Compatible Discretization of the Helmholtz Equation

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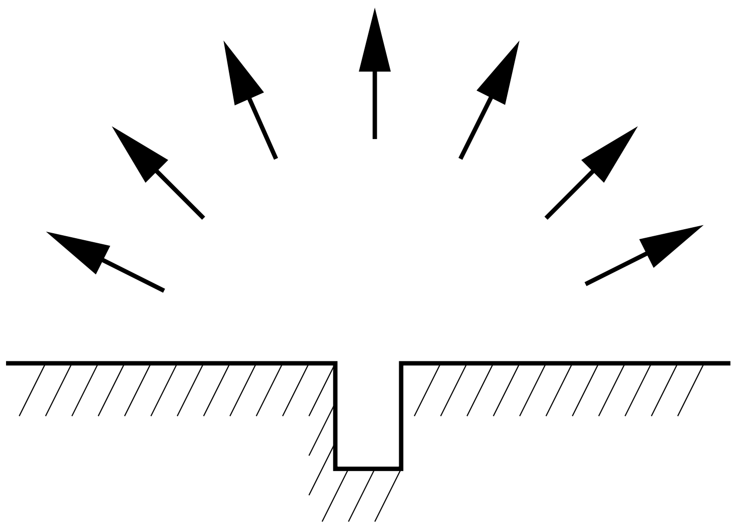
Werner Koch, Stefan Hain
Institute of Aerodynamics and Flow Technology, DLR Göttingen

ICIAM Zürich, 2007

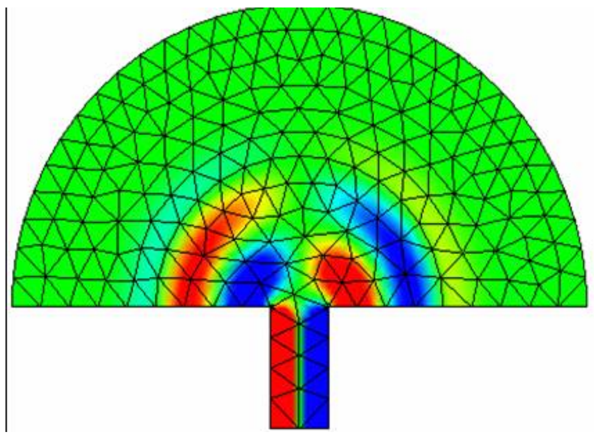
Resonances on the open cavity domain



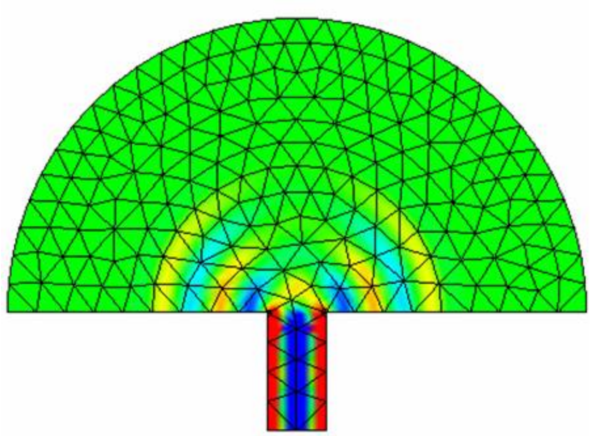
$$
\begin{aligned}
-\Delta u - \omega^2 u &= 0 && \text{Helmholtz equation} \\
\frac{\partial u}{\partial n} &= 0 && \text{Wall boundary} \\
\frac{\partial u}{\partial r} - i\omega u &= o(r^{-(d-1)/2}) && \text{Radiation condition}
\end{aligned}
$$

Find discrete resonances (eigenvalues) ω !

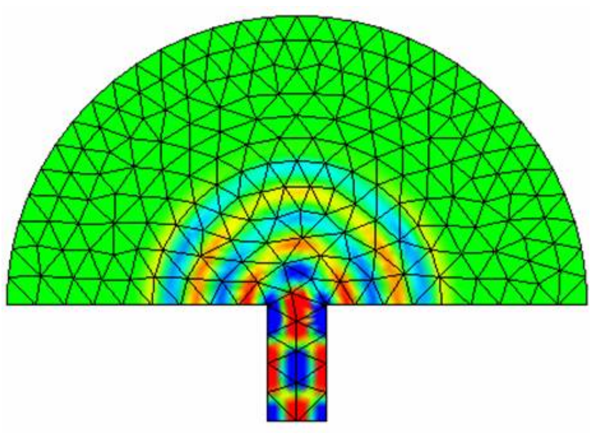
Solutions:



Mode 1-0



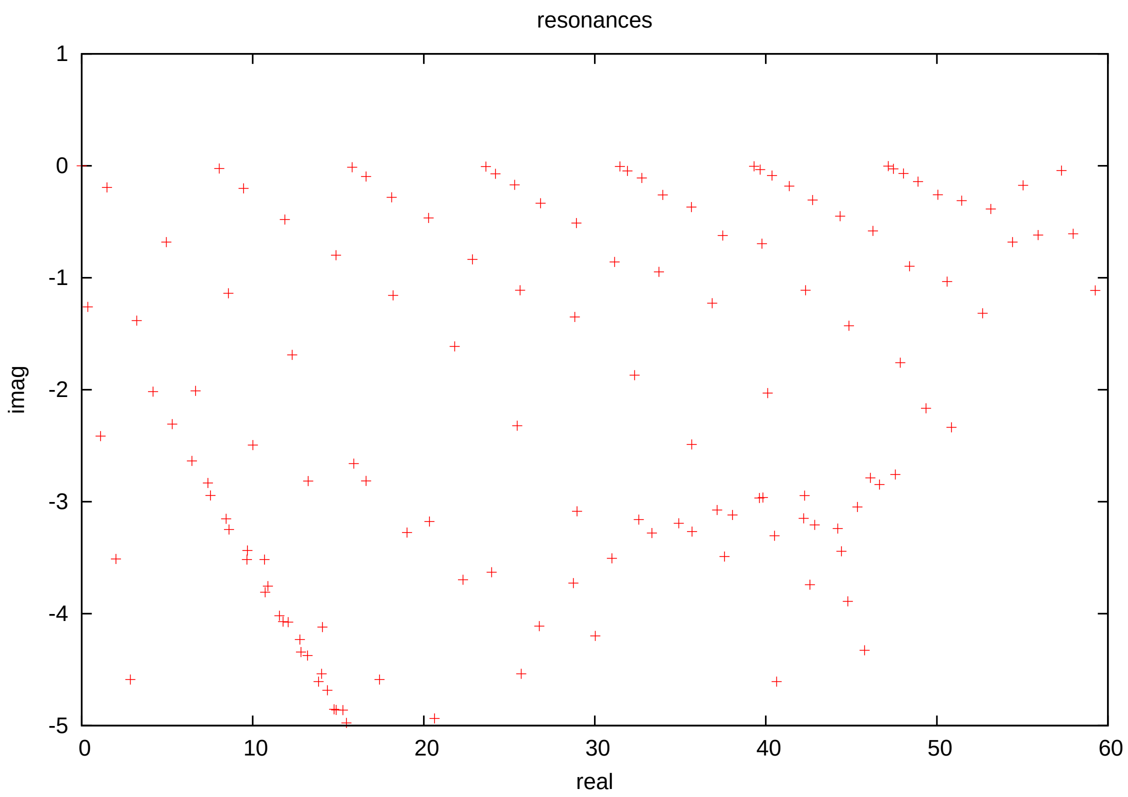
Mode 2-0



Mode 2-2

The spectrum from finite element simulation

Implementation of radiation condition by the PML (perfectly matched layer) method.



