

# Divergence-free Hybrid Discontinuous Galerkin Finite Elements for Incompressible Navier Stokes Equations, Augmented Lagrangians, and Robust Preconditioners

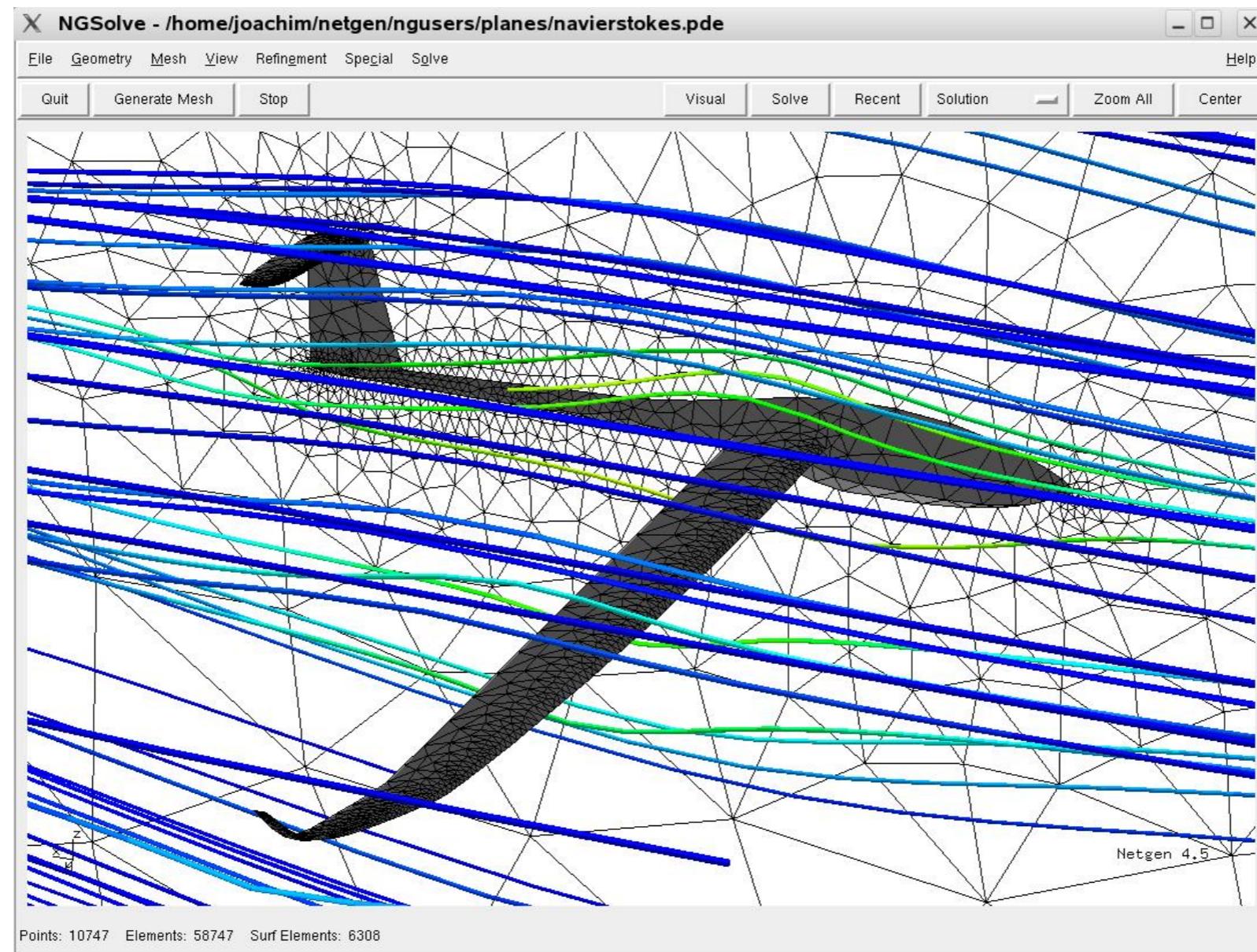
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## Toy Example: Sailplane



Stokes Flow, 2<sup>nd</sup>-order HDG elements, 59E3 elements, 1.65E6 dofs, 2GB RAM, 300 sec (2-core 1.8GHz)

## Contents

- Augmented Lagrangian Formulation for Navier Stokes Equations
- Preconditioning for Parameter Dependent Problem
- Hybrid Discontinuous Galerkin Finite Elements for Navier Stokes

## Navier Stokes Equations

$$\begin{aligned}\frac{\partial u}{\partial t} - \operatorname{div}(2\nu\nabla^S u - u \otimes u - pI) &= f \\ \operatorname{div} u &= 0 \\ +b.c.\end{aligned}$$

space discretization, simple time-stepping:

$$\begin{aligned}\left(\frac{1}{\tau}M + A^\nu\right)\hat{u} + B^T\hat{p} &= \frac{1}{\tau}Mu + f - A^c(u) \\ B\hat{u} &= 0\end{aligned}$$

nonlinear convection term treated explicitly

## Augmented Lagrangian Formulation

$$\begin{pmatrix} A + \frac{1}{\varepsilon} B^T C^{-1} B & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

