Hybrid Discontinuous Galerkin Methods for Fluid Dynamics and Solid Mechanics

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based on contributions by
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Start project “hp-FEM”

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Incompressible flows

Stokes Equation:

$\Omega \subset \mathbb{R}^d$. Find velocity $u \in [H^1]^d$ such that $u = u_D$ on Γ_D, and pressure $p \in Q := L_2$ such that

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega \operatorname{div} v \, p = \int_\Omega f v \qquad \forall \, v \in V_0$$

and incompressibility constraint

$$\int \operatorname{div} u \, q = 0 \qquad \forall \, q \in Q$$

with Dirichlet b.c. (no slip and inflow), point-wise mixed b.c. (slip) and Neumann (outflow).

Difficulty: Incompressibility constraint

Mixed finite elements: continuous pressure ? discontinuous pressure ? stabilized methods ?

Linear Elasticity

$\Omega \subset \mathbb{R}^d$. Find displacement $u \in [H^1]^d$ such that $u = u_D$ on Γ_D and

$$\int_\Omega D\varepsilon(u) : \varepsilon(v) = \int_\Omega f v \qquad \forall\, v \in V_0$$

with the linear strain operator $\varepsilon(\cdot) : [H^1]^d \to [L_2]^{d \times d, sym}$

$$\varepsilon(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big) = \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)_{i,j=1,..d}$$

and the isotropic material operator $D : [L_2]^{d\times d} \to [L_2]^{d \times d}$

$$D\varepsilon = 2\mu\varepsilon + \lambda \operatorname{tr}(\varepsilon) I$$

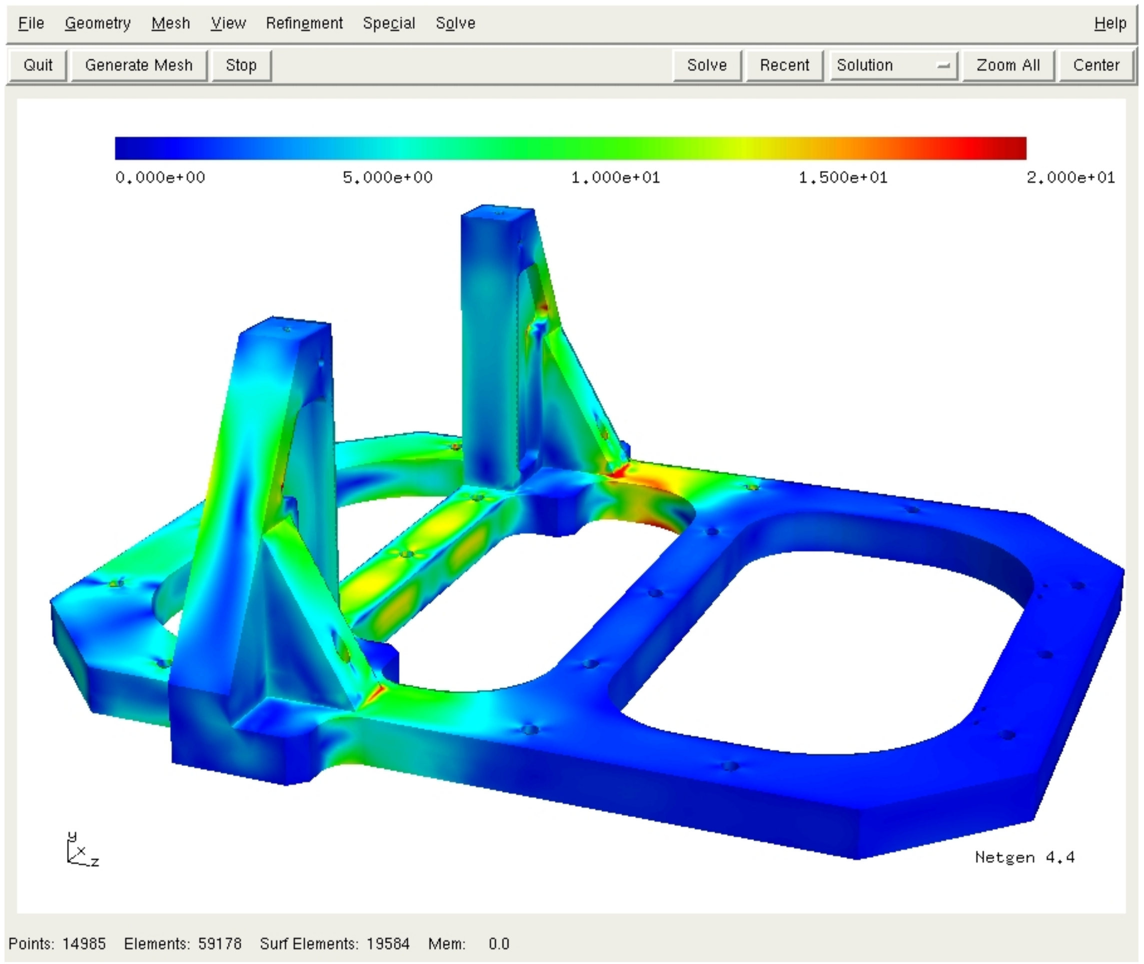
The stress tensor is

$$\sigma = D\varepsilon(u)$$

Continuous and elliptic in $[H^1]^d$

BUT: Constants depend on λ/μ, and on the domain (Korn's inequality) LOCKING !!

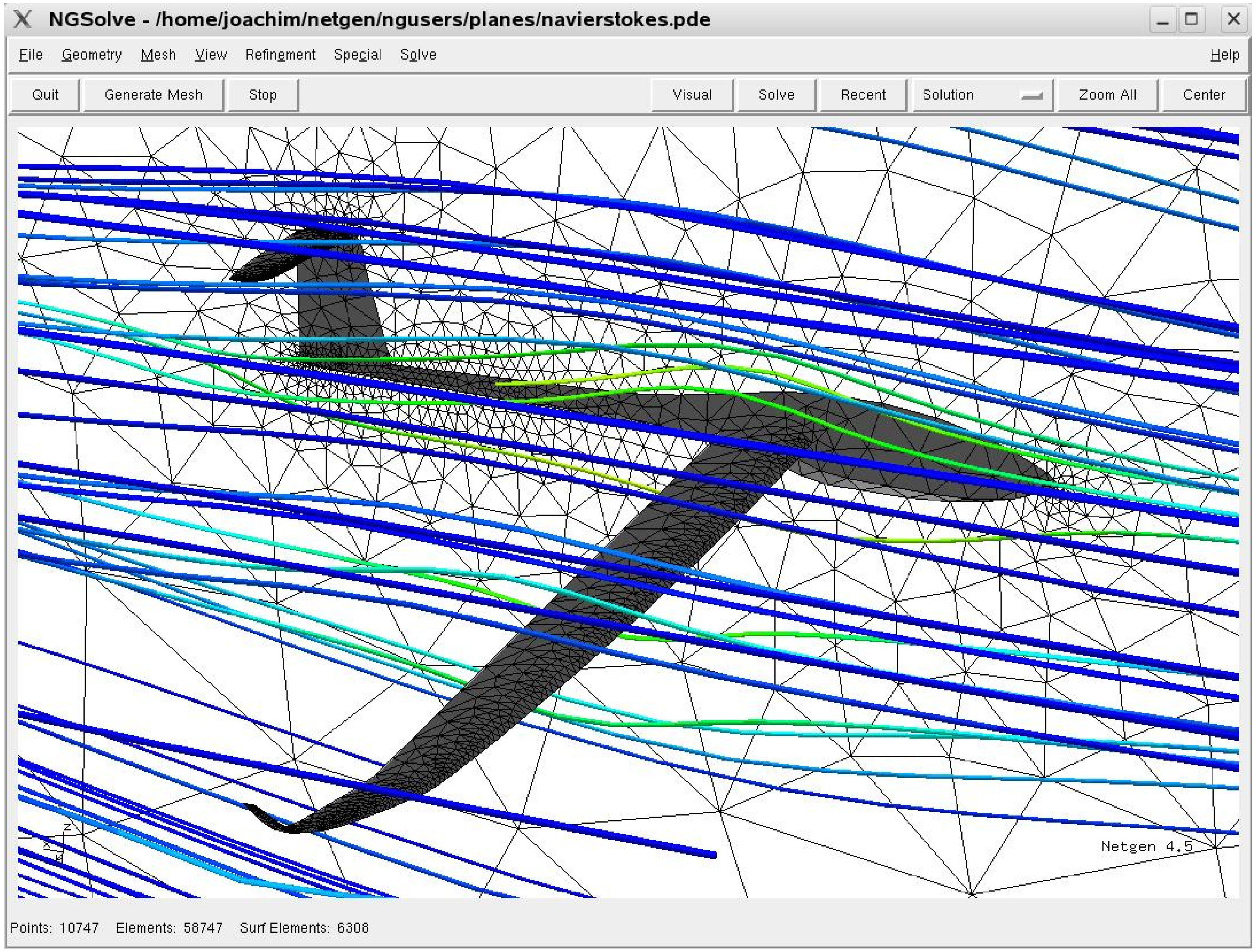
Von-Mises Stresses in a Machine Frame (linear elasticity)



Simulation with Netgen/NGSolve

45553 tets, p = 5, 3×1074201 unknowns, 5 min on 8 core 2.4 GHz 64-bit PC 6 GB RAM

Toy Example: Sailplane



Incomp. N.-St., 2^{nd}-order HDG elements, 59E3 elements, 1.65E6 dofs, 2GB RAM, 5 min (2-core 1.8GHz)

Function spaces $H(\mathrm{curl})$ and $H(\mathrm{div})$

$$
\begin{aligned}
H(\mathrm{curl}) &= \{u \in [L_2]^d : \mathrm{curl}\, u \in L_2^{d\times d, skew}\} \\
H(\mathrm{div}) &= \{u \in [L_2]^d : \mathrm{div}\, u \in L_2\}
\end{aligned}
$$

Piece-wise smooth functions in

- $H(\mathrm{curl})$ have continuous tangential components,
- $H(\mathrm{div})$ have continuous normal components.

Important for constructing conforming finite elements such as Raviart Thomas, Brezzi-Douglas-Marini, and Nedelec elements.

Natural function space for Maxwell equations: Find $A \in H(\mathrm{curl})$ such that

$$
\int_\Omega \mu^{-1} \,\mathrm{curl}\, A \;\mathrm{curl}\, v + \int_\Omega (i\sigma\omega - \varepsilon\omega^2) A v = \int j v \qquad \forall\, v \in H(\mathrm{curl})
$$

Contents

- Introduction
- Hybrid Discontinuous Galerkin Method
- Finite Elements for $H(\mathrm{div})$ and $H(\mathrm{curl})$
- Tangential-continuous finite elements for elasticity
- Normal-continuous finite elements for Stokes

Hybrid Discontinuous Galerkin (HDG) Method

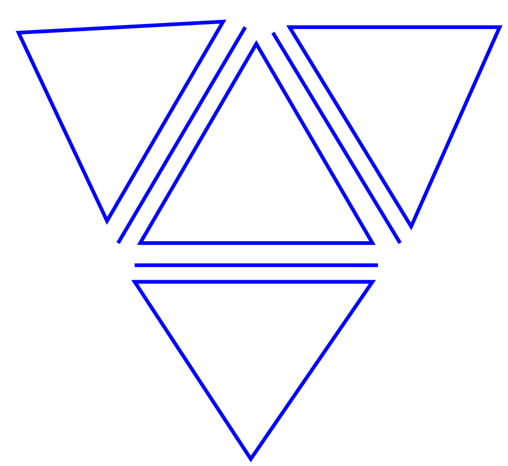
Model problem: $-\Delta u = f$

A mesh consisting of elements and facets (= edes in 2D and faces in 3D)

$$\mathcal{T} = \{T\} \qquad \mathcal{F} = \{F\}$$

Goal: Approximate u with piece-wise polynomials on elements and additional polynomials on facets:

$$u_N \in P^p(\cup T) \qquad \lambda_N \in P^p(\cup F)$$



HDG - Derivation

Exact solution u, traces on element boundaries: $\lambda := u|_{\cup F}$

Integrate against discontinuous test-functions $v \in H^1(\cup T)$, element-wise integration by parts:

$$\sum_T \Big\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} v \Big\} = \int_\Omega f v$$

Use continuity of $\frac{\partial u}{\partial n}$, and test with single-valued functions $\mu \in L_2(\cup F)$:

$$\sum_T \Big\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} (v - \mu) \Big\} = \int_\Omega f v$$

Use consistency $u = \lambda$ on ∂T to symmetrice, and stabilize ...

$$\sum_T \Big\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} (v - \mu) - \int_{\partial T} \frac{\partial v}{\partial n} (u - \lambda) + \alpha \, (u - \lambda, v - \mu)_{j, \partial T} \Big\} = \int_\Omega f v$$

Dirichlet b.c.: Imposed on λ, Neumann b.c.: $\int_{\Gamma_N} g \mu$

Interior penalty method

Stabilization with α **suff large**

$$\alpha\,(u-\lambda, v-\mu)_{j,\partial T} = \frac{\alpha p^2}{h}(u-\lambda, v-\mu)_{L_2(\partial T)}$$

Norm:

$$\|(u,\lambda)\|_{1,HDG}^2 := \|\nabla u\|_{L_2(T)}^2 + \|u-\lambda\|_{j,T}^2$$

Stability is proven by Young's inequality and inverse inequality $\|\frac{\partial u}{\partial n}\|_{L_2(\partial T)}^2 \leq c_{inv}\frac{p^2}{h}\|\nabla u\|_{L_2(T)}^2$:

$$\begin{array}{rcl} A^T(u,\lambda;u,\lambda) & = & \|\nabla u\|_{L_2(T)}^2 - \underbrace{2\int_{\partial T}\frac{\partial u}{\partial n}(u-\lambda)}_{\leq\sqrt{\frac{c_{inv}}{\alpha}}\|\nabla u\|_{L_2(T)}^2+\sqrt{c_{inv}\alpha}\frac{p^2}{h}\|u-\lambda\|_{L_2(\partial T)}^2} + \frac{\alpha p^2}{h}\|u-\lambda\|_{L_2(\partial T)}^2 \\ & \simeq & \|(u,\lambda)\|_{1,HDG}^2 \end{array}$$

for $\alpha > c_{inv}$.

