On Equilibrated Residual Error Estimates

Joachim Schöberl

Center for Computational Engineering Science (CCES)
RWTH Aachen University

Dietrich Braess

Faculty of Mathematics
Ruhr University Bochum

Veronika Pillwein

Research institute for symbolic computation (RISC)
Johannes Kepler University Linz

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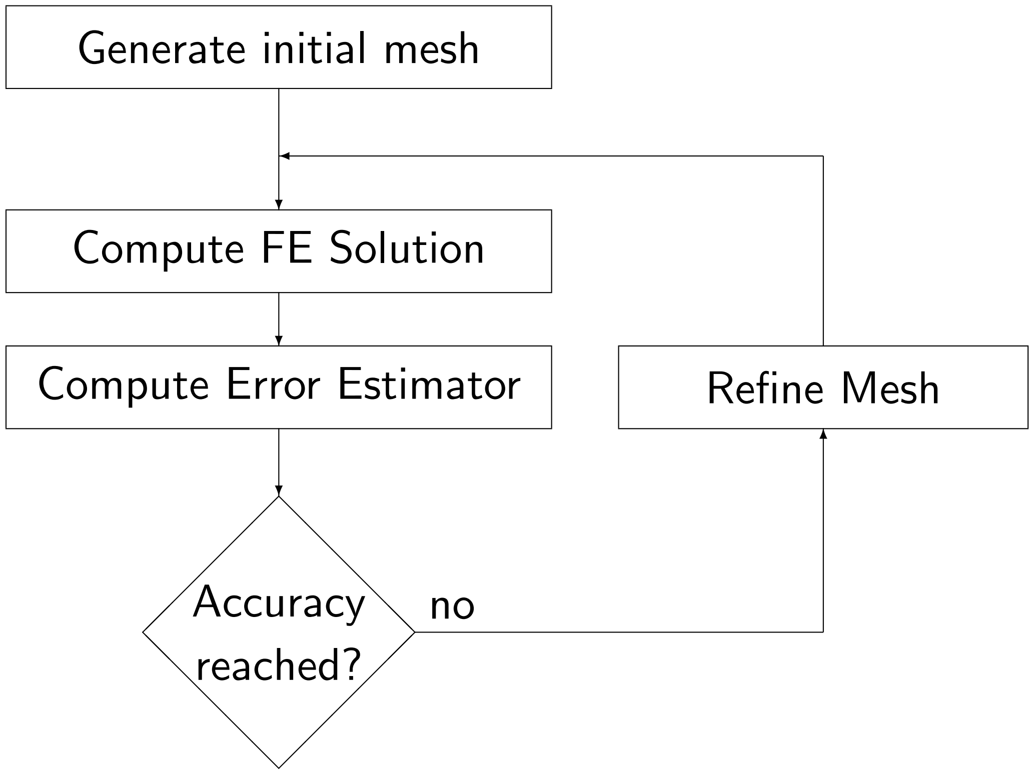
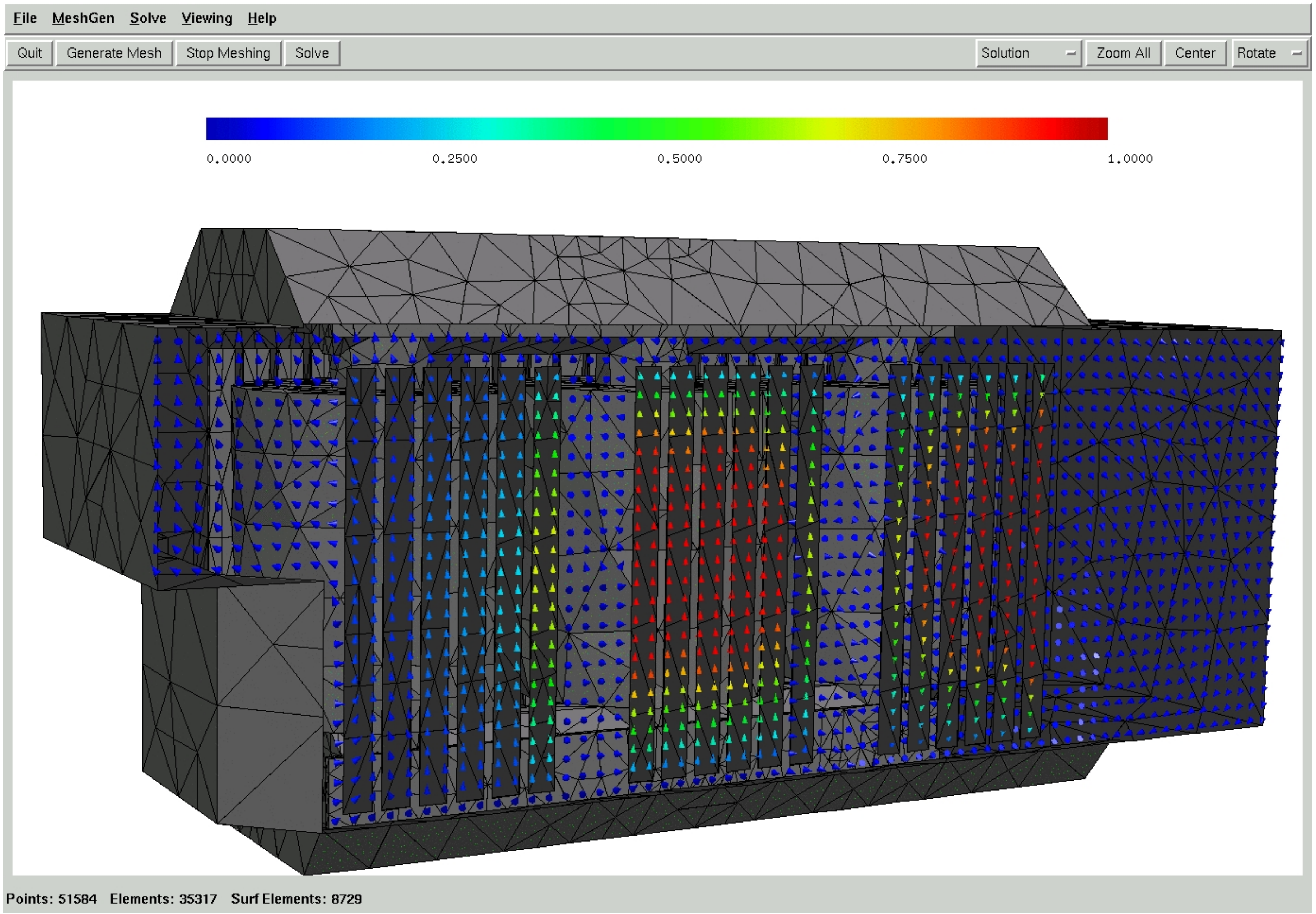
Outline

