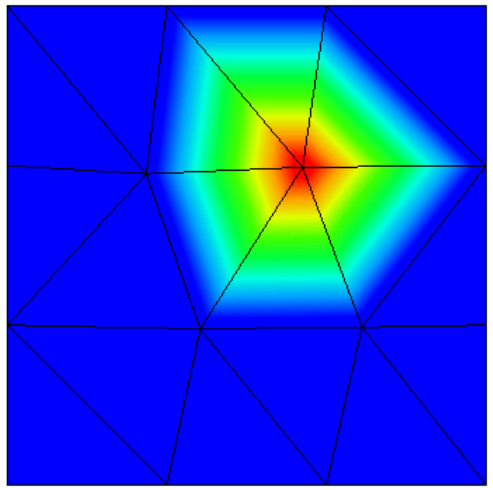
Collapse the quadrilateral to the triangle by $x \rightarrow (1-y)x$

$$
\begin{aligned}
\varphi^{E_1}_i(x,y) &= P_i^0(\tfrac{x}{1-y})\,(1-y)^i \\
\varphi^F_{i\,j}(x,y) &= \underbrace{P_i^0(\frac{x}{1-y})(1-y)^i}_{u_i(x,y)}\ \underbrace{P_j(2y-1)y}_{v_j(y)}
\end{aligned}
$$

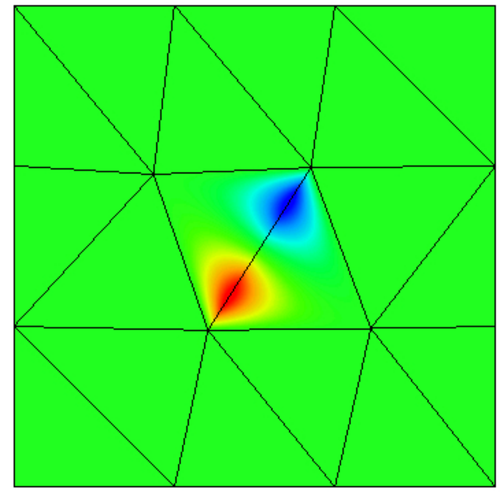
Remark: Implementation is free of divisions

The deRham Complex tells us that $\nabla H^1 \subset H(\mathrm{curl})$, as well for discrete spaces $\nabla W^{p+1} \subset V^p$.

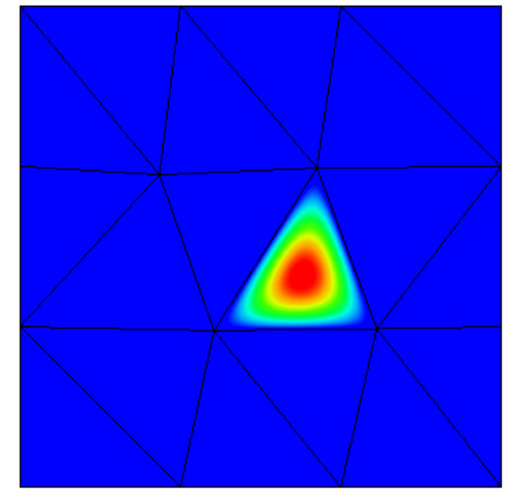
Vertex basis function



Edge basis function p=3

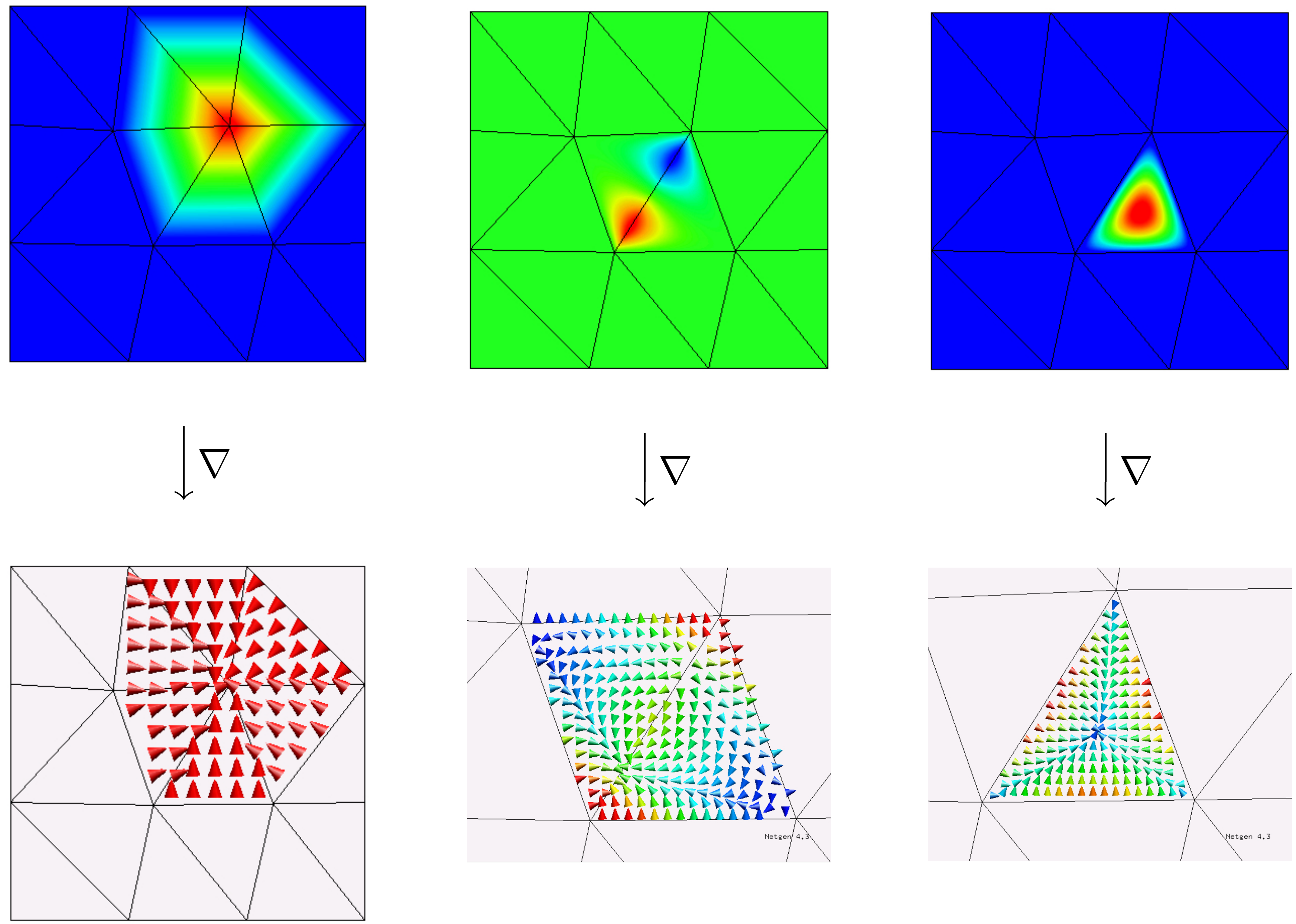


Inner basis function p=3



The deRham Complex tells us that $\nabla H^1 \subset H(\mathrm{curl})$, as well for discrete spaces $\nabla W^{p+1} \subset V^p$.

Vertex basis function | Edge basis function p=3 | Inner basis function p=3



$$\nabla W_{V_i} \subset V_{\mathcal{N}_0} \qquad \nabla W^{p+1}_{E_k} = V^p_{E_k} \qquad \nabla W^{p+1}_{F_k} \subset V^p_{F_k}$$

$H(\mathrm{curl})$-conforming face shape functions with $\nabla W_F^{p+1} \subset V_F^p$

We use inner H^1-shape functions spanning $W_F^{p+1} \subset H^1$ of the structure

$$\varphi_{i,j}^{F,\nabla} = u_i(x,y)\, v_j(y).$$

We suggest the following $H(\mathrm{curl})$ face shape functions consisting of 3 types:

- **Type 1: Gradient-fields**
$$\varphi_{1,\,i,j}^{F,curl} = \nabla\, \varphi_{i,j}^{F,\nabla} = \nabla(u_i\, v_j) = u_i\, \nabla v_j + v_j\, \nabla u_i$$
- **Type 2: other combination**
$$\varphi_{2,\,i,j}^{F,\mathrm{curl}} = u_i\, \nabla v_j - v_j\, \nabla u_i$$
- **Type 3:** to achieve a base spanning V_F $(p-1)$ lin. independent functions are missing
$$\varphi_{3,\,j}^{F,\mathrm{curl}} = \mathcal{N}_0(x,y)\, v_j(y).$$

Localized complete sequence property

We have constructed **V**ertex-**E**dge-**F**ace-**C**ell shape functions satisfying

$$
\begin{array}{lllllll}
W^V_{h,\,p+1=1} & \xrightarrow{\nabla} & V_h^{\mathcal{N}_0} & \xrightarrow{\mathrm{curl}} & Q_h^{\mathcal{RT}_0} & \xrightarrow{\mathrm{div}} & S_{h,\,0} \\
W^E_{p_E+1} & \xrightarrow{\nabla} & V^E_{p_E} & & & & \\
W^F_{p_F+1} & \xrightarrow{\nabla} & V^F_{p_F} & \xrightarrow{\mathrm{curl}} & Q^F_{p_F-1} & & \\
W^C_{p_C+1} & \xrightarrow{\nabla} & V^C_{p_C} & \xrightarrow{\mathrm{curl}} & Q^C_{p_C-1} & \xrightarrow{\mathrm{div}} & S^C_{p_C-2}.
\end{array}
$$