Bassi-Rebay type method

Stabilization term is

$$\alpha\,(u-\lambda, v-\mu)_{j,\partial T} = \alpha\,\big(r(u-\lambda), r(v-\mu)\big)_{L_2(T)}$$

with lifting operator $r : P^p(\mathcal{F}_T) \to [P^p(T)]^d$ such that

$$(r(u-\lambda), \sigma)_{L_2(T)} = (u-\lambda, \sigma_n)_{L_2(\partial T)} \qquad \forall\, \sigma \in [P^p(T)]^d$$

The corresponding jump-norm is

$$\|u-\lambda\|_{j,\partial T} = \sup_{\sigma \in [P^p(T)]^d} \frac{(u-\lambda, \sigma_n)_{L_2(\partial T)}}{\|\sigma\|_{L_2(T)}}$$

Stability for any $\alpha > 1$:

$$\begin{aligned} A^T(u,\lambda;u,\lambda) &= \|\nabla u\|^2_{L_2(T)} - \underbrace{2\int_{\partial T} \frac{\partial u}{\partial n}(u-\lambda)}_{\le \|\nabla u\|_{L_2(T)} \sup_{\sigma\in[P^p]^d} \frac{\int_{\partial T} \sigma_n (u-\lambda)}{\|\sigma\|_{L_2(T)}}} + \alpha\|u-\lambda\|^2_{j,T} \\ &\simeq \|(u,\lambda)\|^2_{1,HDG} \end{aligned}$$

Error estimates

Follows from consistency and discrete stability:

$$
\begin{aligned}
\|(u-u_N, u-\lambda_N)\|_{1,HDG} &\preceq \inf_{v_N,\mu_N} \Big\{ \|\nabla(u-v_N)\|_{L_2(\mathcal{T})} + \|u_N-\lambda_N\|_j + \|\partial_n u - \partial_n u_N\|_{j^*} \\
&\preceq p^\gamma \frac{h^s}{p^s} \|u\|_{H^{1+s}}
\end{aligned}
$$

- for $1 \leq s \leq p$
- with $\gamma = 1/2$ or $\gamma = 0$ depending on mesh-conformity, and jump-term.

Convection - Diffusion Problems

$$\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u &= f \qquad \text{in } \partial\Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

HDG Formulation:

$$A^d(u,\lambda;v,\mu) + A^c(u,\lambda;v,\mu) = \int fv$$

with diffusive term $A^d(.,.)$ from above and upwind-discretization for convective term

$$A^c(u,\lambda;v,\mu) = \sum_T \left\{ -\int bu\cdot\nabla v + \int_{\partial T} b_n\{u/\lambda\}v \right\}$$

with upwind choice

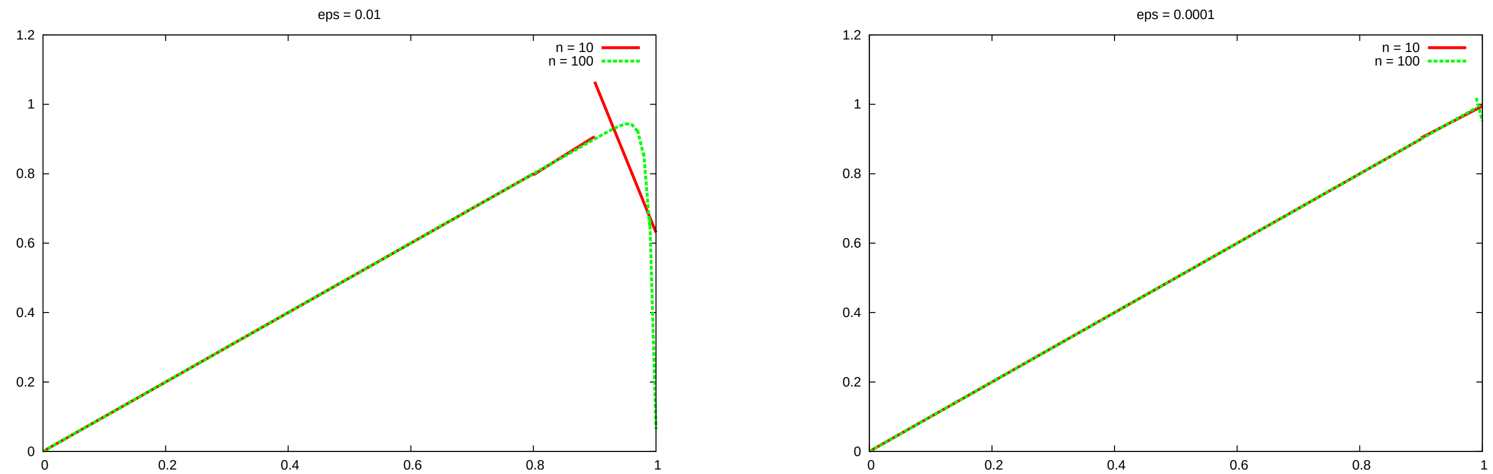
$$\{u/\lambda\} = \left\{ \begin{array}{ll} \lambda & \text{if } b_n < 0, \text{ i.e. inflow edge} \\ u & \text{if } b_n > 0, \text{ i.e. outflow edge} \end{array} \right.$$

assuming $\operatorname{div} b = 0$. Then $A^c(u,\lambda;u,\lambda) \geq 0$ (and $\inf - \sup$ stability)

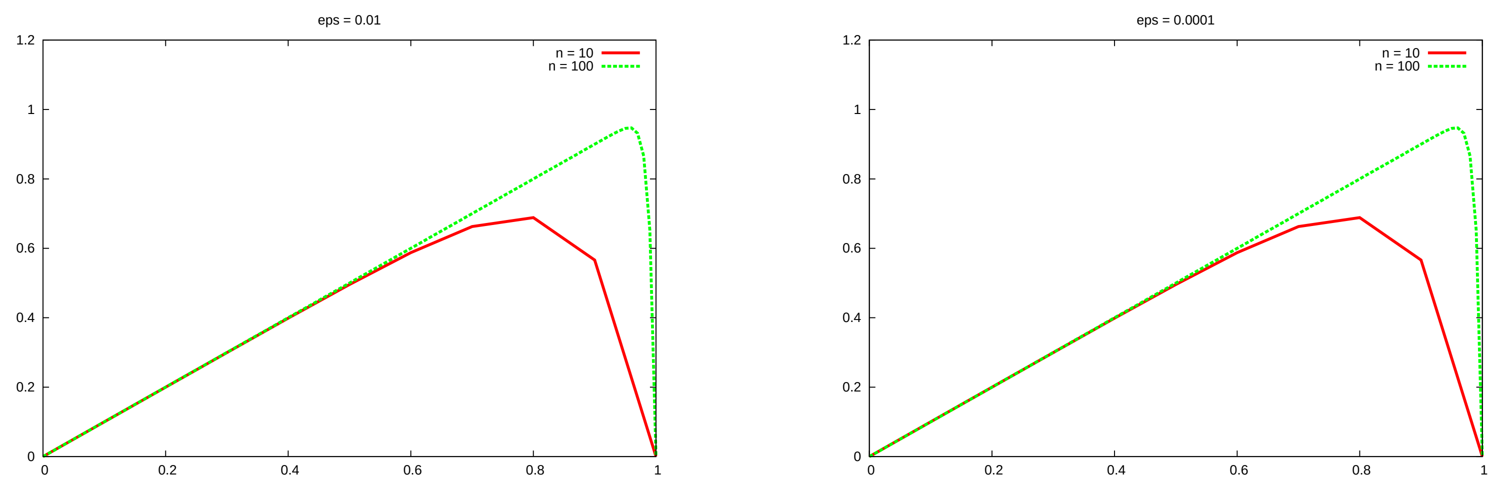
Results for 1D

$$-\varepsilon u'' + u' = 1, \qquad u(0) = u(1) = 0$$

HDG Discretization:
left: $\varepsilon = 10^{-2}$
right: $\varepsilon = 10^{-4}$



conforming elements with SUPG stabilization



Relation to standard Interior Penalty DG method

DG - space

$$V_N := P^p(\cup T)$$

Bilinearform

$$A^{DG}(u,v) = \sum_T \left\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n}[v] - \int_{\partial T} \frac{\partial v}{\partial n}[u] + \frac{\alpha p^2}{h} \int_{\partial T} [u][v] \right\}$$

Hybrid DG has

- even more unknowns, but less matrix entries
- allows element-wise assembling
- allows static condensation of element unknowns

Hybridization of standard DG methods [Cockburn+Gopalakrishnan+Lazarov]

Relation to classical hybridization of mixed methods

First order system

$$A\sigma - \nabla u = 0 \qquad \text{and} \qquad \operatorname{div}\sigma = -f$$

Mixed method: Find $\sigma \in H(\operatorname{div})$ and $u \in L_2$ such that

$$\begin{array}{llllll} \int A\sigma\tau & - & \int \operatorname{div}\tau\, u & = & 0 & \forall\, \tau \in H(\operatorname{div}) \\ \int \operatorname{div}\sigma\, v & & & = & -\int fv & \forall\, v \in L_2 \end{array}$$

Breaking normal-continuity of σ_n, and reinforcing it by another Lagrange parameter [Arnold-Brezzi, 86]

Find $\sigma \in H(\operatorname{div})$, $u \in L_2$, and $\lambda \in L_2(\cup F)$ such that

$$\begin{array}{llllll} \int A\sigma\tau & + & \sum_T \int_T \operatorname{div}\tau\, u + \sum_F \int_F [\tau_n]\lambda & = & 0 & \forall\, \tau \in H(\operatorname{div}) \\ \sum_T \int_T \operatorname{div}\sigma\, v & & & = & -\int fv & \forall\, v \in L_2 \\ \sum_F \int_F [\sigma_n]\mu & & & = & 0 & \forall \mu \in L_2(\cup F) \end{array}$$

Allows to eliminate σ (and also u) leading to a coercive system in u and λ (or, only λ).

Comparison to mixed hybrid system

HDG method needs facet variable of one order higher ???

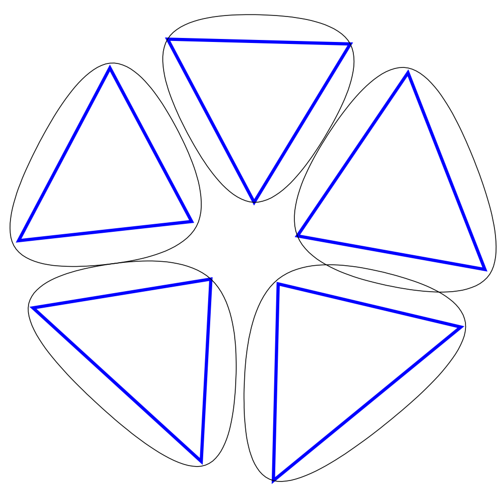
$\lambda \in P^{p-1}(\cup F)$ is enough when inserting a projector:

$$
\begin{aligned}
A^{HDG}(u,\lambda;v,\mu) &= \sum_T \Big\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n}(v-\mu) \\
&\quad - \int_{\partial T} \frac{\partial v}{\partial n}(u-\lambda) + \frac{\alpha p^2}{h} \int_{\partial T} \Pi^{p-1}(u-\lambda)\,\Pi^{p-1}(v-\mu) \Big\}
\end{aligned}
$$

Implementation of the projector by an EAS - like method.

How to solve ?

