

Hybrid (DG) Methods for the Helmholtz Equation

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Helmholtz Equation

Helmholtz Equation

$$\Delta u + \omega^2 u = f$$

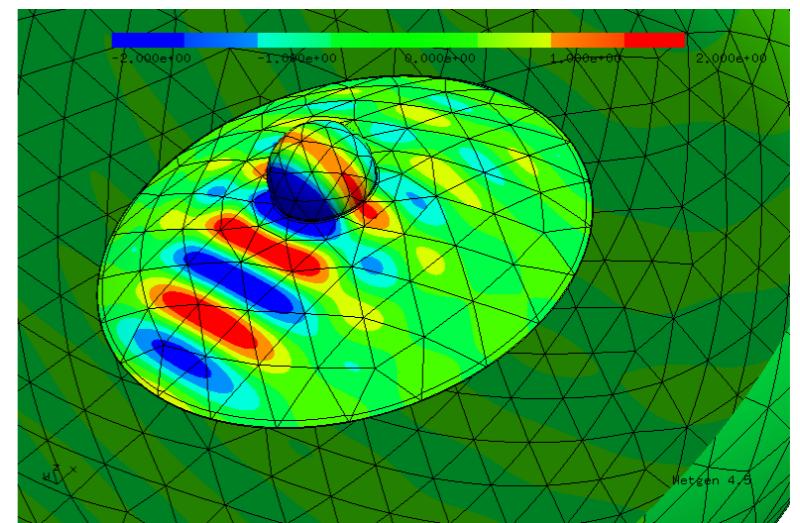
Impedance trace boundary condition

$$\frac{\partial u}{\partial n} + i\omega u = i\omega g_{in}$$

for convenience.

better: PML, pole-condition, ...

Weak form: Find $u \in H^1$ s.t.



$$\int_{\Omega} \nabla u \nabla v - \omega^2 u v + i\omega \int_{\partial\Omega} u v = \int_{\partial\Omega} g v \quad \forall v \in H^1$$

Weak and dual formulation

Mixed formulation. Introduce velocity $\sigma := \frac{1}{i\omega} \nabla u$:

$$\begin{aligned}\nabla u - i\omega\sigma &= 0 \\ \operatorname{div} \sigma - i\omega u &= \frac{1}{i\omega} f \\ \sigma_n + u &= g_{in}\end{aligned}$$

Dual formulation. Eliminate u :

$$\nabla \operatorname{div} \sigma + \omega^2 \sigma = \nabla f$$

Weak form:

$$\frac{i}{\omega} \int \operatorname{div} \sigma \operatorname{div} \tau - i\omega \int \sigma \tau + \int_{\partial\Omega} \sigma_n \tau_n = \int f \operatorname{div} \tau + \int_{\partial\Omega} g_{in} \tau_n$$

Function space $H(\text{div})$

Dual formulation requires the function space

$$H(\text{div}) = \{\boldsymbol{\tau} \in [L_2]^d : \text{div } \boldsymbol{\tau} \in L_2\}$$

has well-defined normal-trace operator

Raviart-Thomas Finite Elements:

$$V_{RT^k} = \{\vec{a} + b\vec{x} : \vec{a} \in [P^k]^d, b \in P^k\}$$

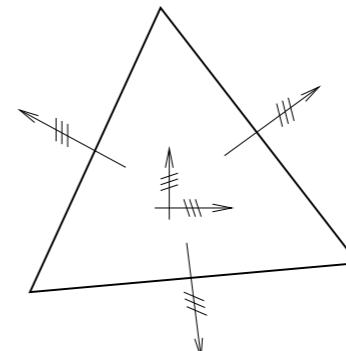
There holds

$$[P^k]^d \subset V_{RT^k} \subset [P^{k+1}]^d, \quad \text{div } V_{RT^k} = P^k, \quad \text{tr}_{n,F} V_{RT^k} \in P^k(F)$$