- Artificial eigenvalues due to the PML
- Artificial eigenvalues due to the FEM discretization

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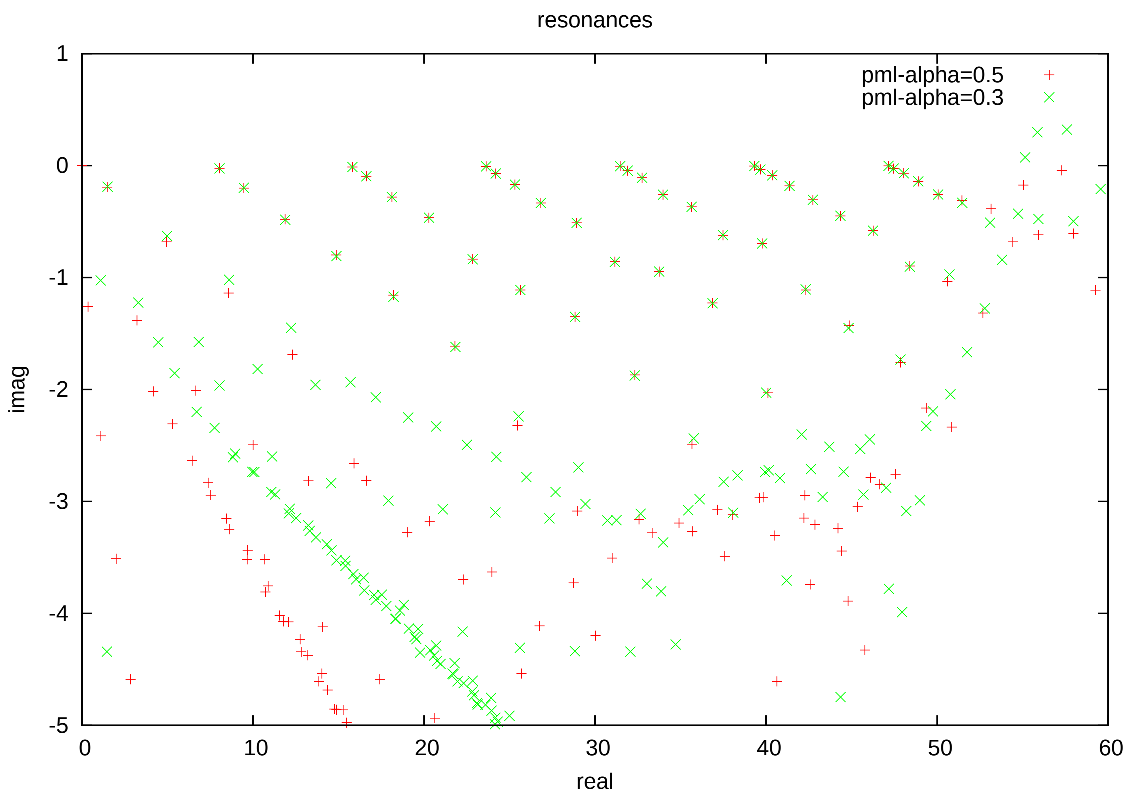

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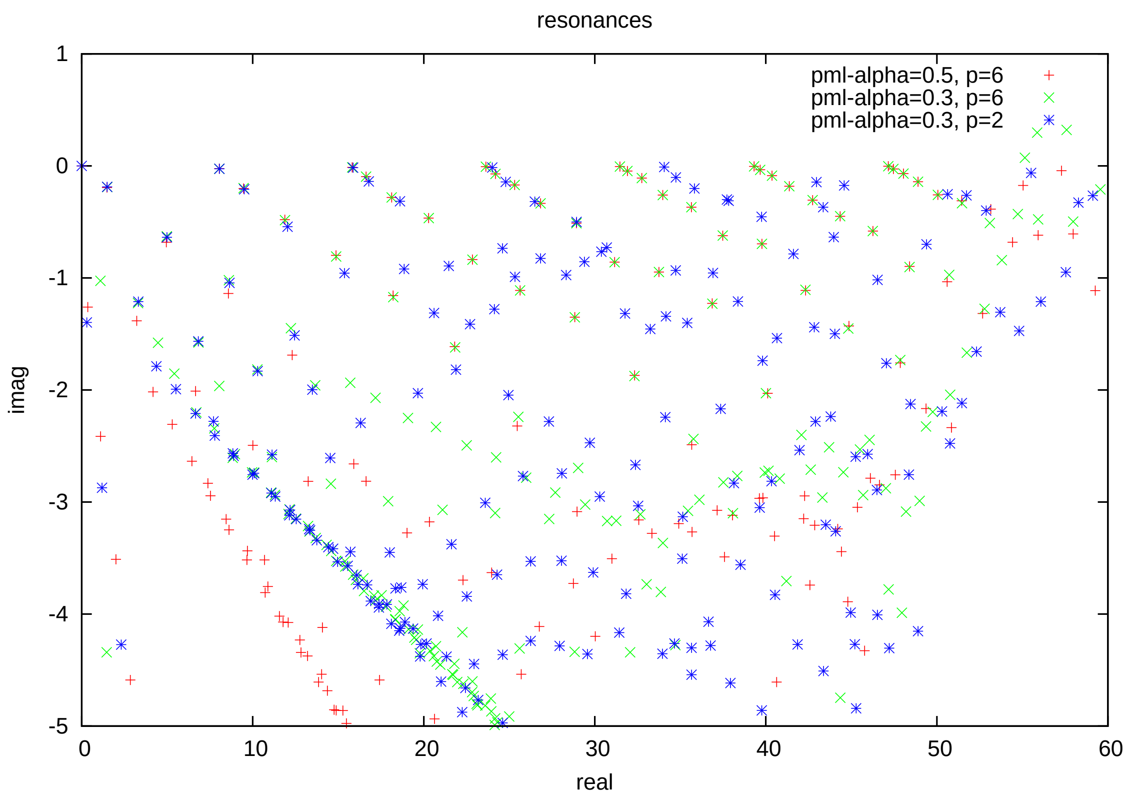

