

Domain Decomposition Methods for Hybrid Discontinuous Galerkin Methods

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Hybrid Discontinuous Galerkin (HDG) Method

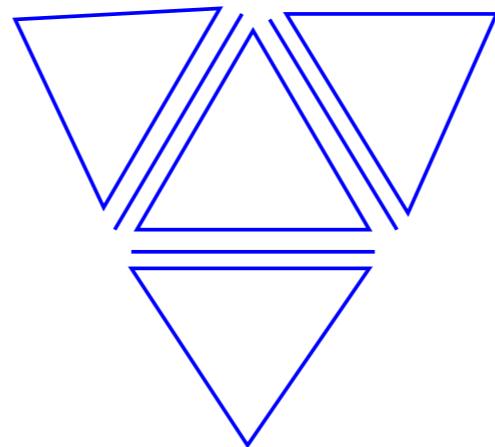
Model problem: $-\Delta u = f$ with mixed boundary conditions

A mesh consisting of elements and facets (= edges in 2D and faces in 3D)

$$\mathcal{T} = \{T\} \quad \mathcal{F} = \{F\}$$

Goal: Approximate u with piece-wise polynomials on elements and additional polynomials on facets:

$$u_N \in P^p(\mathcal{T}) \quad \lambda_N \in P^p(\mathcal{F})$$



HDG - Derivation

Exact solution u , traces on facets: $\lambda := u|_{\mathcal{F}}$

Integrate against discontinuous test-functions $v \in H^1(\mathcal{T})$, element-wise integration by parts:

$$\sum_T \left\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} v \right\} = \int_{\Omega} f v$$

Use continuity of $\frac{\partial u}{\partial n}$, and test with single-valued functions $\mu \in L_2(\mathcal{F})$:

$$\sum_T \left\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} (v - \mu) \right\} = \int_{\Omega} f v$$

Use consistency $u = \lambda$ on ∂T to symmetrize, and stabilize ...

$$\sum_T \left\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} (v - \mu) - \int_{\partial T} \frac{\partial v}{\partial n} (u - \lambda) + \alpha (u - \lambda, v - \mu)_{j, \partial T} \right\} = \int_{\Omega} f v$$

Dirichlet b.c.: Imposed on λ , Neumann b.c.: $\int_{\Gamma_N} g \mu$

Interior penalty method

Stabilization with α suff large

$$\alpha (u - \lambda, v - \mu)_{j,\partial T} = \frac{\alpha p^2}{h} (u - \lambda, v - \mu)_{L_2(\partial T)}$$

Norm:

$$\|(u, \lambda)\|_{1,HDG}^2 := \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,T}^2$$

Stability is proven by Young's inequality and inverse inequality $\|\frac{\partial u}{\partial n}\|_{L_2(\partial T)}^2 \leq c_{inv} \frac{p^2}{h} \|\nabla u\|_{L_2(T)}^2$:

$$\begin{aligned} A^T(u, \lambda; u, \lambda) &= \|\nabla u\|_{L_2(T)}^2 - \underbrace{2 \int_{\partial T} \frac{\partial u}{\partial n} (u - \lambda)}_{\leq \sqrt{\frac{c_{inv}}{\alpha}} \|\nabla u\|_{L_2(T)}^2 + \sqrt{c_{inv} \alpha} \frac{p^2}{h} \|u - \lambda\|_{L_2(\partial T)}^2} + \frac{\alpha p^2}{h} \|u - \lambda\|_{L_2(\partial T)}^2 \\ &\simeq \|(u, \lambda)\|_{1,HDG}^2 \end{aligned}$$

for $\alpha > c_{inv}$.

Bassi-Rebay type method

Stabilization term is

$$\alpha(u - \lambda, v - \mu)_{j,\partial T} = \alpha(r(u - \lambda), r(v - \mu))_{L_2(T)}$$

with lifting operator $r : P^p(\mathcal{F}_T) \rightarrow [P^p(T)]^d$ such that

$$(r(u - \lambda), \sigma)_{L_2(T)} = (u - \lambda, \sigma_n)_{L_2(\partial T)} \quad \forall \sigma \in [P^p(T)]^d$$

The corresponding jump-norm is

$$\|u - \lambda\|_{j,\partial T} = \sup_{\sigma \in [P^p(T)]^d} \frac{(u - \lambda, \sigma_n)_{L_2(\partial T)}}{\|\sigma\|_{L_2(T)}}$$

Stability for any $\alpha > 1$:

$$\begin{aligned} A^T(u, \lambda; u, \lambda) &= \|\nabla u\|_{L_2(T)}^2 - \underbrace{2 \int_{\partial T} \frac{\partial u}{\partial n} (u - \lambda)}_{\leq \|\nabla u\|_{L_2(T)} \sup_{\sigma \in [P^p]_T^d} \frac{\int_{\partial T} \sigma_n (u - \lambda)}{\|\sigma\|_{L_2(T)}}} + \alpha \|u - \lambda\|_{j,T}^2 \\ &\simeq \|(u, \lambda)\|_{1,HDG}^2 \end{aligned}$$

Facet-wise Bassi-Rebay type method

Stabilization term is

$$\alpha (u - \lambda, v - \mu)_{j,\partial T} = \alpha \sum_{F \in \mathcal{F}_T} (r_F(u - \lambda), r_F(v - \mu))_{L_2(T)}$$

with lifting operator $r_F : P^p(F) \rightarrow [P^p(T)]^d$ such that

$$(r(u - \lambda), \sigma)_{L_2(T)} = (u - \lambda, \sigma_n)_{L_2(F)} \quad \forall \sigma \in [P^p(T)]^d$$

The corresponding jump-norm is (here we assume non-curved elements)

$$\|u - \lambda\|_{j,\partial T} = \sup_{\sigma \in [P^p(T)]^d} \frac{(u - \lambda, \sigma_n)_{L_2(\partial T)}}{\|\sigma\|_{L_2(T)}} = \sup_{\sigma \in P^p(T)} \frac{(u - \lambda, \sigma)_{L_2(\partial T)}}{\|\sigma\|_{L_2(T)}}$$

Stability for any $\alpha > |\mathcal{F}_T|$.

Error estimates

Follows from consistency and discrete stability:

$$\begin{aligned}\| (u - u_N, u - \lambda_N) \|_{1,HDG} &\preceq \inf_{v_N, \mu_N} \left\{ \| \nabla(u - v_N) \|_{L_2(\mathcal{T})} + \| u_N - \lambda_N \|_j + \| \partial_n u - \partial_n u_N \|_{j^*} \right. \\ &\preceq p^\gamma \frac{h^s}{p^s} \| u \|_{H^{1+s}}\end{aligned}$$

- for $1 \leq s \leq p$
- with $\gamma = 1/2$ or $\gamma = 0$ depending on mesh-conformity, and jump-term.