Advantages are

- allows arbitrary and variable polynomial order on each edge, face and cell
- Additive Schwarz Preconditioning with cheap $\mathcal{N}_0 - E - F - C$ blocks gets efficient
- Reduced-basis gauging by skipping higher-order gradient bases functions
- discrete differential operators B_∇, B_{curl}, B_{div} are trivial

Vector transformations

Element transformation Φ, Jacobian $F = \Phi'$, and $J = \det F$
Transformation of scalar functions:

$$w(\Phi(\hat{x})) = \hat{w}(\hat{x}) \qquad \Rightarrow \qquad (\nabla w)(\Phi(\hat{x})) = F^{-T}(\nabla \hat{w})(\hat{x})$$

Transformation of $H(\mathrm{curl})$ functions

$$u(\Phi(\hat{x})) = F^{-T}\hat{u}(\hat{x}) \qquad \Rightarrow \qquad (\mathrm{curl}\, u)(\Phi(\hat{x})) = J^{-1}F(\mathrm{curl}\, \hat{u})(\hat{x})$$

Preserves line integrals:

$$\int_E u_\tau = \int_{\hat{E}} \hat{u}_\tau$$

Transformation of $H(\mathrm{div})$ functions (Piola transformation)

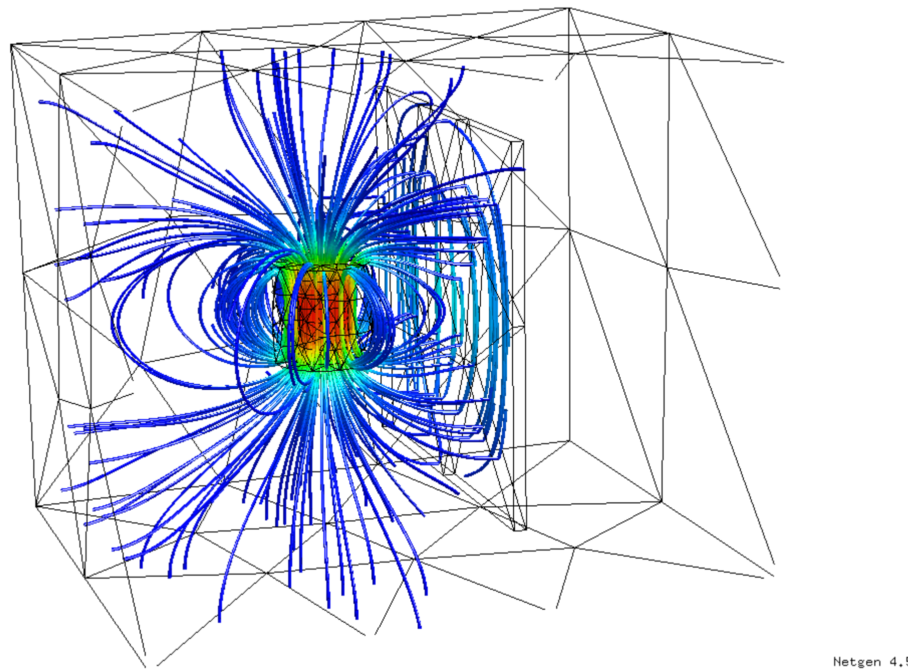
$$q(\Phi(\hat{x})) = J^{-1}F\hat{q}(\hat{x}) \qquad \Rightarrow \qquad (\mathrm{div}\, q)(\Phi(\hat{x})) = J^{-1}(\mathrm{div}\, \hat{q})(\hat{x})$$

Preserves face integrals:

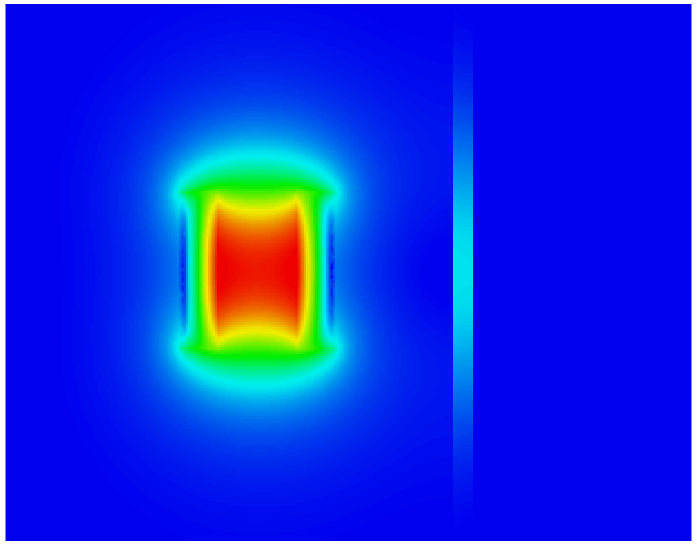
$$\int_F q_n = \int_{\hat{F}} \hat{q}_n$$

Magnetostatic BVP - The shielding problem

Simulation of the magnetic field induced by a coil with prescribed currents:



Electromagnetic shielding problem: magnetic field, p=5

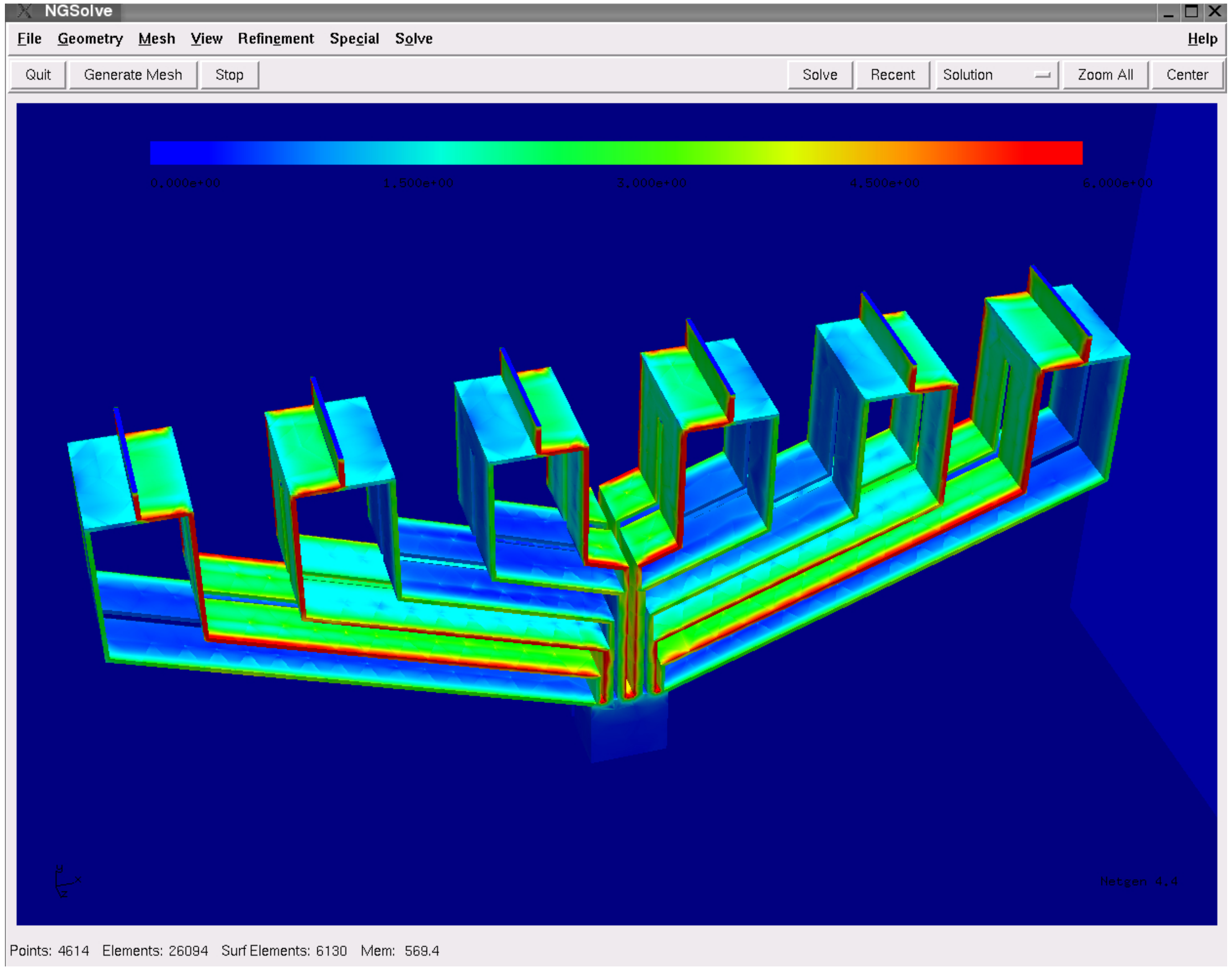


Absolute value of magnetic flux, p=5

... prism layer in shield, unstructured mesh (tets, pyramids) in air/coil.