Degrees of freedom are

- moments of normal traces on facets F of order k
- moments on the element T of order $k - 1$

dofs ensure continuity of normal-trace



Hybridization of RT -elements

Arnold-Brezzi hybridization technique: Do not incorporate normal-continuity

$$\sigma|_{T_1} \cdot n_1 + \sigma|_{T_2} \cdot n_2 = 0$$

into the finite element space, but enforce it by additional equations (Lagrange multiplier):

$$\int_F [\sigma_n] \hat{v} = 0 \quad \forall \hat{v} \in P^k(F)$$

Hybrid system: Find $\sigma \in V_{RT^k}^{dc}$, $\hat{u} \in P^k(F)$ such that

$$\begin{aligned} \sum_T \frac{i}{\omega} \int_T \operatorname{div} \sigma \operatorname{div} \tau - i\omega \int_T \sigma \tau &+ \sum_T \int_{\partial T} \tau_n \hat{u} = 0 & \forall \tau \\ \sum_T \int_{\partial T} \sigma_n \hat{v} &+ \int_{\partial \Omega} \hat{u} \hat{v} = \int_{\partial \Omega} g_{in} \hat{v} & \forall \hat{v} \end{aligned}$$

\hat{u} approximates the primal variable on the facet.

Wish to eliminate σ from first equation (small local problems !)

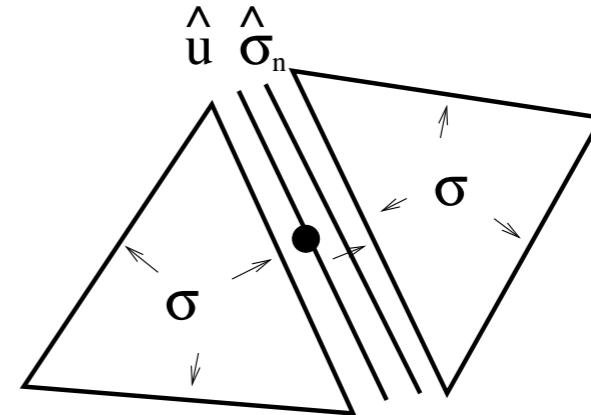
BUT: local Dirichlet - problems are unstable for $h\omega \geq 1$

Hybridization of Helmholtz equation

Add a second Lagrange multiplier [Monk+Sinwell+JS 11]

$$\hat{\sigma}_n := \sigma|_F \cdot n_F$$

and stabilize by adding $0 = \int_{\partial T} (\sigma_n - \hat{\sigma}_n)(\tau_n - \hat{\tau}_n)$



$$\begin{aligned} \sum_T \frac{i}{\omega} \int_T \operatorname{div} \sigma \operatorname{div} \tau - i\omega \int_T \sigma \tau + \int_{\partial T} \sigma_n \tau_n &+ \sum_T \int_{\partial T} \tau_n \hat{u} - \tau_n \hat{\sigma}_n = 0 \quad \forall \tau \\ \sum_T \int_{\partial T} \sigma_n \hat{v} - \sigma_n \hat{\tau}_n &+ \sum_T \int_{\partial T} \hat{\sigma}_n \hat{\tau}_n + \int_{\partial \Omega} \hat{u} \hat{v} = \int_{\partial \Omega} g_{in} \hat{v} \quad \forall \hat{v} \end{aligned}$$

The element-variable σ can now be eliminated from the stable local equation (Robin - b.c.):

$$\frac{i}{\omega} \int_T \operatorname{div} \sigma \operatorname{div} v - i\omega \int_T \sigma \tau + \int_{\partial T} \sigma_n \tau_n = \int_{\partial T} (\hat{\sigma}_n - \hat{u}) \tau_n$$

Analysis

- Unique discrete solution and convergence rate estimates by usual duality technique (h sufficiently small)
- Qualitative property: Impedance trace isometry
- conjecture: local solvability of discrete problem. brute-force proof for lowest order

Impedance trace isometry

Element equation reads as

$$\frac{i}{\omega} \int_T \operatorname{div} \sigma \operatorname{div} \tau - i\omega \int \sigma \tau + \int_{\partial T} \sigma_n \tau_n = \int_{\partial T} (\underbrace{\widehat{\sigma}_n - \widehat{u}}_{g_{in}}) \tau_n \quad \forall \tau$$

define element-wise out-going impedance trace $g_{out} \in P^k(F)$ via

$$\frac{i}{\omega} \int_T \operatorname{div} \sigma \operatorname{div} v - i\omega \int \sigma \tau - \int_{\partial T} \sigma_n \tau_n = \int_{\partial T} g_{out} \tau_n$$

Choose $\tau = \bar{\sigma}$ and take real parts: $\|\sigma_n\|_{\partial T}^2 = \operatorname{Re} \left\{ \int g_{in} \bar{\sigma}_n \right\}$ So we get

$$\|g_{out}\|_{\partial T}^2 = \|g_{in} - 2\sigma_n\|_{\partial T}^2 = \|g_{in}\|^2 - 4\operatorname{Re} \int g_{in} \bar{\sigma}_n + 4\|g_{in}\|^2 = \|g_{in}\|_{\partial T}^2$$