1. Energy Error estimates
2. Error estimates for the Poisson Equation
3. Error estimates for Maxwell Equations
4. The High Order Case (Poisson)

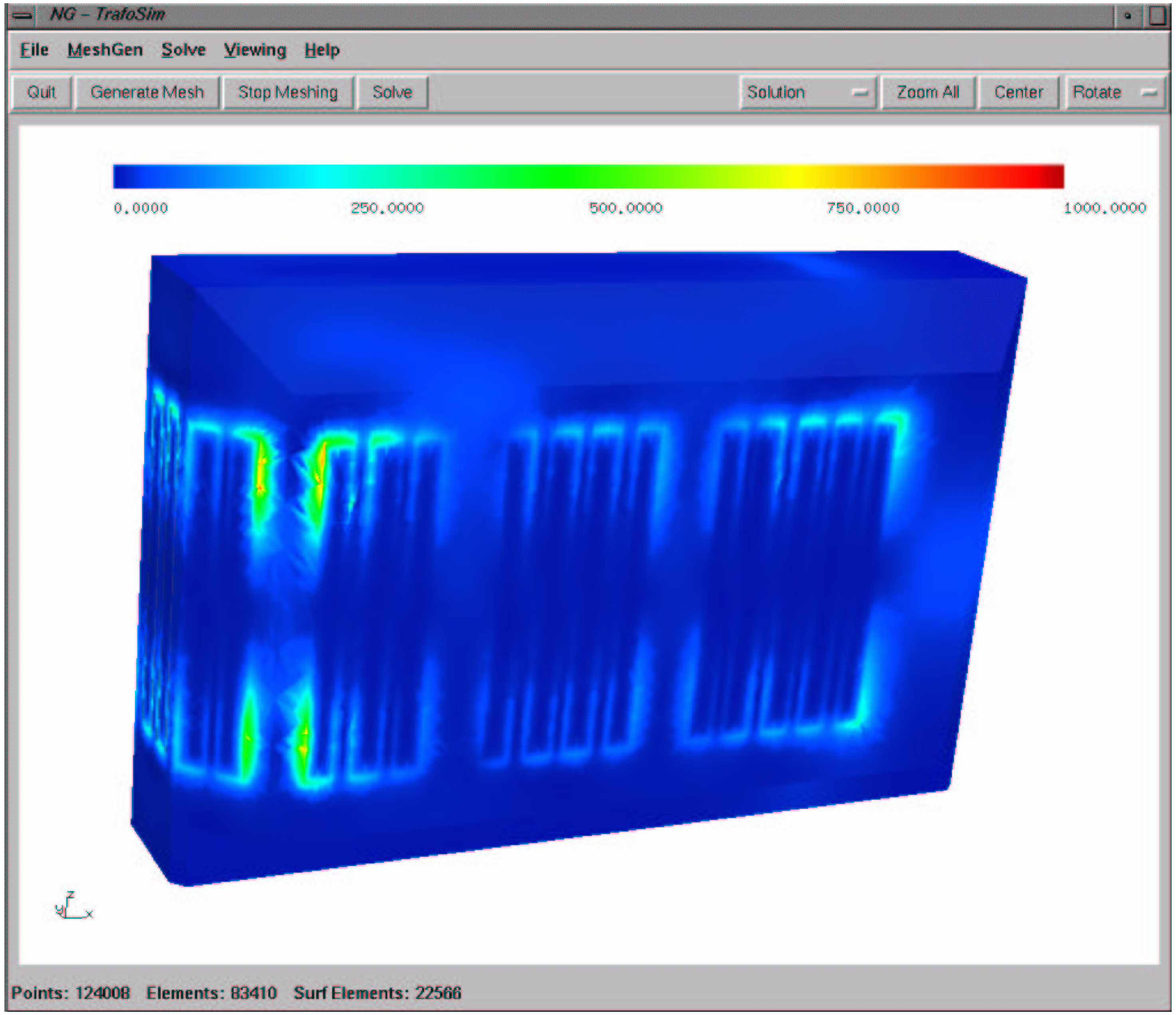
A posteriori Error Estimates and Adaptive Refinement