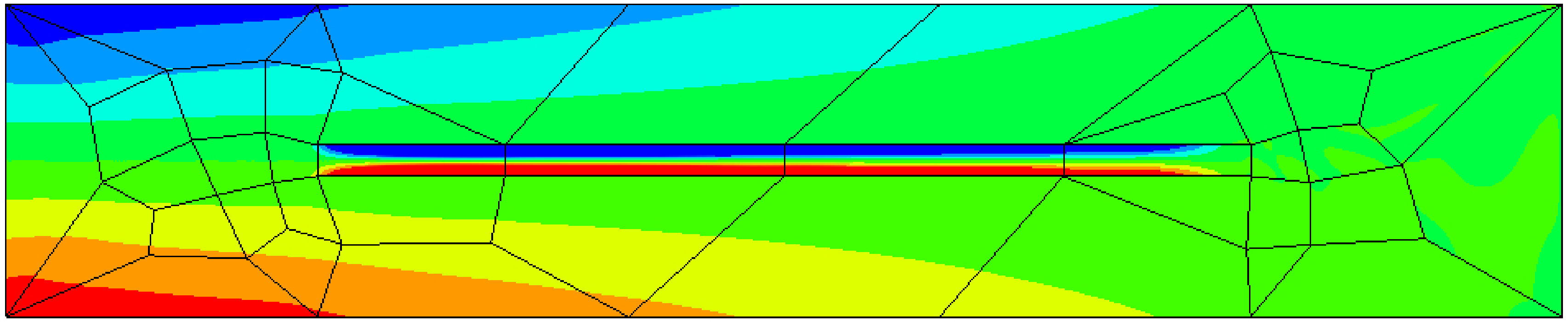
p	dofs	grads	$\kappa(C^{-1}A)$	iter	solvertime
4	96870	yes	34.31	37	24.9 s
4	57602	no	31.14	36	6.6 s
7	425976	yes	140.74	63	241.7 s
7	265221	no	72.63	51	87.6 s

Application: Simulation of eddy-currents in bus bars

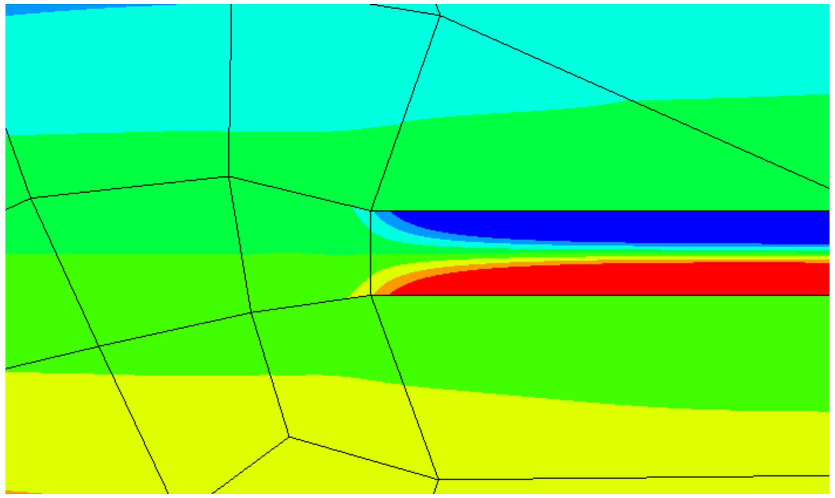


Full basis for $p = 3$ in conductor, reduced basis for $p = 3$ in air

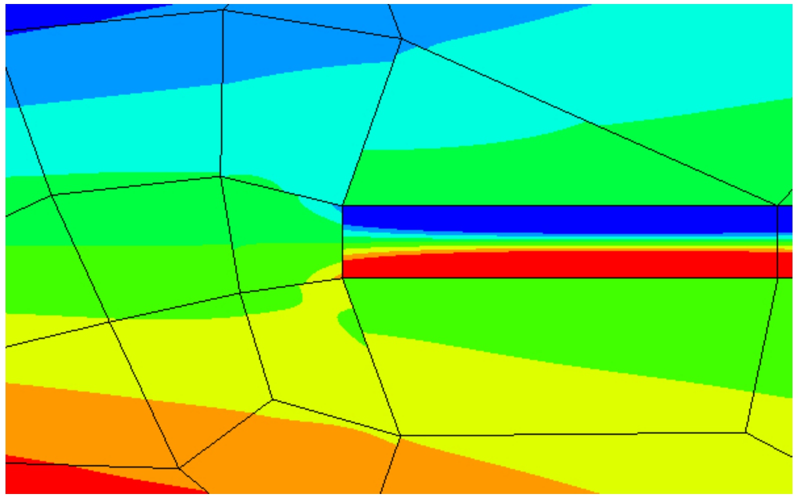
Elasticity: A beam in a beam



Reenforcement with $E = 50$ in medium with $E = 1$.



New mixed FEM, $p = 2$



Primal FEM, $p = 3$

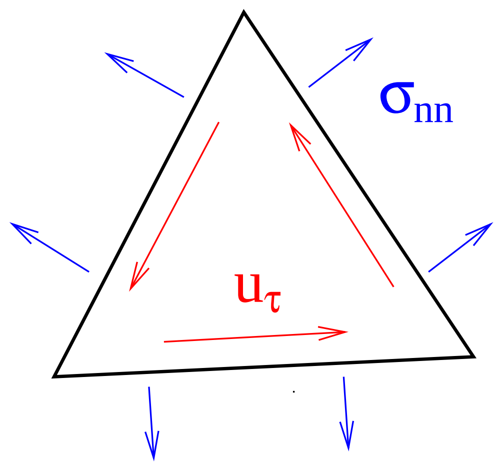
Tangential displacement - normal normal stress constinuous mixed method

[Phd thesis A. Sinwel 09]

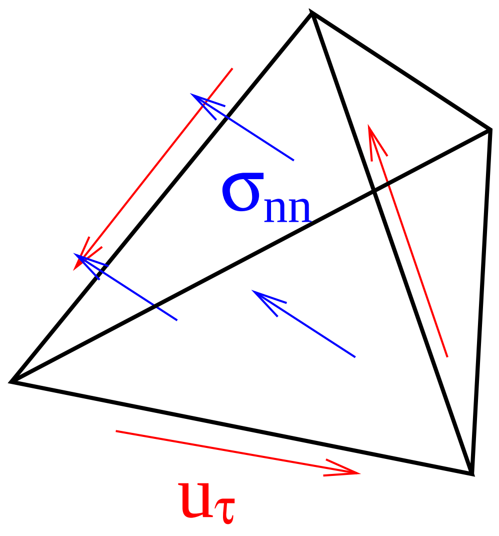
Mixed elements for approximating displacements and stresses.

- tangential components of displacement vector
- normal-normal component of stress tensor

Triangular Finite Element:

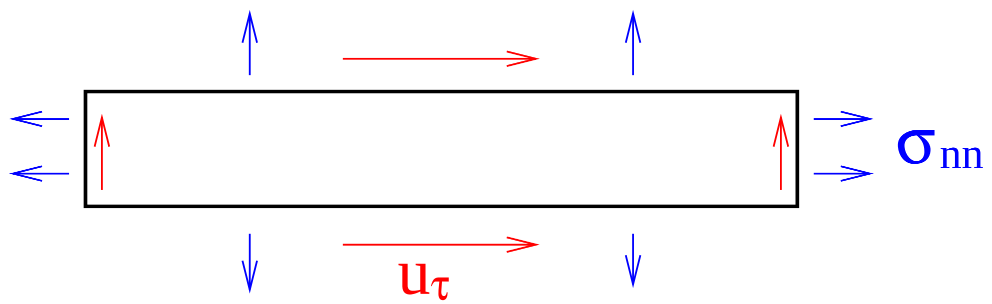


Tetrahedral Finite Element:



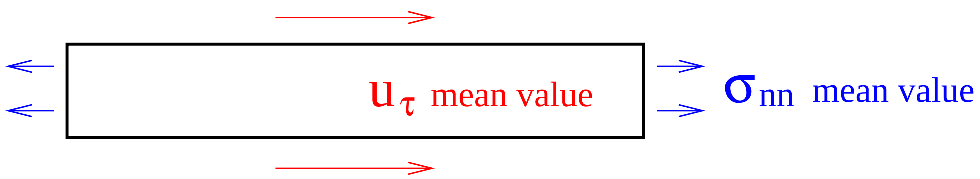
The quadrilateral element

Dofs for general quadrilateral element:

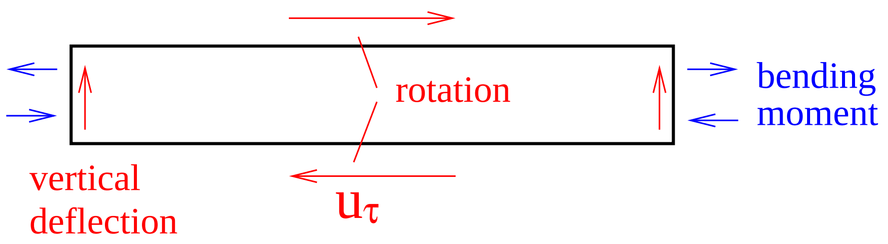


Thin beam dofs ($\sigma_{nn} = 0$ on bottom and top):

Beam stretching components:



Beam bending components:



Hellinger Reissner mixed methods for elasticity

Primal mixed method:

Find $\sigma \in L_2^{sym}$ and $u \in [H^1]^2$ such that

$$\begin{array}{rclll} \int A\sigma : \tau & - & \int \tau : \varepsilon(u) & = & 0 & \forall\, \tau \\ -\int \sigma : \varepsilon(v) & & & = & -\int f \cdot v & \forall\, v \end{array}$$

Dual mixed method:

Find $\sigma \in H(\mathrm{div})^{sym}$ and $u \in [L_2]^2$ such that

$$\begin{array}{rclll} \int A\sigma : \tau & + & \int \operatorname{div} \tau \cdot u & = & 0 & \forall\, \tau \\ \int \operatorname{div} \sigma \cdot v & & & = & -\int f \cdot v & \forall\, v \end{array}$$

[Arnold+Falk+Winther]

Reduced Symmetry mixed methods

Decompose

$$\varepsilon(u) = \nabla u + \tfrac{1}{2}\operatorname{Curl} u = \nabla u + \omega$$

with $\operatorname{Curl} u = 2\operatorname{skew}(\nabla u) = \big(\partial_{x_i} u_j - \partial_{x_j} u_i\big)_{i,j=1,\dots d}$

Impose symmetry of the stress tensor by an additional Lagrange parameter:

Find $\sigma \in [H(\operatorname{div})]^d$, $u \in [L_2]^d$, and $\omega \in L_2^{d\times d, skew}$ such that

$$\begin{array}{lclcll}
\int A\sigma : \tau & + & \int u \operatorname{div} \tau + \int \tau : \omega & = & 0 & \forall\, \tau \\
\int v \operatorname{div} \sigma & & & = & -\int f v & \forall\, v \\
\int \sigma : \gamma & & & = & 0 & \forall\, \gamma
\end{array}$$

The solution satisfies $u \in L_2$ and $\omega = \operatorname{Curl} u \in L_2^{d\times d, skew}$, i.e.,

$$u \in H(\operatorname{curl})$$

Arnold+Brezzi, Stenberg,... 80s

Choices of spaces

$\int \operatorname{div} \sigma \cdot u$ understood as

$\langle \operatorname{div} \sigma, u \rangle_{H^{-1} \times H^1} = -(\varepsilon(u), \sigma)_{L_2}$

$(\operatorname{div} \sigma, u)_{L_2}$

Displacement

$u \in [H^1]^2$
continuous f.e.