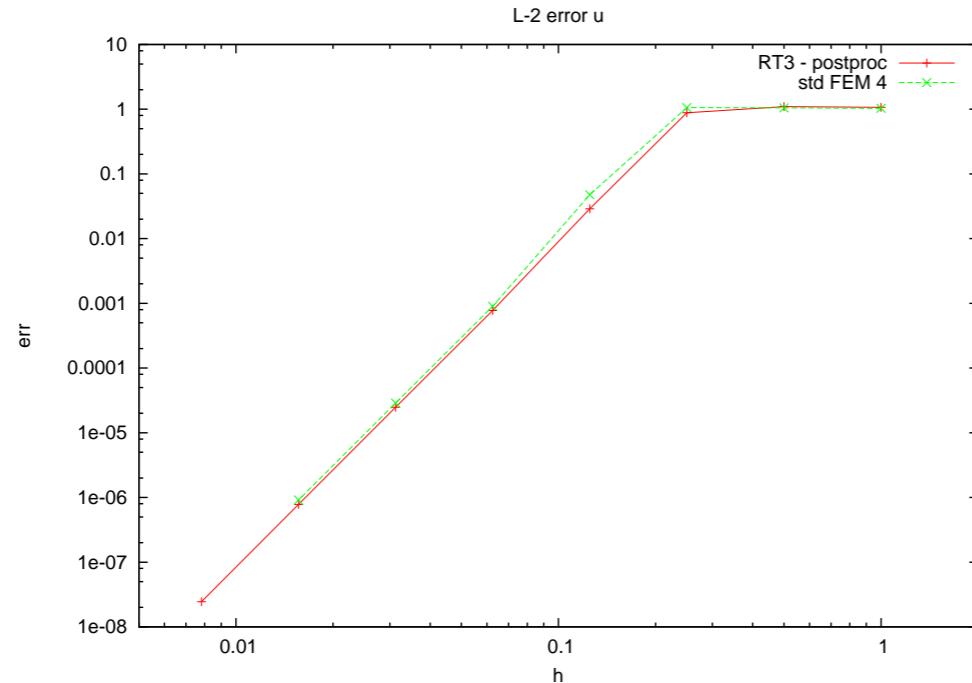
The facet-equations can be rephrased as (for neighbour elements \tilde{T}).

$$g_{in} = g_{out}(\sigma^{\tilde{T}})$$

Convergence plots

Comparing Hybrid-Helmholtz solution with conforming H^1 -FEM.

$\sigma \in RT^k$, $\hat{\sigma}_n, \hat{u} \in P^k(F)$. Postprocessing to $\tilde{u} \in P^{k+1}(T)$.



Hybrid-Helmholtz has about twice the global dofs of standard FEM.

How to solve ???

Want to iterate for facet-variables only.

Idea:

In the wave-regime the wave-relaxation

$$g_{in} := g_{out}(\sigma^{\hat{T}})$$

converges well. But, does not work for the elliptic regime when $h\omega$ is small.

Try to combine with an elliptic preconditioner.

Facet dofs

Facet - dofs are Dirichlet and Neumann data

$$\hat{u}, \hat{\sigma}_n,$$

or left- and right-going traces

$$g^l, g^r$$

transformation

$$\begin{pmatrix} g^l \\ g^r \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{\sigma}_n \end{pmatrix}$$

holds point-wise since all variables are in $P^k(F)$

Motivation: 1D system

The 1D system (with exact local problems) reads as

$$\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_{j-1}^l \\ g_{j-1}^r \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g_j^l \\ g_j^r \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} g_{j+1}^l \\ g_{j+1}^r \end{pmatrix}$$

left and right-going plane waves with phase-shift γ .

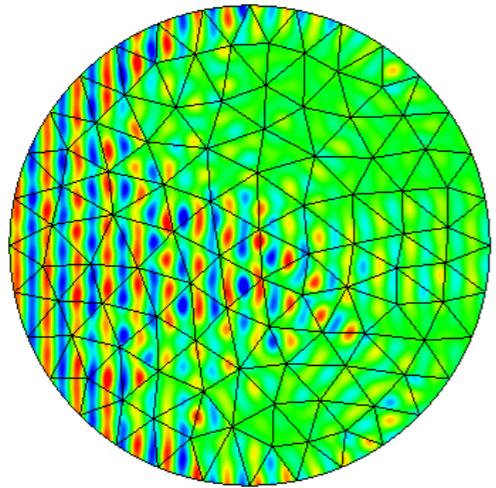
left (right) going Gauss-Seidel for g^l (g^r) is an exact solve.