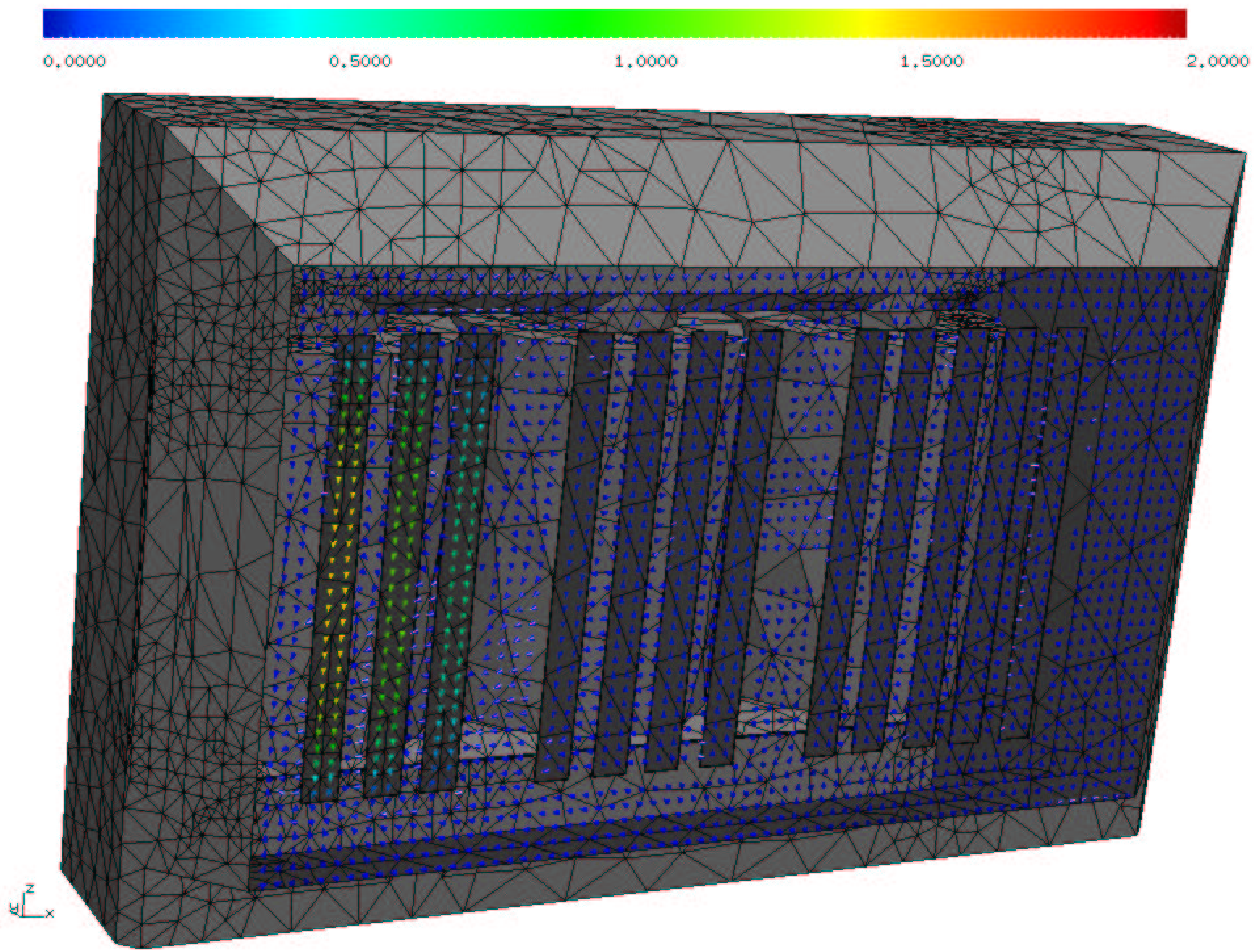
Magnetic flux density in a power transformer:



Eddy losses in casing:



Magnetic flux density:



Energy Error Estimates

Bilinear form $a(.,.)$ and linear form $f(.)$:

$$a(u,v) = (\nabla u, \nabla v) \qquad \text{and} \qquad f(v) = \sum_T (f_T, v)_T$$

Exact solution $u \in V \subset H^1$ and FEM solution $u_h \in V_h$ satisfy

$$a(u,v) = f(v) \quad \forall\, v \in V \qquad \text{and} \qquad a(u_h, v_h) = f(v_h) \quad \forall\, v_h \in V_h.$$

The residual in V^* is

$$\langle r, v \rangle = a(u - u_h, v) = \sum_T (f_T + \Delta u_h, v)_T + \sum_E ([\partial_n u_h], v)_E$$

It satisfies $\langle r, \varphi_V \rangle = 0$.

Residual a posteriori Error Estimates

$$
\begin{aligned}
\|\nabla(u-u_h)\| &= \|r\|_{V^*} = \sup_{\|\nabla v\|\le 1} (\nabla(u-u_h), \nabla v) \\
&= \sup_{\|\nabla v\|\le 1} (\nabla(u-u_h), \nabla(v-\Pi_h v)) \\
&= \sup_{\|\nabla v\|\le 1} \sum_T (f+\Delta u_h, v-\Pi_h v)_T + \sum_E ([\partial_n u_h], v-\Pi_h v)_T \\
&\le \sup_{\|\nabla v\|\le 1} \sum_T \|f+\Delta u_h\|_{L_2(T)} \|v-\Pi_h v\|_{L_2(T)} + \sum_E \|[\partial_n u_h]\|_{L_2(E)} \, \|v-\Pi_h v\|_{L_2(E)} \\
&\le \sup_{\|\nabla v\|\le 1} \sum_T \|f+\Delta u_h\|_{L_2(T)} \, ch \, \|\nabla v\|_{L_2(\omega_T)} + \sum_E \|[\partial_n u_h]\|_{L_2(E)} \, ch^{1/2} \|\nabla v\|_{L_2(\omega_E)} \\
&\le C \left\{ \sum_T h^2 \|f+\Delta u_h\|^2_{L_2(T)} + \sum_E h \|[\partial_n u_h]\|^2_{L_2(E)} \right\}^{1/2}
\end{aligned}
$$

For Maxwell: Monk 98, Hiptmair 99, JS 08

How big is the constant C ?

The Hypercircle Method

For any flux $\sigma \in H(\mathrm{div})$ there holds

$$
\begin{aligned}
\|\nabla(u-u_h)\| &\leq \sup_{\|\nabla v\|\leq 1} (\nabla(u-u_h), \nabla v) \\
&\leq \sup_{\|\nabla v\|\leq 1} (\nabla u - \sigma, \nabla v) + \sup_{\|\nabla v\|\leq 1} (\sigma - \nabla u_h, \nabla v) \\
&\leq \sup_{\|\nabla v\|\leq 1} (f + \operatorname{div}\sigma, v) + \|\sigma - \nabla u_h\| \\
&= \|f + \operatorname{div}\sigma\|_{H^{-1}} + \|\sigma - \nabla u_h\|
\end{aligned}
$$

- Estimate $\|f + \operatorname{div}\sigma\|_{H^{-1}}$: Neitaanmäki + Repin 04, Vejchodsky 04
- Ignore $\|f + \operatorname{div}\sigma\|_{H^{-1}}$: Gradient recovery methods, Zienkiewicz+Zhou (ZZ) - estimators
- Let $\|f + \operatorname{div}\sigma\|$ disappear: Equilibrate residuals. Ainsworth + Oden 2000, Demkowicz 90++

A lifting for the residual

Goal: Find a flux $\sigma^\Delta \in [L_2]^d$ such that

$$\operatorname{div} \sigma^\Delta = r \quad \in V^* \qquad \text{i.e.} \qquad -(\sigma^\Delta, \nabla v) = \langle r, v\rangle$$

Then there holds

$$\|\nabla(u-u_h)\|_{L_2} = \sup_{\|\nabla v\|_{\le 1}} \langle r, v\rangle = \sup_{\|\nabla v\|_{\le 1}} (\sigma^\Delta, \nabla v) \le \|\sigma^\Delta\|_{L_2}$$

This is an a posteriori error estimate providing a true upper bound without generic constant !