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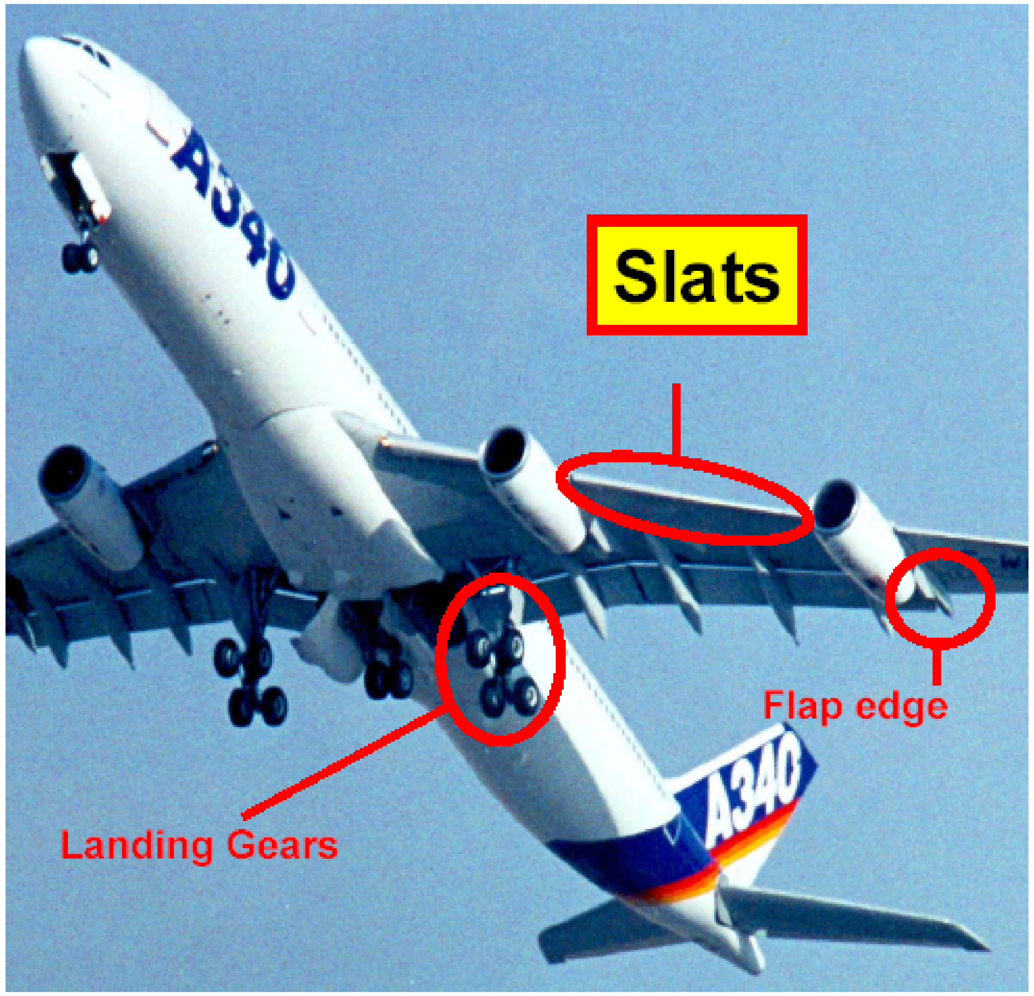
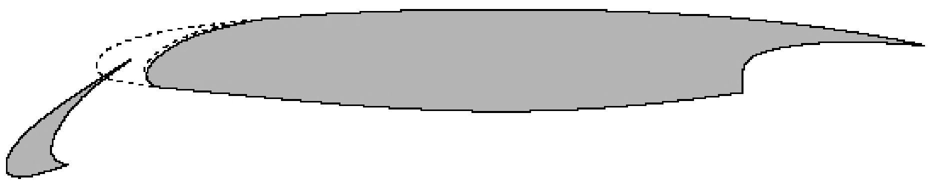
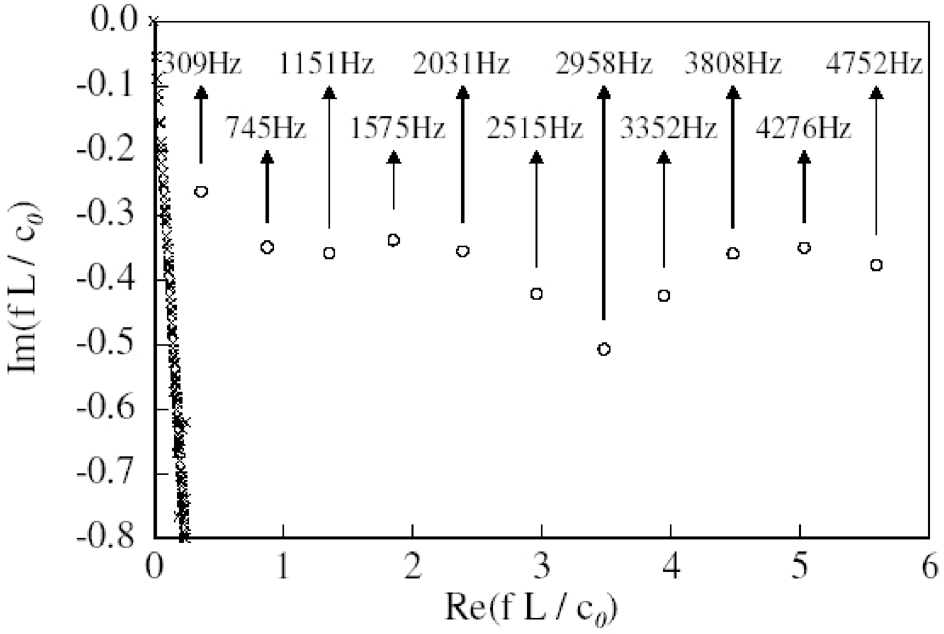
Major Sources of Airframe Noise

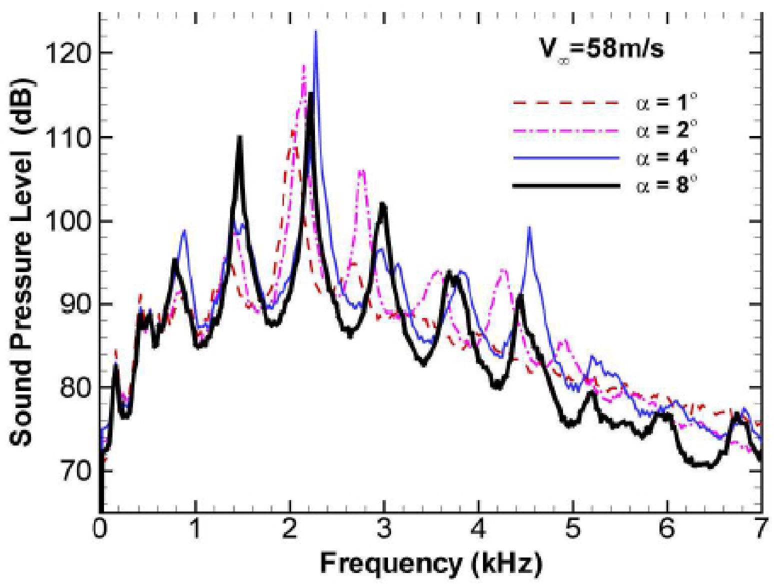


Source: U. Michel *International Symposium Arcachon, France* (2002)

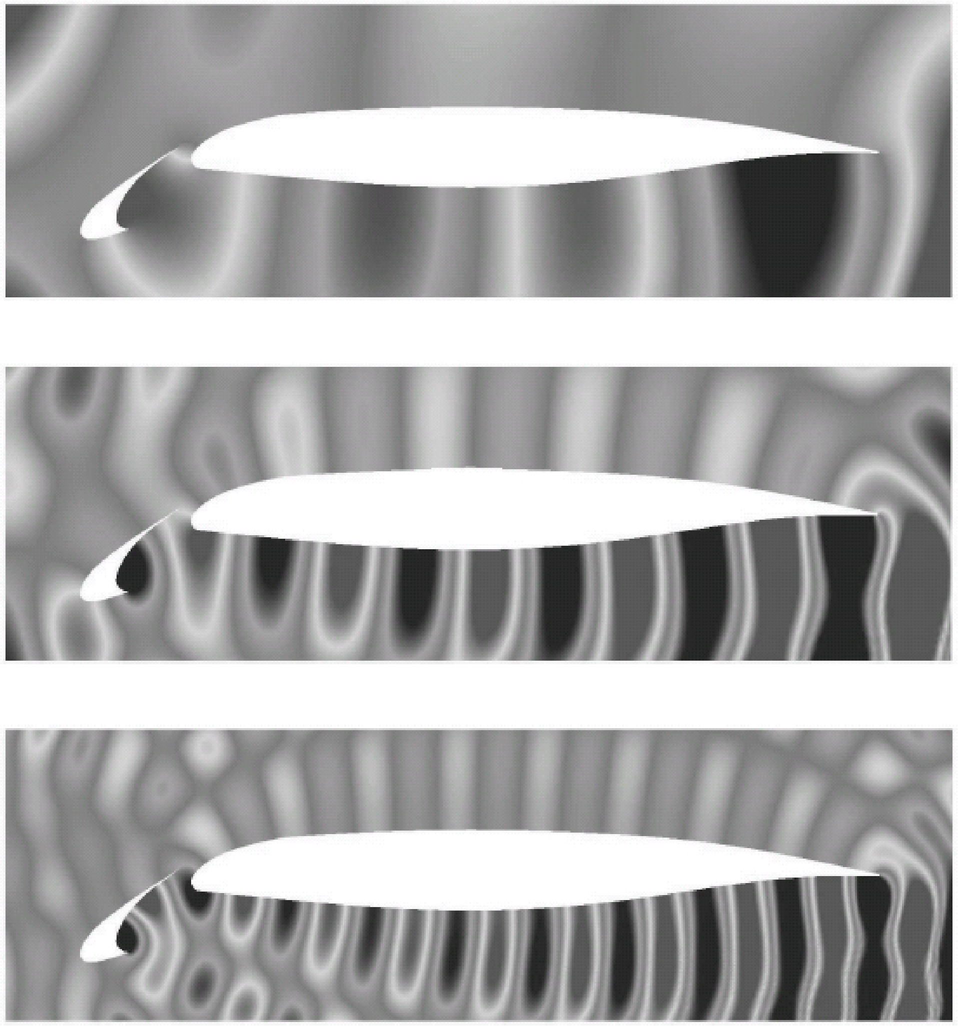
Slat-resonances: Simulation and Measurement (DLR)