$u \in [L_2]^2$
non-continuous f.e.

Stress

$\sigma \in L_2^{sym}$
non-continuous f.e.

$\sigma \in H(\operatorname{div})^{sym}$
normal continuous (σ_n) f.e.

The mixed system is well posed for all of these pairs.

Choices of spaces

$\int \operatorname{div} \sigma \cdot u$ understood as

$\langle \operatorname{div} \sigma, u \rangle_{H^{-1} \times H^1} = -(\varepsilon(u), \sigma)_{L_2}$	$\langle \operatorname{div} \sigma, u \rangle_{H(\mathrm{curl})^* \times H(\mathrm{curl})}$	$(\operatorname{div} \sigma, u)_{L_2}$

Displacement

$u \in [H^1]^2$ continuous f.e.	$u \in H(\mathrm{curl})$ tangential-continuous f.e.	$u \in [L_2]^2$ non-continuous f.e.

Stress

$\sigma \in L_2^{sym}$ non-continuous f.e.	$\sigma \in L_2^{sym}, \operatorname{div} \operatorname{div} \sigma \in H^{-1}$ normal-normal continuous (σ_{nn}) f.e.	$\sigma \in H(\operatorname{div})^{sym}$ normal continuous (σ_n) f.e.

The mixed system is well posed for all of these pairs.

The space $H(\operatorname{div}\operatorname{div})$

The dual space of $H(\mathrm{curl})$ is $H^{-1}(\mathrm{div})$:

$$\begin{aligned}\|f\|_{H(\mathrm{curl})^*} &= \sup_{v\in H(\mathrm{curl})}\frac{\langle f,v\rangle}{\|v\|_{H(\mathrm{curl})}} \simeq \sup_{\varphi\in H^1, z\in[H^1]^2}\frac{\langle f,\nabla\varphi+z\rangle}{\|\varphi\|_{H^1}+\|z\|_{H^1}} \simeq \|\operatorname{div} f\|_{H^{-1}}+\|f\|_{H^{-1}}\\ &\simeq H^{-1}(\mathrm{div})\end{aligned}$$

We search for $\sigma\in L_2^{sym}$ and $\operatorname{div}\sigma\in H^{-1}(\mathrm{div})$. This is equivalent to

$$\sigma\in H(\operatorname{div}\operatorname{div}) := \{\sigma\in L_2^{sym} : \operatorname{div}\operatorname{div}\sigma\in H^{-1}\},$$

where

$$\operatorname{div}\operatorname{div}\begin{pmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{pmatrix} = \operatorname{div}\begin{pmatrix}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}\\ \frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}\end{pmatrix} = \sum_{ij}\frac{\partial^2\sigma_{ij}}{\partial x_i\partial x_j}\in H^{-1}$$

This implies continuity and LBB for $\langle\operatorname{div}\sigma,u\rangle$.

Continuity properties of the space $H(\operatorname{div}\operatorname{div})$

Lemma: Let σ be a piece-wise smooth tensor field on the mesh $\mathcal{T} = \{T\}$ such that $\sigma_{nt} \in H^{1/2}(\partial T)$. Assume that $\sigma_{nn} = n^T \sigma n$ is continuous across element interfaces. Then there holds $\operatorname{div}\sigma \in H(\operatorname{curl})^*$.

The 3-step 'exact sequence'

$$H^1 \xrightarrow{\nabla} H(\mathrm{curl}) \xrightarrow{\sigma(\cdot)} H(\mathrm{div}\,\mathrm{div}) \xrightarrow{\mathrm{div}} H^{-1}(\mathrm{div}) \xrightarrow{\mathrm{div}} H^{-1}$$

with the stress operator

$$\sigma(v) = \begin{pmatrix} \frac{\partial v_y}{\partial y} & -\frac{1}{2}\left\{\frac{\partial v_y}{\partial x} + \frac{\partial v_y}{\partial x}\right\} \\ sym & \frac{\partial v_x}{\partial x} \end{pmatrix}.$$

The composite operators are

$$\begin{aligned} \mathrm{airy}(w) = \sigma(\nabla w) &= \begin{pmatrix} \frac{\partial^2 w}{\partial y^2} & -\frac{\partial w}{\partial x \partial y} \\ sym & \frac{\partial w}{\partial x^2} \end{pmatrix} \\ \mathrm{div}\,\sigma(v) &= \frac{1}{2}\mathrm{Curl}\,\mathrm{curl}\,v \end{aligned}$$

There holds

$$\begin{aligned} \mathrm{range}(\sigma(\nabla\cdot)) &= \ker(\mathrm{div}) \\ \mathrm{range}(\mathrm{div}\,\sigma(\cdot) &= \ker(\mathrm{div}) \end{aligned}$$

related to the Elasticiy complex by Arnold-Falk-Winther

Finite elements for $H(\operatorname{div}\operatorname{div})$

Start with C^0-continuous finite elements for $H^1 \cap H^2(\mathcal{T})$

Finite elements for $H(\operatorname{div}\operatorname{div})$ can be built with

edge-based basis functions: $\sigma(\nabla \varphi^E)$

cell basis functions: Sym$[\nabla \lambda_\alpha^\perp \otimes \nabla \lambda_\beta^\perp]\, \lambda_\gamma P^{k-1}$

Potential to save dofs for subdomains with $\operatorname{div} \sigma = 0$.

Finite Element Analysis

Analysis in discrete norms:

$$\begin{aligned}
\|v\|_{V_h}^2 &= \sum_T \|\varepsilon(v)\|_T^2 + \sum_E h^{-1} \|[v_n]\|_{L_2(E)}^2 \\
\|\tau\|_{\Sigma_h}^2 &= \|\tau\|_{L_2}^2 + \sum_E h\, \|\tau_{nn}\|_{L_2(E)}^2.
\end{aligned}$$

Continuous and inf-sup stable. By saddle-point theory:

$$\|u - u_h\|_{V_h} + \|\sigma - \sigma_h\|_{\Sigma_h} \leq c h^m \|\varepsilon(u)\|_{H^m}$$

for $m \leq k$.

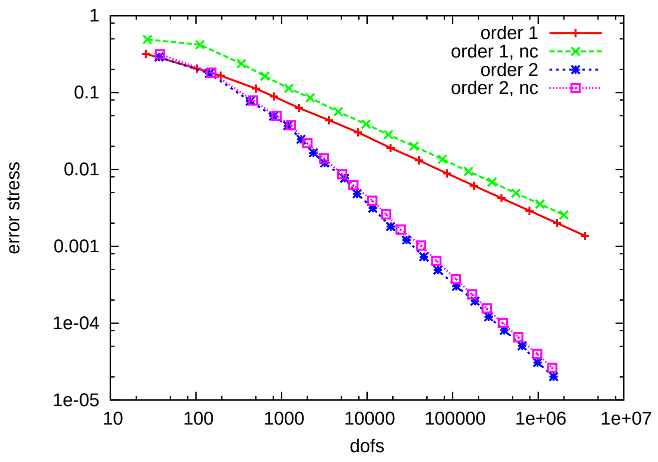
By adding a local stabilization term, the method is robust for $\nu \to \frac{1}{2}$.

Unit square, left side fixed, vertical load, adaptive refinement

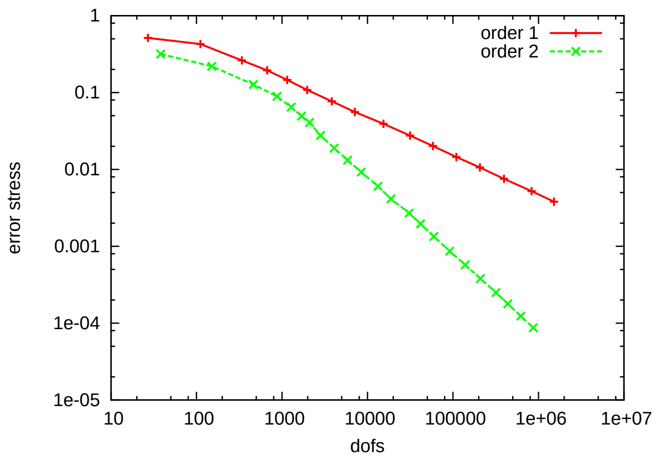
$\sigma \in P^1$
2 dof σ_{nn} per edge

	conforming	non-conforming
$u \in P^1$	2 dof u_τ	1 dof u_τ
$u \in P^2$	3 dof u_τ	2 dof u_τ

$\nu = 0.3$:

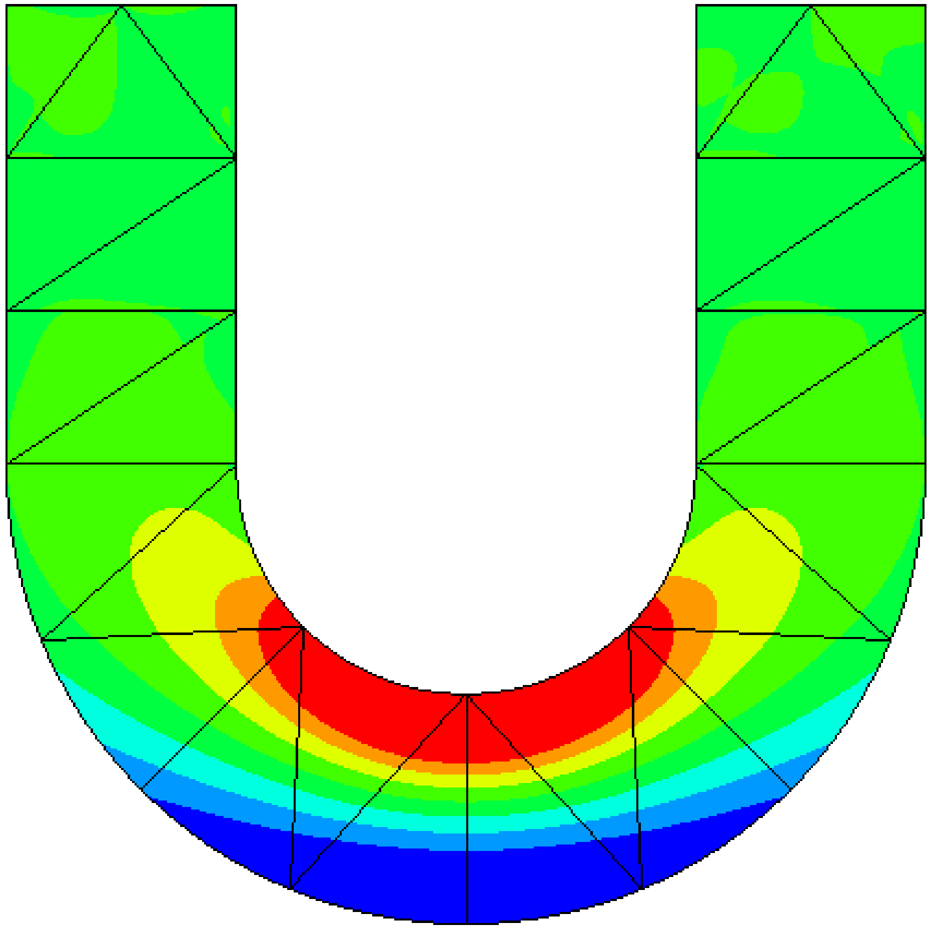


$\nu = 0.4999$:



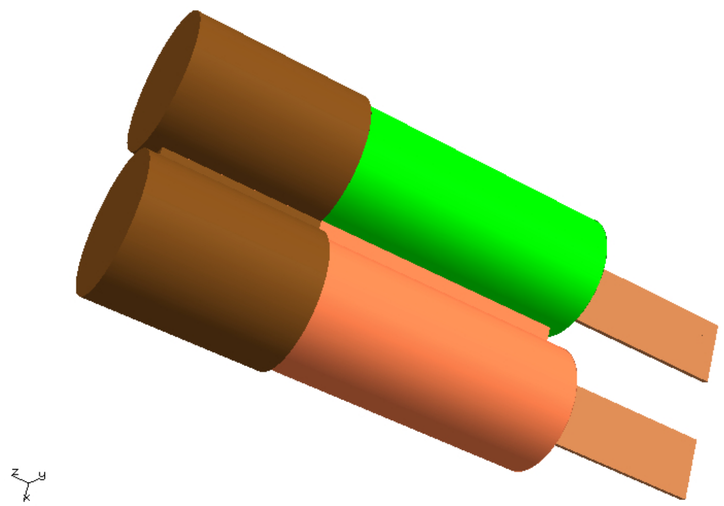
Curved elements

fixed left top, pull right top
Elements of order 5

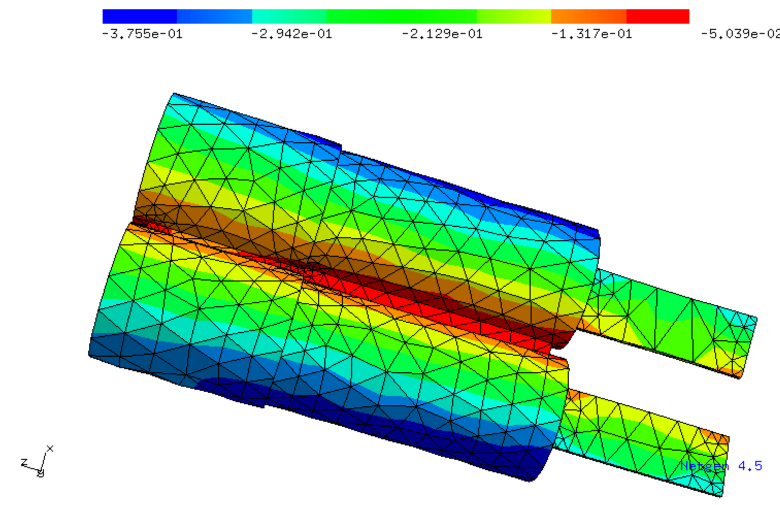


σ_{xx}

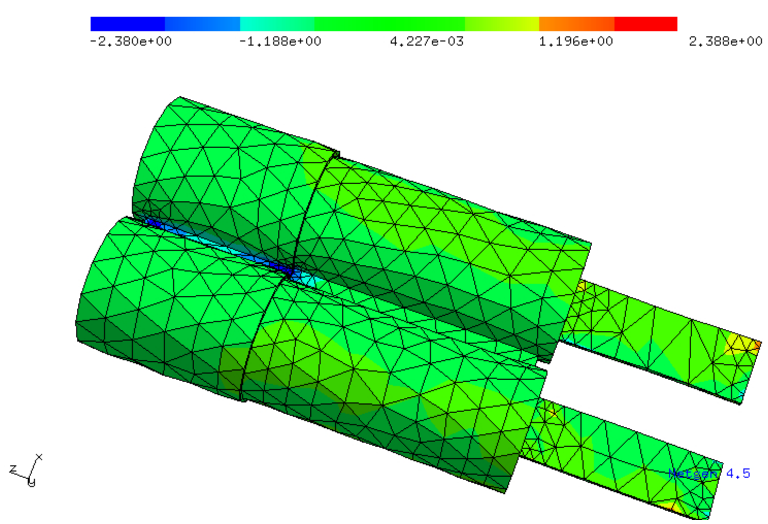
For Hot Days ...