Relation to standard IP/BR DG method

DG - space

$$V_N := P^p(\mathcal{T})$$

Bilinear-form

$$A^{DG}(u, v) = \sum_T \left\{ \int_T \nabla u \nabla v - \frac{1}{2} \int_{\partial T} \frac{\partial u}{\partial n} [v] - \frac{1}{2} \int_{\partial T} \frac{\partial v}{\partial n} [u] + \alpha ([u], [v])_j \right\}$$

Hybrid DG has

- even more unknowns, but less matrix entries
- allows element-wise assembling
- allows static condensation of element unknowns

Hybridization of standard DG methods [Cockburn+Gopalakrishnan+Lazarov]

Good properties

mathematical:

- conservative fluxes: σ_n defined by $(\sigma_n, \mu) = A_T(u, \lambda; 0, \mu)$ $\forall \mu$ satisfies

$$\int_{\partial T} \sigma_n = \int_T f$$

- allows upwinding for convection dominated problems
- allows interesting elements with partial continuity (LBB, ...)

computational:

- only polynomial spaces on T and F , no bubbles for T, F, E and V .
- only direct neighbour communication in parallel computing

Incompressible Navier Stokes Equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(u \otimes u) - \nu \Delta u + \nabla p = f, \quad \operatorname{div} u = 0$$

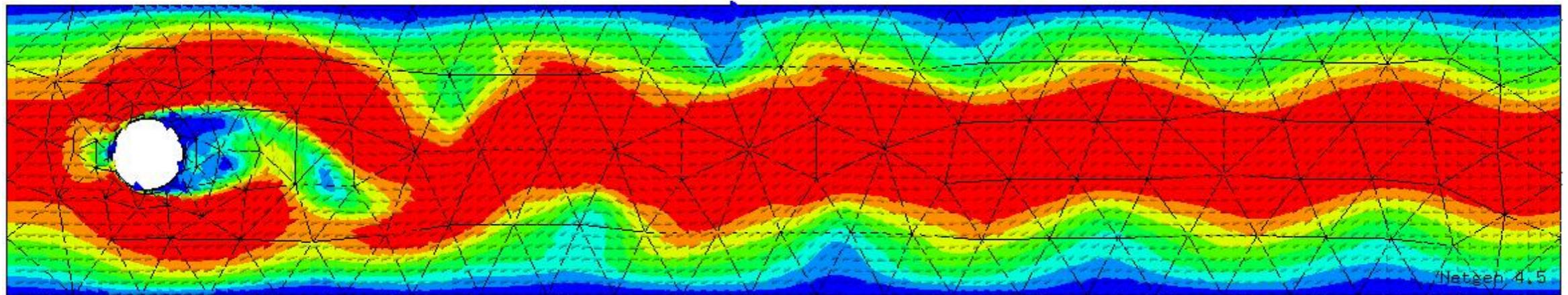
Semi-Implicit time discretization

$$\begin{aligned} \frac{1}{\tau} M(\hat{u} - u) - \nu \Delta_h \hat{u} + B\hat{p} &= f - \operatorname{div}_h(u \otimes u) \\ B\hat{u} &= 0 \end{aligned}$$

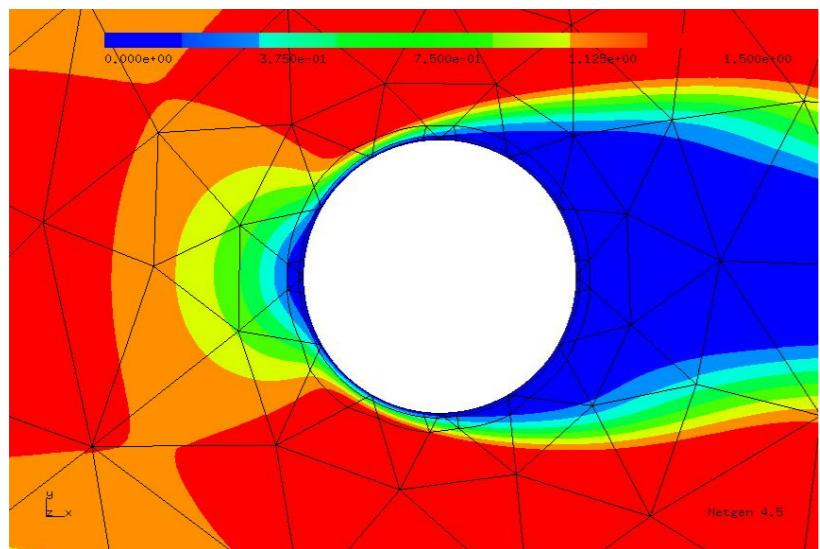
- $u_h \in V_h := BDM_k \subset H(\operatorname{div}) \times V_{facet,\tau}$, $p \in Q_h := P_{k-1} \subset L_2$
hybrid form of [Cockburn+Kanschat+Schötzau, 07]
- u_h is exactly div-free
- viscosity term by hybrid DG (facet element with tangential component)
- convective term by upwinding
- bound for kinetic energy ($\frac{d}{dt} \|u\|_0^2 \leq \frac{1}{\nu} \|f\|_{L_2}^2$)

Flow around a disk, 2D

$Re = 100$, 5th-order elements

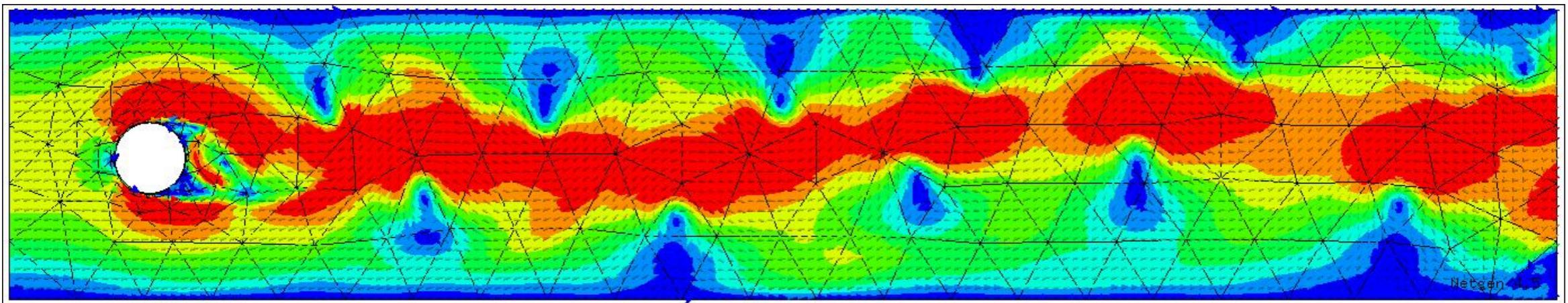


Boundary layer mesh around cylinder:

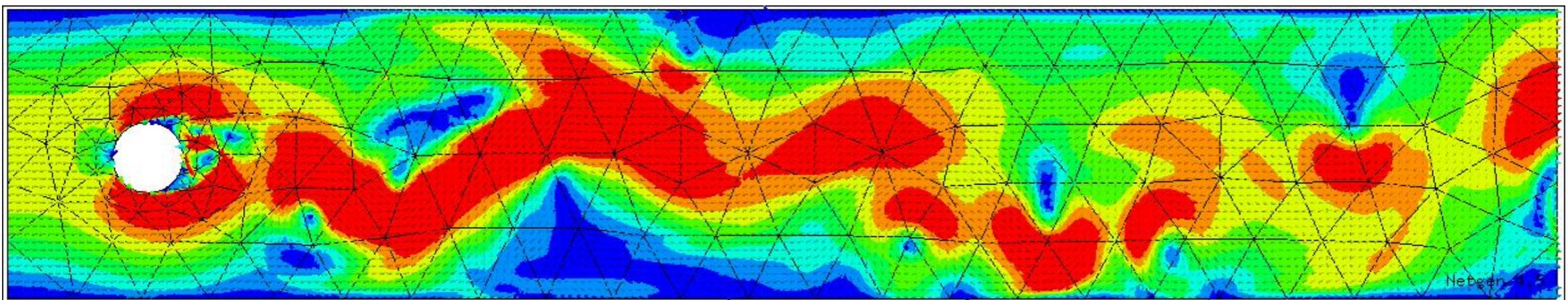


Flow around a disk, 2D

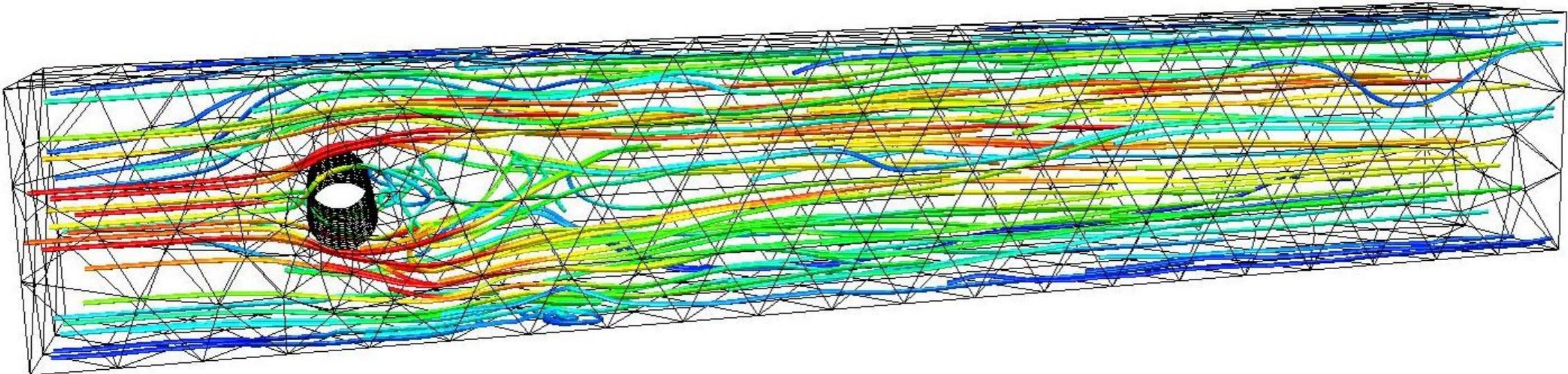
$Re = 1000$:



$Re = 5000$:



Flow around a cylinder, $Re = 100$



$p = 5$, $N \approx 5 \cdot 10^6$, 20 sec per timestep

on 4×10 -core Intel Xeon server

Master's thesis Christoph Lehrenfeld, 2011, [H. Egger+C. Waluga]

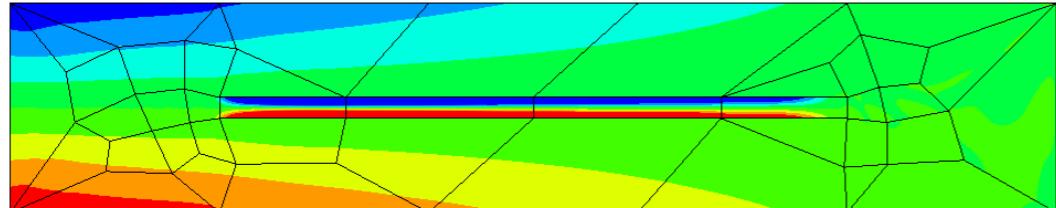
Hybrid DG in elasticity

Tangential components continuous, normal component by HDG

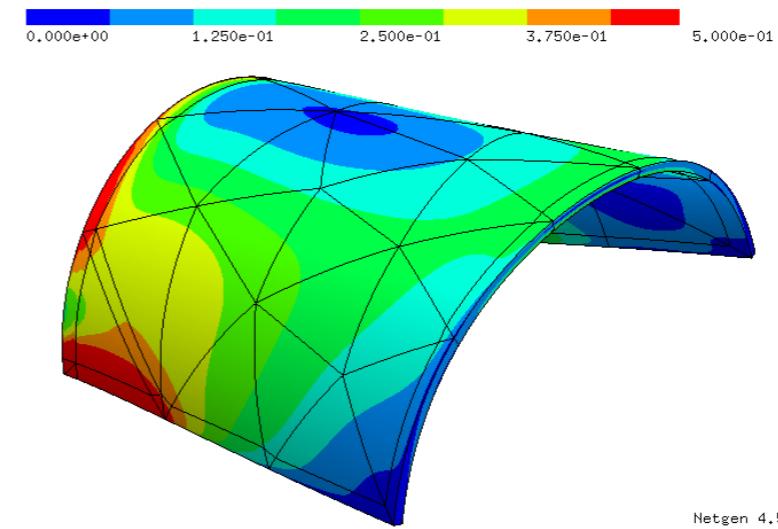
HDG version of tangential-displacement normal-normal-stress (TD-NNS) mixed method

Anisotropic error estimates:

$$\sum_T \|\varepsilon(u - u_h)\|_T^2 + \sum_F h_{op}^{-1} \| [u_n] \|^2_F + \|\sigma - \sigma_h\|^2 \leq c \{ h_x^p \|\partial_x^p \varepsilon(u)\| + h_y^p \|\partial_y^p \varepsilon(u)\| \}^2$$



Reinforcement with $E = 50$ in medium
with $E = 1$.