Matrix is not diagonal dominant.

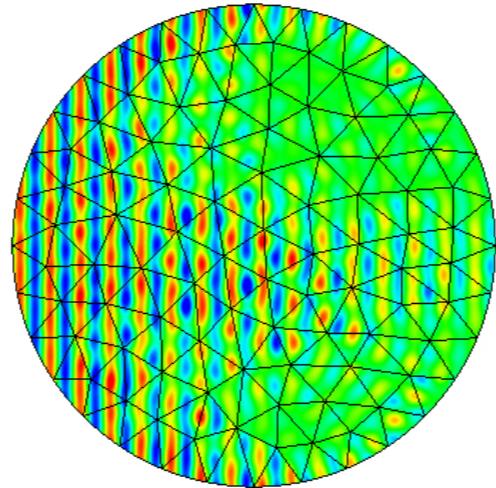
But, exchanging rows pairwise gives a diagonal-dominant, non-symmetric matrix.

Krylov-space methods with most diagonal-like preconditioners do converge.

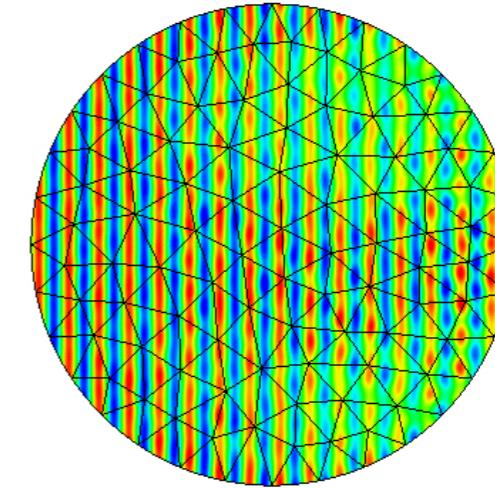
Facet-wise multiplicative Schwarz iteration



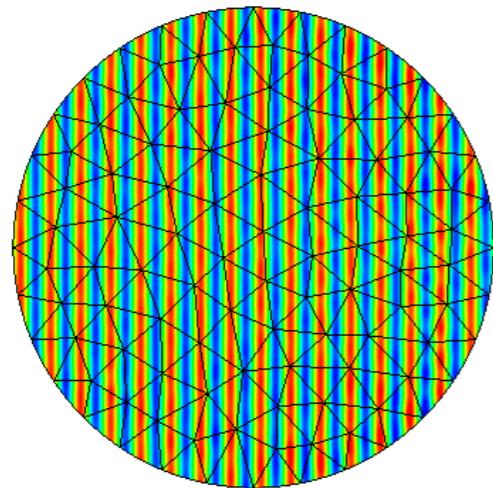
1 iteration



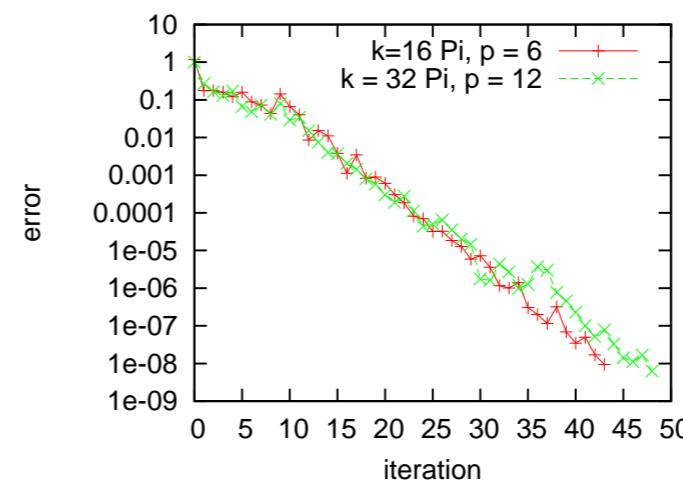
2 iterations



5 iterations



10 iterations



Balancing domain decomposition methods

Domain decomposition, $\Omega = \cup \Omega_i$, distribute elements

Sub-assemble fe matrices with sub-domain matrix A_i

$$A = \sum A_i$$

Balancing DD Preconditioner:

$$\hat{A}^{-1} = \sum R_i A_i^{-1} R_i^T$$

Elliptic case: local Neumann problems are singular !

