

Nitsche-type Mortaring for Maxwell's Equations

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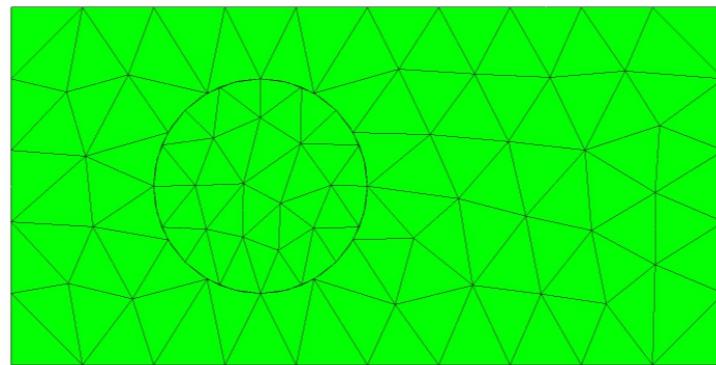


CME

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Problem setup

Domain decomposition on non-matching meshes:



Contents:

- Method for scalar model problem
- Nitsche-method for Maxwell's equation
- Numerical results

A model problem

Poisson equation:

$$\begin{aligned}-\Delta u &= f && \text{on } \Omega_1 \cup \Omega_2 \\ u &= 0 && \text{at } \partial(\overline{\Omega_1 \cup \Omega_2})\end{aligned}$$

Interface conditions on Γ :

$$[u] := u_1 - u_2 = 0$$

$$\partial_{n_1} u_1 + \partial_{n_2} u_2 = 0$$

Mortar method

Pose the constraint as additional equation.

Find $u \in H_{0,D}^1(\Omega_1) \times H_{0,D}^1(\Omega_2)$ and $\lambda \in H^{-1/2}(\Gamma)$ such that

$$\begin{aligned}\int_{\Omega_1 \cup \Omega_2} \nabla u \cdot \nabla v + \int_{\Gamma} [v]\lambda &= \int f v & \forall v \\ \int_{\Gamma} [u]\mu &= 0 & \forall \mu\end{aligned}$$

The Lagrange parameter λ is the normal flux $\partial_n u$.

Requires stability condition for finite element spaces (LBB).

Leads to an indefinite system matrix.

Nitsche / Discontinuous Galerkin method

Allows discontinuous approximation by keeping extra boundary terms:

Find $u \in H_{0,D}^1(\Omega_1) \times H_{0,D}^1(\Omega_1)$ such that

$$\int_{\Omega_1 \cup \Omega_2} \nabla u \cdot \nabla v - \int_{\Gamma} \partial_n u [v] - \int_{\Gamma} \partial_n v [u] + \alpha \int_{\Gamma} [u] [v] = \int f v \quad \forall v$$

with soft penalty term $\alpha \sim p^2/h$ sufficiently large.

No extra stability condition is required.

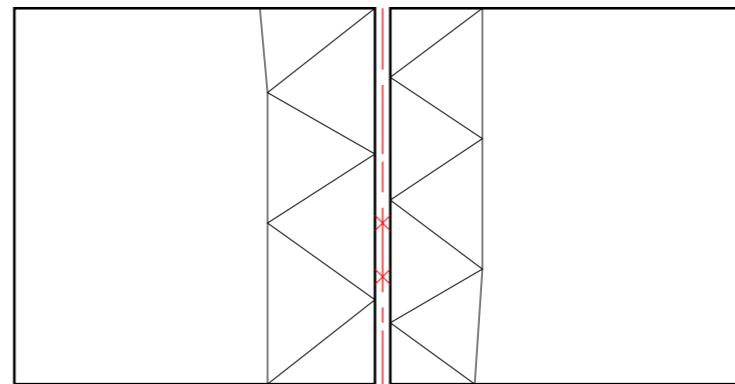
Leads to a symmetric positive definite stiffness matrix.