Slat-resonances

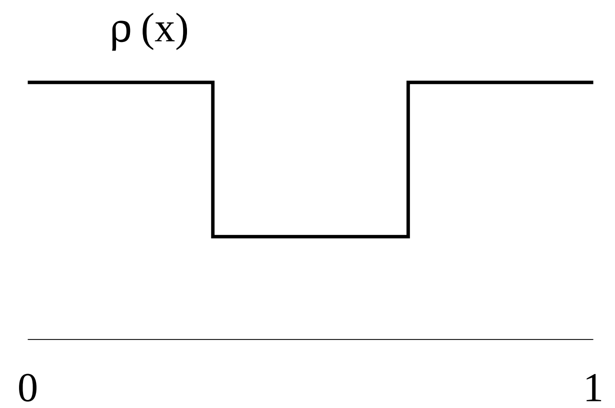


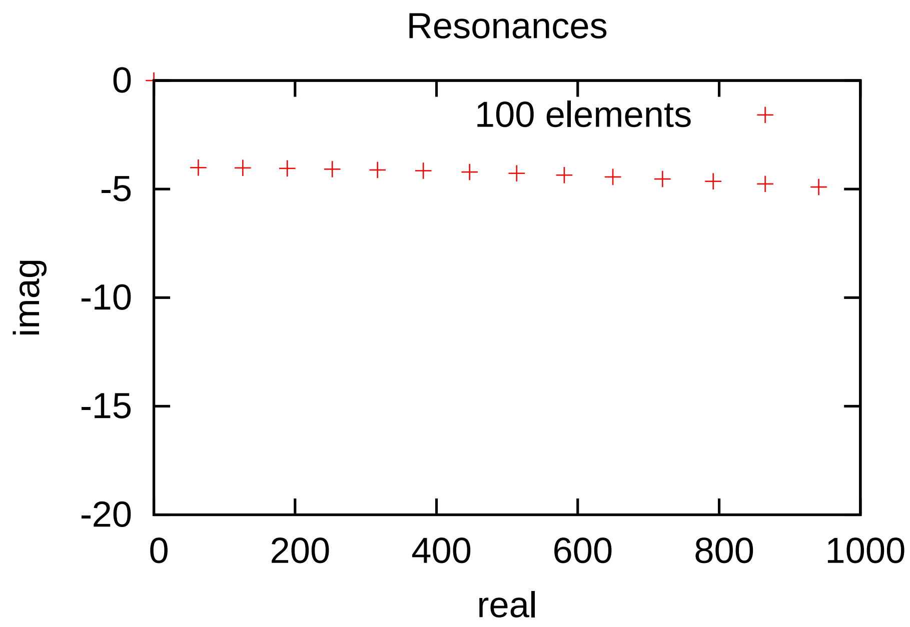
S. Hein, T. Hohage, W. Koch, and J. Schöberl: *Acoustic Resonances in a High-Lift Configuration*, J. Fluid Mech., 2007

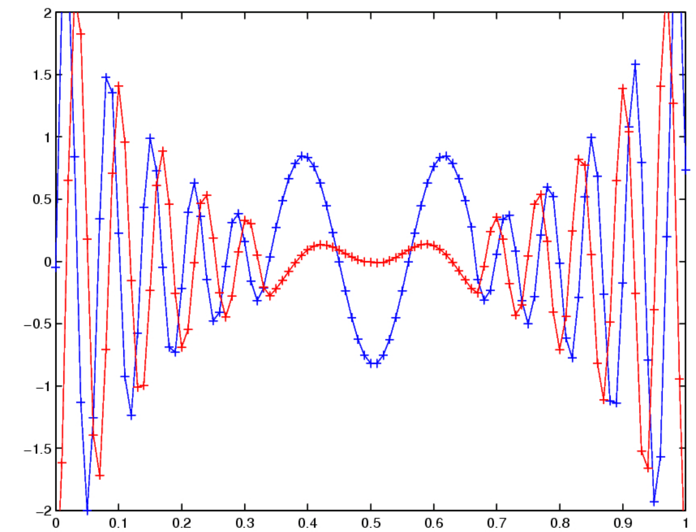
A one-dimensional Example

Schrödinger-like equation:

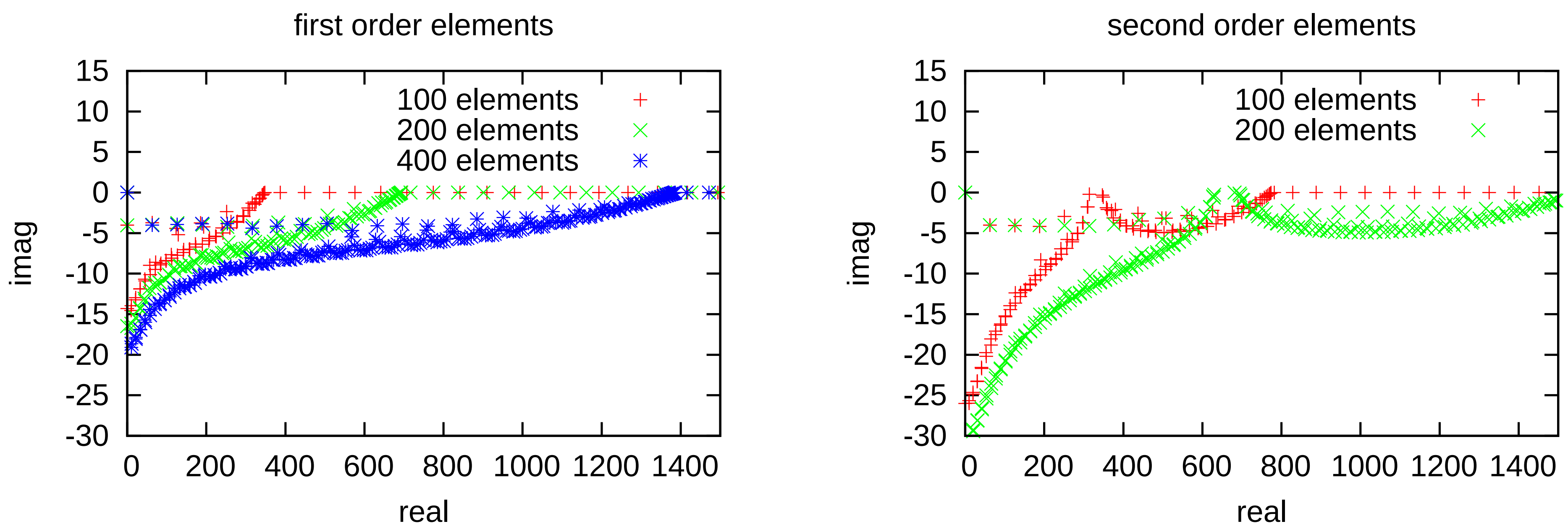
$$
\begin{aligned}
-u'' - \omega^2 \rho(x) u &= 0 \qquad \text{in } \; I = (0,1) \\
\frac{\partial u}{\partial n} - i\omega u &= 0 \qquad \text{on } \; \partial I = \{0,1\}
\end{aligned}
$$







Finite Element Spectra



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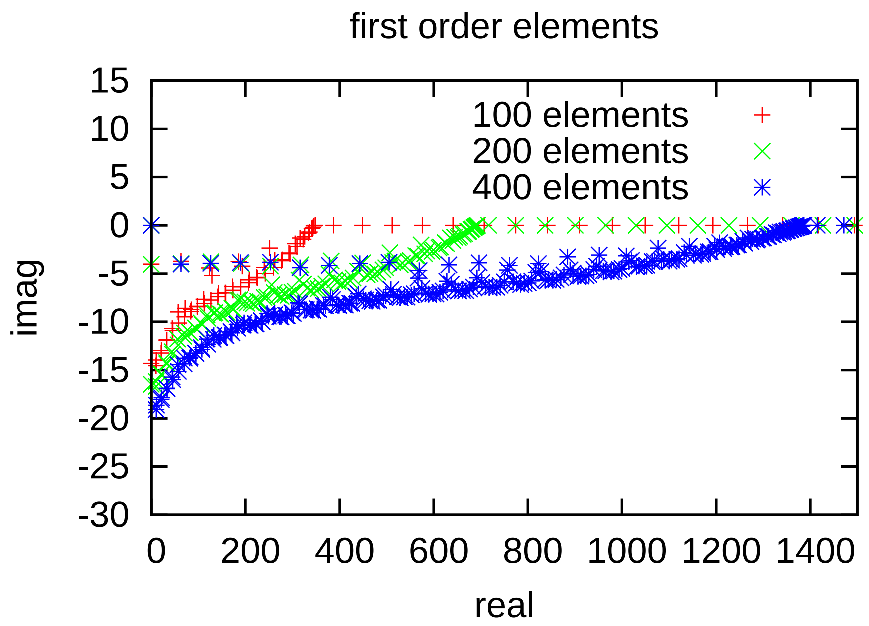