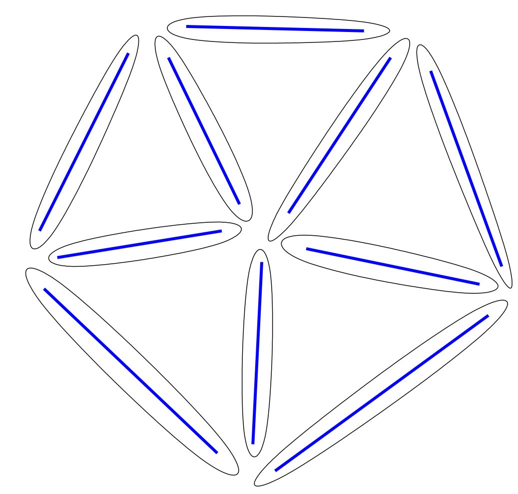
Standard DG



$$\kappa\{C_{ASM}^{-1} A\} \simeq p^2$$

for element-by-element Schwarz preconditioner C_{ASM} plus coarse grid [Antonietti+Houston,11]

Hybrid DG
with facet Schur-complement S



$$\kappa\{C_{ASM}^{-1} S\} \simeq (\log p)^\gamma$$

for facet-by-facet Schwarz preconditioner C_{ASM} plus coarse grid

Trace norms inequality

For $\lambda \in P^p(F)$ define semi-norm and norm

$$\begin{aligned} |\lambda|_F^2 &:= \inf_{u \in P^p} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 \right\} \\ \|\lambda\|_{F,0}^2 &:= \inf_{u \in P^p} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 + \|u - 0\|_{j,\partial T \setminus F}^2 \right\} \end{aligned}$$

mimic $|\cdot|_{H^{1/2}(F)}$ and $\|\cdot\|_{H_{00}^{1/2}(F)}$.

Theorem: For $\lambda \in P^p(F)$ with $\int_F \lambda = 0$ there holds

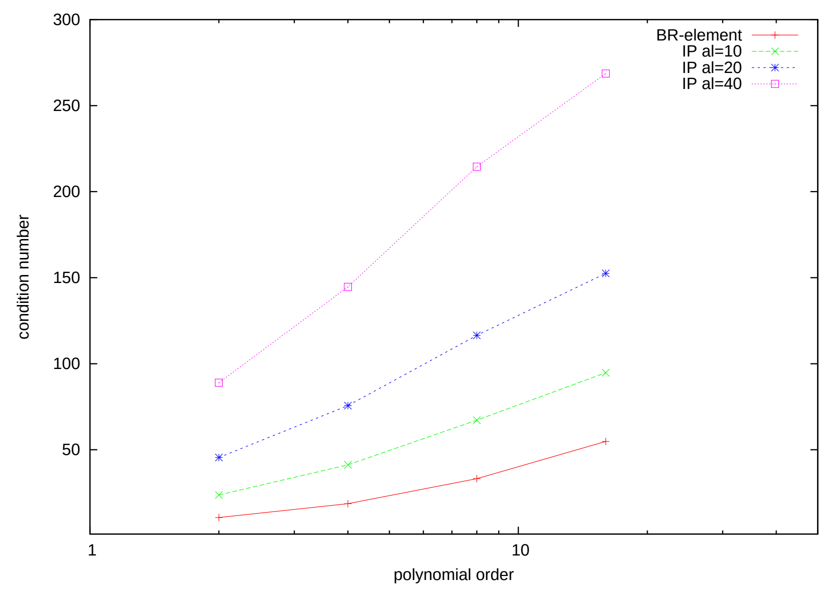
$$\|\lambda\|_{F,0}^2 \preceq (\log p)^\gamma |\lambda|_F^2 \qquad \text{with } \gamma = 3$$

- if T is a trig, quad, or hex, and $\|\cdot\|_j$ is IP or BR
- if T is a tet, and $\|\cdot\|_j$ is BR

From the trace norms inequality we get immediately condition number estimates for Schwarz methods and BDDC preconditioners

Condition numbers for BDDC preconditioner

Laplace equation, mesh consisting of 184 tetrahedra, HDG discretization



- Bassi-Rebay with $\alpha = 1.5$ (proven to be $O(\log^3 p)$)
- interior penalty with $\alpha = 10, 20, 40$ (only $O(p)$ is proven)

Mixed Continuous / Hybrid Discontinuous Galerkin method

Vector-valued spaces with partial continuity and partial components on facets:

$$
\begin{aligned}
V_{\mathcal{T},n} &= \{v \in [P^p(\cup T)]^d : [v_n] = 0\} \qquad & V_{\mathcal{T},\tau} &= \{v \in [P^p(\cup T)]^d : [v_\tau] = 0\} \\
V_{\mathcal{F},n} &= \{v \in [P^p(\cup F)]^d : v_\tau = 0\} \qquad & V_{\mathcal{F},\tau} &= \{v \in [P^p(\cup F)]^d : v_n = 0\}
\end{aligned}
$$

$H(\mathrm{curl})$ - based formulation for elasticity: Find $u \in V_{\mathcal{T},\tau}$ and $\lambda \in V_{\mathcal{F},n}$ such that

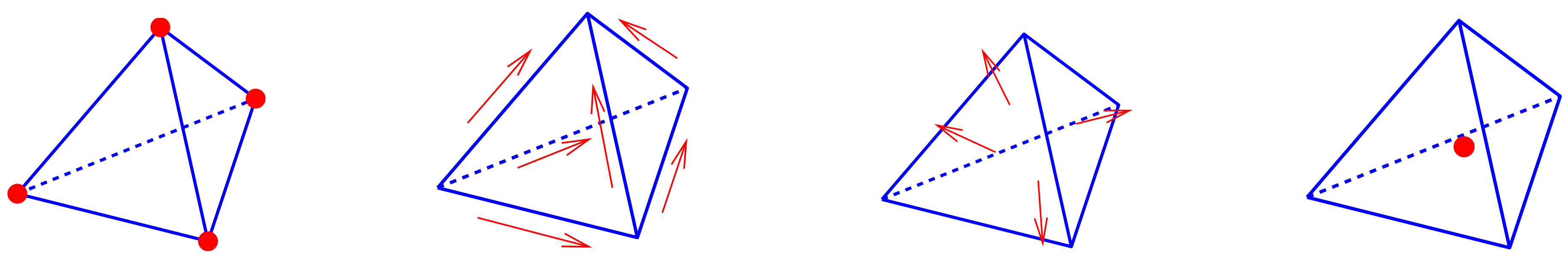
$$
A^\tau(u, \lambda; v, \mu) = \int f v \qquad \forall\, v \in V_{\mathcal{T},\tau}\ \forall \mu \in V_{\mathcal{F},\nu}
$$

$$
\begin{aligned}
A^\tau(u, \lambda; v, \mu) \;=\; & \sum_T \Big\{ \int_T D\varepsilon(u) : \varepsilon(v) - \int_{\partial T} (D\varepsilon(u))_{nn} (v - \mu)_n \\
& - \int_{\partial T} (D\varepsilon(v))_{nn} (u - \lambda)_n + \frac{\alpha p^2}{h} \int_{\partial T} (u - \lambda)_n (v - \mu)_n \Big\}
\end{aligned}
$$

Or, vice versa ...

The de Rham Complex

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\nabla} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
W_h & \xrightarrow{\nabla} & V_h & \xrightarrow{\mathrm{curl}} & Q_h & \xrightarrow{\mathrm{div}} & S_h
\end{array}
$$



satisfies the **exact sequence property**

$$
\begin{aligned}
\mathrm{range}(\nabla) &= \ker(\mathrm{curl}) \\
\mathrm{range}(\mathrm{curl}) &= \ker(\mathrm{div})
\end{aligned}
$$

on the continuous and the discrete level.

Important for stability, error estimates, preconditioning, ...

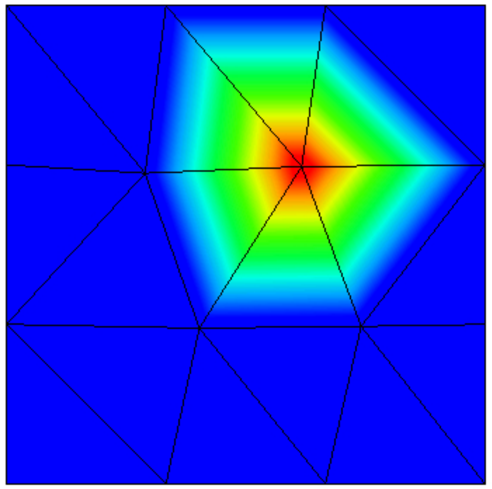
Construction of high order $H(\mathrm{curl})$ and $H(\mathrm{div})$ finite elements

- [Dubiner, Karniadakis+Sherwin] H^1-conforming shape functions in tensor product structure
  $\rightarrow$ allows fast summation techniques
- [Webb] $H(\mathrm{curl})$ hierarchical shape functions with local exact sequence property
  convenient to implement up to order 4
- [Demkowicz+Monk] Based on global exact sequence property construction of Nédélec elements of variable order (with constraints on order distribution) for hexahedra
- [Ainsworth+Coyle] Systematic construction of $H(\mathrm{curl})$-conforming and $H(\mathrm{div})$-conforming elements of arbitrarily high order for tetrahedra
- [JS+Zaglmayr] Based on **local exact sequence property** and by using **tensor-product structure** we achieve a **systematic strategy** for the construction of $H(\mathrm{curl})$-conforming hierarchical shape functions of **arbitrary** and **variable order for common element geometries** (segments, quadrilaterals, triangles, hexahedra, tetrahedra, prisms, pyramids).
  [COMPEL, 2005], PhD-Thesis Zaglmayr 2006

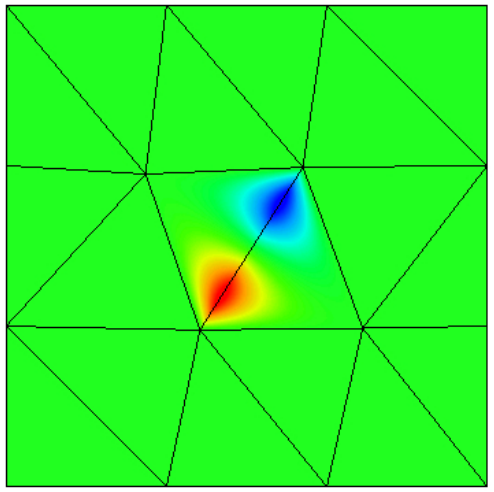
Hierarchical $VEFC$ basis for H^1-conforming Finite Elements

The high order elements have basis functions connected with the vertices, edges, (faces,) and cell of the mesh:

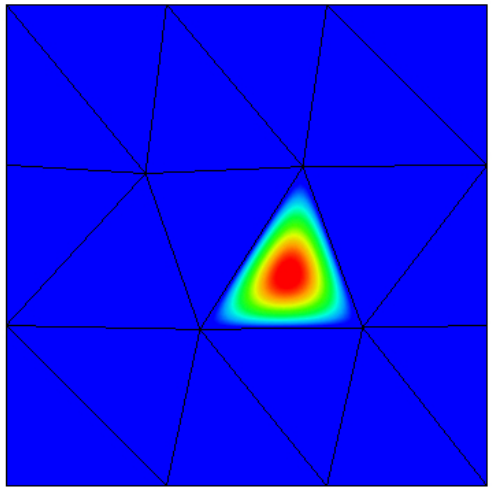
Vertex basis function



Edge basis function p=3



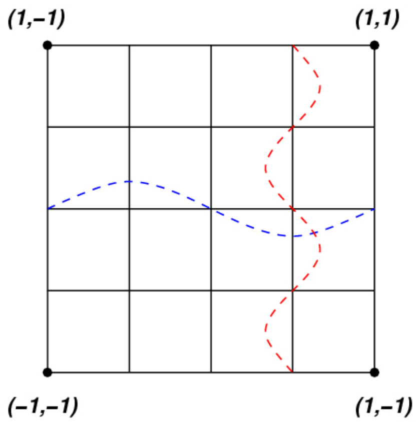
Inner basis function p=3



This allows an individual polynomial order for each edge, face, and cell..