The equilibrated flux

$$\sigma := \nabla u_h - \sigma^\Delta$$

satisfies

$$\operatorname{div} \sigma = \operatorname{div} \nabla u_h + \operatorname{div} \sigma^\Delta = -f_h - (f - f_h) = -f$$

[Equilibration by postprocessing: Ladeveze + Leguillon, 83]

Local construction of the lifting σ^{Δ}

We decompose the residual into local contributions on vertex patches:

$$r = \sum r_V \qquad \text{such that} \qquad \langle r_V, 1\rangle = 0$$

For each patch, we solve a local problem with boundary conditions $\sigma_V \cdot n = 0$ and

$$\operatorname{div} \sigma_V = r_V$$

The global lifting is obtained as

$$\sigma^{\Delta} = \sum \sigma_V$$

Two principles:

1. Decomposition of the residual

2. Solvability of the local problems : Exact sequences

Decomposition of the residual

Lowest order case: u_h is p.w. linear, and f_T is piecewise constant.

$$\langle r, v \rangle = \sum_T (r_T, v)_T + \sum_E (r_E, v)_E$$

with p.w. constants $r_T = f_T$ and $r_E = [\partial_n u_h]$. The degrees of freedom are

$$\widehat{r}^T := \int_T r_T \qquad \widehat{r}^E := \int_E r_E$$

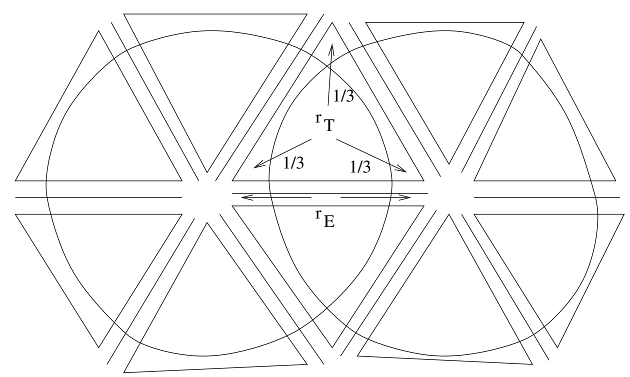
The Galerkin condition $\langle r, \varphi_V \rangle = 0$ reads as

$$\sum_{T \subset \omega_V} \int_T r_T \varphi_V + \sum_{E \subset \omega_V} \int_E r_E \varphi_V = \frac{1}{3} \sum_{T \subset \omega_V} \widehat{r}^T + \frac{1}{2} \sum_{E \subset \omega_V} \widehat{r}^E = 0$$

Decomposition of the residual

From Galerkin orthogonality:

$$\frac{1}{3}\sum_{T\subset\omega_V}\widehat{r}^T+\frac{1}{2}\sum_{E\subset\omega_V}\widehat{r}^E=0$$



Define the localized residual on the vertex patch with dofs

$$\widehat{r_V}^T:=\frac{1}{3}\widehat{r}^T\qquad\widehat{r_V}^E:=\frac{1}{2}\widehat{r}^E$$

This is a decomposition of the residual, i.e. $\sum_V r_V=r$ which satisfies

$$\langle r_V,1\rangle=\sum_{T\subset\omega_V}\widehat{r_V}^T+\sum_{E\subset\omega_V}\widehat{r_V}^E=\frac{1}{3}\sum_{T\subset\omega_V}\widehat{r}^T+\frac{1}{2}\sum_{E\subset\omega_V}\widehat{r}^E=0.$$

de Rham Sequences

Let $\Omega \subset \mathbb{R}^2$ be contractible. Then

$$\mathbb{R} \xrightarrow{\text{id}} H^1 \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L_2 \longrightarrow 0$$

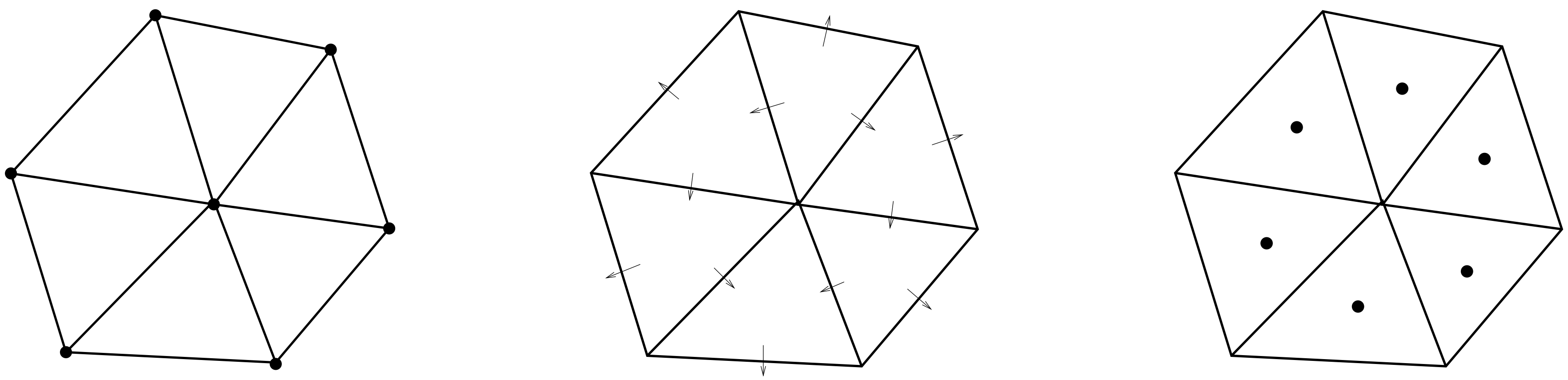
is an exact sequence. This means that

- the kernel $\{u \in H^1 : \operatorname{curl} u = 0\}$ are constant functions
- the kernel $\{\sigma \in H(\text{div}) : \operatorname{div} \sigma = 0\}$ of the operator div is exactly the range of the operator curl
- the range of the operator div is exactly L_2.