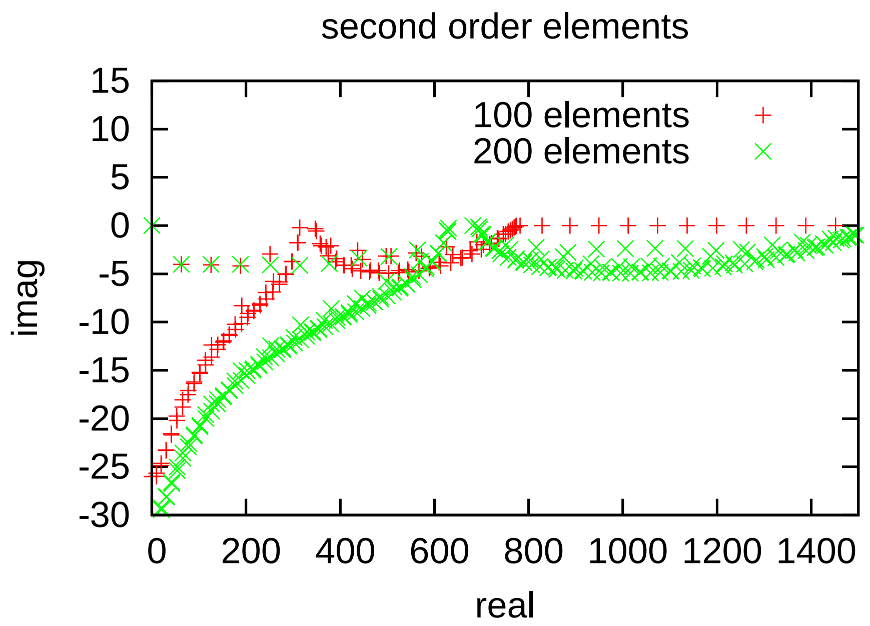

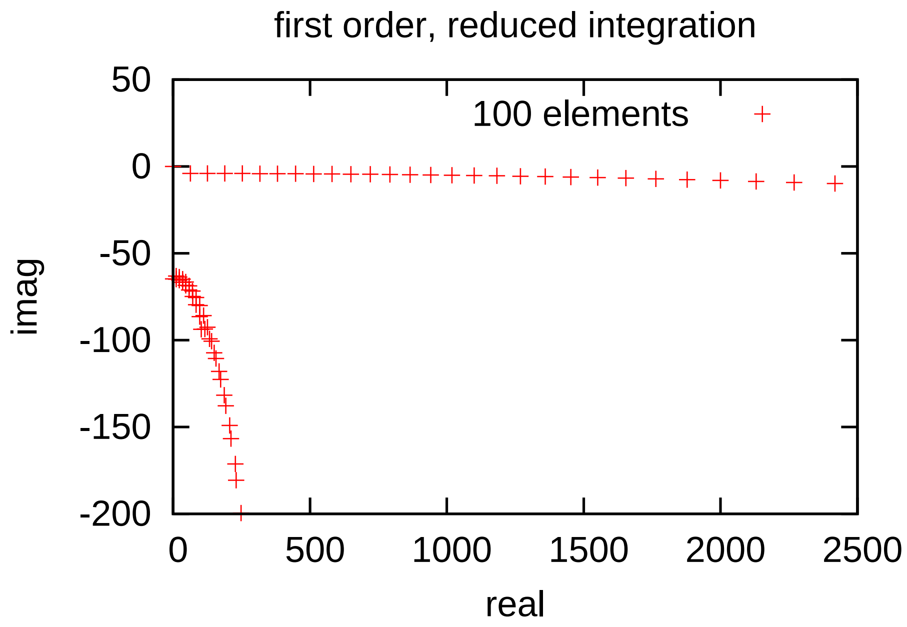

Reduced Integration Finite Element Method

Finite element space $V_h = \{v \in C : v_T \in P^1\}$.

Finite element method: Find $(\omega, u_h) \in \mathbb{C} \times V_h$:

$$\int_I u_h' v_h' \; - \; \omega^2 \int_I \rho u_h v_h \; - \; i\omega \int_{\partial I} u_h v_h = 0 \qquad \forall\, v_h \in V_h$$

Reduced integration method:

$$\int_I u_h' v_h' \; - \; \omega^2 \int\limits_{\substack{mid-point \\ rule}} \rho u_h v_h \; - \; i\omega \int_{\partial I} u_h v_h = 0 \qquad \forall\, v_h \in V_h$$

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Equivalent: L_2-projection into piecewise constants, i.e., $P_h : V_h \to Q_h$, with $Q_h = \{v : v_T \in P^0\}$:

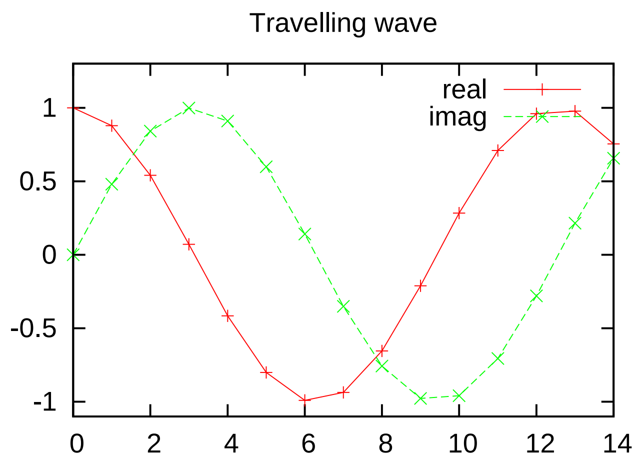
$$\int_I u_h' v_h' \; - \; \omega^2 \int_I \rho P_h u_h P_h v_h \; - \; i\omega \int_{\partial I} u_h v_h = 0 \qquad \forall\, v_h \in V_h$$

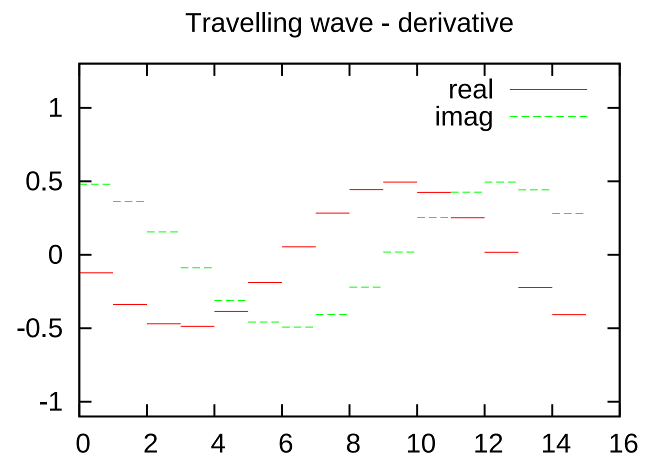
Travelling waves

An outgoing wave satisfies

$$\frac{\partial u}{\partial |x|} - i\omega u = 0.$$

This relation cannot be satisfied exactly by finite element functions, since u_h is continuous and piecewise P^1, while u' is discontinuous and piecewise P^0:





Compatible Discretization: It can be satisfied in reduced sense:

$$\frac{\partial u_h}{\partial |x|} - i\omega P_h u_h = 0.$$

Dispersion relation

On a uniform, infinite grid, the stiffness and mass matrices have the structure

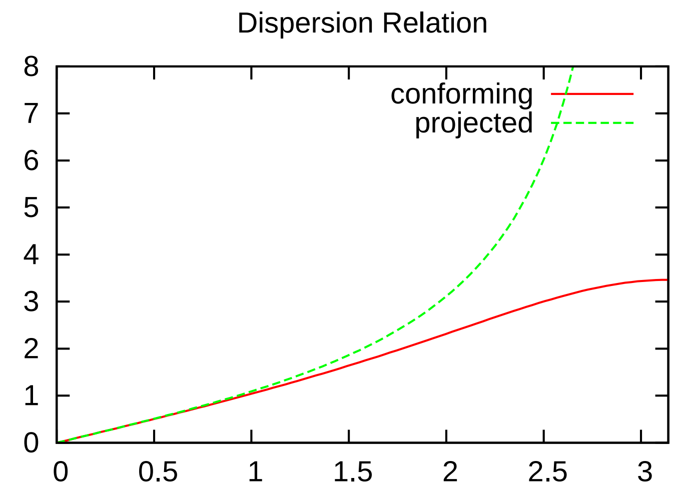
$$K = \frac{1}{h} \operatorname{tridiag}\left[-1,\ 2,\ -1\right]$$

$$M_{conf} = h \operatorname{tridiag}\left[\frac{1}{6},\ \frac{2}{3},\ \frac{1}{6}\right] \qquad M_{proj} = h \operatorname{tridiag}\left[\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4}\right]$$

All operators have the eigenfunctions $u = (u_j)_{j\in\mathbb{Z}} = (cos(jkh))_{j\in\mathbb{Z}}$, where k is the wave vector. The dispersion relation is to find $\omega(k)$ such that $Ku - \omega(k)^2 Mu = 0$.

$$\omega_{conf}(k) = \frac{\sqrt{6}}{h}\sqrt{\frac{1-\cos kh}{2+\cos kh}}$$

$$\omega_{proj}(k) = \frac{\sqrt{4}}{h}\sqrt{\frac{1-\cos kh}{1+\cos kh}}$$



Reduction property - Matrix level

The reduced integration FEM leads to the matrix eigenvalue problem:

$$\left\{ \frac{1}{h} \begin{pmatrix} \ddots & \ddots & & \\ \ddots & \ddots & \ddots & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{pmatrix} - \omega^2 h \begin{pmatrix} \ddots & \ddots & & \\ \ddots & \ddots & \ddots & \\ & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ & & \frac{1}{4} & \frac{1}{4} \end{pmatrix} - i\omega \begin{pmatrix} \ddots & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \right\} \begin{pmatrix} \vdots \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = 0$$

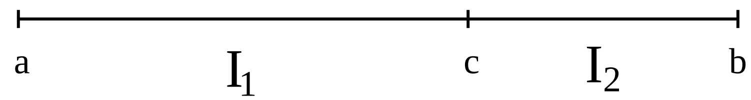
The last equation allows to express u_n from u_{n-1} via

$$\left(\frac{1}{h} - \omega^2 h \frac{1}{4} - i\omega \right) u_n = \left(\frac{1}{h} + \omega^2 h \frac{1}{4} \right) u_{n-1}.$$

Plugging into the previous equation leads to the reduced evp:

$$\left\{ \frac{1}{h} \begin{pmatrix} \ddots & \ddots & & \\ \ddots & \ddots & \ddots & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{pmatrix} - \omega^2 h \begin{pmatrix} \ddots & \ddots & & \\ \ddots & \ddots & \ddots & \\ & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ & & \frac{1}{4} & \frac{1}{4} \end{pmatrix} - i\omega \begin{pmatrix} \ddots & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \right\} \begin{pmatrix} \vdots \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = 0$$

Reduction property - Variational formulation



Theorem: V_h is a continuous FE space of order p, and P is the projection into discontinuous elements of order $p-1$. Let $\rho = 1$ in I_2, and c be a point in the mesh.

If u_h solves

$$\int_I u_h' v_h' - \omega^2 \int_I \rho P u_h P v_h - i\omega \int_{\partial I} u_h v_h = 0,$$

then the same u_h solves also

$$\int_{I_1} u_h' v_h' - \omega^2 \int_{I_1} \rho P u_h P v_h - i\omega \int_{\partial I_1} u_h v_h = 0.$$

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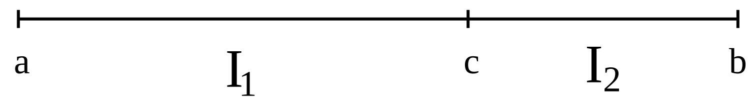

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$$\int_{I_1} u_h' v_h' - \omega^2 \int_{I_1} \rho P u_h P v_h - i\omega \int_{\partial I_1} u_h v_h = 0.$$

Proof: The difference is

$$\int_{I_2} u_h' v_h' - \omega^2 \int_{I_2} \rho P u_h P v_h - i\omega \{u_h(b) v_h(b) - u_h(c) v_h(c)\}$$

From the variational equation on I there follows that this term vanishes for v_h satisfying $v_h(c) = 0$. We have to show that the term vanishes for all v_h.

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The term is rewritten as

$$\begin{aligned}
& \int_{I_2} (u_h' - i\omega P u_h)(v_h' - i\omega P v_h) + i\omega \int_{I_2} P u_h v_h' + P v_h u_h' - i\omega\{u_h(b)v_h(b) - u_h(c)v_h(c)\} \\
= \quad & \int_{I_2} (u_h' - i\omega P u_h)(v_h' - i\omega P v_h) + i\omega \int_c^b (u_h v_h)' - i\omega\{u_h(b)v_h(b) - u_h(c)v_h(c)\} \\
= \quad & \int_{I_2} (u_h' - i\omega P u_h)(v_h' - i\omega P v_h).
\end{aligned}$$

One verifies that

$$\{v_h' - i\omega P v_h : v_h \in P^p, v_h(c) = 0\} = P^{p-1}.$$

Since $u_h' + i\omega P u_h$ is in P^{p-1}, and is orthogonal to P^{p-1}, it is 0. One can test against any v_h ! (together with Mark Ainsworth)

The Effect of PML

PML formulation:

$$\int_I \frac{1}{\sigma} u'v' - \omega^2 \int_I \sigma uv + i\omega \int_{\partial I} Ruv = 0$$

Where $\sigma = 1$ inside, and σ is a complex number in the PML layer.

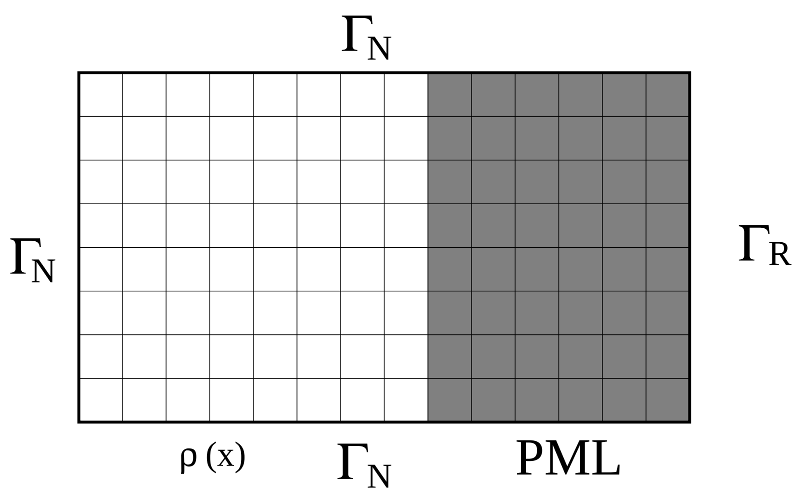
Let R be some absorbing coefficient. Repeating the matrix-reduction process leads to a new absorbing coefficient for the reduced problem as

$$\hat{R} = \frac{1 + Rc}{c + R} \qquad \text{with} \qquad c = \frac{1}{i\omega h\sigma} + \frac{i\omega h\sigma}{4}$$

If R is the correct coefficient (=1), it is the fixed point of the reduction formula. Otherwise, the reduction process converges to the correct coefficient with asymptotic rate

$$\left|\frac{\hat{R} - R^*}{R - R^*}\right| \approx \left|\frac{i\omega h\sigma - 2}{i\omega h\sigma + 2}\right|$$

The Channel Problem



2D PML Problem:

$$\int_\Omega \frac{1}{\sigma}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \sigma \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,dx - \omega^2 \int_\Omega \rho\sigma uv\,dx - i\omega \int_{\Gamma_R} uv = 0$$

Separation of variables based on eigensystem in y-direction:

$$(\varphi_i'(y), \psi'(y)) = \lambda_i^2(\varphi_i(y), \psi(y))$$

The expansion $u = \sum u_i(x)\varphi_i(y)$ leads to the decoupled problems

$$\int_I \frac{1}{\sigma} u_i' v_i' + \sigma(\lambda_i^2 - \omega^2\rho)uv\,dx - i\omega R u(1)v(1) = 0.$$

The right a.b.c for the i^{th} mode would be $\omega R = \sqrt{\omega^2\rho - \lambda_i^2}$. We do not want to adjust the R for each mode. The PML lets the R converge to the right R_i for the reduced system.