High-order H^1-conforming shape functions in tensor product structure

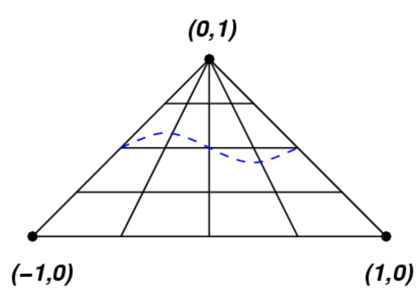
Exploit the tensor product structure of quadrilateral elements to build edge and face shapes



Family of orthogonal polynomials $(\, P_k^0[-1,1]\,)_{2 \leq k \leq p}$ vanishing in ± 1.

$$\begin{aligned} \varphi^F_{i\,j}(x,y) &= P_i^0(x)\, P_j^0(y), \\ \varphi^{E_1}_i(x,y) &= P_i^0(x)\, \tfrac{1-y}{2}. \end{aligned}$$

Tensor-product structure for triangle [Dubiner, Karniadakis+Sherwin]:



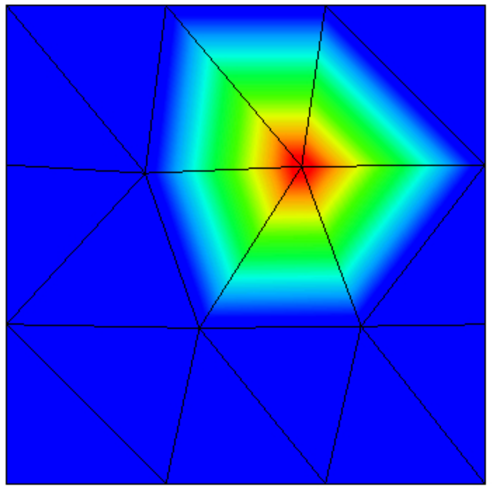
Collapse the quadrilateral to the triangle by $x \rightarrow (1-y)x$

$$\begin{aligned} \varphi^{E_1}_i(x,y) &= P_i^0(\tfrac{x}{1-y})\,(1-y)^i \\ \varphi^F_{i\,j}(x,y) &= \underbrace{P_i^0(\frac{x}{1-y})(1-y)^i}_{u_i(x,y)}\, \underbrace{P_j(2y-1)y}_{v_j(y)} \end{aligned}$$

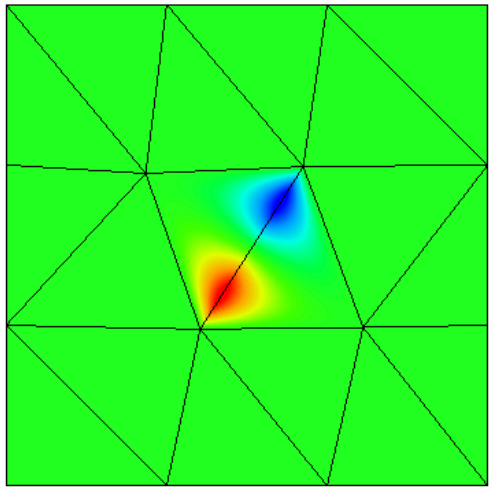
Remark: Implementation is free of divisions

The deRham Complex tells us that $\nabla H^1 \subset H(\mathrm{curl})$, as well for discrete spaces $\nabla W^{p+1} \subset V^p$.

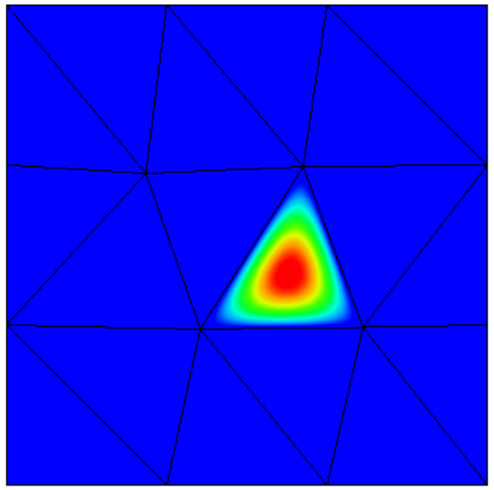
Vertex basis function



Edge basis function p=3

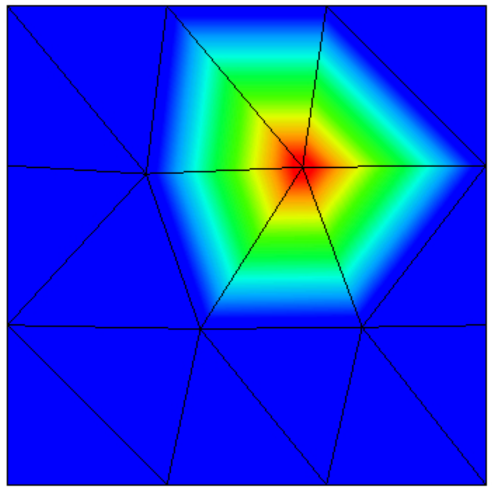


Inner basis function p=3

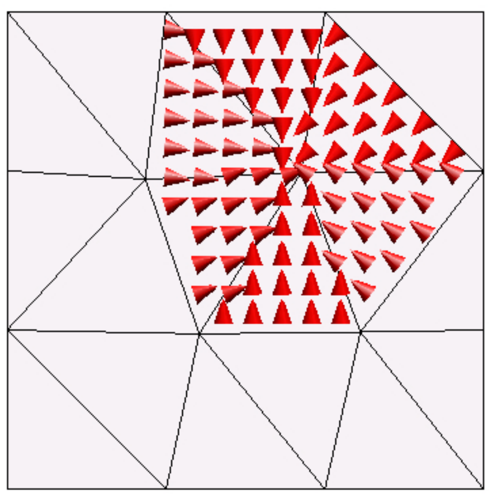


The deRham Complex tells us that $\nabla H^1 \subset H(\mathrm{curl})$, as well for discrete spaces $\nabla W^{p+1} \subset V^p$.

Vertex basis function

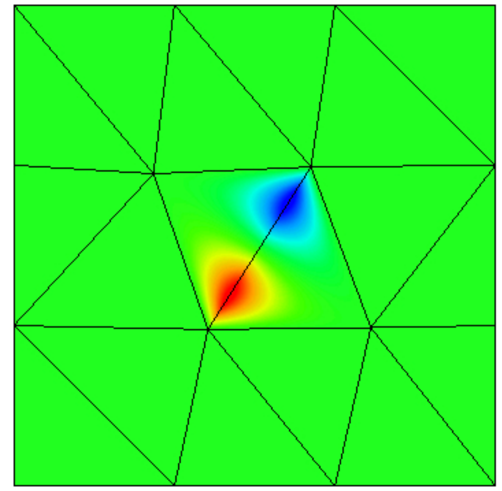


$\downarrow \nabla$

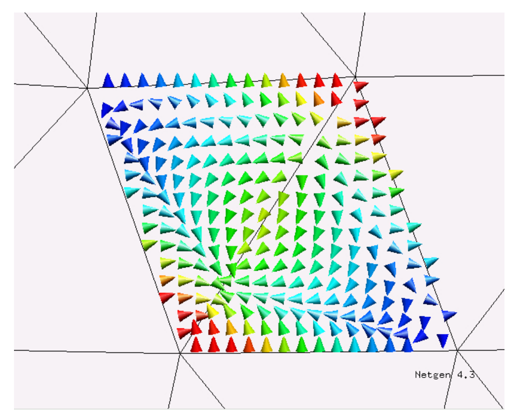


$$\nabla W_{V_i} \subset V_{\mathcal{N}_0}$$

Edge basis function p=3

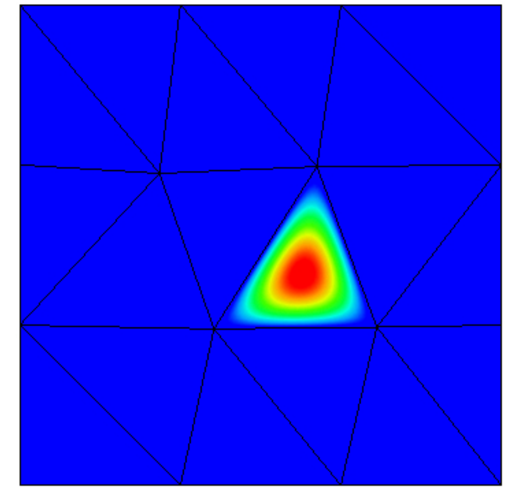


$\downarrow \nabla$

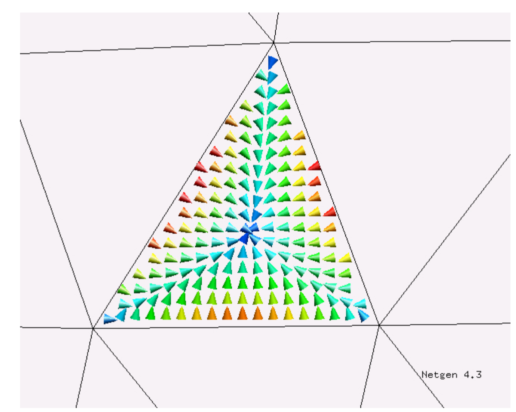


$$\nabla W_{E_k}^{p+1} = V_{E_k}^p$$

Inner basis function p=3



$\downarrow \nabla$



$$\nabla W_{F_k}^{p+1} \subset V_{F_k}^p$$

$H(\mathrm{curl})$-conforming face shape functions with $\nabla W_F^{p+1} \subset V_F^p$

We use inner H^1-shape functions spanning $W_F^{p+1} \subset H^1$ of the structure

$$\varphi_{i,j}^{F,\nabla} = u_i(x,y)\, v_j(y).$$

We suggest the following $H(\mathrm{curl})$ face shape functions consisting of 3 types:

- **Type 1: Gradient-fields**
$$\varphi_{1,\,i,j}^{F,curl} = \nabla\, \varphi_{i,j}^{F,\nabla} = \nabla(u_i\, v_j) = u_i\, \nabla v_j + v_j\, \nabla u_i$$
- **Type 2: other combination**
$$\varphi_{2,\,i,j}^{F,\mathrm{curl}} = u_i\, \nabla v_j - v_j\, \nabla u_i$$
- **Type 3:** to achieve a base spanning V_F $(p-1)$ lin. independent functions are missing
$$\varphi_{3,\,j}^{F,\mathrm{curl}} = \mathcal{N}_0(x,y)\, v_j(y).$$

Similar in 3D and for $H(\mathrm{div})$.

Localized exact sequence property

We have constructed **V**ertex-**E**dge-**F**ace-**C**ell shape functions satisfying

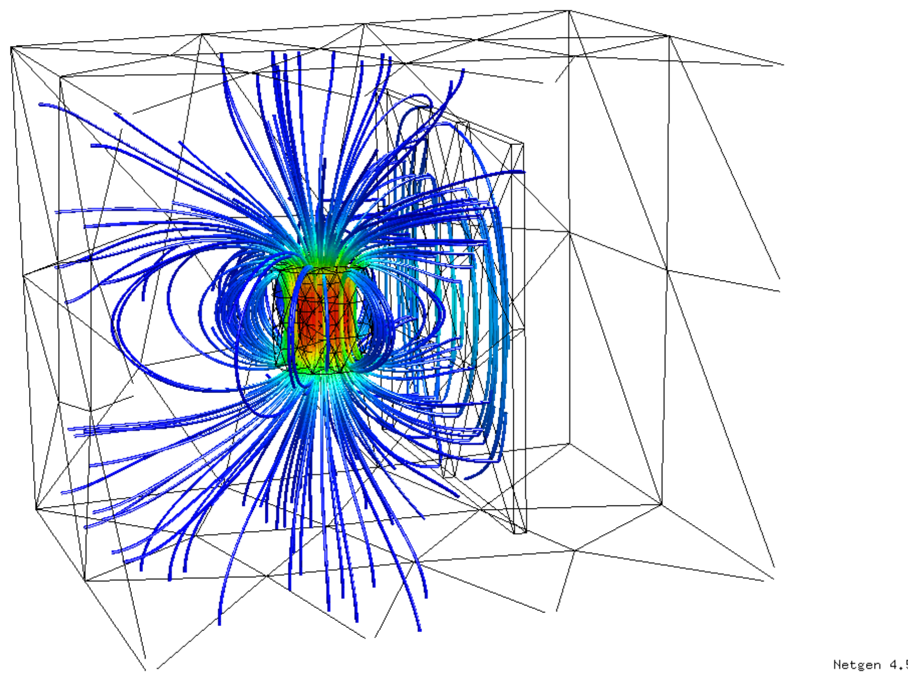
$$\begin{array}{lllllll}
W^{V}_{h,\,p+1=1} & \xrightarrow{\nabla} V_h^{\mathcal{N}_0} & \xrightarrow{\text{curl}} & Q_h^{\mathcal{RT}_0} & \xrightarrow{\text{div}} & S_{h,\,0} \\
W^{E}_{p_E+1} & \xrightarrow{\nabla} V^{E}_{p_E} & & & & \\
W^{F}_{p_F+1} & \xrightarrow{\nabla} V^{F}_{p_F} & \xrightarrow{\text{curl}} & Q^{F}_{p_F-1} & & \\
W^{C}_{p_C+1} & \xrightarrow{\nabla} V^{C}_{p_C} & \xrightarrow{\text{curl}} & Q^{C}_{p_C-1} & \xrightarrow{\text{div}} & S^{C}_{p_C-2}.
\end{array}$$

Advantages are

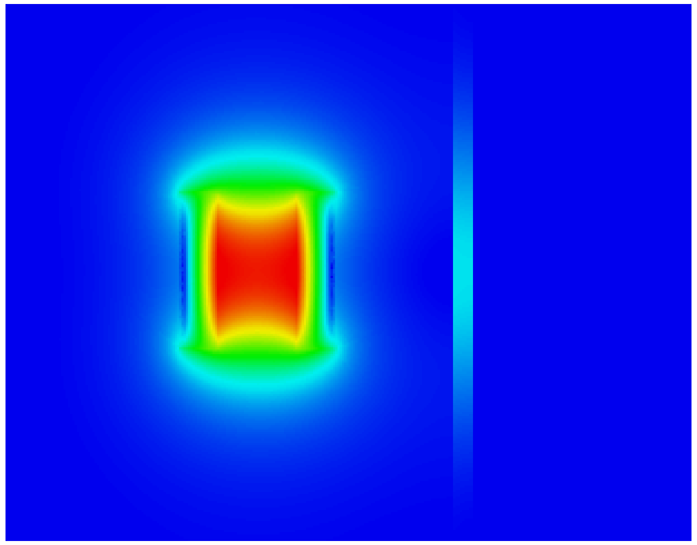
- allows arbitrary and variable polynomial order on each edge, face and cell
- Additive Schwarz Preconditioning with cheap $\mathcal{N}_0 - E - F - C$ blocks gets efficient
- Reduced-basis gauging by skipping higher-order gradient bases functions
- discrete differential operators B_∇, B_{curl}, B_{div} are trivial