Solution: BDD with constraints \Rightarrow BDDC (Dohrmann, Widlund, ...)

$$R \begin{pmatrix} A_{cc} & A_{c1} & \cdots & A_{cm} \\ A_{1c} & A_{11} & & \\ \vdots & & \ddots & \\ A_{mc} & & & A_{mm} \end{pmatrix}^{-1} R^T$$

Analysis for the elliptic case with \log^α condition numbers.

BDDC for the Hybrid Helmholtz system

Bilinearform

$$\begin{aligned} B(\sigma, \hat{\sigma}, \hat{u}; \tau, \hat{\tau}_n, \hat{v}) = & \sum_T i \frac{1}{\omega} \int \operatorname{div} \sigma \operatorname{div} \tau - i \omega \int \sigma \tau + \int_{\partial T} \sigma_n \hat{v} + \tau_n \hat{u} + \\ & + \int_{\partial T} (\sigma_n - \hat{\sigma}_n)(\tau_n - \hat{\tau}_n) - \gamma \int_{\partial T} \hat{\sigma}_n \hat{v} + \hat{\tau}_n \hat{u} \end{aligned}$$

The new blue term is consistent

Apply BDDC preconditioner with mean-value facet-dof constraints

Good values for γ are found by experiments.

[M. Huber+JS, Hybrid Domain Decomposition Solvers for the Helmholtz Equation, DD XXI, 2014]

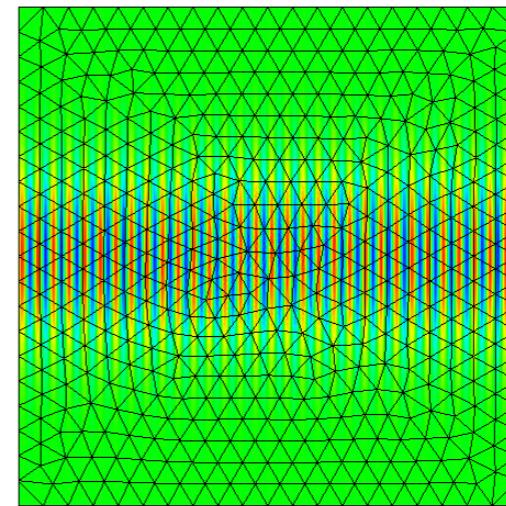
Iteration numbers

with coarse grid, 4 domains

L/λ	2	8	32
p=2	21	25	-
p=4	27	30	42
p=8	38	46	41
p=16	59	73	60

with coarse grid, 16 domains

L/λ	2	8	32
p=2	23	32	-
p=4	33	41	76
p=8	42	57	71
p=16	65	87	100



no coarse grid, 4 domains

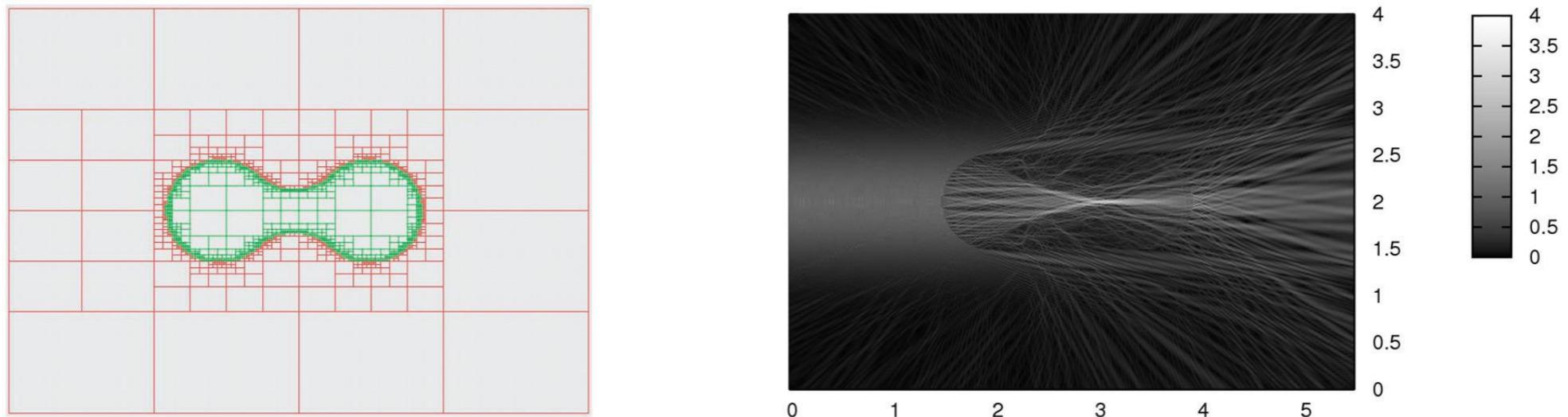
L/λ	2	8	32
p=2	51	40	-
p=4	64	48	40
p=8	96	61	43
p=16	132	84	63