The projection operator

The 2D projection should map into the space which is

- continuous and of order k in y-direction
- discontinuous and of order $k-1$ in x direction

Then, the 2D problem is equivalent to the family of 1D problems.

The projection is not local on the element-level and thus cannot be realized by a reduced integration formula anymore.

The mixed method

Original equation:

$$\int \nabla u \nabla v \, dx + (i\omega)^2 \int \rho u v \, dx - i\omega \int_{\Gamma_R} u v \, dx = 0$$

Introduce a new variable $p = i\omega\rho u$ and obtain:

$$\begin{array}{rcll} \int \nabla u \nabla v \, dx & = & -i\omega \int p v \, dx + i\omega \int_{\Gamma_R} u v \, dx & \forall\, v \\ -\int \rho^{-1} p q \, dx & = & -i\omega \int q u \, dx & \forall q \end{array}$$

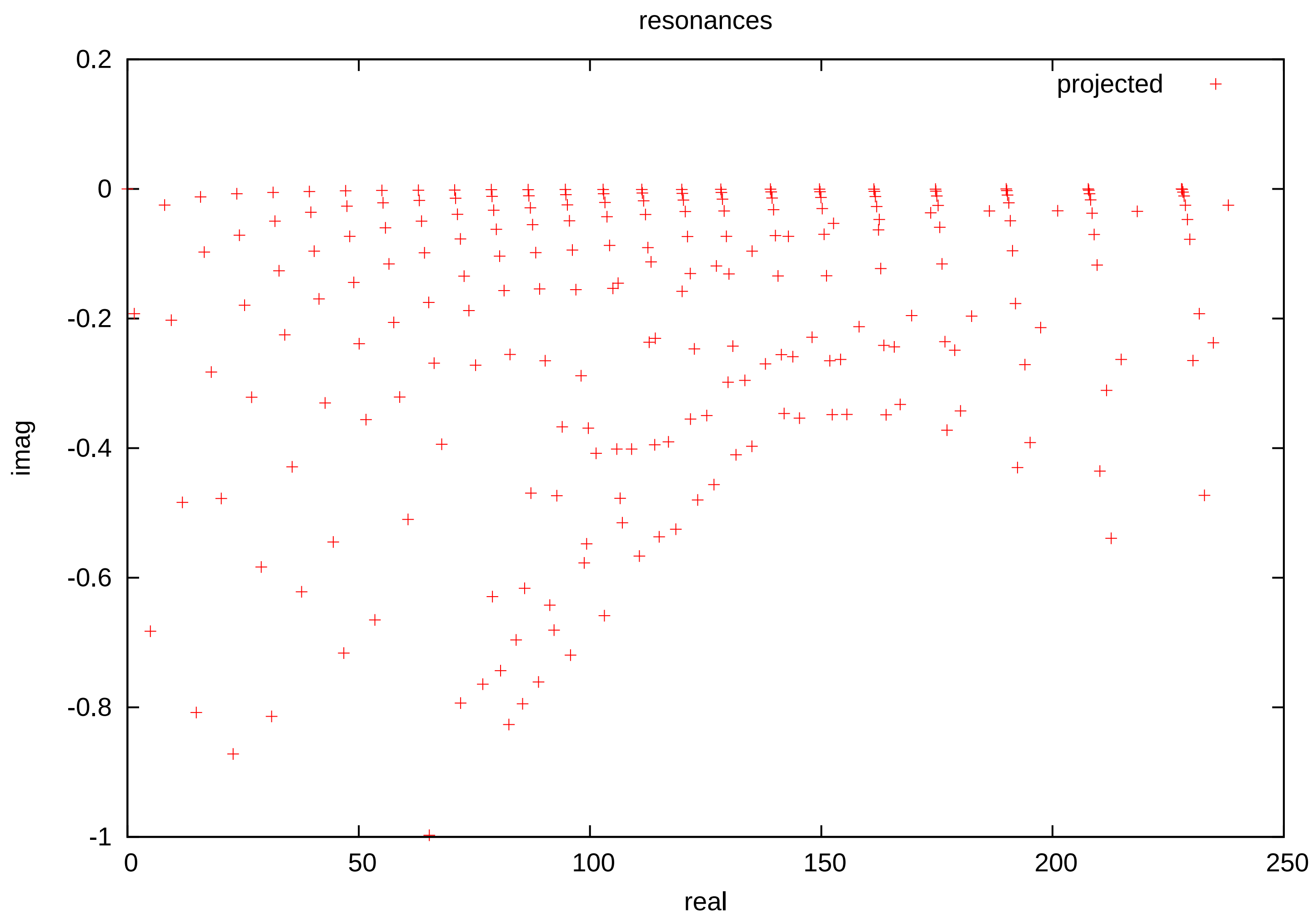
The finite element method with $u_h \in V_h$ and $p_h \in Q_h$ satisfies

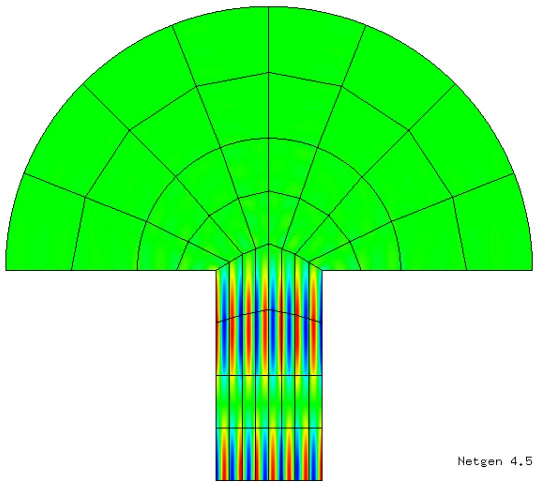
$$p_h = i\omega P_h(\rho u_h).$$

It is a way to implement the projection. The fe space Q_h must be

- continuous and of order k in y-direction
- discontinuous and of order $k-1$ in x direction