Magnetostatic BVP - The shielding problem

Simulation of the magnetic field induced by a coil with prescribed currents:



Electromagnetic shielding problem: magnetic field, p=5

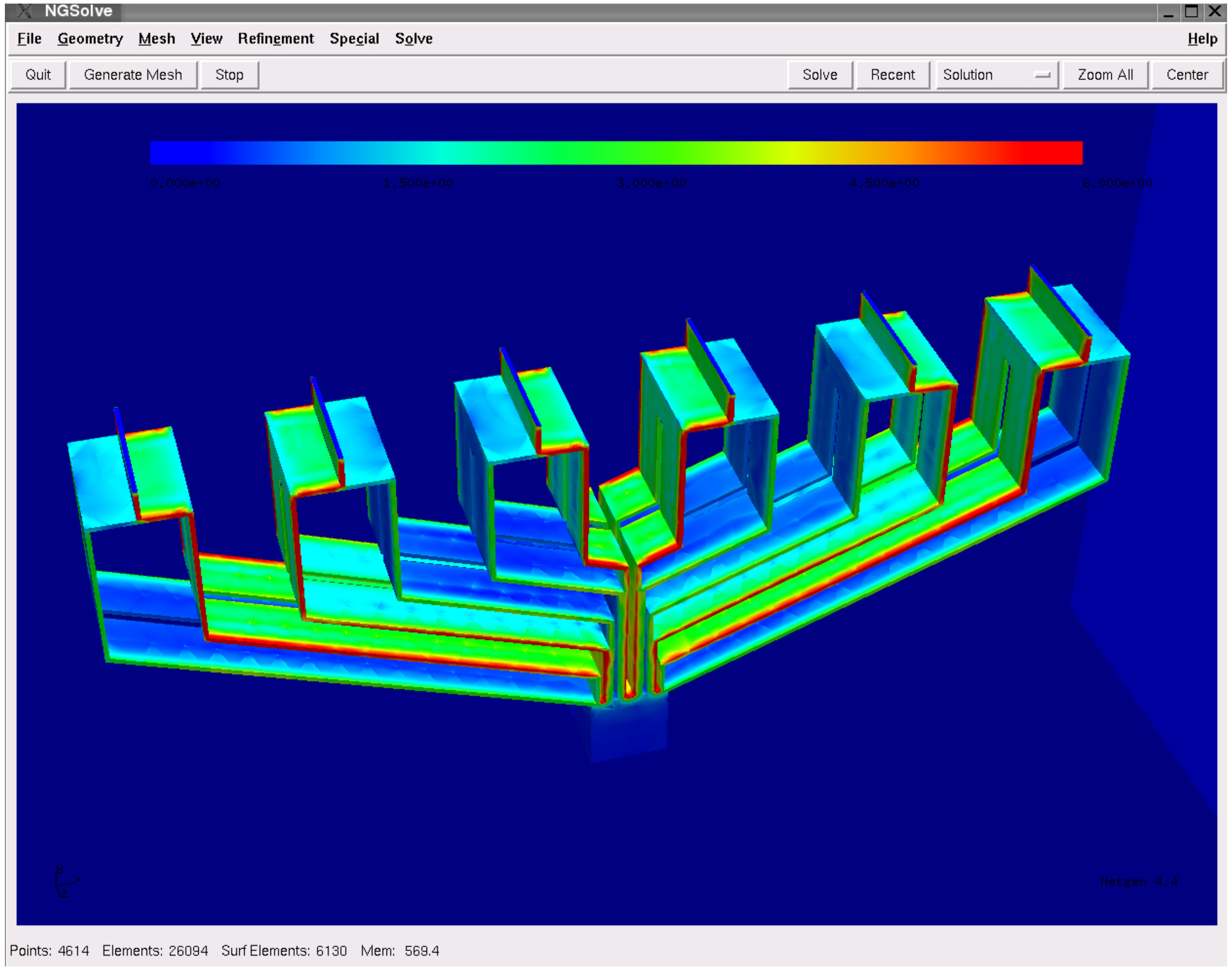


Absolute value of magnetic flux, p=5

... prism layer in shield, unstructured mesh (tets, pyramids) in air/coil.

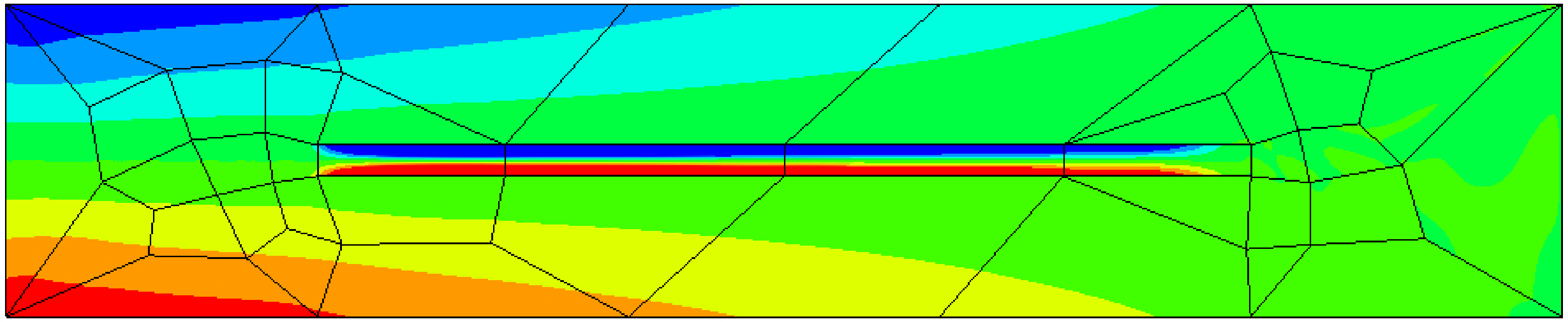
p	dofs	grads	$\kappa(C^{-1}A)$	iter	solvertime
4	96870	yes	34.31	37	24.9 s
4	57602	no	31.14	36	6.6 s
7	425976	yes	140.74	63	241.7 s
7	265221	no	72.63	51	87.6 s

Application: Simulation of eddy-currents in bus bars

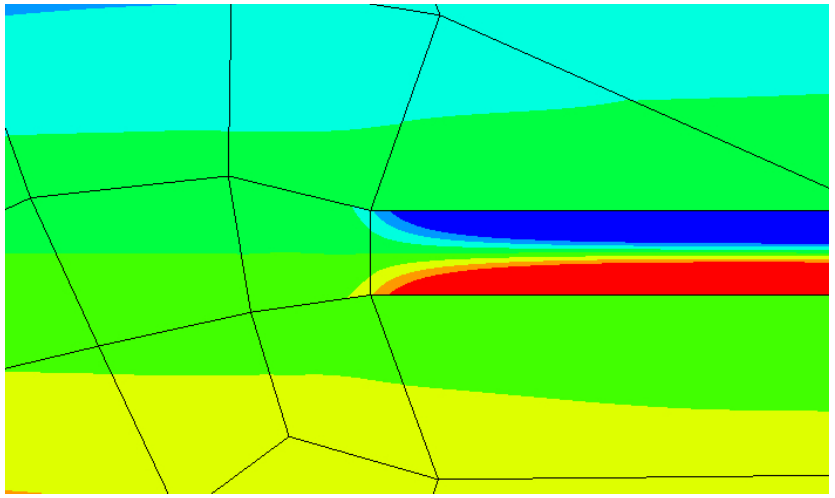


Full basis for $p = 3$ in conductor, reduced basis for $p = 3$ in air

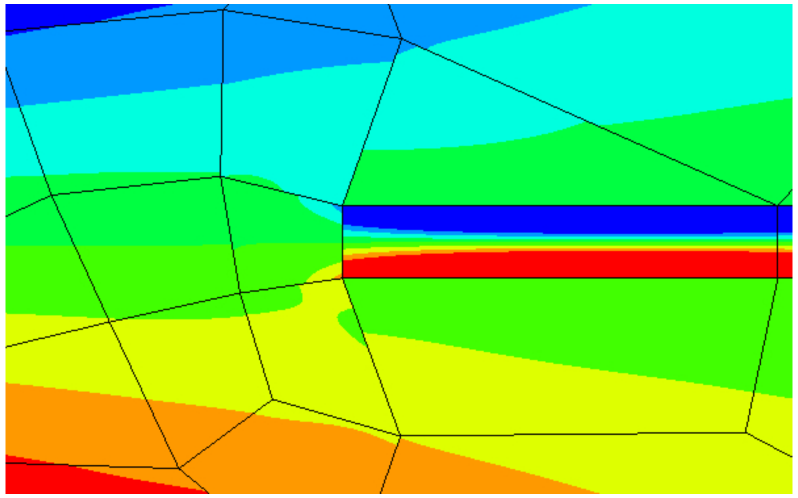
Elasticity: A beam in a beam



Reenforcement with $E = 50$ in medium with $E = 1$.



HDG FEM, $p = 3$



Primal FEM, $p = 3$

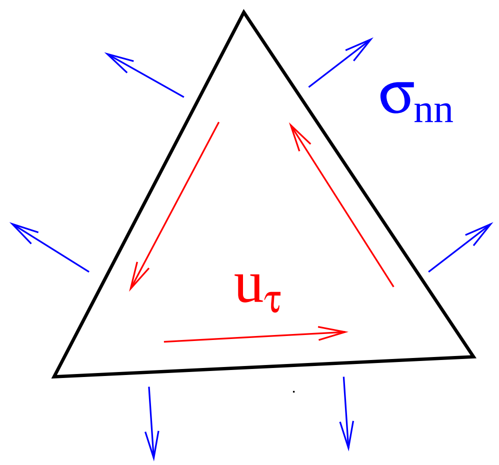
Tangential displacement - normal normal stress constinuous mixed method

[Phd thesis Astrid Sinwel 09 (now Astrid Pechstein)], [A. Pechstein + JS 2011]

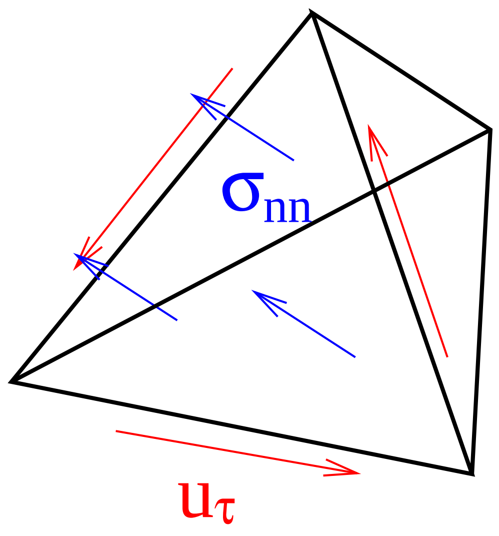
Mixed elements for approximating displacements and stresses.