Assumption:  $p$  discontinuous  $\Rightarrow C$  is (block)diagonal

For  $\varepsilon \rightarrow 0$ :

$$S := B(A + \frac{1}{\varepsilon} B^T C^{-1} B)^{-1} B^T \rightarrow \varepsilon C$$

Plan:  $\varepsilon \approx 10^{-6}\nu$  very small, then 1 or 2 Uzawa iterations are sufficient

But: Very ill-conditioned system in the 1-1 block

Exact Schur complement formulation for weakly compressible flows

## III-conditioned Systems

Bilinear-form for continuous problem:

$$A^\varepsilon(u, v) = \int \nabla u \nabla v + \frac{1}{\varepsilon} \int \operatorname{div} u \operatorname{div} v$$

Second term has large kernel of div-free functions, but not enough div-free low order fe functions.

Mesh-dependent bilinear-forms:

$$A_h^\varepsilon(u, v) = \int \nabla u \nabla v + \frac{1}{\varepsilon} \int P_h \operatorname{div} u P_h \operatorname{div} v$$

with projector  $P_h : L_2 \rightarrow Q_h$ .

Robust error estimates by equivalence to mixed system:

Find  $u_h \in V_h$  and  $p_h \in Q_h$ :

$$\begin{array}{lclcl} \int \nabla u_h \nabla v_h & + & \int \operatorname{div} v_h p_h & = & \int f v_h & \forall v_h \in V_h \\ \int \operatorname{div} u_h q_h & - & \varepsilon \int p_h q_h & = & 0 & \forall q_h \in Q_h \end{array}$$

theory of 80s

## Some more parameter dependent problems

Reissner Mindlin plate equation:

$$A(w, \beta; v, \delta) = \int_{\Omega} \nabla \beta \nabla \delta + \frac{1}{t^2} \int_{\Omega} (\nabla w - \beta)^T (\nabla v - \delta)$$

As  $t \rightarrow 0$ : penalty formulation for  $\nabla w - \beta = 0$ .

Non-conforming discretization with shear-reduction operator  $R_h$  to allow  $\nabla w - R_h \beta \approx 0$ .

Maxwell equations in vector-potential formulation:

$$A(E, v) = \int_{\Omega} \mu^{-1} \operatorname{curl} E \operatorname{curl} v + \int_{\Omega} (i\omega\sigma - \omega^2 \varepsilon) Ev$$

$\operatorname{curl}$  operator has larger kernel of gradient fields

Conforming discretization with  $H(\operatorname{curl})$ -conforming Nedelec elements.

## Additive Schwarz Preconditioning

symmetric, coercive bilinear-form

$$A(., .) : V \times V \rightarrow \mathbb{R}$$

sup-space decomposition

$$V = \sum V_i$$

local blocks, multi-level, domain decomposition, ...

Additive Schwarz Preconditioner:

$$C^{-1} : V^* \rightarrow V : d(.) \mapsto w$$

by solving local problems:

$$w_i \in V_i : A(w_i, v_i) = d(v_i) \quad \forall v_i \in V_i$$

$$w = \sum w_i$$

multiplicative version similar

## Additive Schwarz Lemma

Lemma of many fathers (Lions, Nepomnyaschikh, Dryja+Widlund, Xu, Griebel+Oswald, ...)

$$\|u\|_C^2 = \inf_{\substack{u=\sum u_i \\ u_i \in V_i}} \sum \|u_i\|_A^2$$

Goal: Find sub-spaces with good spectral estimates

$$\gamma_1 \|u\|_C^2 \leq \|u\|_A^2 \leq \gamma_2 \|u\|_C^2$$

Local sub-spaces for  $2^{nd}$ -order problems:  $\gamma_1 = O(h^2)$ ,  $\gamma_2 \approx 1$

Multi-level decomposition:  $\gamma_1 \approx \gamma_2 \approx 1$

## Robust decomposition

III-conditioned bilinear-form:

$$A_h^\varepsilon(u, v) = a(u, v) + \varepsilon^{-1}c(P_hBu, P_hBv)$$

Kernel

$$V_{h,0} = \{v_h : P_hBv_h = 0\}$$

Quadratic form of preconditioner:

$$\|u\|_C^2 = \inf_{\substack{u=\sum u_i \\ u_i \in V_i}} \sum \|u_i\|_{A_h^\varepsilon}^2$$

For  $u_h \in V_{h,0}$ :

$$\|u_h\|_{A_h^\varepsilon} = O(1) \quad \text{but} \quad \|u_h\|_C = O(\varepsilon^{-1}) \quad \text{in general}$$