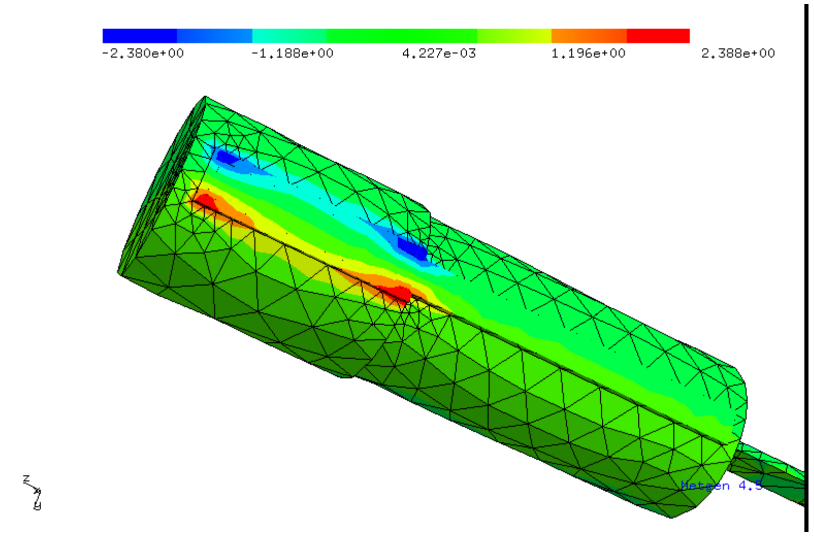
Geometry



Displacement u_y

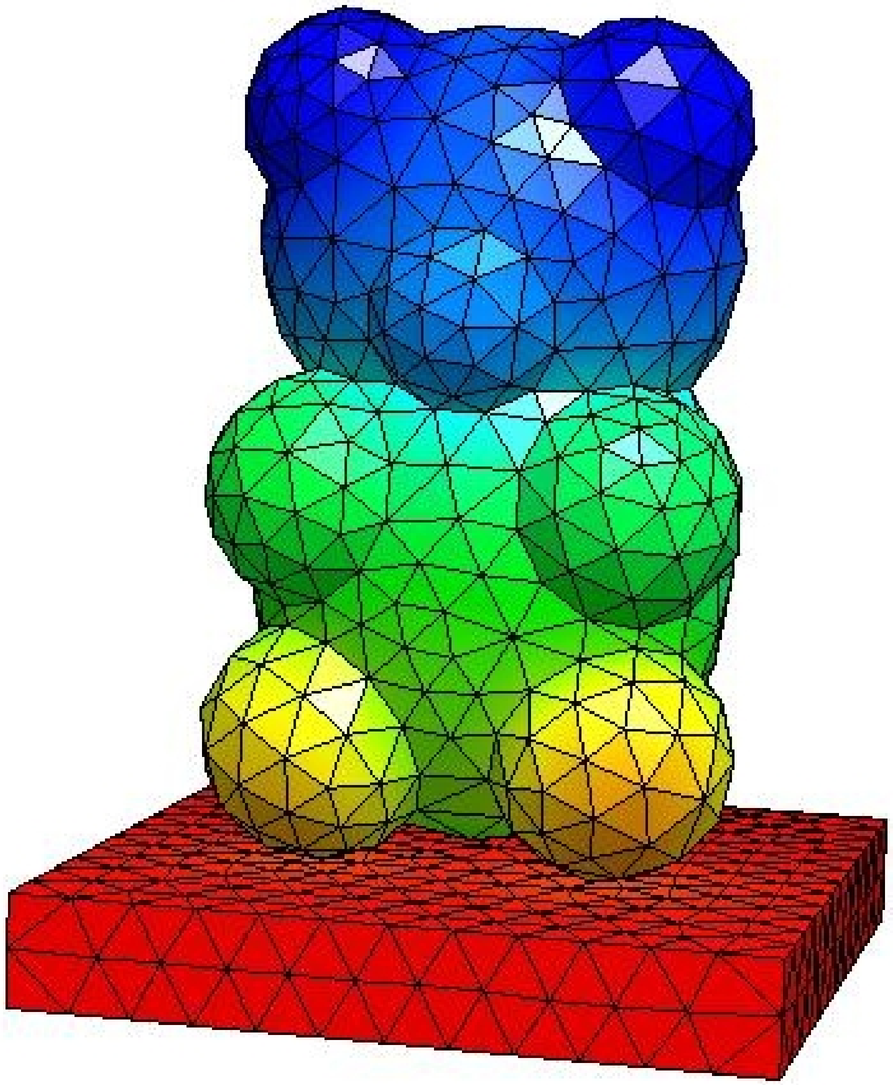


Deformed geometry, stress σ_{xx}

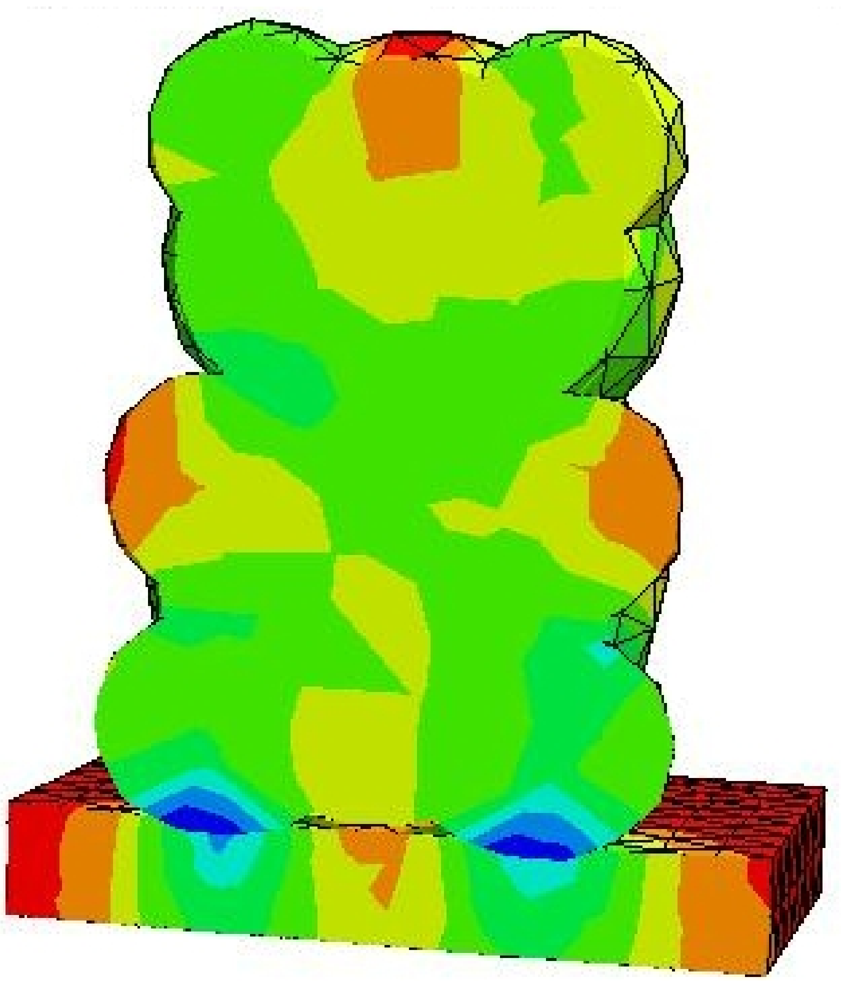


Interior stress

Contact problem with friction



Undeformed bear



Stress, component σ_{33}

Reissner Mindlin Plates

Energy functional for vertical displacement w and rotations β:

$$\|\varepsilon(\beta)\|_{A^{-1}}^2 + t^{-2}\|\nabla w - \beta\|^2$$

MITC elements with Nédélec reduction operator:

$$\|\varepsilon(\beta)\|_{A^{-1}}^2 + t^{-2}\|\nabla w - R_h\beta\|^2$$

Mixed method with $\sigma = A^{-1}\varepsilon(\beta) \in H(\operatorname{div}\operatorname{div})$, $\beta \in H(\operatorname{curl})$, and $w \in H^1$:

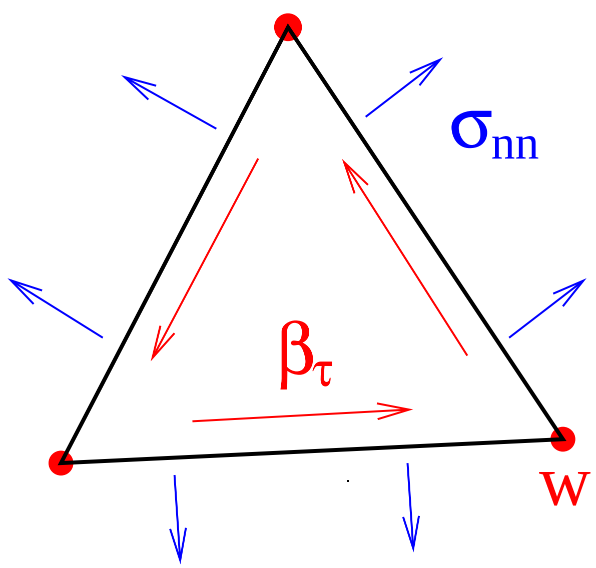
$$L(\sigma; \beta, w) = \|\sigma\|_A^2 + \langle \operatorname{div}\sigma, \beta\rangle - t^{-2}\|\nabla w - \beta\|^2$$

Reissner Mindlin Plates and Thin 3D Elements

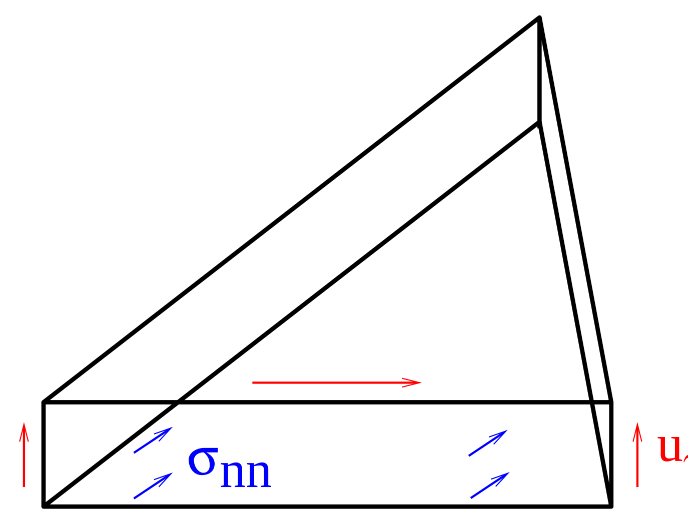
Mixed method with $\sigma = A^{-1}\varepsilon(\beta) \in H(\operatorname{div}\operatorname{div})$, $\beta \in H(\operatorname{curl})$, and $w \in H^1$:

$$L(\sigma;\beta,w) = \|\sigma\|_A^2 + \langle \operatorname{div}\sigma, \beta\rangle - t^{-2}\|\nabla w - \beta\|^2$$

Reissner Mindlin element:



3D prism element:



Tensor-product Finite Elements

Thin domain: $\omega \subset \mathbb{R}^2$, $I = (-t/2, t/2)$, $\Omega = \omega \times I$. FE-space for displacement:

$$\begin{aligned} \mathcal{L}^{xy}_{k+1} &= \{v \in H^1(\omega) : v|_T \in P^{k+1}\} \qquad & \mathcal{L}^{z}_{k+1} &= \{v \in H^1(I) : v|_T \in P^{k+1}\} \\ \mathcal{N}^{xy}_{k} &= \{v \in H(\mathrm{curl}, \omega) : v|_T \in P^{k}\} \qquad & \mathcal{N}^{z}_{k} &= \{v \in L_2(I) : v|_T \in P^{k}\} \end{aligned}$$

Tensor-product Nédélec space:

$$V_k = \underbrace{\mathcal{N}^{xy}_k \otimes \mathcal{L}^z_{k+1}}_{u_{xy}} \times \underbrace{\mathcal{L}^{xy}_{k+1} \otimes \mathcal{N}^z_k}_{u_z}$$

Regularity-free quasi-interpolation operators (Clement) which commute (JS 2001):

$$I^{xy}_{k+1} : L_2(\omega) \to \mathcal{L}^{xy}_{k+1}, \qquad Q^{xy}_k : L_2(\omega) \to \mathcal{N}^{xy}_k : \qquad \nabla I^{xy}_{k+1} = Q^{xy}_k \nabla$$

Tensor product interpolation operator:

$$Q_k = \underbrace{Q^{xy}_k \otimes I^z_{k+1}}_{u_{xy}} \times \underbrace{I^{xy}_{k+1} \otimes Q^z_k}_{u_z}$$