Integration of boundary terms

Both methods require to compute integrals of finite element functions from different meshes:

Mortar method: $\int_{\Gamma} u_2 \mu$

Nitsche method: $\int_{\Gamma} v_1 \partial_n u_2$



Requires the calculation of an intersection mesh

Complicated implementation, in particular on curved interfaces in 3D

Hybrid Nitsche method - derivation

Introduce a new variable for the primal unknown on the interface:

$$\lambda := u|_{\Gamma}$$

Multiply by test-functions, and integrate by parts:

$$\sum_i \int_{\Omega_i} \nabla u \cdot \nabla v - \int_{\Gamma} \partial_n u v = \int f v \quad \forall v \in H^1(\Omega_1) \times H^1(\Omega_2)$$

Use continuity of $\partial_n u$ and introduce single-valued test function μ on interface:

$$\sum_i \int \nabla u \cdot \nabla v - \int_{\Gamma} \partial_n u (v - \mu) = \int f v \quad \forall v \forall \mu$$

Use $u = \lambda$ on Γ to symmetrize and stabilize with $\alpha \sim p^2/h$.

$$\sum_i \int \nabla u \cdot \nabla v - \int_{\Gamma} \partial_n u (v - \mu) - \int_{\Gamma} \partial_n v (u - \lambda) + \alpha \int_{\Gamma} (u - \lambda)(v - \mu) = \int f v \quad \forall v \forall \mu$$

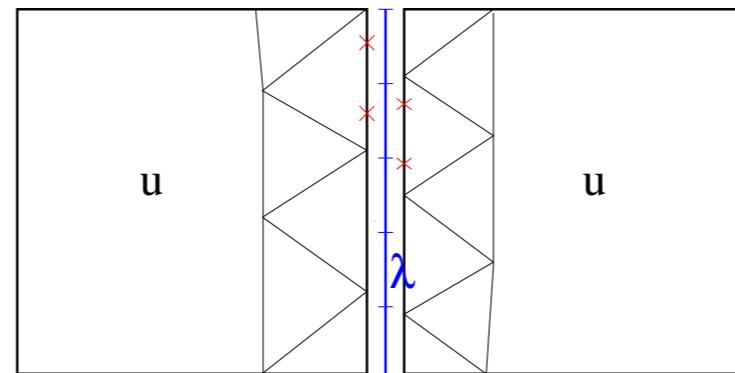
For α chosen right, the discrete formulation is stable independent of the choice of fe spaces for u and λ .

Discretizing and numerical integration

In general, numerical integration is still difficult.

We propose to use smooth B-spline functions for discretizing the hybrid variable λ . This allows

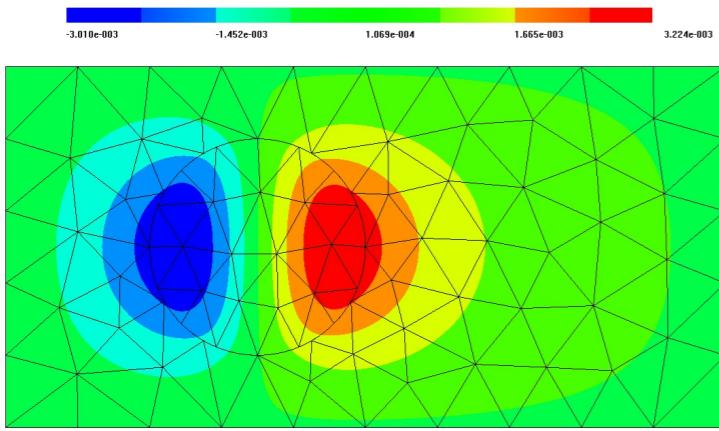
- evaluation in global coordinates
- efficient numerical integration by Gauss-rules on the surface elements