An exact sequence with boundary conditions is

$$0 \longrightarrow H_0^1 \xrightarrow{\text{curl}} H_0(\text{div}) \xrightarrow{\text{div}} L_2 \xrightarrow{\int 1} \mathbb{R} \longrightarrow 0.$$

Finite Element de Rham Sequences



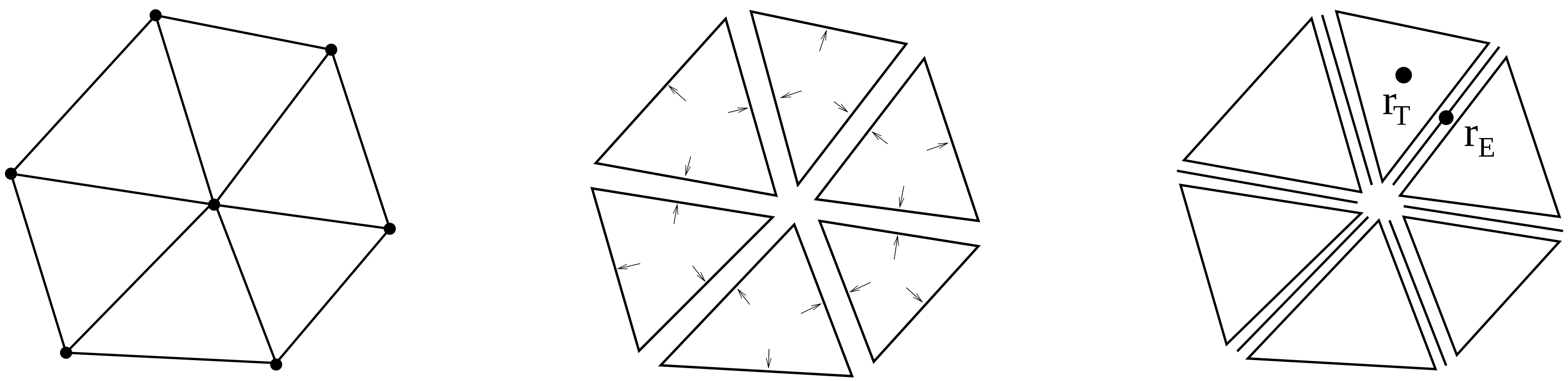
$$\mathcal{M}_0^1 \quad \xrightarrow{\text{curl}} \quad \mathcal{RT} \quad \xrightarrow{\text{div}} \quad \mathcal{M}_{-1}^0$$

Discrete calculus:

$$\sigma = \operatorname{curl} u \quad \text{reads as} \quad \widehat{\sigma}^{\,E} = \widehat{u}^{\,V_{E,1}} - \widehat{u}^{\,V_{E,2}},$$

$$f = \operatorname{div} \sigma \quad \text{reads as} \quad \widehat{f}^{\,T} = \sum_{E \subset T} \pm \widehat{\sigma}^{\,E},$$

First Distributional de Rham Sequences



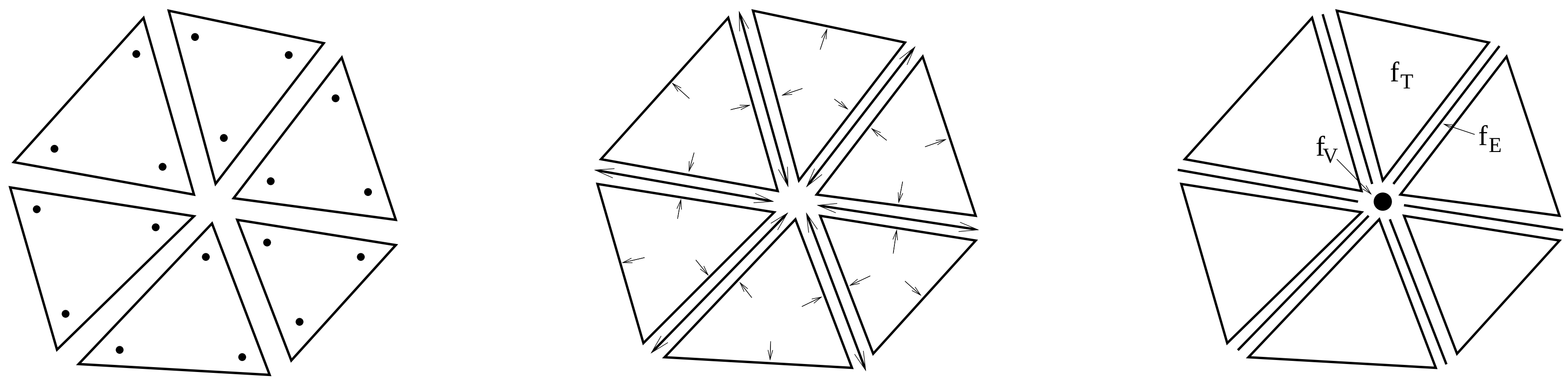
$$\mathcal{M}_0^1 \xrightarrow{\text{curl}} \mathcal{R}T_{-1} \xrightarrow{\text{div}} \mathcal{M}^0_{-2}$$

Discrete calculus:

$$\sigma = \operatorname{curl} u \quad \text{reads as} \quad \widehat{\sigma_T}^{\,E} = \widehat{u}^{\,V_{E,1}} - \widehat{u}^{\,V_{E,2}},$$

$$f = \operatorname{div} \sigma \quad \text{reads as} \quad \widehat{f}^{\,T} = \sum_{E \subset T} \widehat{\sigma_T}^{\,E} \qquad \text{and} \qquad \widehat{f}^{\,E} = -\sum_{T: E \subset T} \widehat{\sigma_T}^{\,E}.$$