PhD thesis A. Pechstein (born Sinwel) '09, [A. Pechstein + JS, '11]

Hybrid method for the Helmholtz Equation

$$\begin{aligned}-\Delta u - \omega^2 u &= f \\ \frac{\partial u}{\partial n} + i\omega u &= g \quad \text{impedance b.c.}\end{aligned}$$

Hybrid discretization with left- and right-going traces on element-interfaces:

$$g_l = \frac{\partial u}{\partial n} + i\omega u \quad g_r = \frac{\partial u}{\partial n} - i\omega u$$

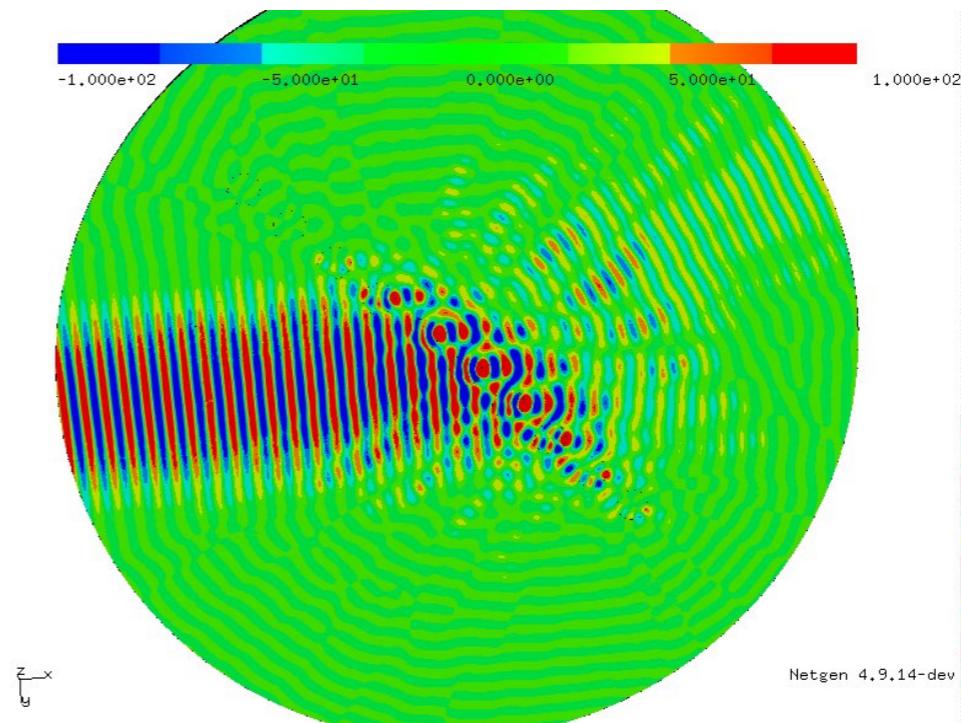
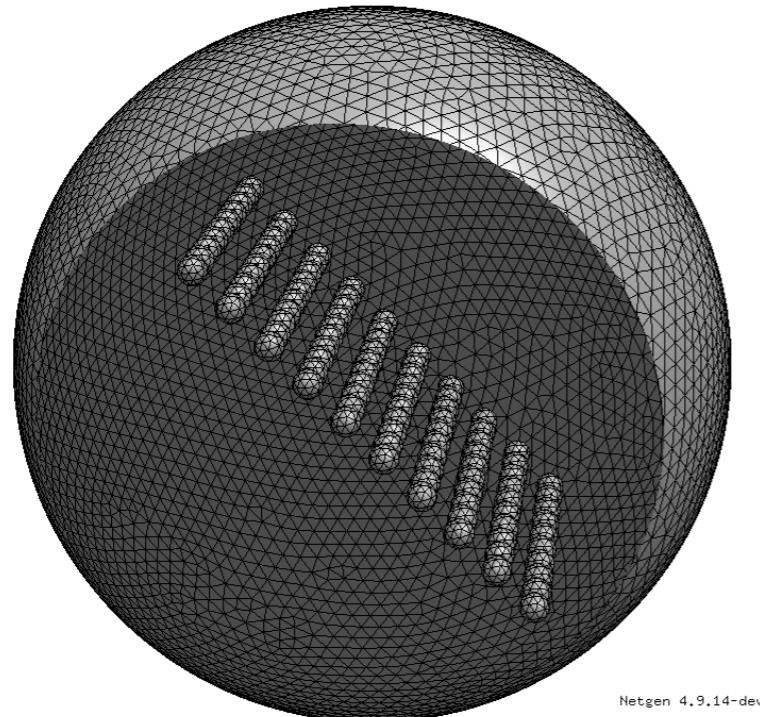
Goal: scalable iterative solver [P. Monk, A. Sinwel (now Pechstein), JS, 2010]

Diffraction from a grating

Sphere with $D = 40\lambda$, $127k$ elements, $h \approx \lambda$, $p = 5$, $39M$ dofs (corresponds to $9.4M$ primal dofs)

78 sub-domains / processes

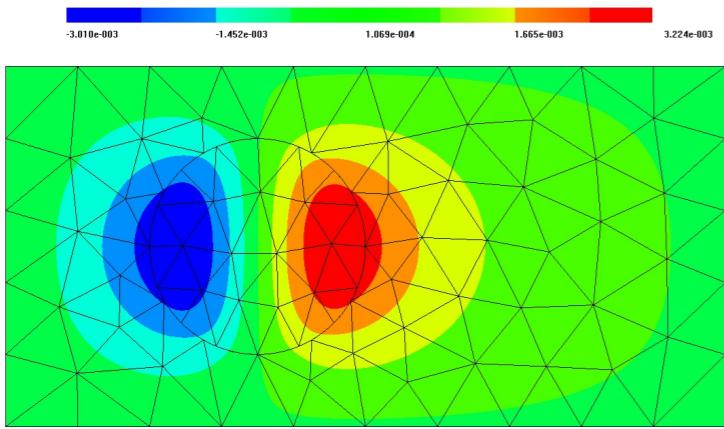
$T_{ass} = 9m$, $T_{pre} = 12m$, $T_{solve} = 21m$, 156 its



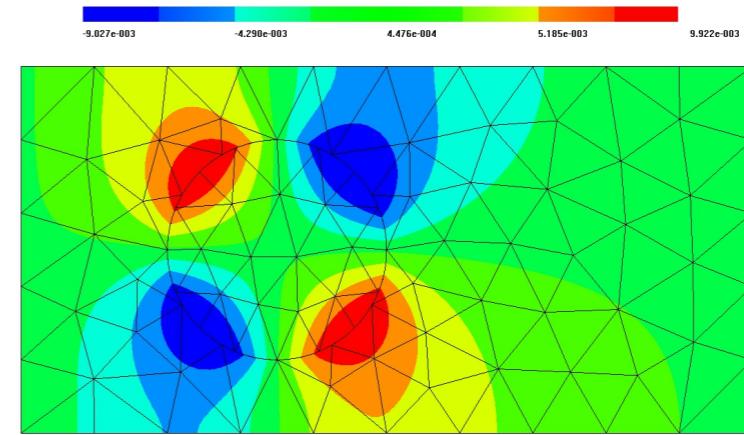
HDG/Nitsche for non-matching meshes for 2D Laplace

$f = x$ in circle, else $f = 0$.