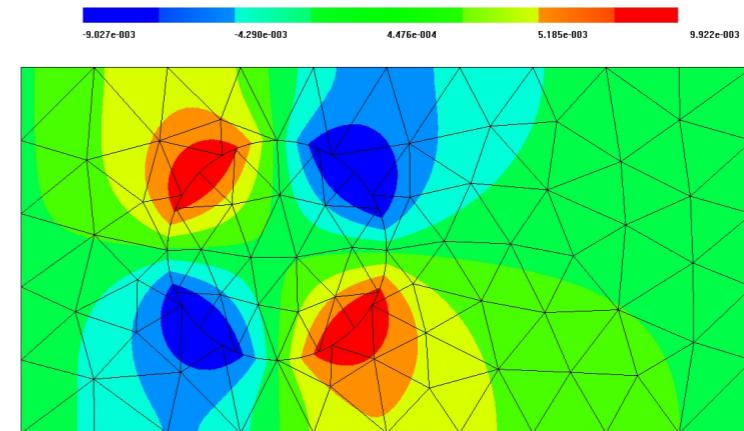
Numerical experiments in 2D

$f = x$ in circle, else $f = 0$.

Solution u :



Solution $\partial u / \partial x$:

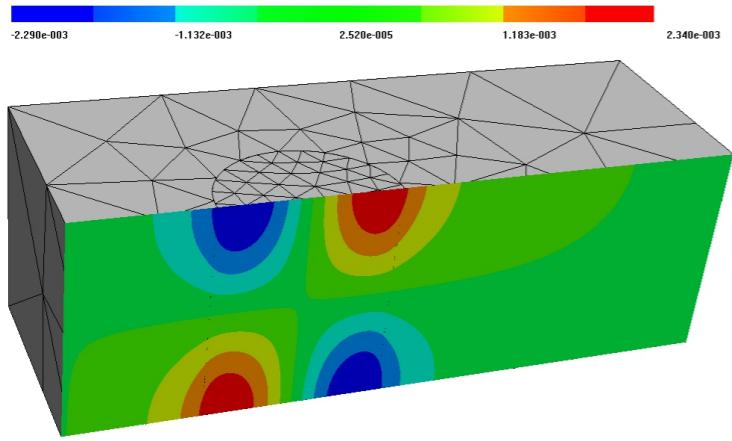


Finite element order $p = 5$.

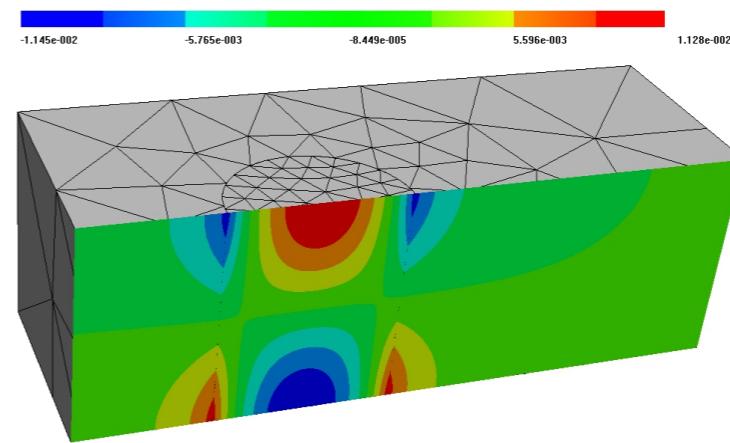
Numerical experiments in 3D

$f = xz$ in cylinder, else $f = 0$.

Solution u :



Solution $\partial u / \partial x$:



Finite element order $p = 4$.