- tangential components of displacement vector
- normal-normal component of stress tensor

Triangular Finite Element:

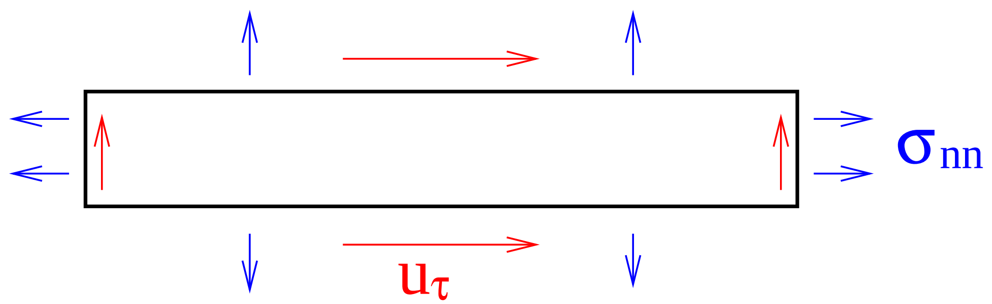


Tetrahedral Finite Element:



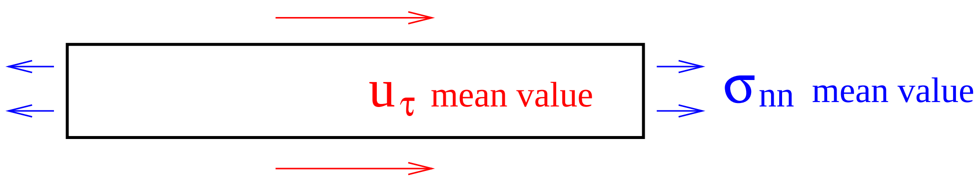
The quadrilateral element

Dofs for general quadrilateral element:

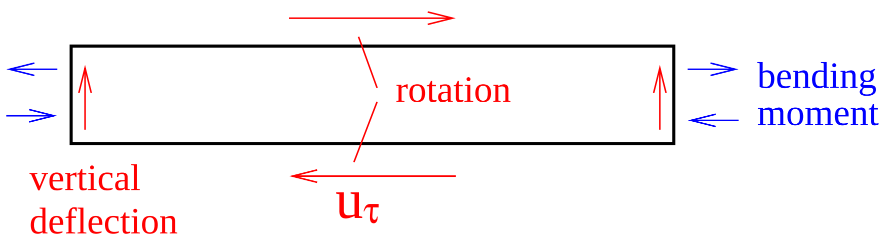


Thin beam dofs ($\sigma_{nn} = 0$ on bottom and top):

Beam stretching components:



Beam bending components:



Hellinger Reissner mixed methods for elasticity

Primal mixed method:

Find $\sigma \in L_2^{sym}$ and $u \in [H^1]^2$ such that

$$\begin{array}{rcrcl l}
\int A\sigma : \tau & - & \int \tau : \varepsilon(u) & = & 0 & \forall\, \tau \\
-\int \sigma : \varepsilon(v) & & & = & -\int f \cdot v & \forall\, v
\end{array}$$

Dual mixed method:

Find $\sigma \in H(\mathrm{div})^{sym}$ and $u \in [L_2]^2$ such that

$$\begin{array}{rcrcl l}
\int A\sigma : \tau & + & \int \mathrm{div}\,\tau \cdot u & = & 0 & \forall\, \tau \\
\int \mathrm{div}\,\sigma \cdot v & & & = & -\int f \cdot v & \forall\, v
\end{array}$$

[Arnold+Falk+Winther]

Reduced Symmetry mixed methods

Decompose

$$\varepsilon(u) = \nabla u + \tfrac{1}{2}\operatorname{Curl} u = \nabla u + \omega$$

with $\operatorname{Curl} u = 2\,\operatorname{skew}(\nabla u) = \big(\partial_{x_i} u_j - \partial_{x_j} u_i\big)_{i,j=1,\ldots d}$

Impose symmetry of the stress tensor by an additional Lagrange parameter:

Find $\sigma \in [H(\operatorname{div})]^d$, $u \in [L_2]^d$, and $\omega \in L_2^{d\times d, skew}$ such that

$$\begin{array}{lclcll}
\int A\sigma : \tau & + & \int u \operatorname{div} \tau + \int \tau : \omega & = & 0 & \forall\, \tau \\
\int v \operatorname{div} \sigma & & & = & -\int f v & \forall\, v \\
\int \sigma : \gamma & & & = & 0 & \forall\, \gamma
\end{array}$$

The solution satisfies $u \in L_2$ and $\omega = \operatorname{Curl} u \in L_2^{d\times d, skew}$, i.e.,

$$u \in H(\operatorname{curl})$$

Arnold+Brezzi, Stenberg,... 80s

Choices of spaces

$\int \operatorname{div} \sigma \cdot u$ understood as

$\langle \operatorname{div} \sigma, u \rangle_{H^{-1} \times H^1} = -(\varepsilon(u), \sigma)_{L_2}$

$(\operatorname{div} \sigma, u)_{L_2}$

Displacement

$u \in [H^1]^2$
continuous f.e.

$u \in [L_2]^2$
non-continuous f.e.

Stress

$\sigma \in L_2^{sym}$
non-continuous f.e.

$\sigma \in H(\operatorname{div})^{sym}$
normal continuous (σ_n) f.e.

The mixed system is well posed for all of these pairs.

Choices of spaces

$\int \operatorname{div} \sigma \cdot u$ understood as

$\langle \operatorname{div} \sigma, u \rangle_{H^{-1} \times H^1} = -(\varepsilon(u), \sigma)_{L_2}$ $\quad$ $\langle \operatorname{div} \sigma, u \rangle_{H(\mathrm{curl})^* \times H(\mathrm{curl})}$ $\quad$ $(\operatorname{div} \sigma, u)_{L_2}$

Displacement

$u \in [H^1]^2$	$u \in H(\mathrm{curl})$	$u \in [L_2]^2$
continuous f.e.	tangential-continuous f.e.	non-continuous f.e.

Stress

$\sigma \in L_2^{sym}$	$\sigma \in L_2^{sym}, \operatorname{div} \operatorname{div} \sigma \in H^{-1}$	$\sigma \in H(\operatorname{div})^{sym}$
non-continuous f.e.	normal-normal continuous (σ_{nn}) f.e.	normal continuous (σ_n) f.e.

The mixed system is well posed for all of these pairs.

The TD-NNS-continuous mixed method

Assuming piece-wise smooth solutions, the elasticity problem is equivalent to the following mixed problem: Find $\sigma \in H(\operatorname{div}\operatorname{div})$ and $u \in H(\operatorname{curl})$ such that

$$\begin{array}{ccccc} \int A\sigma : \tau & + & \sum_T \left\{ \int_T \operatorname{div}\tau \cdot u - \int_{\partial T} \tau_{n\tau} u_\tau \right\} & = & 0 & \forall\,\tau \\ \sum_T \left\{ \int_T \operatorname{div}\sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \right\} & & & = & -\int f \cdot v & \forall\,v \end{array}$$

Proof: The second line is equilibrium, plus tangential continuity of the normal stress vector:

$$\sum_T \int_T (\operatorname{div}\sigma + f) v + \sum_E \int_E [\sigma_{n\tau}] v_\tau = 0 \qquad \forall\,v$$

Since the space requires continuity of σ_{nn}, the normal stress vector is continuous.
Element-wise integration by parts in the first line gives

$$\sum_T \int_T (A\sigma - \varepsilon(u)) : \tau + \sum_E \int_E \tau_{nn} [u_n] = 0 \qquad \forall\,\tau$$

This is the constitutive relation, plus normal-continuity of the displacement. Tangential continuity of the displacement is implied by the space $H(\operatorname{curl})$.

Reissner Mindlin Plates

Energy functional for vertical displacement w and rotations β:

$$\|\varepsilon(\beta)\|^2_{A^{-1}} + t^{-2}\|\nabla w - \beta\|^2$$

MITC elements with Nédélec reduction operator:

$$\|\varepsilon(\beta)\|^2_{A^{-1}} + t^{-2}\|\nabla w - R_h\beta\|^2$$

Mixed method with $\sigma = A^{-1}\varepsilon(\beta) \in H(\operatorname{div}\operatorname{div})$, $\beta \in H(\operatorname{curl})$, and $w \in H^1$:

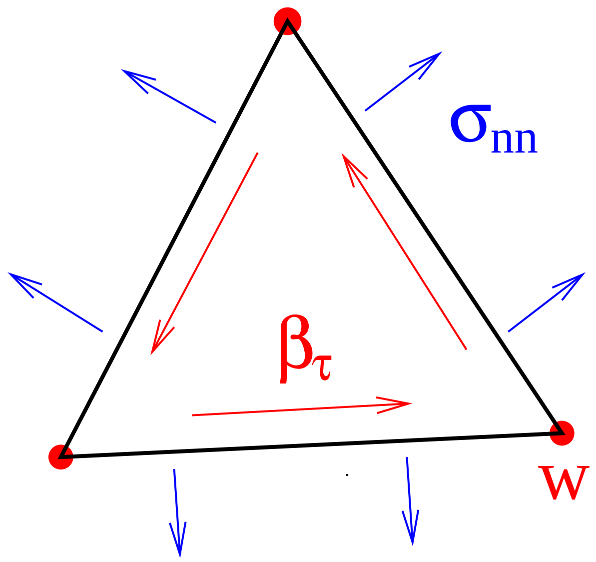
$$L(\sigma;\beta,w) = \frac{1}{2}\|\sigma\|^2_A + \langle \operatorname{div}\sigma, \beta\rangle - t^{-2}\|\nabla w - \beta\|^2$$

Reissner Mindlin Plates and Thin 3D Elements

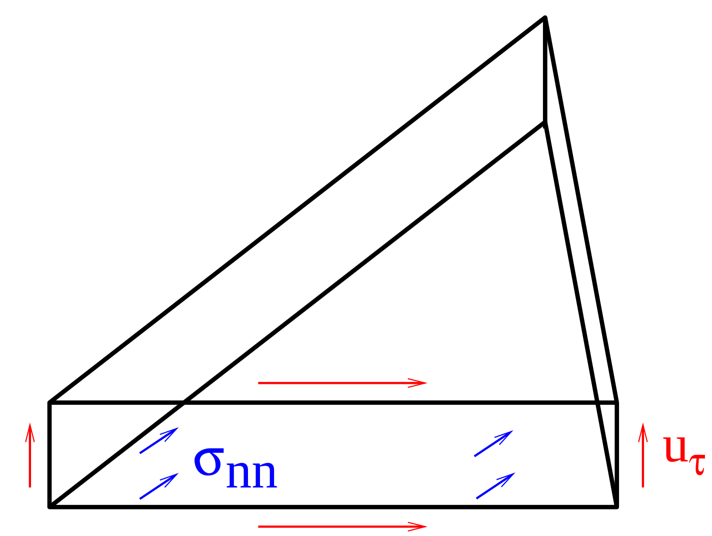
Mixed method with $\sigma = A^{-1}\varepsilon(\beta) \in H(\operatorname{div}\operatorname{div})$, $\beta \in H(\operatorname{curl})$, and $w \in H^1$:

$$L(\sigma; \beta, w) = \|\sigma\|_A^2 + \langle \operatorname{div}\sigma, \beta\rangle - t^{-2}\|\nabla w - \beta\|^2$$

Reissner Mindlin element:



3D prism element:



Anisotropic Estimates

Thm: There holds

$$\sum_T \|\varepsilon(u-u_h)\|_T^2 + \sum_F h_{op}^{-1} \|[u_n]\|_F^2 + \|\sigma - \sigma_h\|^2 \leq c \left\{ h_{xy}^m \|\nabla_{xy}^m \varepsilon(u)\| + h_z^m \|\nabla_z^m \varepsilon(u)\| \right\}^2$$

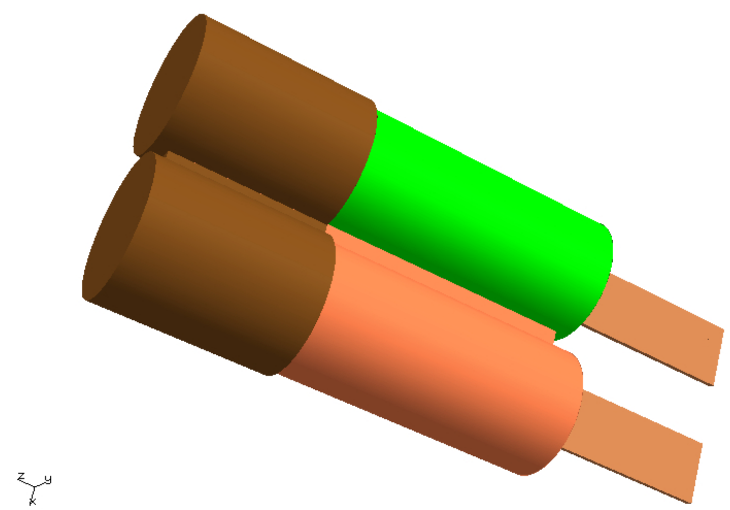
Proof: Stability constants are robust in aspect ratio (for tensor product elements)

Anisotropic interpolation estimates (H^1: Apel). Interpolation operators commute with the strain operator:

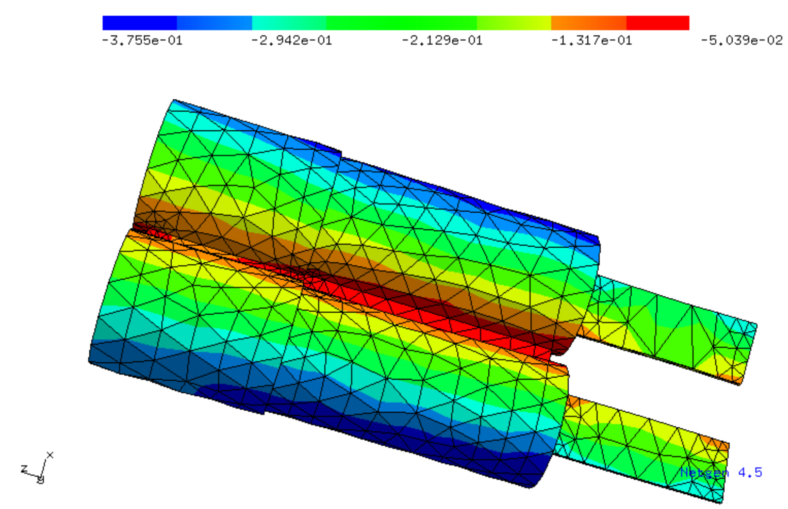
$$\begin{aligned} \|\varepsilon(u - Qu)\|_{L_2} &= \|(I - \tilde{Q})\varepsilon(u)\|_{L_2} \\ &\preceq h_{xy}^m \|\nabla_x^m \varepsilon_{xy,z}(u)\|_0 + h_z^m \|\nabla_z^m \varepsilon_{xy,z}(u)\|_{L_2} \end{aligned}$$

[A. Pechstein + JS, 2011]

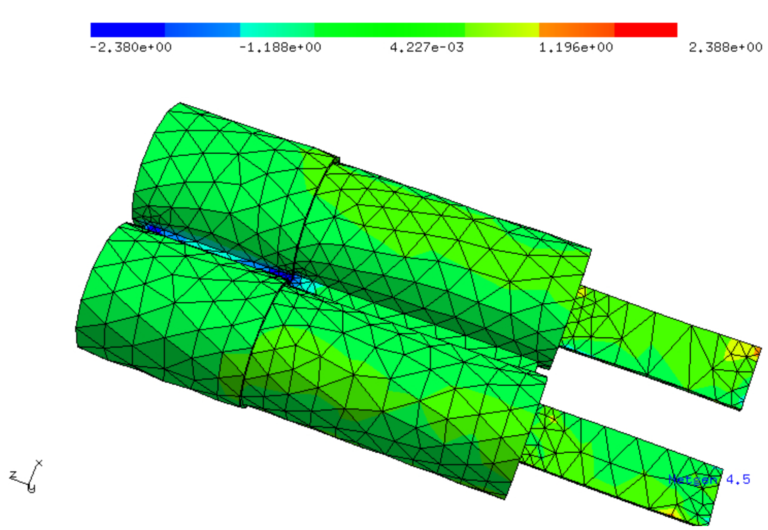
For Hot Days ...