Solution u :



Solution $\partial u / \partial x$:

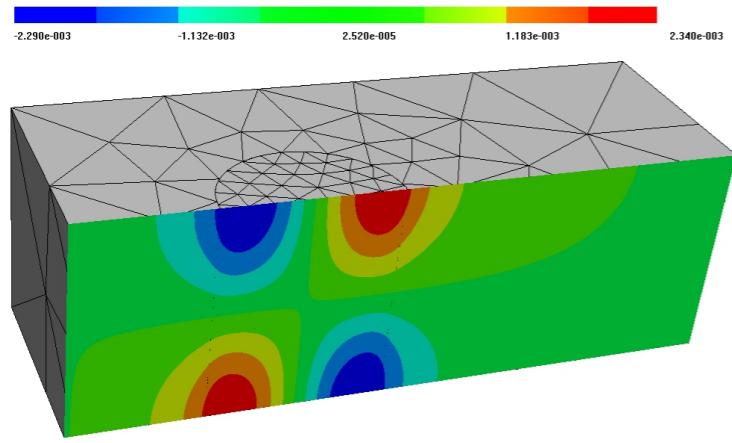


Finite element order $p = 5$.

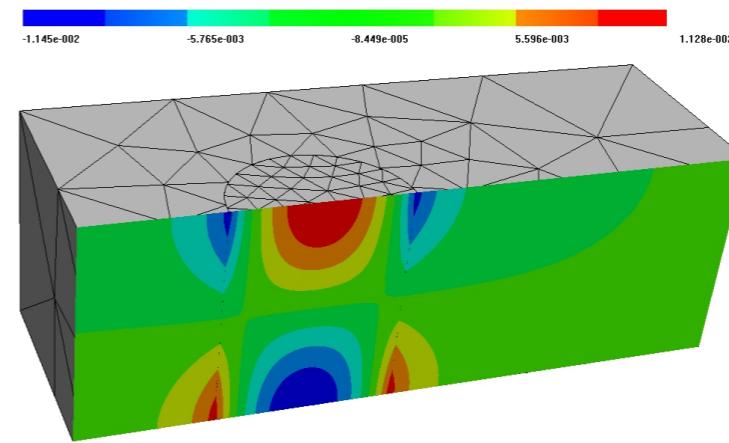
HDG/Nitsche for non-matching meshes for 3D Laplace

$f = xz$ in cylinder, else $f = 0$.

Solution u :



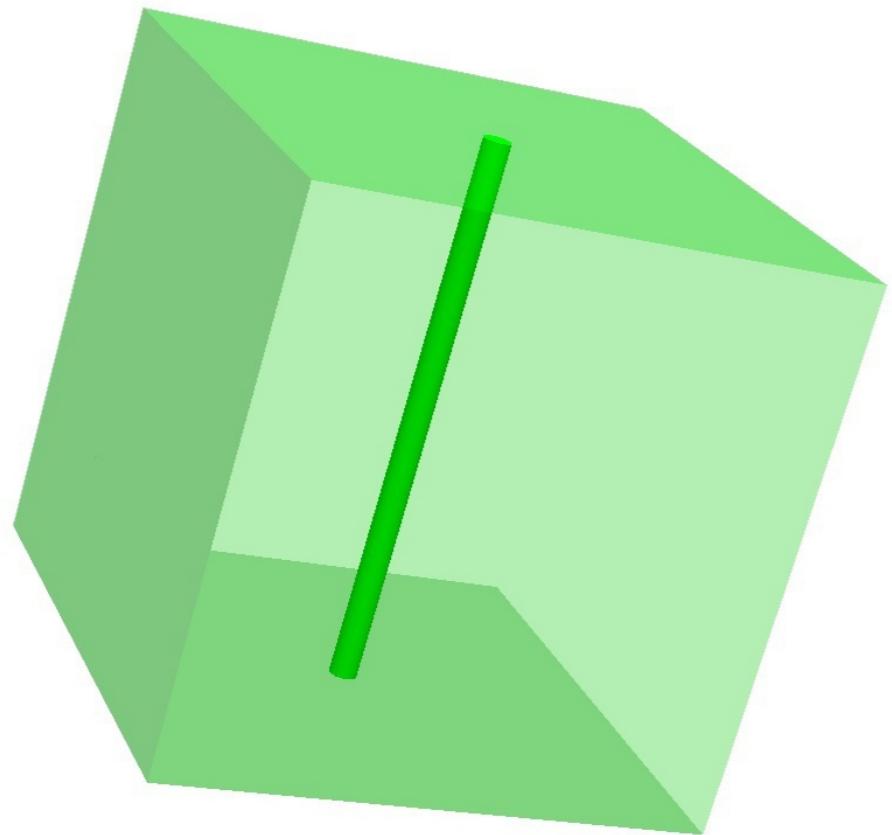
Solution $\partial u / \partial x$:



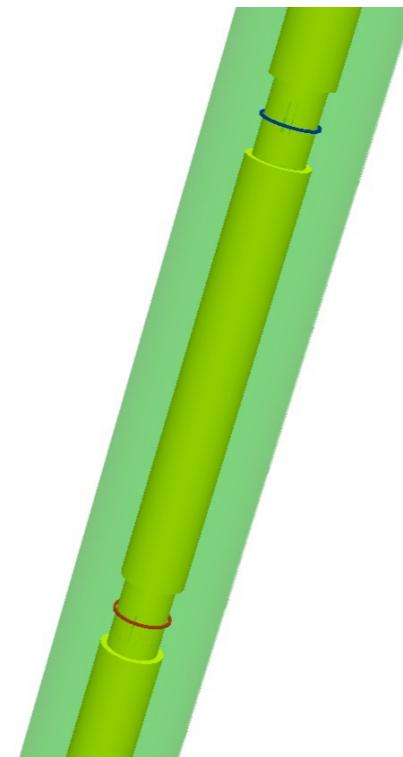
Finite element order $p = 4$.