Maxwell's Equations

Time harmonic Maxwell's equations

$$\operatorname{curl} \mu^{-1} \operatorname{curl} u + \kappa u = j \quad \text{in } \Omega_i$$

with $\kappa = i\omega\sigma - \omega^2\epsilon$, and

$$E = -i\omega u, \quad H = \mu^{-1} \operatorname{curl} u.$$

Transmission conditions

$$u_1 \times n_1 + u_2 \times n_2 = 0,$$

$$\mu_1^{-1} \operatorname{curl} u_1 \times n_1 + \mu_2^{-1} \operatorname{curl} u_2 \times n_2 = 0.$$

Hybrid Nitsche formulation

proceed as in the scalar case:

$$\int_{\Omega_i} \{\mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} v + \kappa u \cdot v\} + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u \cdot (v \times n) = \int_{\Omega_i} j \cdot v$$

add symmetry and penalty terms: find (u, λ) such that

$$\sum_{i=1}^2 \left\{ \int_{\Omega_i} \mu^{-1} \{\operatorname{curl} u \cdot \operatorname{curl} v + \kappa u \cdot v\} + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u \cdot [(v - \mu) \times n] \right. \\ \left. + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} v \cdot [(u - \lambda) \times n] + \frac{\alpha p^2}{\mu h} \int_{\partial\Omega_i} [(u - \lambda) \times n] \cdot [(v - \mu) \times n] \right\} = \int_{\Omega} j \cdot v,$$

where $u, v \in H(\operatorname{curl}, \Omega_1) \times H(\operatorname{curl}, \Omega_2)$, and λ, μ are tangential vector valued fields on the interface.

Overpenalization of gradient fields

The natural energy norm is

$$\|u\|^2 = \mu^{-1} \|\operatorname{curl} u\|_{L_2}^2 + |\kappa| \|u\|_{L_2}^2$$

For gradient fields $u = \nabla\phi$, this norms scales as

$$\|\nabla\phi\|^2 = O(\kappa)$$

This is small for small frequencies/conductivities.

The norm for the Nitsche method is

$$|(u, \lambda)|^2 = \sum_{i=1}^2 \left\{ \mu^{-1} \|\operatorname{curl} u\|_{\Omega_i}^2 + \kappa \|u\|_{\Omega_i}^2 + \alpha \mu^{-1} \|(u - \lambda) \times n\|_{\Gamma}^2 \right\}$$

But, the last term of this norm does not scale with κ for gradient fields.

Thus, the penalty term $\|u - \lambda\|$ leads to an overpenalization of the jump for gradient fields.

Scalar potential at the boundary

Goal: Want to replace the continuity condition

$$(u_i - \lambda) \times n_i = 0 \quad i = 1, 2$$

by

$$\begin{aligned}(u_i - \nabla\phi_i - \lambda) \times n_i &= 0 \\ \phi_i - \phi_\Gamma &= 0\end{aligned}$$

with arbitrary scalar fields $\phi_1 = \phi_2 = \phi_\Gamma$ on the boundary.

This allows to scale the penalty terms for gradients and rotations differently.

Variational formulation

$$\sum_{i=1}^2 \left\{ \int_{\Omega_i} \{\mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} v + \kappa u v\} + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u [(v - \mu) \times n] + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} v [(u - \nabla\phi - \lambda) \times n] + \alpha \int_{\partial\Omega_i} \mu^{-1} [(u - \nabla\phi - \lambda) \times n][(v - \nabla\psi - \mu) \times n] + \alpha \int_{\partial\Omega_i} \kappa(\phi - \phi_\Gamma)(\psi - \psi_\Gamma) \right\} = \int_{\Omega} j v$$

A boundary identity

Testing the weak form with $v = \nabla\psi$ gives

$$\int_{\Omega_i} \kappa u \cdot \nabla\psi + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u \cdot (\nabla\psi \times n) = \int_{\Omega} j \cdot \nabla\psi$$

Taking the divergence in the strong form, and integrating by parts leads to

$$\begin{aligned} \int_{\Omega_i} \operatorname{div}(\kappa u) \psi &= \int_{\Omega_i} \operatorname{div} j \psi \\ - \int_{\Omega_i} \kappa u \cdot \nabla\psi + \int_{\partial\Omega_i} \kappa u_n \psi &= - \int_{\Omega_i} j \cdot \nabla\psi + \int_{\partial\Omega_i} j_n \psi \end{aligned}$$