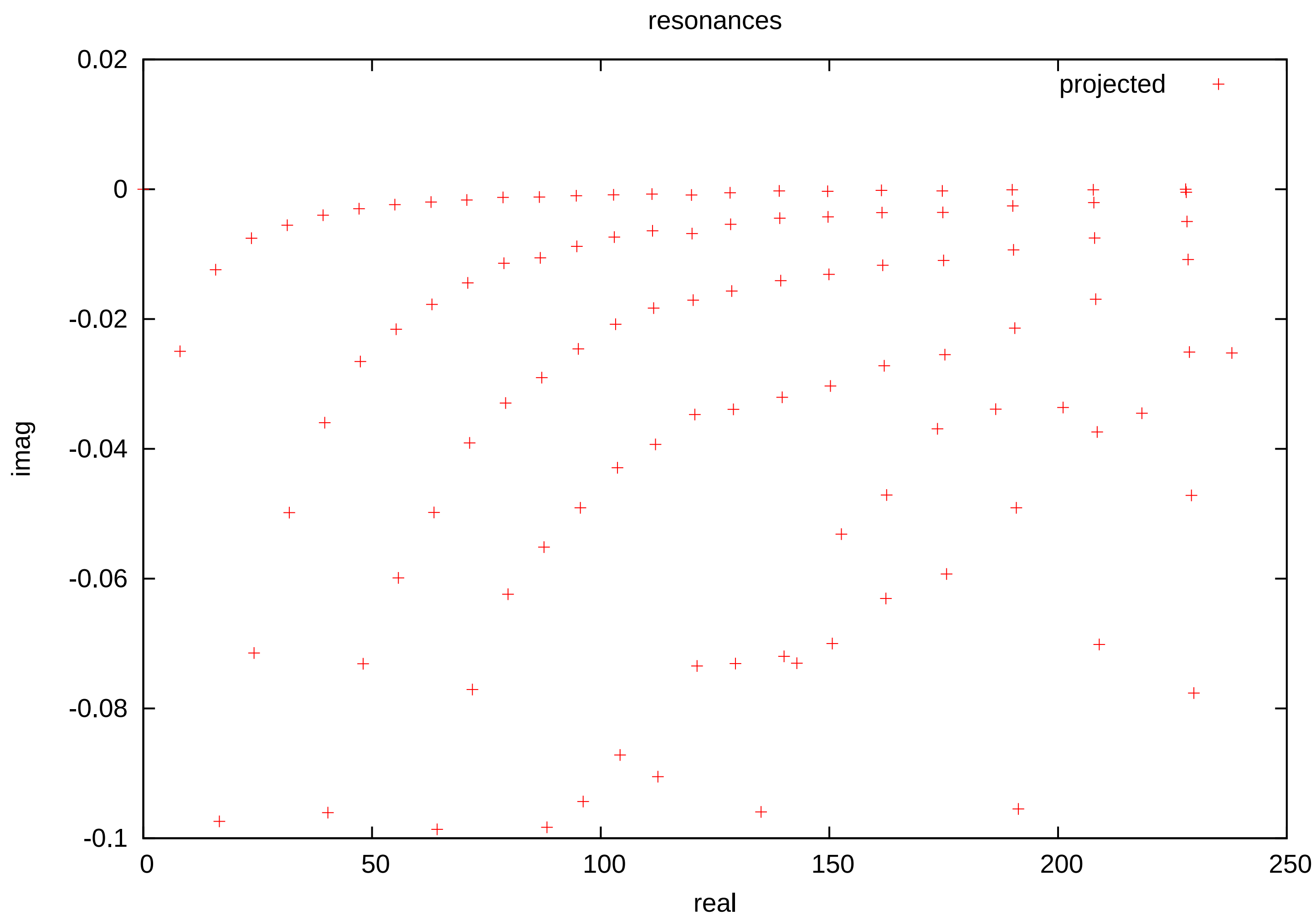
Cavity Resonances

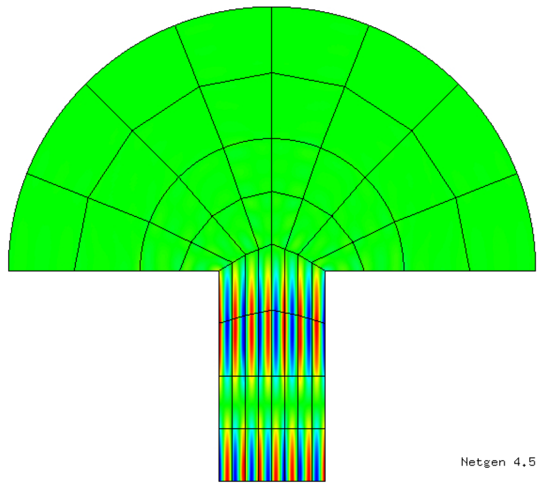




64 elements, p=4
N=2145

Cavity Resonances





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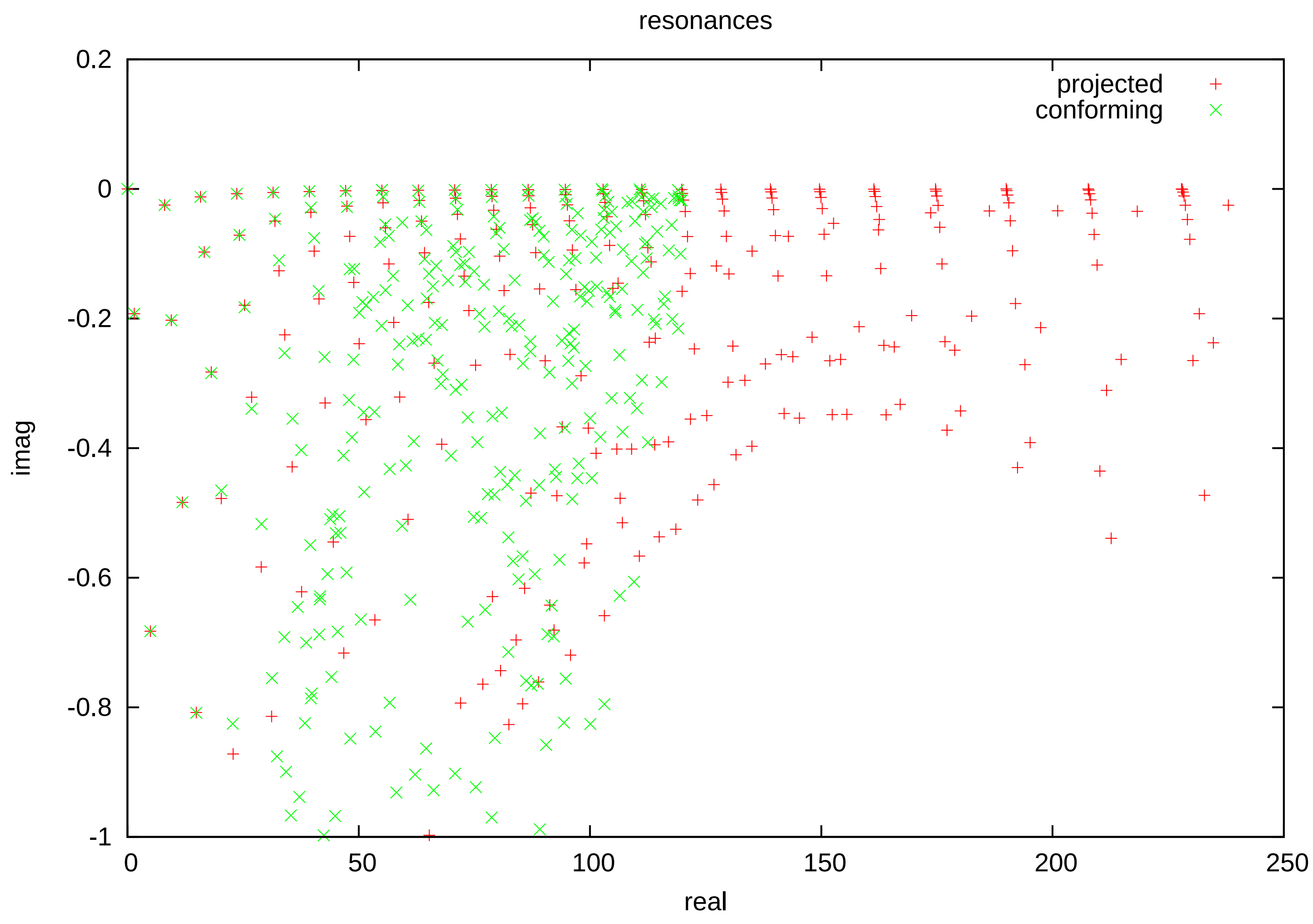

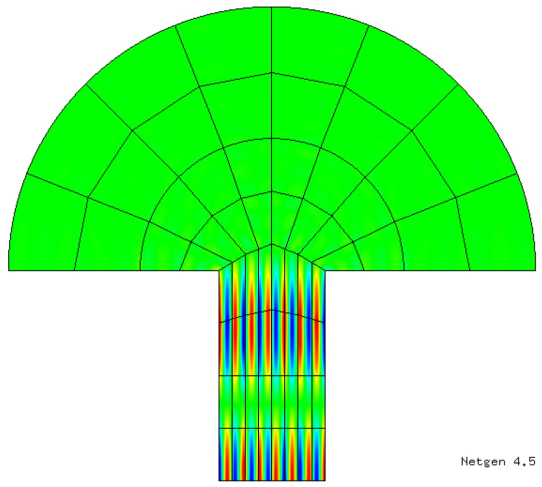

64 elements, p=4
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Concluding Remarks

- Compatible discretization avoiding spurious FE eigenvalues
- 1D: implementation by reduced integration formulas
- 2D/3D on topological tensor product meshes: mixed method

Ongoing work

- Extension to unstructured meshes (distance layers, projection space)
- Structure preserving eigenvalue solver

References:

- S. Hein, T. Hohage, W. Koch, J. Schöberl: Acoustic Resonances in a High Lift Configuration, J. Fluid Mech., 2007
- M. Rechberger: Numerical Methods for the Simulation of Acoustic Resonances, Master Th. JKU Linz