Bore-hole electromagnetics

borehole with soil

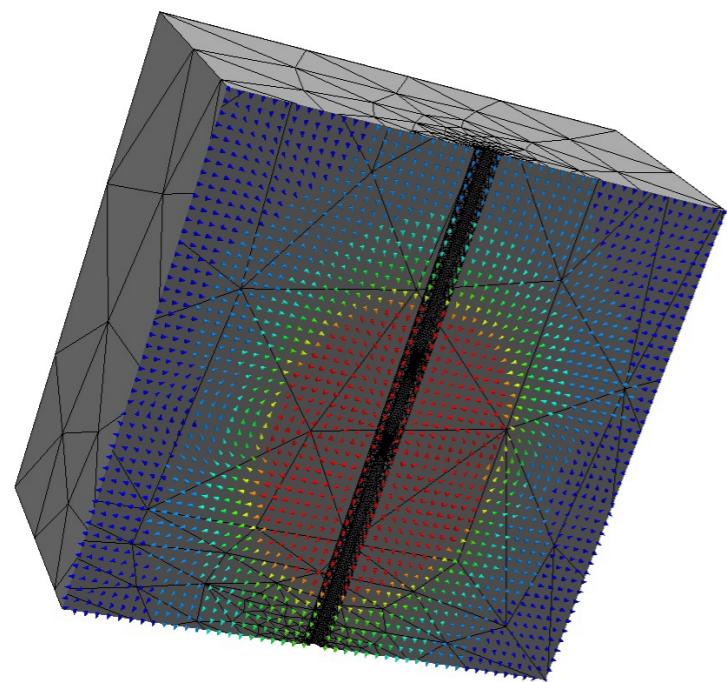


tool with antennas

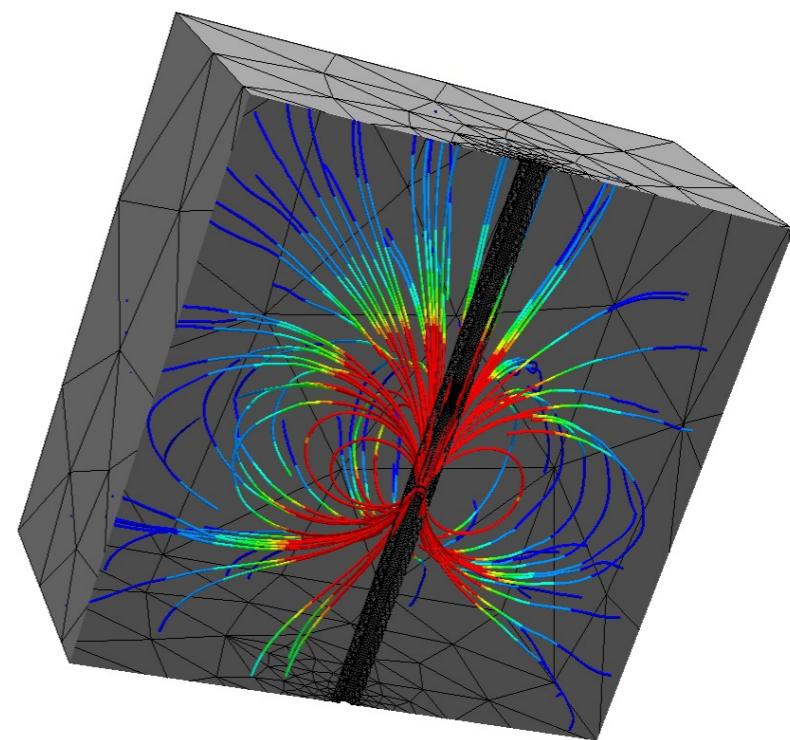


Bore-hole electromagnetics

B-field



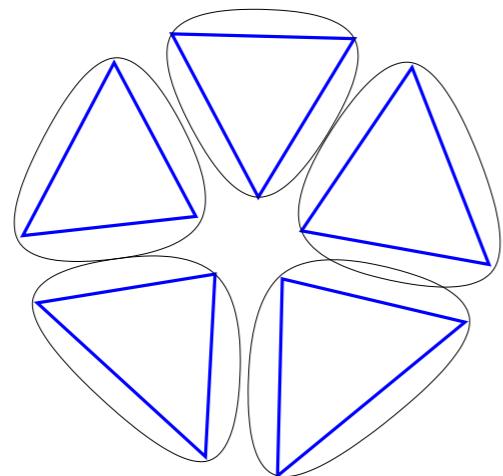
B-field, field-lines:



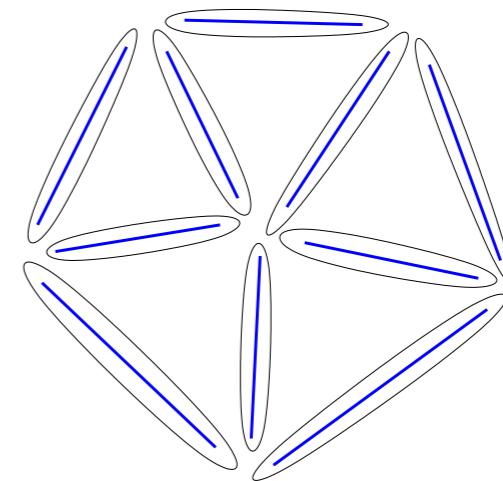
Master's thesis Daniel Feldengut '11, [Hollaus,Feldengut,JS,Wabro,Omeragic '11]

How to solve ?

Standard DG



Hybrid DG
with facet Schur-complement S



$$\kappa\{C_{ASM}^{-1}A\} \simeq p^2$$

for element-by-element Schwarz
preconditioner C_{ASM} plus coarse grid
[Antonietti+Houston,11]

$$\kappa\{C_{ASM}^{-1}S\} \simeq (\log p)^\gamma$$

for facet-by-facet Schwarz preconditioner
 C_{ASM} plus coarse grid

Trace norms inequality

For $\lambda \in P^p(F)$ define semi-norm and norm

$$|\lambda|_F^2 := \inf_{u \in P^p} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 \right\}$$
$$\|\lambda\|_{F,0}^2 := \inf_{u \in P^p} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 + \|u - 0\|_{j,\partial T \setminus F}^2 \right\}$$

mimic $|\cdot|_{H^{1/2}(F)}$ and $\|\cdot\|_{H_{00}^{1/2}(F)}$.

Theorem: For $\lambda \in P^p(F)$ with $\int_F \lambda = 0$ there holds

$$\|\lambda\|_{F,0}^2 \leq (\log p)^\gamma |\lambda|_F^2 \quad \text{with } \gamma = 3$$

- if T is a trig, quad, or hex, and $\|\cdot\|_j$ is IP or BR
- if T is a tet, and $\|\cdot\|_j$ is BR

From the trace norms inequality we get immediately condition number estimates for Schwarz methods and BDDC preconditioners

Meta-proof of trace norms inequality

scale to reference element, given $\lambda \in P^p(F)$ with $\int_F \lambda = 0$. Take $u \in P^p(T)$ such that

$$\|u\|_{H^1(T)}^2 + \|u - \lambda\|_j^2 \preceq |\lambda|_F^2$$

(Poincare-type estimate). Projection-based interpolation technique

$$\begin{aligned} u_2 &= u - \sum_{V \in F} \mathcal{E}_{V \rightarrow T} u(V) \\ u_3 &= u_2 - \sum_{E \subset F} \mathcal{E}_{E \rightarrow T} u_2|_E \quad (3D \text{ only}) \\ \tilde{u} &= \mathcal{E}_{F \rightarrow T} u_3 \end{aligned}$$

gives $\tilde{u} \in P^p$ such that $\tilde{u} = 0$ on $\partial T \setminus F$, and

$$\|\nabla \tilde{u}\|_{L_2(T)}^2 + \|\tilde{u} - u\|_{j,F}^2 \preceq (\log p)^\gamma \|u\|_{H^1(T)}^2,$$

and thus

$$\|\lambda\|_{F,0}^2 \preceq \|\nabla \tilde{u}\|_{L_2(T)}^2 + \|\lambda - \tilde{u}\|_j^2 \preceq (\log p)^\gamma \|u\|_{H^1(T)}^2 + \|\lambda - u\|_j^2 \preceq (\log p)^\gamma \|\lambda\|_F^2$$