Second Distributional de Rham Sequences



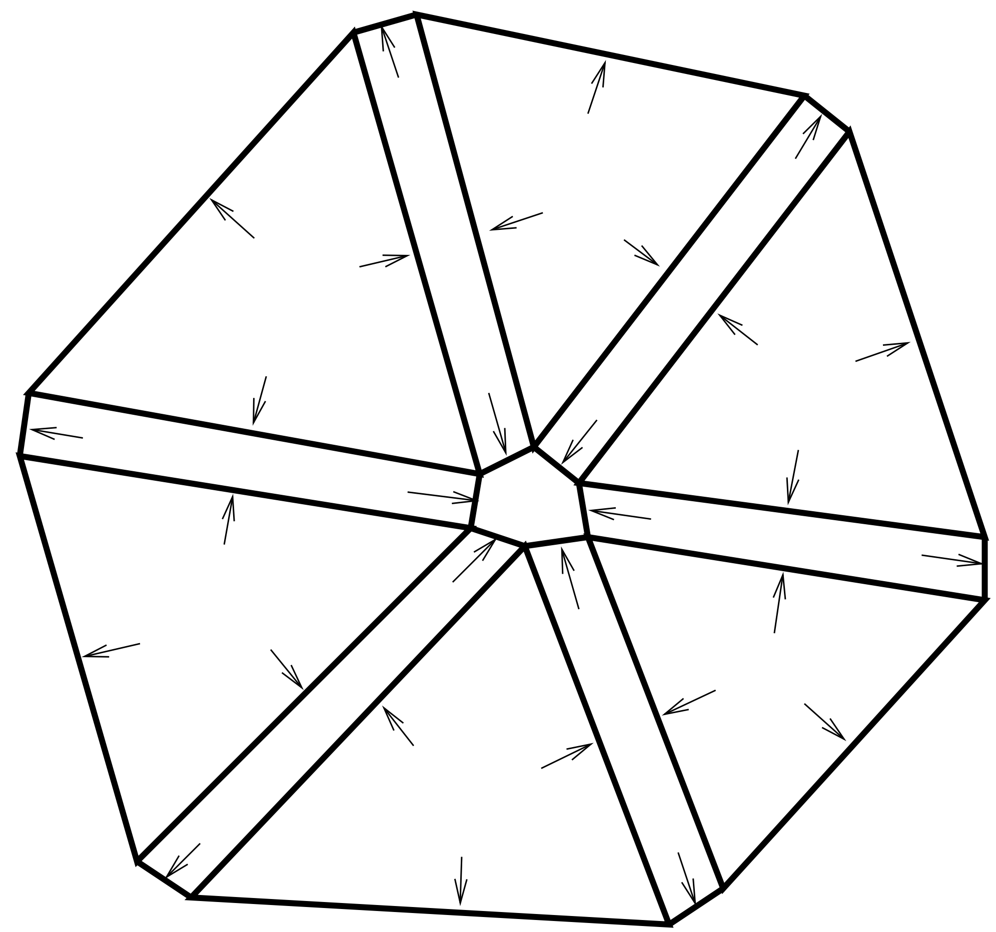
$$\mathcal{M}^1_{-1} \quad \xrightarrow{\text{curl}} \quad \mathcal{RT}_{-2} \quad \xrightarrow{\text{div}} \quad \mathcal{M}^0_{-3}$$

Discrete calculus:

$$\sigma = \operatorname{curl} u \quad \text{reads as} \quad \widehat{\sigma_T}^{E} = \widehat{u_T}^{V_{E,1}} - \widehat{u_T}^{V_{E,2}}, \qquad \widehat{\sigma_E}^{V} = \widehat{u_{T_1}}^{V} - \widehat{u_{T_2}}^{V},$$

$$f = \operatorname{div} \sigma \quad \text{reads as} \quad \widehat{f}^{T} = \sum_{E \subset T} \widehat{\sigma_T}^{E}, \qquad \widehat{f}^{E} = \sum_{V \in E} \widehat{\sigma_E}^{V} - \sum_{T : E \subset T} \widehat{\sigma_T}^{E}, \qquad \widehat{f}^{V} = - \sum_{E : V \in E} \widehat{\sigma_E}^{V}.$$

Regular elements on Slim Rectangles

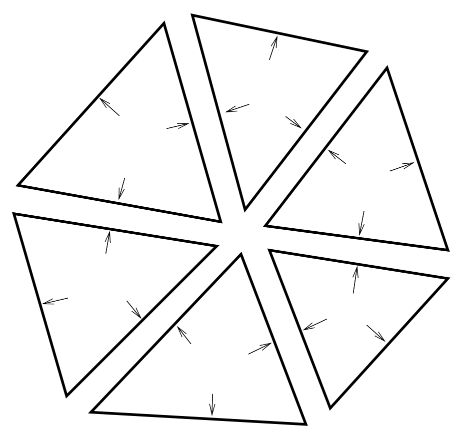


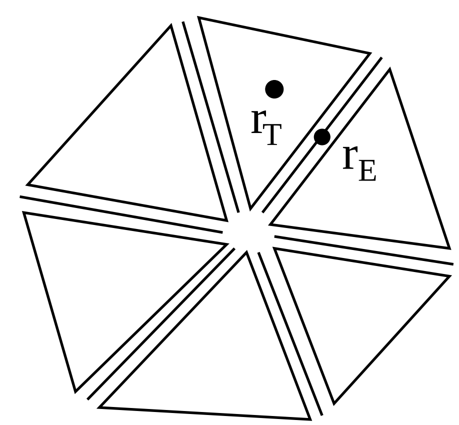
Lifting for scalar equation

Given: Local residual $r_V \in \mathcal{M}^0_{-2}$ with $\langle r_V, 1\rangle = 0$.

Compute $\sigma_V \in \mathcal{R}T_{-1}$ with homogeneous boundary conditions

Solvable by exactness of the sequence





$$\mathcal{R}T_{-1}\ \text{0 b.c.} \xrightarrow{\mathrm{div}} \mathcal{M}^0_{-2} \xrightarrow{\int 1} \mathbb{R}$$

Full reliability and local efficiency

The EE satisfies the reliability estimate

$$\|\nabla(u - u_h)\|_{L_2(\Omega)} \leq \|\sigma^{\Delta}\|_{L_2(\Omega)}$$

The EE satisfies the local efficiency estimate with generic constants depending on the shape of elements:

$$\|\nabla(u - u_h)\|_{L_2(\omega_V)} \geq c_v \|\sigma_V\|_{L_2(\omega_V)}$$

Important for convergence of adaptive process !

Equations of Magnetostatics

Given: Current density j such that $\operatorname{div} j = 0$.
Compute: Vector potential A such that

$$\operatorname{curl} \mu^{-1} \operatorname{curl} A = j$$

Magnetic field intensity

$$H = \mu^{-1} \operatorname{curl} A$$

Assume that j is given in terms of lowest order RT elements.