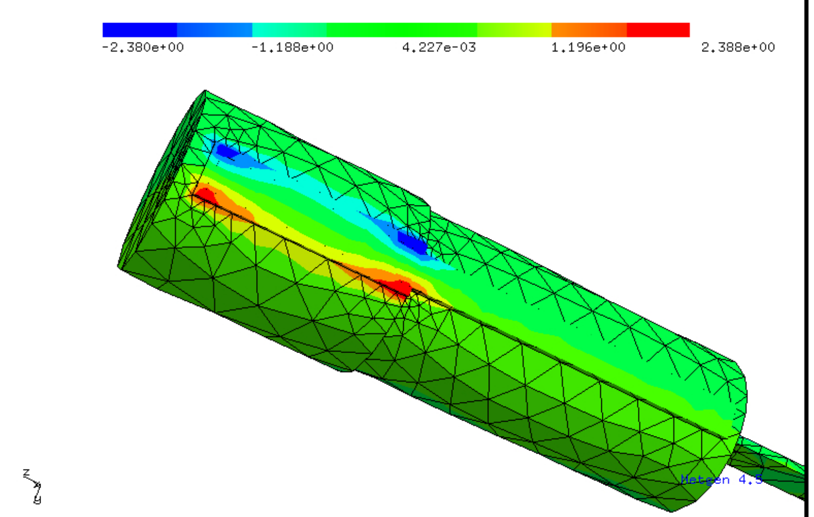
Geometry



Displacement u_y

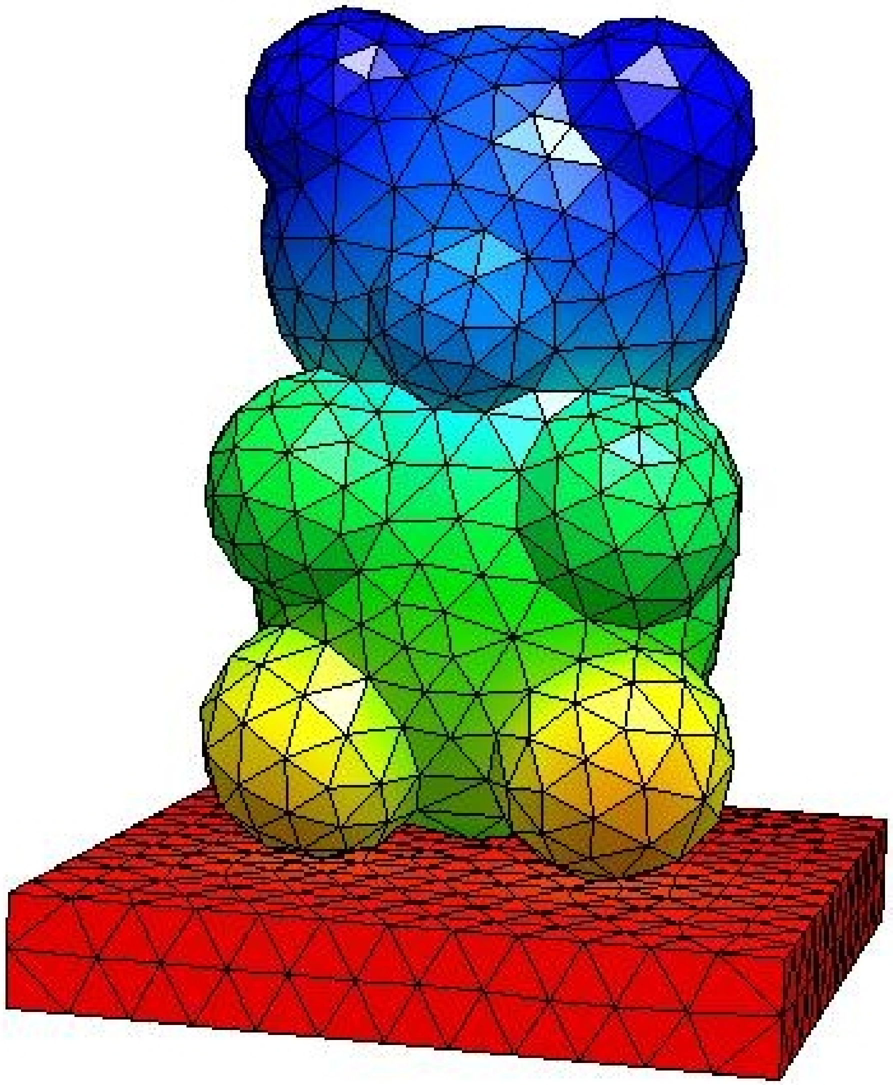


Deformed geometry, stress σ_{xx}

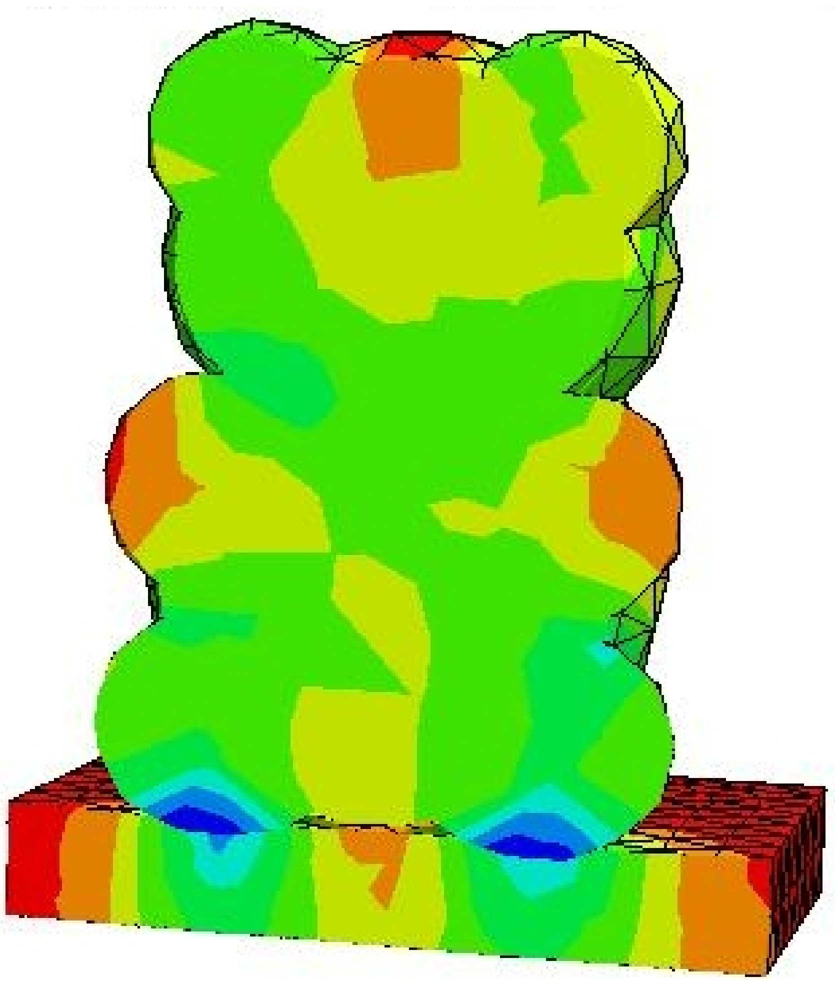


Interior stress

Contact problem with friction



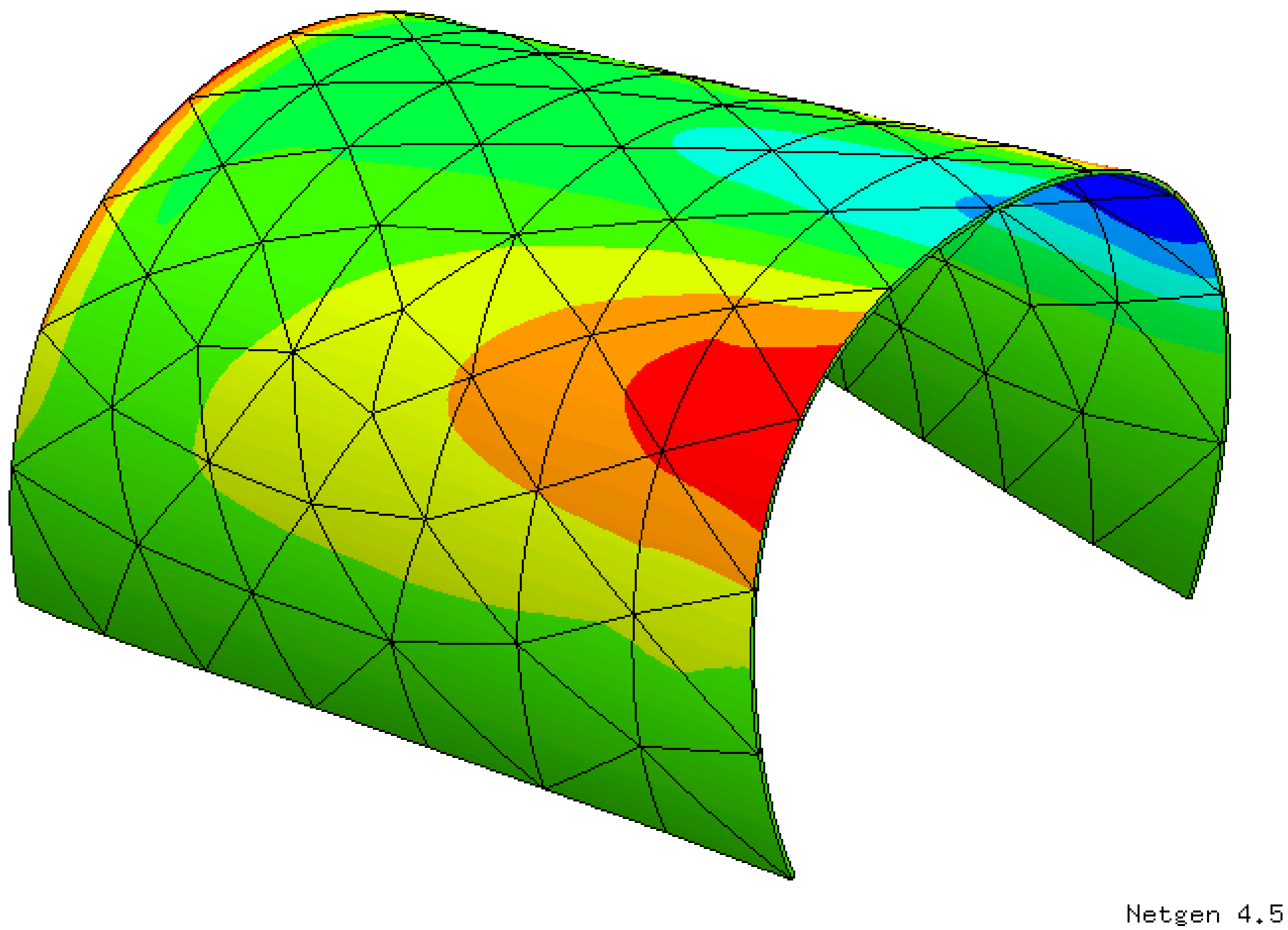
Undeformed bear



Stress, component σ_{33}

Shell structure

R = 0.5, t = 0.005
$\sigma \in P^2$, $u \in P^3$



stress component σ_{yy}

Hybridization: Implementation aspects

Both methods are (essentially) equivalent:

- Classical hybridization of mixed method:

  Introduce Lagrange parameter λ_n to enforce continuity of σ_{nn}. Its meaning is the displacement in normal direction.

- Continuous / hybrid discontinuous Galerkin method:

  Displacement u is strictly tangential continuous, HDG facet variable ($=$ normal displacement) enforces weak continuity of normal component.