Components for quads and trigs

define polynomial fast decaying function (Pavarino+Widlund)

$$l^p(x) = \underset{\substack{v \in P^p \\ v(0)=1, v(1)=0}}{\operatorname{argmin}} \int_0^1 v^2(x) dx$$

there holds

$$\|l^p\|_0^2 \preceq p^{-2} \quad \text{and} \quad \|(l^p)'\|_0^2 \preceq p^2$$

Vertex-to-Element extension:

$$\mathcal{E}_{V \rightarrow T} u = u(V) l^p(1 - \lambda_v)$$

Then

$$\|\mathcal{E}_{V \rightarrow T} u\|_{H^1(T)}^2 + p^2 \|\mathcal{E}_{V \rightarrow T} u\|_{L_2(F)}^2 \preceq u(V)^2 \preceq \log p \|u\|_{H^1(T)}^2$$

Components for hexes

need additionally edge-to-element extension for $u_E \in P_0^p(E)$

$$\mathcal{E}_{E \rightarrow T} u_E := u(x)l^p(1-y)l^p(1-z) \quad \text{for } E = (x, 1, 1)$$

satisfies

$$\|\mathcal{E}_{E \rightarrow T} u\|_{H^1(T)}^2 + p^2 \|\mathcal{E}_{E \rightarrow T} u\|_{L_2(F)}^2 \preceq \|u\|_{L_2(E)}^2 \preceq \log p \|u\|_{H^1(T)}^2$$

Difficulty on tets

- Cannot multiply with fast decaying function in polynomial space of total order p
- Babuška et al averaging - extension, from $E = (x, 0, 0)$:

$$\mathcal{E}_{E \rightarrow T} u(x, y, z) := \frac{1}{y + z} \int_{x-y-z}^{x+y+z} u(s) ds$$

does not give low energy in jump-norm !

- On triangles, the estimate

$$\min_{\substack{v \in P^p(T) \\ v=u \text{ on } E}} p \|v\|_{L_2(T)}^2 \preceq \|u\|_{L_2(E)}^2$$

is sharp (in contrast to p^2 on quads!).

Way out on tets

- Bassi-Rebay stabilization is essentially weaker than interior penalty !
- Construct new fast decaying edge-to-tet extension operator motivated by Pavarino-Widlund construction

Technical tool: trace estimate on the interval

Lemma:

$$\min_{v \in P^n, v(0)=1} \int_0^1 y^\alpha (1-y)^\beta v(y)^2 dy \simeq \frac{1}{n^{\alpha+1} (n+\beta)^{\alpha+1}}$$

Proof: expanding $v = \sum c_k P^{(\alpha, \beta)}(1-2y)$ leads to a quadratic minimization problem with a diagonal matrix and a scalar constraint, direct evaluation of the KKT system gives

$$\frac{1}{\min} = \sum_{k=0}^n (2k + \alpha + \beta + 1) \frac{(k + \alpha)! (k + \alpha + \beta)!}{(\alpha!)^2 k! (k + \beta)!}$$

by means of Gosper's algorithm for hypergeometric summation, friends from computer algebra (Veronika Pillwein) computed the closed form

$$\frac{1}{\min} = \frac{(n + \alpha + 1)! (n + \alpha + \beta + 1)!}{\alpha! (\alpha + 1)! n! (n + \beta)!}$$

Bassi-Rebay jump norm

Lemma: tet T with face F , and $\lambda \in P^p(F)$, \mathbf{P}^p is $L_2(F)$ -orthogonal projection. Then:

$$\sup_{\sigma \in P^p(T)} \frac{(\lambda, \sigma)_{L_2(F)}^2}{\|\sigma\|_{L_2(T)}^2} \simeq \sum_{k=0}^p p(p-k+1) \|(\mathbf{P}^p - \mathbf{P}^{p-1})\lambda\|_{L_2(F)}^2$$

Proof: Expand λ and σ in $L_2(F)$ -orthogonal Dubiner basis:

$$\begin{aligned}\lambda(x, y) &= \sum_{i+j \leq p} \lambda_{ij} \varphi_{ij}(x, y) \\ \sigma(x, y, z) &= \sum_{i+j \leq p} \sigma_{ij}(z) \varphi_{ij}\left(\frac{x}{1-z}, \frac{y}{1-z}\right) (1-z)^{i+j}\end{aligned}$$

with $\sigma_{ij} \in P^{p-i-j}$. Then

$$\|\sigma\|_{L_2(T)}^2 = \sum_{i+j \leq p} \|\varphi_{ij}\|^2 \int_0^1 (1-z)^{2i+2j+2} \sigma_{ij}(z)^2 dz.$$

$$\begin{aligned}
\sup_{\sigma \in P^p(T)} \frac{(\lambda, \sigma)_F^2}{\|\sigma\|_T^2} &= \sup_{\sigma \in P^p} \frac{\left(\sum \lambda_{ij} \sigma_{ij}(0) \|\varphi_{ij}\|^2 \right)^2}{\sum_{ij} \int_0^1 (1-z)^{2i+2j+2} \sigma_{ij}^2(z) dz \|\varphi_{ij}\|^2} \\
&= \sum_{i+j \leq p} \sup_{\sigma_{ij} \in P^{p-i-j}} \frac{\sigma_{ij}(0)^2}{\int_0^1 (1-z)^{2i+2j+2} \sigma_{ij}(z)^2 dz} \lambda_{ij} \|\varphi_{ij}\|_{L_2(F)}^2 \\
&\simeq \sum_{i+j \leq p} p(p-i-j+1) \lambda_{ij} \|\varphi_{ij}\|_{L_2(F)}^2 \\
&= \sum_{k=0}^p p(p-k+1) \sum_{i+j=k} \|\lambda_{ij} \varphi_{ij}\|_{L_2(F)}^2 \\
&= \sum_{k=0}^p p(p-k+1) \|(\mathbf{P}^k - \mathbf{P}^{k-1}) \lambda\|_{L_2(F)}^2
\end{aligned}$$

First idea for simplicial extension operator

Let

$$e_i(y) = \operatorname{argmin}_{v \in P^{p-i}, v(0)=1} \int_0^1 y(1-y)^{2i+1} v(y)^2 dy$$

and $L_i(x) := \int_{-1}^x P_{i-1}(s) ds$ are the integrated Legendre pols.

For an edge-bubble

$$u_E(x) = \sum u_i L_i(x)$$

define the extension

$$\mathcal{E}_{E \rightarrow T} u_E(x, y, z) := \sum_{i=2}^p u_i L_i\left(\frac{x}{1-y-z}\right) (1-y-z)^i e_i(y+z).$$

Then

$$\|\mathcal{E}_{E \rightarrow F} u_E\|_{H^1(T)}^2 \simeq \sum_i \int_0^1 y e_i^2 dy \frac{u_i^2}{i} + \int_0^1 y (\partial_y(e_i - e_{i-2}))^2 dy \frac{(u_i - u_{i-2})^2}{i^3}$$

Problem: Term with differences $e_i - e_{i-2}$ is dominating too much.