Adding up leads to the boundary relation

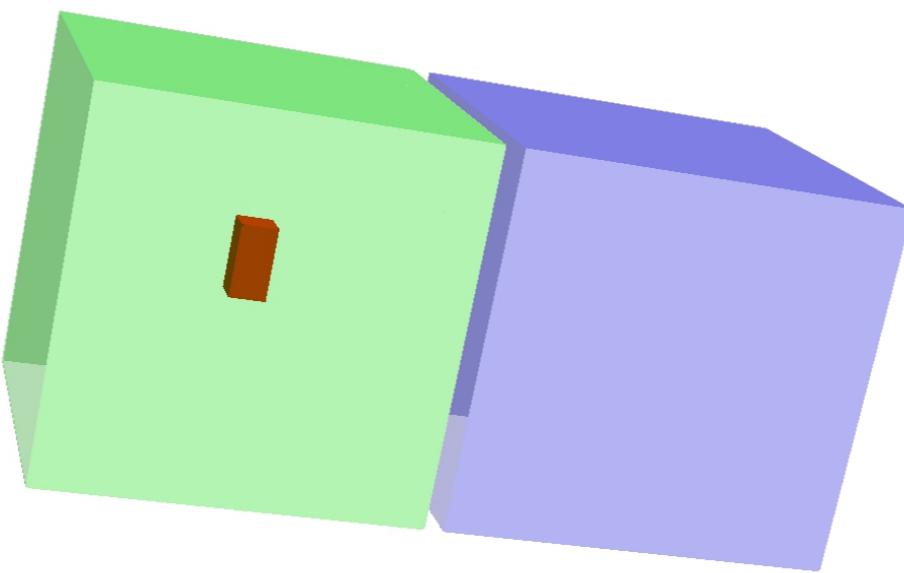
$$\int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u \cdot (\nabla\psi \times n) + \int_{\partial\Omega_i} \kappa u_n \psi = \int_{\partial\Omega_i} j_n \psi$$

Final variational formulation

$$\begin{aligned}
& \sum_{i=1}^2 \left\{ \int_{\Omega_i} \{\mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} v + \kappa u v\} + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u [(v - \nabla\psi - \mu) \times n] \right. \\
& \quad + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} v [(u - \nabla\phi - \lambda) \times n] + \alpha \int_{\partial\Omega_i} \mu^{-1} [(u - \nabla\phi - \lambda) \times n][(v - \nabla\psi - \mu) \times n] \\
& \quad \left. - \int_{\partial\Omega_i} \kappa u_n (\psi - \psi_\Gamma) - \int_{\partial\Omega_i} \kappa v_n (\phi - \phi_\Gamma) + \alpha \int_{\partial\Omega_i} \kappa (\phi - \phi_\Gamma) (\psi - \psi_\Gamma) \right\} = \\
& \quad \sum_{i=1}^2 \left\{ \int_{\Omega_i} j v - \int_{\partial\Omega_i} j_n \psi \right\}
\end{aligned}$$

- $u, v \dots H(\operatorname{curl})$ conforming element basis functions on Ω_i
- $\phi, \psi \dots H^1$ conforming element basis functions on $\Omega_i \cap \Gamma$
- $\lambda, \mu \dots$ tangential vector valued spline functions on Γ
- $\phi_\Gamma, \psi_\Gamma \dots$ scalar spline functions on Γ