Anisotropic Estimates

Thm: There holds

$$\sum_T \|\varepsilon(u-u_h)\|_T^2 + \sum_F h_{op}^{-1} \|[u_n]\|_F^2 + \|\sigma - \sigma_h\|^2 \leq c \left\{ h_{xy}^m \|\nabla_{xy}^m \varepsilon(u)\| + h_z^m \|\nabla_z^m \varepsilon(u)\| \right\}^2$$

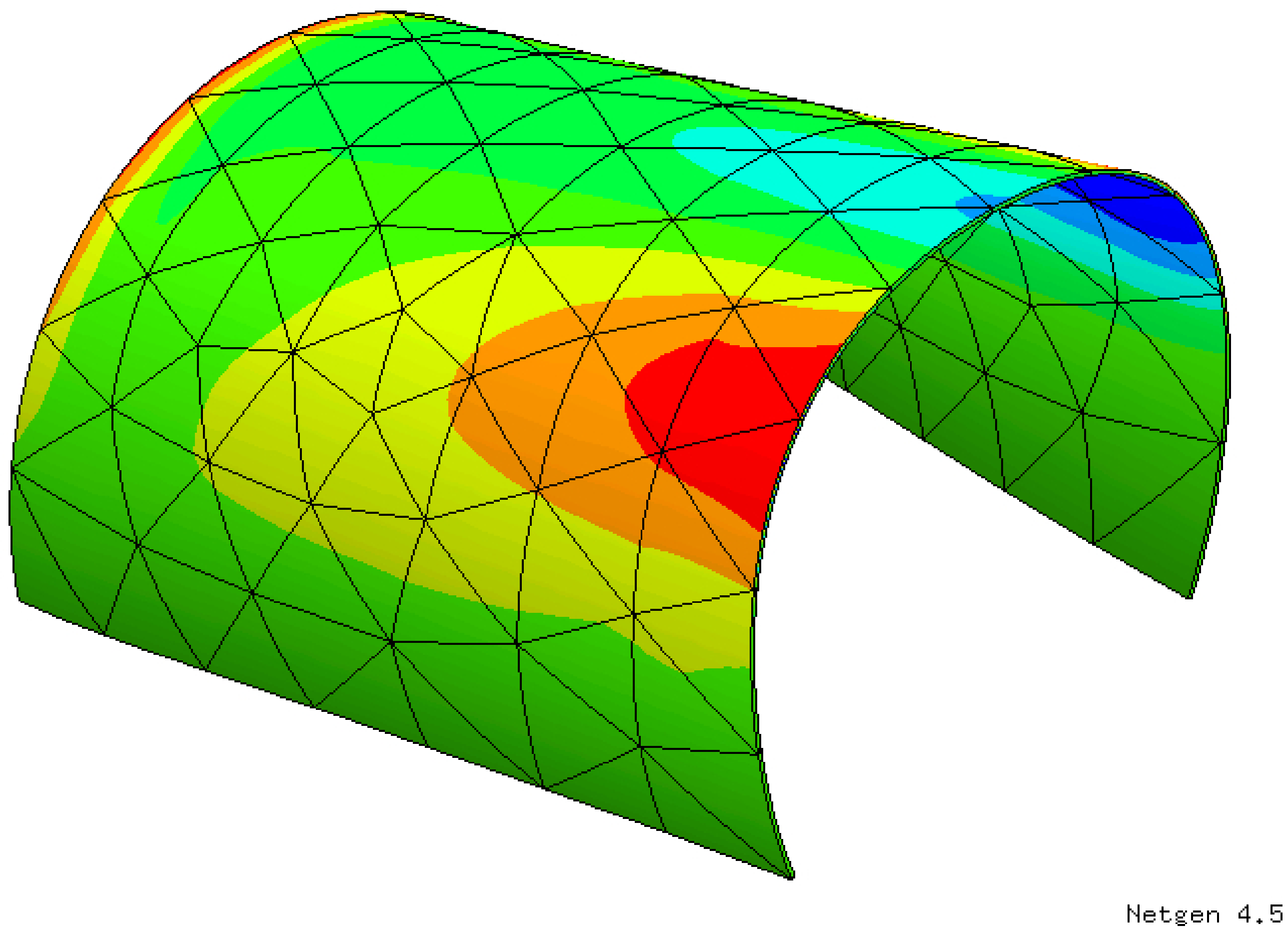
Proof: Stability constants are robust in aspect ratio (for tensor product elements)

Anisotropic interpolation estimates (H^1: Apel). E.g., the shear strain components

$$\begin{aligned}
2\|\varepsilon_{xy,z}(u - Q_k u)\|_{L_2} &= \|\nabla_z(u_{xy} - I^z \otimes Q^{xy} u_{xy}) + \nabla_{xy}(u_z - I^{xy} \otimes Q^z u_z)\|_{L_2} \\
&= \|(I - Q^{xy} \otimes Q^z)\big(\nabla_z u_{xy} + \nabla_{xy} u_z\big)\|_{L_2} \\
&\preceq h_{xy}^m \|\nabla_x^m \varepsilon_{xy,z}(u)\|_0 + h_z^m \|\nabla_z^m \varepsilon_{xy,z}(u)\|_{L_2}
\end{aligned}$$

Shell structure

R = 0.5, t = 0.005
$\sigma \in P^2$, $u \in P^3$



stress component σ_{yy}

Hybridization: Implementation aspects

Both methods are equivalent (for affine element transformations):

- Classical hybridization of mixed method:

  Introduce Lagrange parameter λ_n to enforce continuity of σ_{nn}. Its meaning is the displacement in normal direction.

- Continuous / hybrid discontinuous Galerkin method:

  Displacement u is strictly tangential continuous, HDG facet variable ($=$ normal displacement) enforces weak continuity of normal component.

Anisotropic error estimates from mixed methods can be applied for HDG method !

p-robust anisotropic error estimates

... are on the way:

Key ingredients: Commuting quasi-interpolation operators in 1D:

$$(I_N w)' = \widetilde{I}_N w'$$

such that

$$\|I_N\|_{L_2} \leq c \qquad \|\widetilde{I}_N\|_{L_2} \leq c$$

Based on polynomial δ functions f_p such that

$$\int_{-1}^{1} f_p(x) q(x) = q(0) \qquad \forall\, q \in P^p$$

and

$$\|f_p\|_{L_1} \leq c \qquad \|f_p\|_{L_\infty} \leq cp$$

Computer algebra based construction and proofs by Veronika Pillwein.

Continuous / hybrid discontinuous Galerkin method for Stokes

(with Ch. Lehrenfeld, RWTH)

$H(\mathrm{div})$ - based formulation for Stokes:

Find $u \in V_{\mathcal{T},n} \subset H(\mathrm{div})$, $\lambda \in V_{\mathcal{F},\tau}$ and $p \in P^{p-1}(\mathcal{T})$ such that

$$
\begin{array}{rcll}
A^n(u,\lambda;v,\mu) \quad + \quad \int_\Omega \mathrm{div}\, v\, q & = & \int f v & \forall\, (v,\mu) \\
\int \mathrm{div}\, u\, q & = & 0 & \forall\, q
\end{array}
$$

Provides exactly divergence-free discrete velocity field u

LBB is proven by commuting interpolation operators for de Rham diagram

[Cockburn, Kanschat, Schötzau 2005]

$H(\mathrm{div})$-conforming elements for Navier Stokes

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \mathrm{div}(2\nu\varepsilon(u) - u \otimes u - pI) &= f \\
\mathrm{div}\ u &= 0 \\
+b.c.
\end{aligned}
$$

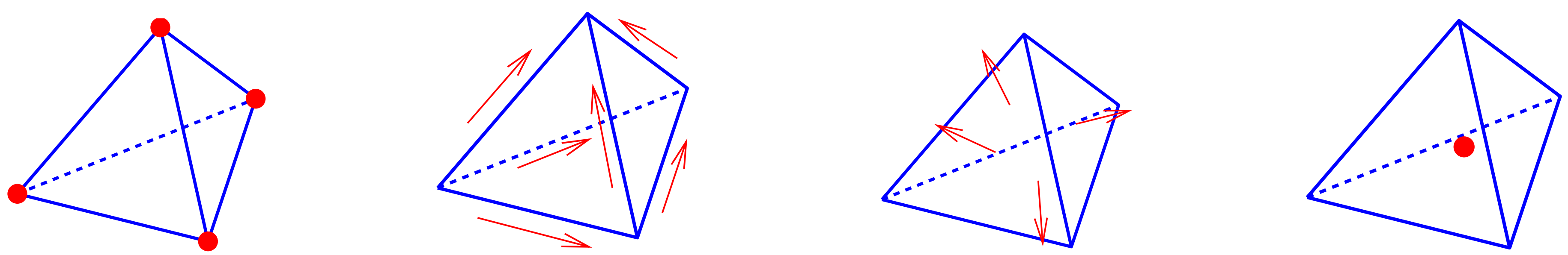
Fully discrete scheme, semi-implicit time stepping:

$$
\begin{aligned}
(\frac{1}{\tau}M + A^{\nu})\hat{u} + B^T\hat{p} &= f - \frac{1}{\tau}Mu - A^c(u) \\
B\hat{u} &= 0
\end{aligned}
$$

- u is exactly div-free $\Rightarrow$ non-negative convective term $\int u\nabla v v \geq 0$
- stability for kinetic energy ($\frac{d}{dt}\|u\|_0^2 \preceq \frac{1}{\nu}\|f\|_{L_2}^2$)
- convective term by upwinding
- allows kernel-preserving smoothing and grid-transfer for fast iterative solver

The de Rham Complex

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\nabla} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
W_h & \xrightarrow{\nabla} & V_h & \xrightarrow{\mathrm{curl}} & Q_h & \xrightarrow{\mathrm{div}} & S_h
\end{array}
$$