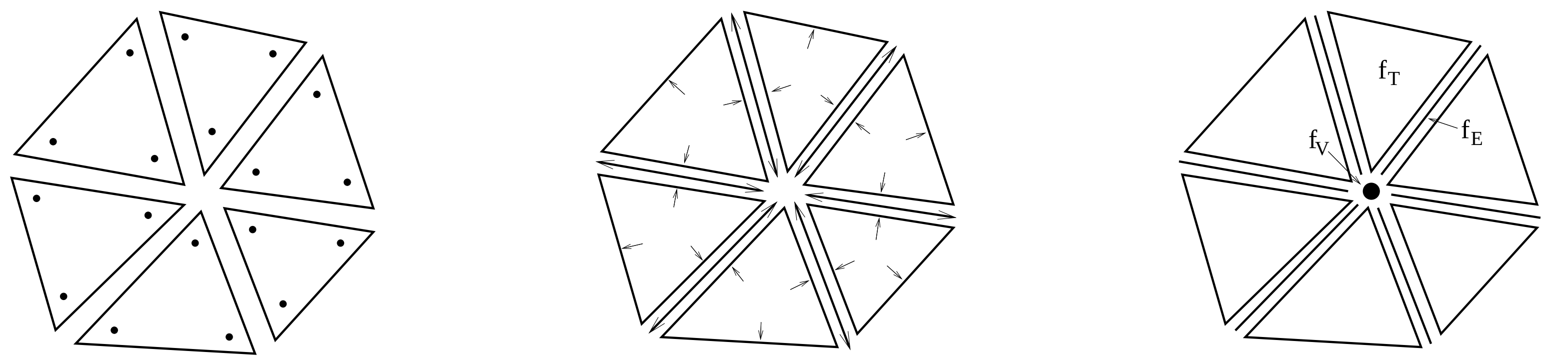
Use $H(\mathrm{curl})$-conforming Nédélec elements for A_h.

- 3D: A, H, and j are vectors
- 2D: A and j are vectors, H is a scalar

The Residual

$$r = \operatorname{curl} \mu^{-1} \operatorname{curl}(A - A_h) = j - \operatorname{curl} H_h$$

The discrete magnetic field H_h is p.w. constant, i.e. in $\mathcal{M}^1_{-1}$. Use distributional f.e. to compute $\operatorname{curl} H_h$:



$$\mathcal{M}^1_{-1} \xrightarrow{\text{curl}} \mathcal{RT}_{-2} \xrightarrow{\text{div}} \mathcal{M}^0_{-3}$$

The residual r is a divergence-free $\mathcal{RT}_{-2}$ distribution.

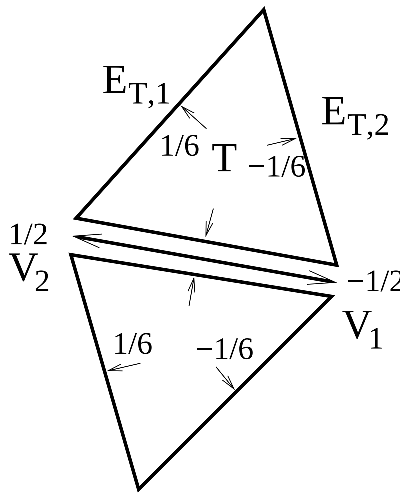
A lifting $H^\Delta \in \mathcal{M}^1_{-1}$ such that

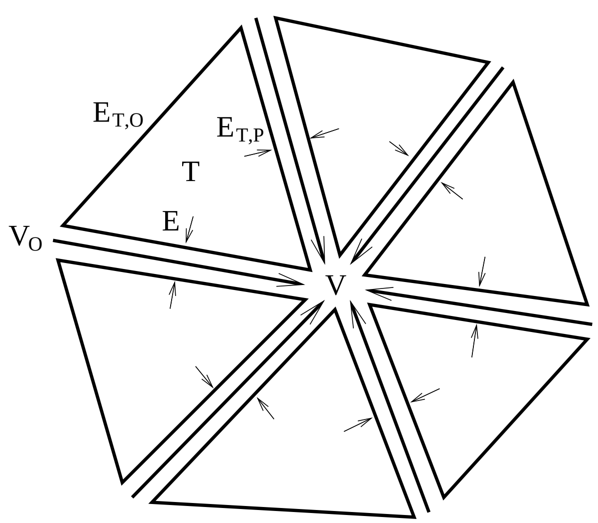
$$\operatorname{curl} H^\Delta = r$$

provides a true upper bound for the error:

$$\|\operatorname{curl}(A - A_h)\|_{\mu^{-1}} \le \|H^\Delta\|_\mu$$

Localization





Galerkin orthogonality leads to one equation for each edge:

$$\frac{1}{6}\sum_{T:E\subset T}\left\{\widehat{r_T}^{E_{T,1}}-\widehat{r_T}^{E_{T,2}}\right\}-\frac{1}{2}\left\{\widehat{r_E}^{V_1}-\widehat{r_E}^{V_2}\right\}=0.$$

Divergence-free local decomposition:

$$\begin{aligned}
\widehat{r_{\omega_V,T}}^{E} &:= \frac{1}{2}\widehat{r_T}^{E}+\frac{1}{6}\left(\widehat{r_T}^{E_{T,O}}-\widehat{r_T}^{E_{T,P}}\right),\\
\widehat{r_{\omega_V,T}}^{E_P} &:= \frac{1}{2}\widehat{r_T}^{E_{T,P}}+\frac{1}{6}\left(\widehat{r_T}^{E_{T,O}}-\widehat{r_T}^{E}\right),\\
\widehat{r_{\omega_V,T}}^{E_O} &:= 0,\\
\widehat{r_{\omega_V,E}}^{V} &:= \widehat{r_E}^{V},\\
\widehat{r_{\omega_V,E}}^{V_O} &:= 0\,.
\end{aligned}$$

High Order Methods - Construction

Residual

$$\langle r, v\rangle = a(u-u_h, v) = \sum_T (f+\Delta u_h, v) + \sum_E ([\partial_n u_h], v) = \sum_T (r_T, v)_T + \sum_E (r_E, v)_E$$

with polynomial element terms r_T and polynomial edge terms r_E.
Localization:

$$\langle r_V, v\rangle := \langle r, \varphi_V v\rangle = \sum_T (\varphi_V r_T, v)_T + \sum_E (\varphi_V r_E, v)_E,$$

i.e. $r_{V,T} = \varphi_V r_T$ and $r_{E,T} = \varphi_V r_E$.