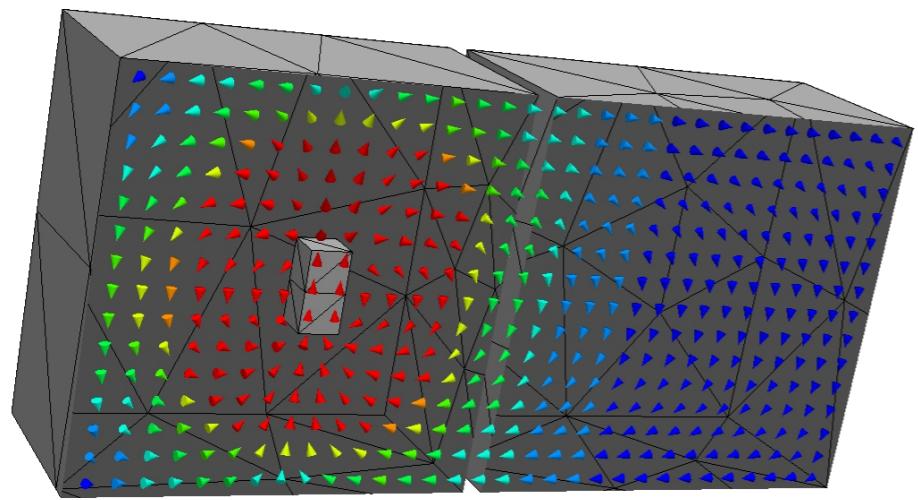
Magnetostatics



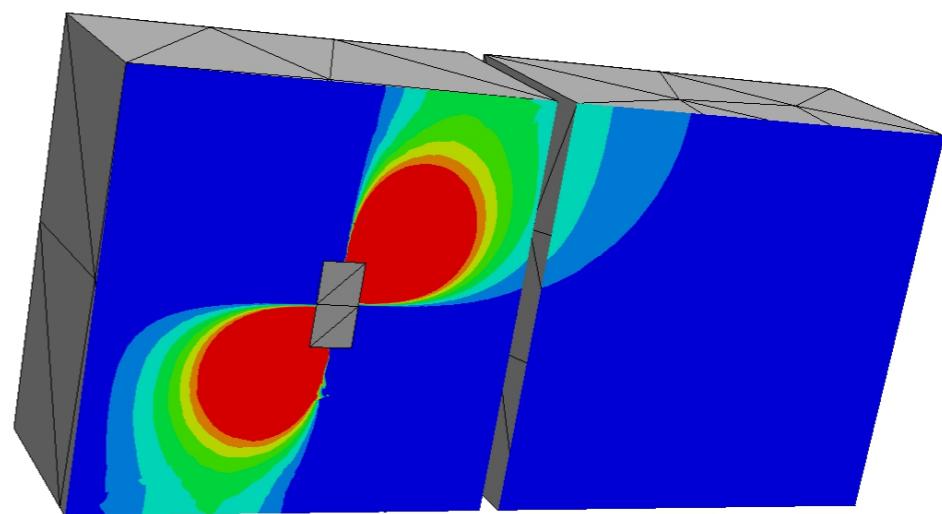
Permanent magnet (red) with two domains (green and blue)