Simplicial extension operator with averaging

Let

$$\tilde{e}_i(y) = \operatorname{argmin}_{v \in P^{p-i}, v(0)=1} \int_0^1 y(1-y)^{2i+1} v(y)^2 dy$$

and

$$e_i = \frac{1}{p-i+1} \sum_{k=i}^p (1-y)^{k-i} \tilde{e}_k(y).$$

Theorem: The extension

$$\mathcal{E}_{E \rightarrow T} u_E(x, y, z) := \sum_{i=2}^p u_i L_i \left(\frac{x}{1-y-z} \right) (1-y-z)^i e_i(y+z).$$

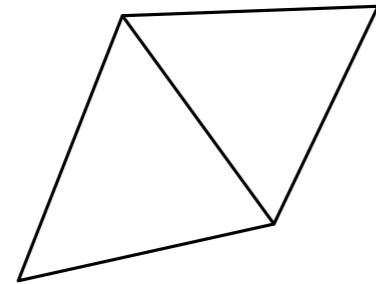
satisfies

$$\|\mathcal{E}_{E \rightarrow T} u_E\|_{H^1(T)}^2 + \|\mathcal{E}_{E \rightarrow F} u_E\|_j^2 \preceq \|u_E\|_{L_2(E)}^2$$

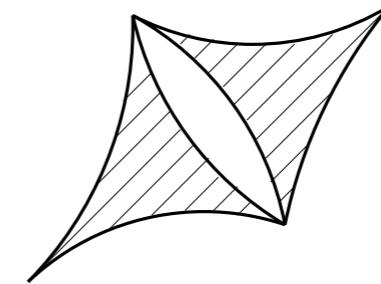
BDDC Preconditioning

[Dohrmann, Mandel, Tezaur, Li, Widlund, Tu, Brenner, Sung, Klawonn, Pavarino, Rheinbach, ... 2003+]

Model problem: Conforming high order finite elements for 2D Laplace:



continuous fe-space V
matrix A



discontinuous fe-space \tilde{V} with vertex constraints
matrix \tilde{A}

averaging operator $R : \tilde{V} \rightarrow V$

BDDC preconditioning action:

$$C_{BDDC}^{-1} = R \tilde{A}^{-1} R^t$$

for HDG: keep mean-value on facets continuous

The algorithm

Preconditioning action: $C_{BDDC}^{-1} : d \mapsto w$

1. $w := A_I^{-1}d$ (local pre-correction step)
2. $\tilde{d} := R_1^t(d - Aw)$ (residual distribution)
3. $\tilde{w} := \tilde{A}^{-1}\tilde{d}$ (global solve)
4. $w := w + R_1\tilde{w}$ (simple averaging of edge dofs)
5. $w := w + A_I^{-1}(d - Aw)$ (local post-correction step)

with

A_I element-wise Dirichlet matrix

R_1 simple averaging of interface-dofs

no local solves are necessary after static condensation

BDDC Analysis

There holds the representation (fictitious space lemma)

$$\|u\|_{C_{BDDC}}^2 = \inf_{\substack{y \in \tilde{V} \\ Ry = u}} \|y\|_{\tilde{A}}^2$$

It implies immediately

$$\sigma\{C_{BDDC}^{-1}A\} \subset [1, \|R\|^2]$$

with the norm of the averaging operator

$$\|R\| := \sup_{y \in \tilde{V}} \frac{\|Ry\|_A}{\|y\|_{\tilde{A}}}$$

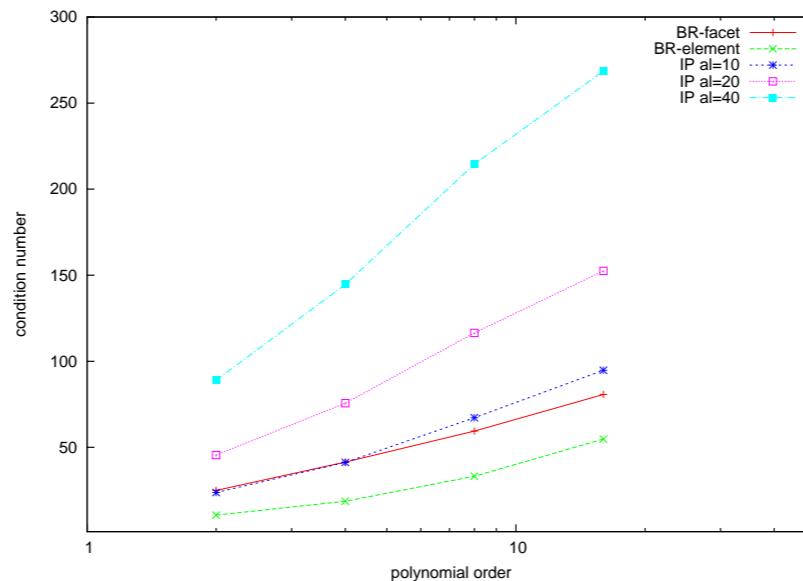
Estimating $\|R\|$

For $y \in \tilde{V}$ there holds

$$\begin{aligned}\|Ry\|_S^2 &= \sum_T |Ry|_{H^{1/2}(\partial T)}^2 \preceq \sum_T |Ry - y|_{H^{1/2}(\partial T)}^2 + \sum_T |y|_{H^{1/2}(\partial T)}^2 \\ &\preceq \sum_T \sum_{F \in \mathcal{F}_T} |Ry - y|_{H_{00}^{1/2}(F)}^2 + \sum_T |y|_{H^{1/2}(\partial T)}^2 \\ &\preceq (\log p)^\gamma \sum_{F \in \mathcal{F}} |[y]|_{H^{1/2}(F)}^2 + \sum_T |y|_{H^{1/2}(\partial T)}^2 \\ &\preceq (\log p)^\gamma \sum_T |y|_{H^{1/2}(\partial T)}^2 = (\log p)^\gamma \|y\|_{\tilde{S}}^2\end{aligned}$$

Condition numbers of BDDC

Laplace equation, mesh consisting of 184 tetrahedra, HDG discretization



- element-wise Bassi-Rebay with $\alpha = 1.5$ (nearly proven to be $O(\log^3 p)$)
- facet-wise Bassi-Rebay with $\alpha = 5$ (proven to be $O(\log^3 p)$)
- interior penalty with $\alpha = 10, 20, 40$ (only $O(p)$ is proven)

Conclusions

- DG and HDG are powerful methods with a lot of tuning possibilities for interesting equations
- HDG allows for fast iterative solvers
- freely available hp-finite element software Netgen/NgSolve from sourceforge
- [Lehrenfeld/JS]-preprint on HDG-DD analysis available from www.asc.tuwien.ac.at/~schoeberl