no coarse grid, 16 domains

L/λ	2	8	32
p=2	85	80	-
p=4	109	88	101
p=8	148	110	89
p=16	214	159	123

p-Version on rectangular grid

p up to 2048 on rectangles (tensor product).



A. Hannukainen, M. Huber, JS, Journal of modern optics, 2011

Infrastructure

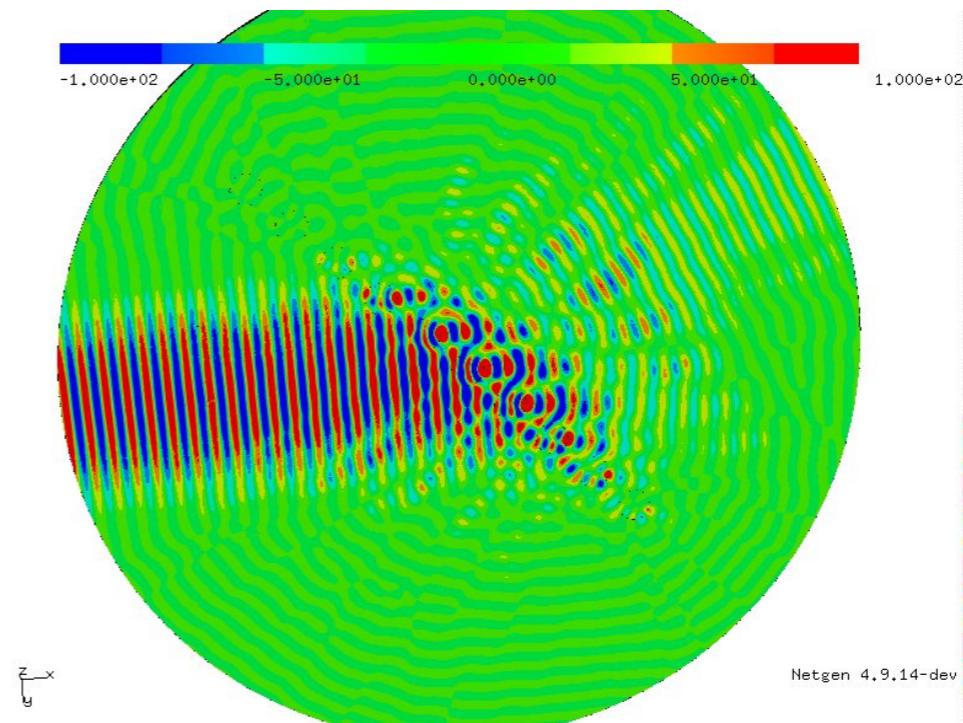
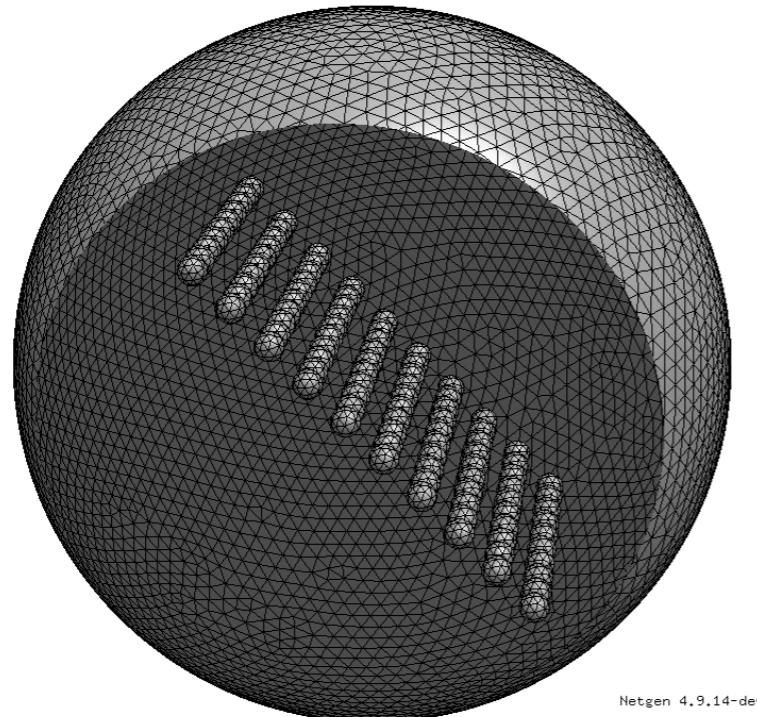
- general purpose hp - Finite Element Code Netgen/NGSolve V 5.1
 - C++, open source
 - MPI - parallel
- Dell R 910 Server
 - 4 Xeon E7 CPUs 10 core @ 2.2 GHz
 - 512 GB RAM shared memory (NUMA)
- Vienna Scientific Cluster 2
 - 20 000 cores, 2 GB RAM / core