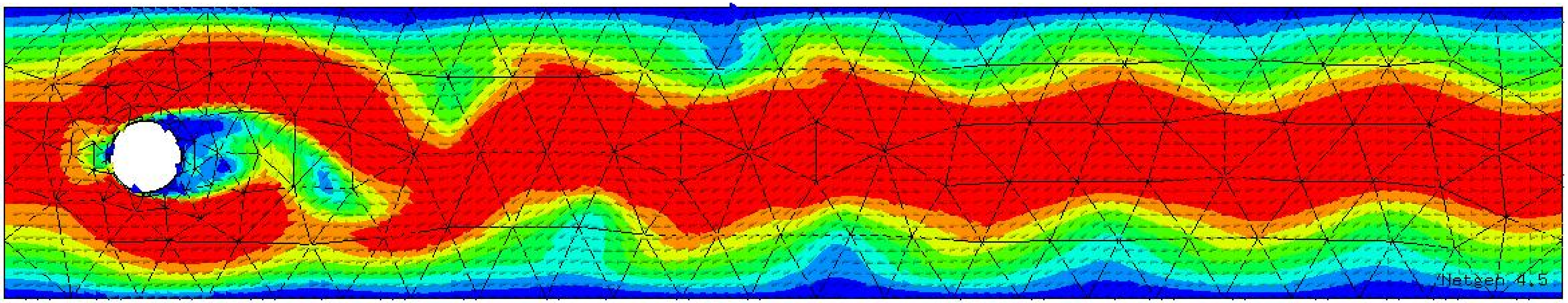
For constructing high order finite elements

$$
\begin{aligned}
W_{hp} &= W_{\mathcal{L}_1} + \mathrm{span}\{\varphi^W_{h.o.}\} \\
V_{hp} &= V_{\mathcal{N}_0} + \mathrm{span}\{\nabla\varphi^W_{h.o.}\} + \mathrm{span}\{\varphi^V_{h.o.}\} \\
Q_{hp} &= Q_{\mathcal{R}T_0} + \mathrm{span}\{\mathrm{curl}\,\varphi^V_{h.o.}\} + \mathrm{span}\{\varphi^Q_{h.o.}\} \\
S_{hp} &= S_{\mathcal{P}_0} + \mathrm{span}\{\mathrm{div}\,\varphi^S_{h.o.}\}
\end{aligned}
$$

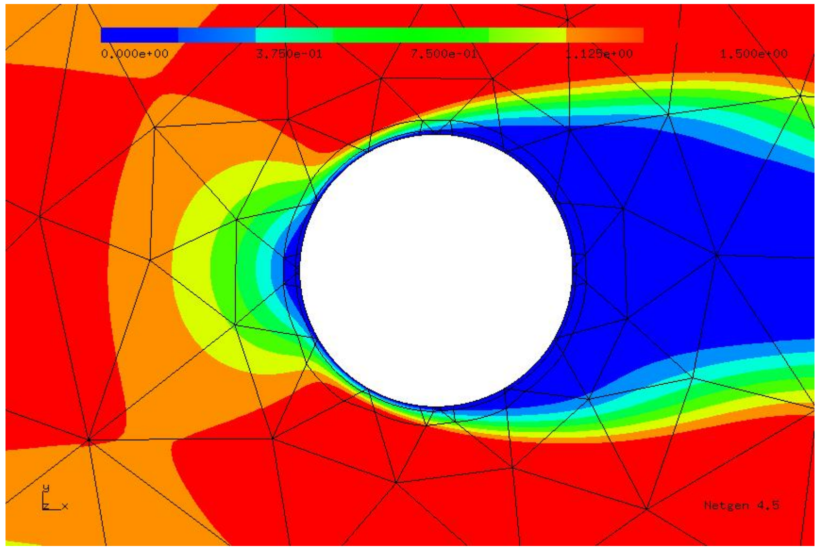
Allows to construct high-order-divergence free elements $\{v \in BDM_k : \mathrm{div}\, v \in P_0\}$

Flow around a disk, 2D

Re $= 100$, 5^{th}-order elements

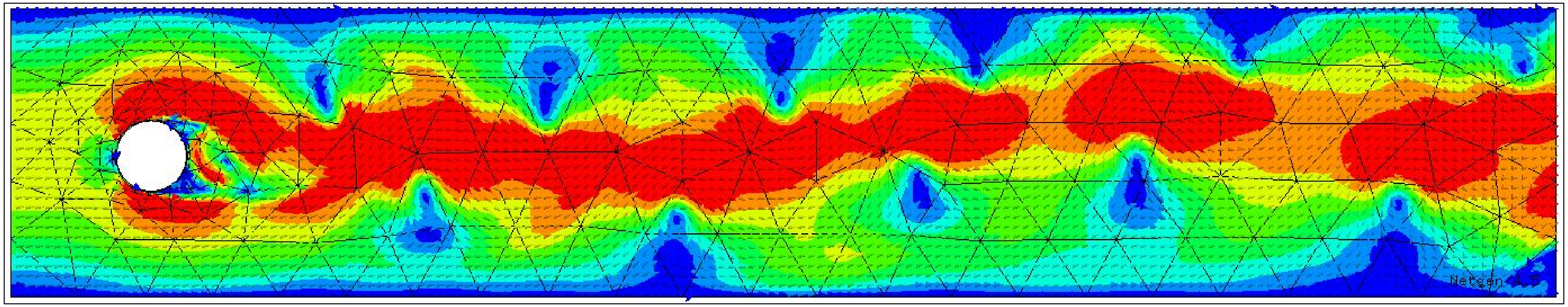


Boundary layer mesh around cylinder:

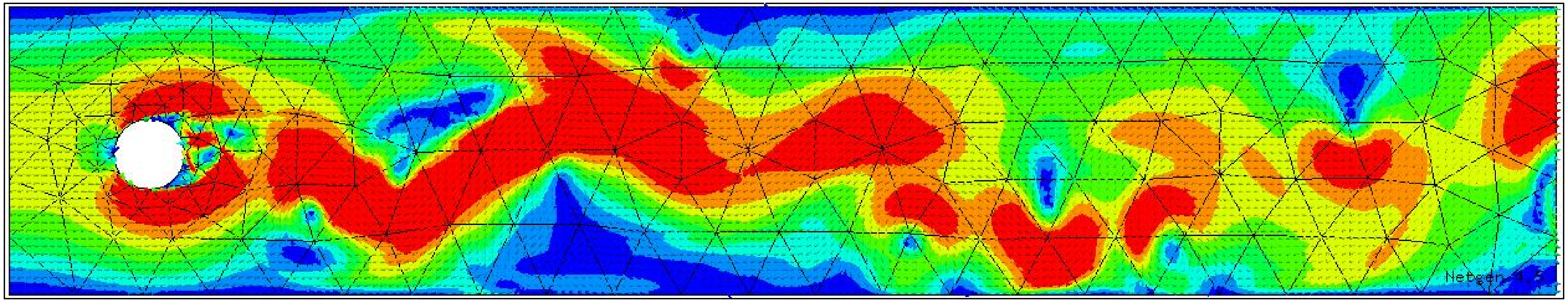


Flow around a disk, 2D

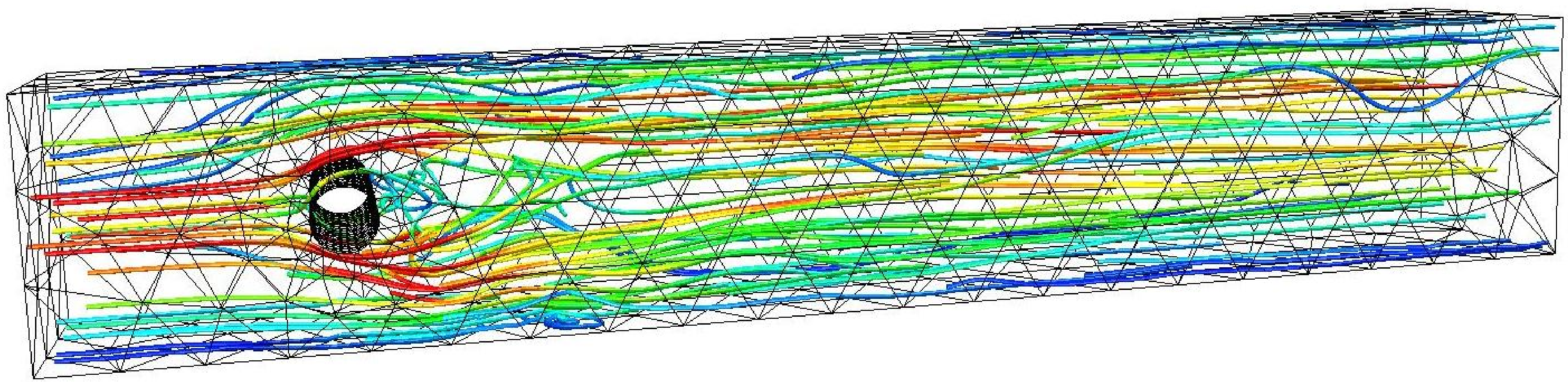
Re = 1000:



Re = 5000:



Flow around a cylinder, Re = 100



Low-order / high-order two-level preconditioning for augmented Lagrangian:

Order	N	$\kappa(C^{-1}A)$	its (1E-8)	N	$\kappa(C^{-1}A)$	its (1E-8)
1	3046	6.2	11	28978	20.6	17
2	9369	21.1	19	92781	45.9	25
3	20052	31.8	22	202080	60.3	29
4	35965	33.9	22	-		

Concluding Remarks

- Hybrid DG is a simple and efficient hp - discretization scheme
- Use of tangential continuous / normal continuous vectorial finite elements
- Robust anisotropic elements for linear elasticity
- Exactly divergence free finite elements for incompressible flows

Ongoing work:

- Geometric non-linear elasticity
- Compressible flows, turbulence models (vms)