Anisotropic error estimates from mixed methods can be applied for HDG method !

Continuous / hybrid discontinuous Galerkin method for Stokes

(Thesis C. Lehrenfeld 2010, RWTH)

$H(\mathrm{div})$ - based formulation for Stokes:

Find $u \in V_{\mathcal{T},n} \subset H(\mathrm{div})$, $\lambda \in V_{\mathcal{F},\tau}$ and $p \in P^{p-1}(\mathcal{T})$ such that

$$\begin{array}{lcll} A^n(u,\lambda; v,\mu) \quad + \quad \int_\Omega \mathrm{div}\, v\, q & = & \int f v & \forall\, (v,\mu) \\ \int \mathrm{div}\, u\, q & = & 0 & \forall\, q \end{array}$$

Provides exactly divergence-free discrete velocity field u

LBB is proven by commuting interpolation operators for de Rham diagram

[Cockburn, Kanschat, Schötzau 2005]

$H(\mathrm{div})$-conforming elements for Navier Stokes

$$\begin{aligned} \frac{\partial u}{\partial t} - \mathrm{div}(2\nu\varepsilon(u) - u\otimes u - pI) &= f \\ \mathrm{div}\ u &= 0 \\ +b.c. \end{aligned}$$

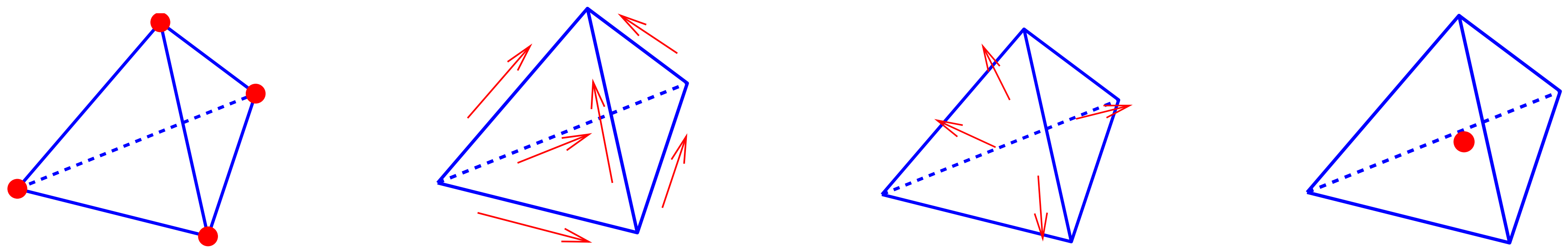
Fully discrete scheme, semi-implicit time stepping:

$$\begin{aligned} (\frac{1}{\tau}M + A^{\nu})\hat{u} + B^T\hat{p} &= f + \frac{1}{\tau}Mu - A^c(u) \\ B\hat{u} &= 0 \end{aligned}$$

- u is exactly div-free $\Rightarrow$ non-negative convective term $\int u\nabla vv \geq 0$
- stability for kinetic energy ($\frac{d}{dt}\|u\|_0^2 \preceq \frac{1}{\nu}\|f\|_{L_2}^2$)
- convective term by upwinding
- allows kernel-preserving smoothing and grid-transfer for fast iterative solver

The de Rham Complex

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\nabla} & H(\text{curl}) & \xrightarrow{\text{curl}} & H(\text{div}) & \xrightarrow{\text{div}} & L^2 \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
W_h & \xrightarrow{\nabla} & V_h & \xrightarrow{\text{curl}} & Q_h & \xrightarrow{\text{div}} & S_h
\end{array}
$$



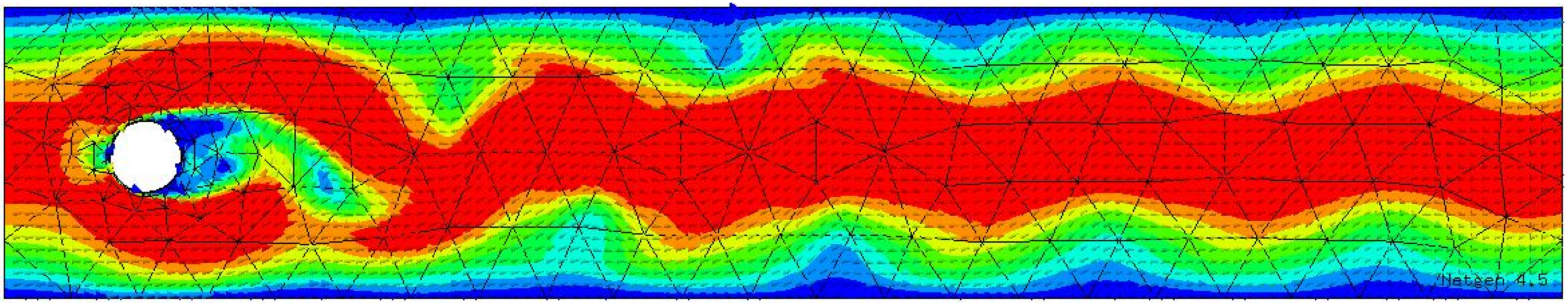
For constructing high order finite elements

$$
\begin{aligned}
W_{hp} &= W_{\mathcal{L}_1} + \text{span}\{\varphi_{h.o.}^W\} \\
V_{hp} &= V_{\mathcal{N}_0} + \text{span}\{\nabla \varphi_{h.o.}^W\} + \text{span}\{\varphi_{h.o.}^V\} \\
Q_{hp} &= Q_{\mathcal{R}T_0} + \text{span}\{\text{curl}\, \varphi_{h.o.}^V\} + \text{span}\{\varphi_{h.o.}^Q\} \\
S_{hp} &= S_{\mathcal{P}_0} + \text{span}\{\text{div}\, \varphi_{h.o.}^S\}
\end{aligned}
$$

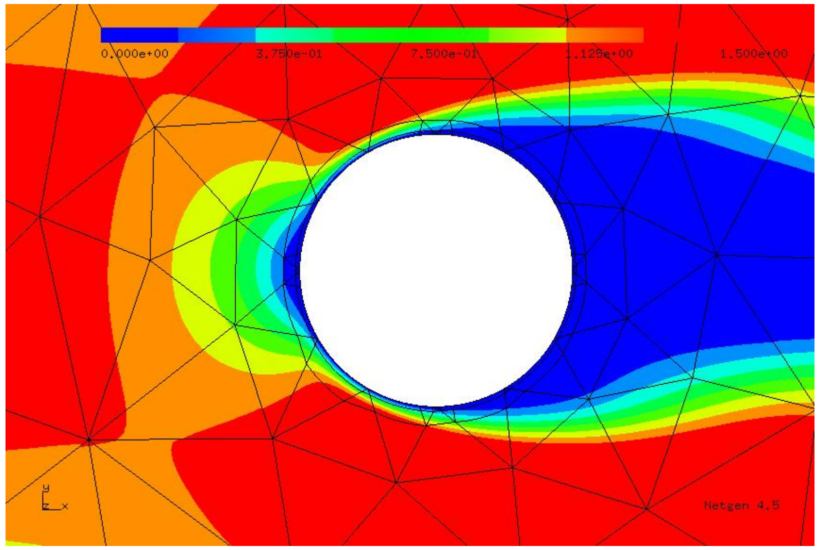
Allows to construct high-order-divergence free elements $\{v \in BDM_k : \text{div}\, v \in P_0\}$

Flow around a disk, 2D

Re $= 100$, 5^{th}-order elements

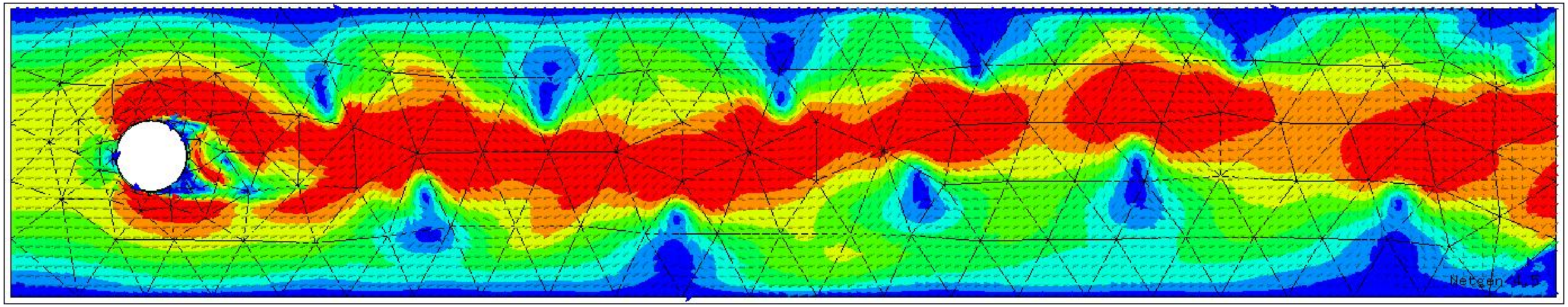


Boundary layer mesh around cylinder:

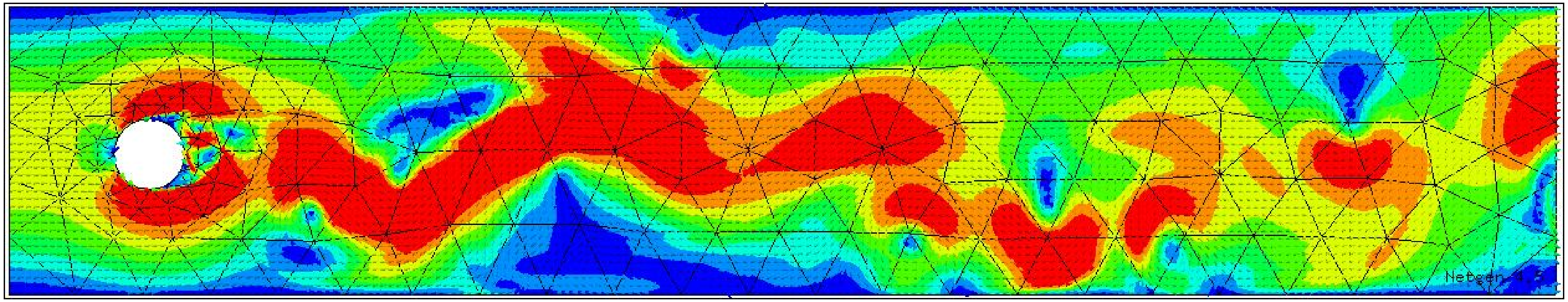


Flow around a disk, 2D

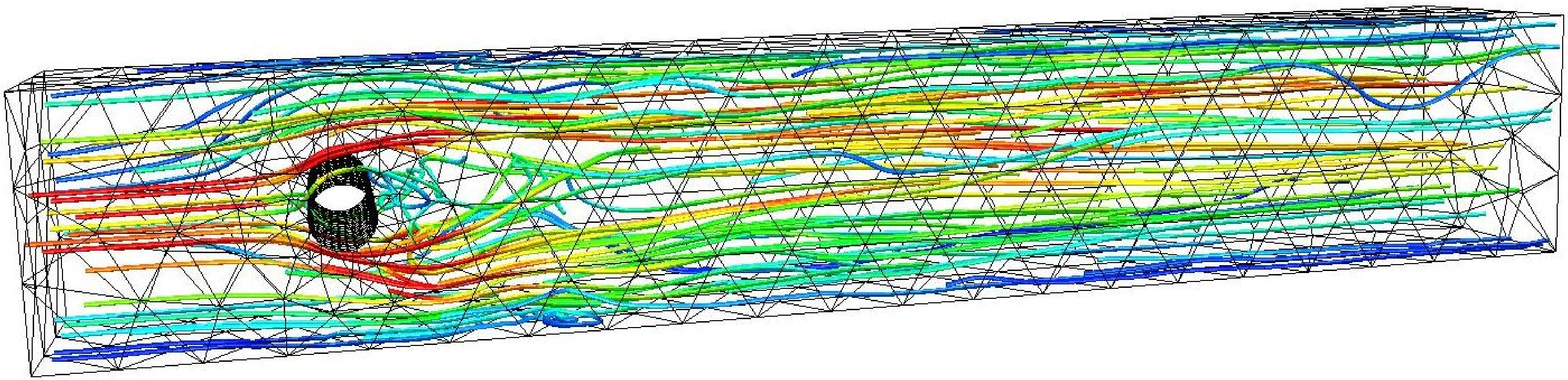
$\mathrm{Re} = 1000$:



$\mathrm{Re} = 5000$:



Flow around a cylinder, $Re = 100$



Concluding Remarks

- Hybrid DG is a simple and efficient hp - discretization scheme
- Robust anisotropic elements for linear elasticity
- Exactly divergence free finite elements for incompressible flows

Ongoing work:

- Operator splitting time integration
- Preconditioning (BDDC element-level domain decomposition)
- MPI-based Parallelization, GPU implementation of explicit time-stepping methods

Open source software on sourceforge:

- Netgen/NGSolve : Mesh generator and general purpose finite element code
- NGS-flow : CFD module for Netgen/NGSolve