Diffraction from a grating

Sphere with $D = 40\lambda$, $127k$ elements, $h \approx \lambda$, $p = 5$, $39M$ dofs (corresponds to $9.4M$ primal dofs)

78 sub-domains / processes, no coarse grid

$T_{ass} = 9m$, $T_{pre} = 12m$, $T_{solve} = 21m$, 156 its

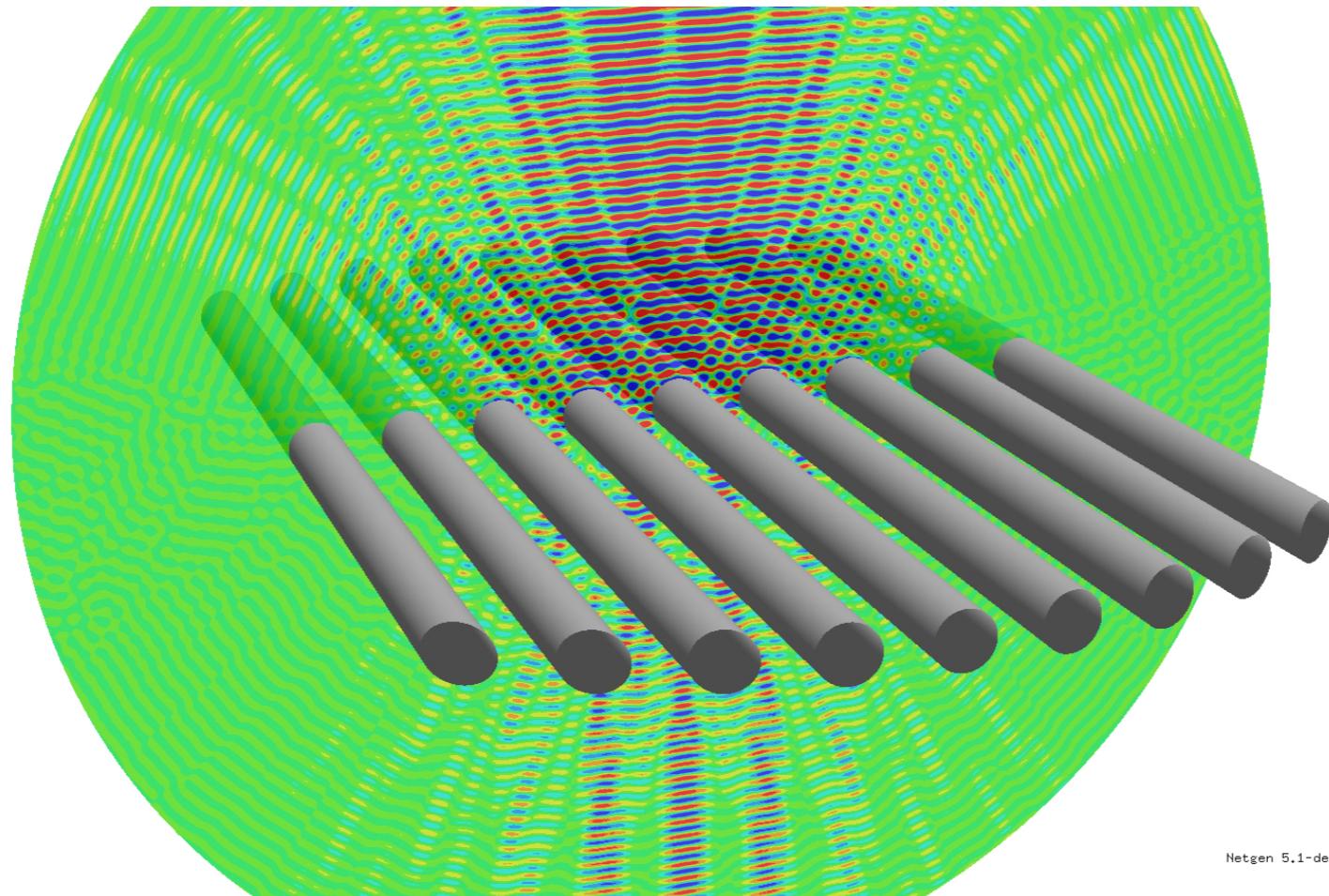


Diffraction from rods

Sphere with $D = 80\lambda$, $2.24M$ elements, $h \approx 0.8\lambda$, $p = 4$, $536M$ dofs

2000 sub-domains / processes, no coarse grid