The r_V form a decomposition of r, i.e.,

$$\left\langle \sum r_V, v \right\rangle = \sum \langle r, \varphi_V v\rangle = \langle r, v\rangle\,,$$

and are bi-orthogonal to constants, i.e.,

$$\langle r_V, 1\rangle = \langle r, \varphi_V\rangle = 0.$$

Thus, there exists a high order, discontinuous RT fe function with homogeneous b.c. σ_V such that $\operatorname{div}\sigma_V = r_V$.

p-robust Efficiency

Step 1: Local decomposition is stable:

$$\sum_V \|r_V\|^2_{[H^1(\omega)]^*} \preceq \|r\|^2_{[H^1_{0,D}(\Omega)]^*}$$

Step 2: Find polynomial right inverse to div on patches, uniformly bounded in $H^{-1} \to L_2$:

$$\sigma \in RT^p_{-1} : \operatorname{div} \sigma = r_V, \qquad \|\sigma\|_{L_2} \leq c \|r\|_{[H^1(\omega)]^*}$$

Requires

a) continuous right inverses on elements
tensor product elements: Braess, Pillwein, JS
simplicial elements: Costabel, McIntosh
b) div-preserving extension operators from element-boundaries
tensor product elements: Costabel, Dauge, Demkowicz
triangles: Ainswoth, Demkowicz
tetrahedral elements: Demkowicz, Gopalakrishnan, JS (preprint)

Continuous right inverse on the quadrilateral

Problem: given $f_p \in P^{p,p}(Q)$, find $\sigma_p \in RT^p$ such that $\operatorname{div} \sigma = f$.

Construction: Solve Dirichlet problem:

$$-\Delta u = f_p, \quad u = 0 \text{ on } \partial Q, \qquad \sigma := \nabla u$$

need commuting projection operators in 1D which are L_2-bounded:

$$(P^{p+1} v)' = \widetilde{P}^p (v')$$

Project σ back to polynomials:

$$\sigma_p = (P^{p+1} \otimes \widetilde{P}^p \sigma_x, \widetilde{P}^p \otimes P^{p+1} \sigma_y)$$

Then

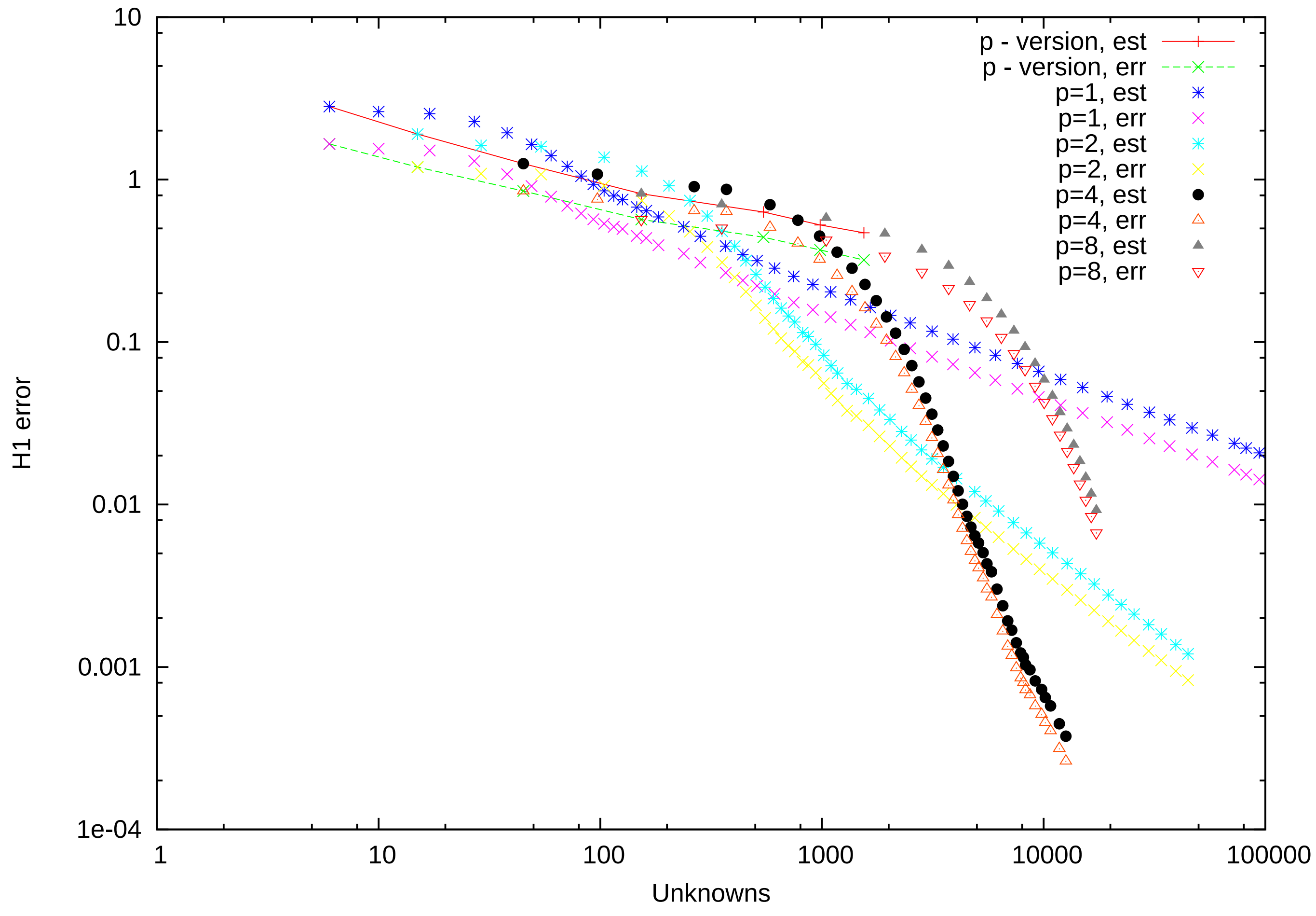
$$\operatorname{div} \sigma_p = \widetilde{P}^p \otimes \widetilde{P}^p \, \operatorname{div} \sigma = \widetilde{P}^p \otimes \widetilde{P}^p \, f_p = f_p$$

and

$$\|\sigma_p\|_{L_2} \preceq \|\sigma\|_{L_2}$$

Numerical Experiments

L-shape domain, mixed b.c. in non-convex vertex, $f = 1$,



Summary

We have

- Fully reliable and locally efficient error estimator for scalar and magnetostatic equations with lowest order elements

- Fully reliable EE for scalar equation with high order elements with p-robust efficiency.

D. Braess, J.S: Equilibrated Residual Error Estimates for Maxwell's Equations, Math. Comp., 2008

D. Braess, V. Pillwein, J.S.: Equilibrated Residual Error Estimates are p-robust, Comp. Meth. Appl Mech. Eng, 2009

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Happy Birthday, Ragnar !