Magnetic flux

magnetic flux



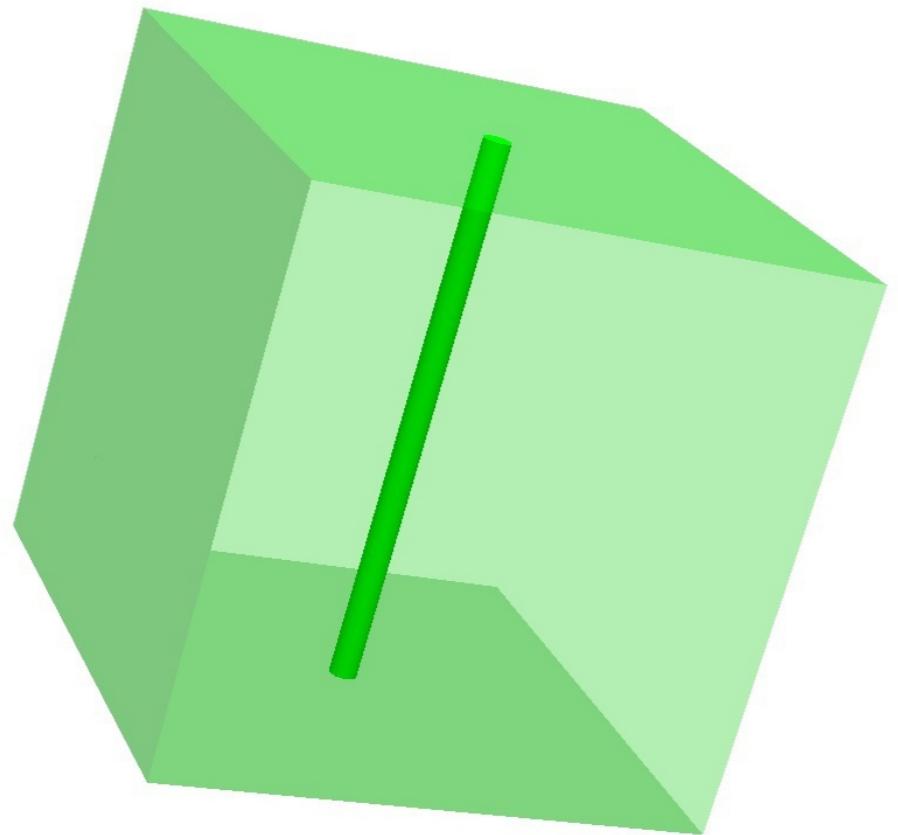
x -component of magnetic flux



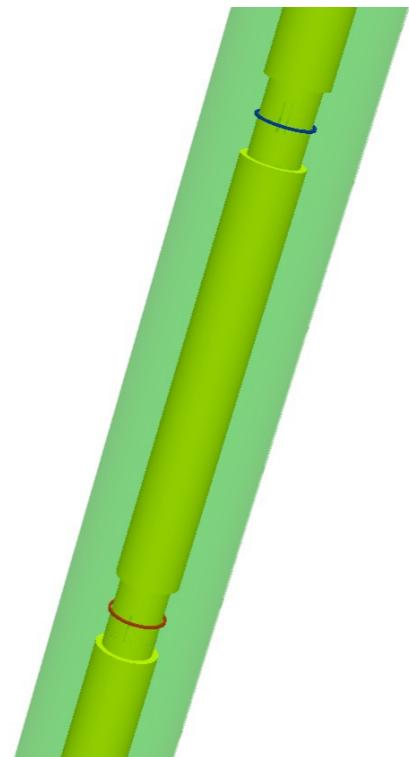
Finite element order $p = 4$.