$T_{ass} = 50s$, $T_{pre} = 30s$, $T_{solve} = 12m$, 496 its



Computational Optics

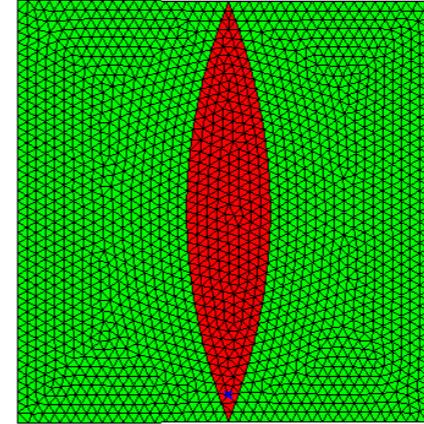
factor out wavy part

$$u = e^{ik \cdot x} \tilde{u}$$

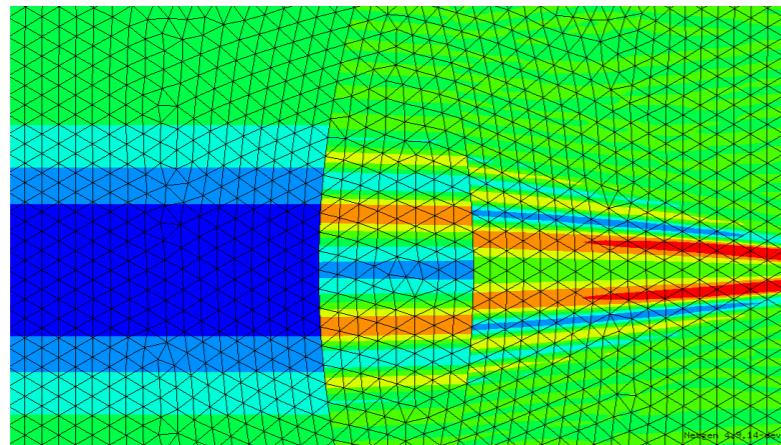
simple guess: $k = n \begin{pmatrix} \omega \\ 0 \end{pmatrix}$

better: ray-tracing, Eikonal equation

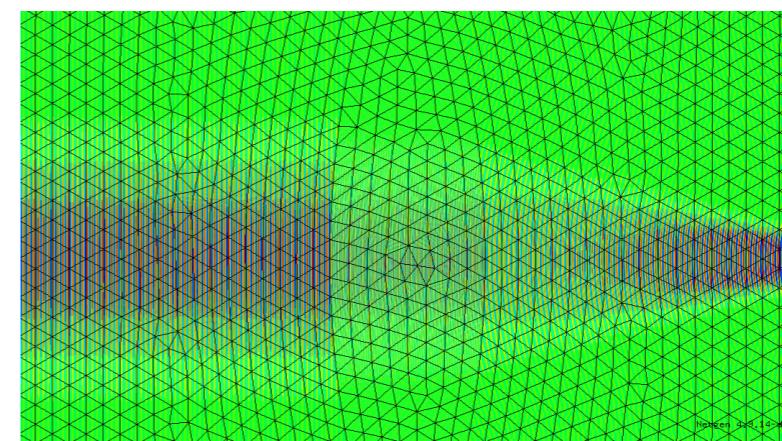
lense, $n = 2$



smooth \tilde{u}



wavy u



with LightTrans GmbH, Jena