LWD-Tool

borehole with soil

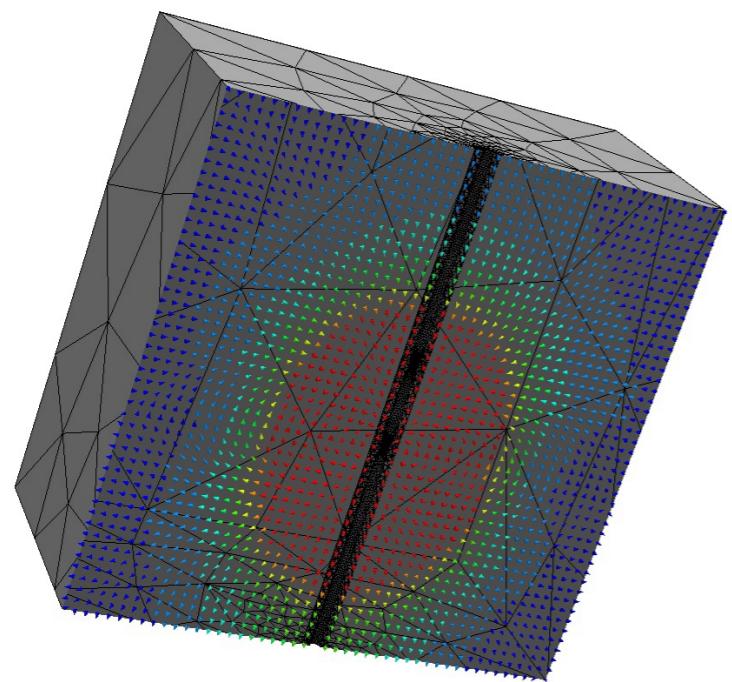


tool with antennas

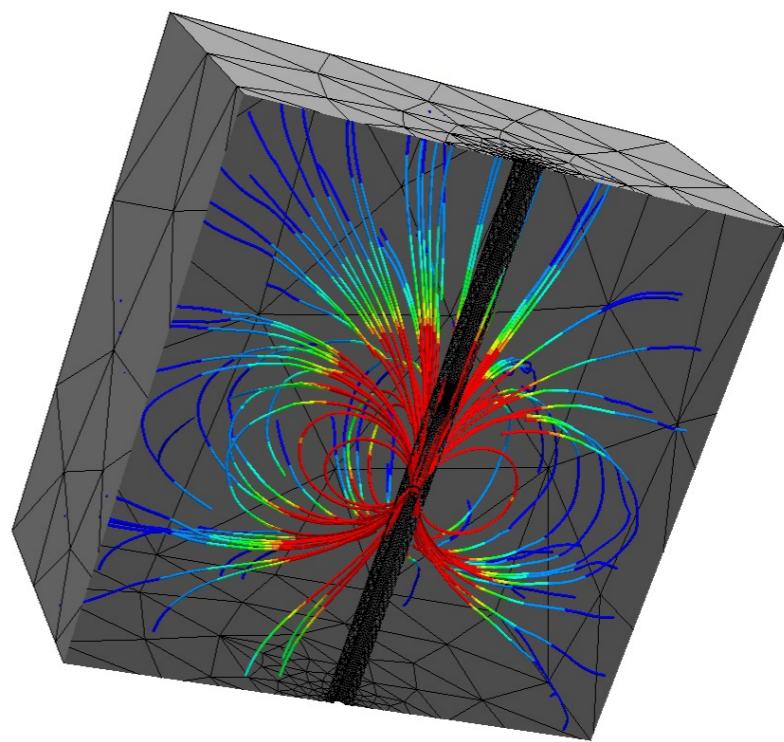


LWD-Tool

B-field



B-field, field-lines:



LWD-Tool

Numerical results for first order elements:

| frequency [kHz] | standard, rec volt [nV] 185 810 dofs | Nitsche, rec volt [nV] 195 383 dofs |
|-----------------|---|--|
| 20 | $25.44 - i 18.38$ | $25.43 - i 18.37$ |
| 100 | $71.68 - i 197.5$ | $71.65 - i 197.3$ |
| 400 | $124.9 - i 963.0$ | $124.9 - i 962.3$ |
| 2000 | $-635.9 - i 5295$ | $-634.8 - i 5255$ |

Numerical results for second order elements:

| frequency [kHz] | standard, rec volt [nV] 733 881 dofs | Nitsche, rec volt [nV] 736 939 dofs |
|-----------------|---|--|
| 20 | $24.99 - i 18.47$ | $24.98 - i 18.47$ |
| 100 | $70.25 - i 196.7$ | $70.23 - i 196.7$ |
| 400 | $121.7 - i 957.9$ | $121.7 - i 957.7$ |
| 2000 | $-648.1 - i 5256$ | $-647.9 - i 5255$ |

Conclusions and Ongoing Work

We have

- Hybrid Nitsche-type mortaring for scalar equation
- Stable transmission conditions for low frequency Maxwell's equations
- Interface fields discretized by smooth B-spline spaces for simple numerical integration

We work on

- Error estimators and adaptivity for spline space and numerical integration
- Iterative solvers, in particular BDDC domain decomposition methods

The methods are implemented within Netgen/NGSolve software

[K. Hollaus, D. Feldengut, J. Schberl, M. Wabro, D. Omeragic: Nitsche-type Mortaring for Maxwell's Equations PIERS Proceedings, 397 - 402, July 5-8, Cambridge, USA 2010]