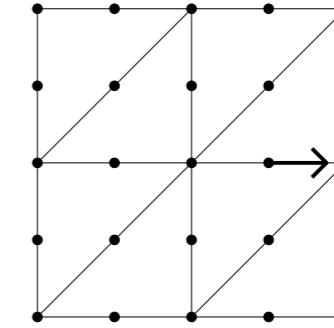
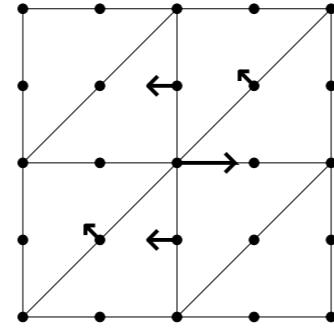
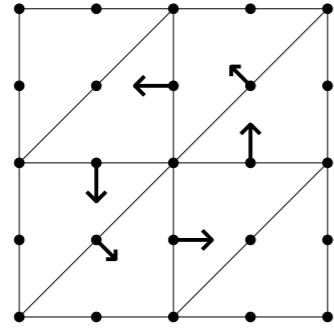
Robust only for kernel compatible decomposition:

$$V_{h,0} = \sum V_i \cap V_{h,0}$$

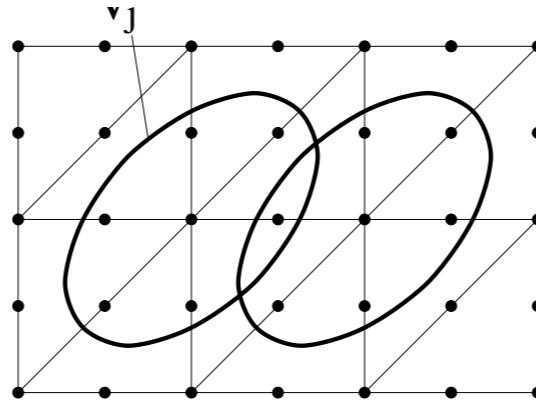
## Local sub-spaces for augmented Stokes with $P^2 - P^0$ elements

$$P_h \operatorname{div} u_h = 0 \Leftrightarrow \int_T \operatorname{div} u_h = 0 \Leftrightarrow \int_{\partial T} n^T u \, ds = 0 \quad \forall T \in \mathcal{T}_h$$

Discrete divergence-free base functions:



Sub-space covering:



## $\varepsilon$ -Robust local preconditioner

$$A_h^\varepsilon(u, v) = a(u, v) + \varepsilon^{-1}c(P_hBu, P_hBv)$$

Space splitting  $V = \sum V_i$  fulfilling the decomposition inequalities

$$\inf_{\substack{u_h=\sum u_i \\ u_i \in V_i}} \sum \|u_i\|_V^2 \leq c_1(h) \|u_h\|_V^2 \quad \forall u_h \in V_h$$

$$\inf_{\substack{u_{h,0}=\sum u_i \\ u_i \in V_i \cap V_{h,0}}} \sum \|u_i\|_a^2 \leq c_2(h) \|u_{h,0}\|_V^2 \quad \forall u_{h,0} \in V_{h,0}$$

LBB condition

$$\sup_{v_h \in V_h} \frac{c(Bv_h, q_h)}{\|v_h\|_V} \geq c_3(h) \|q_h\|_c \quad \forall q_h \in Q_h := P_hBV_h$$

Then the (local) additive Schwarz preconditioner  $D$  fulfills the  $\varepsilon$ -robust spectral estimates

$$\{c_2(h) + c_1(h)/c_3(h)^2\}^{-1} \|u_h\|_D^2 \preceq \|u_h\|_{A_h^\varepsilon}^2 \preceq \|u_h\|_D^2$$

Similar  $H(\text{div})$  and  $H(\text{curl})$ : Vassilevski-Wang, Cai-Goldstein-Pasciak, Arnold-Falk-Winther, Hiptmair,

## Two-level preconditioner

Preconditioning operation:

$$C^{-1} = E_H A_H^{-1} E_H^T + D^{-1}$$

with local preconditioner  $D$

2-level norm:

$$\|u_h\|_C^2 = \inf_{u_h = E_H u_H + \sum u_i} \left\{ \|u_H\|_{A_H}^2 + \sum \|u_i\|_{A_h}^2 \right\}$$

Norm equivalence  $C \simeq A_h$  requires:

- Continuous prolongation operator  $E_H : (V_H, \|\cdot\|_{A_H}) \rightarrow (V_h, \|\cdot\|_{A_h})$
- Existence of continuous interpolation operator  $\Pi_H : (V_h, \|\cdot\|_{A_h}) \rightarrow (V_H, \|\cdot\|_{A_H})$

## $\varepsilon$ -Robust two-Level preconditioner

Coarse grid bilinear form:

$$A_H^\varepsilon(u_H, v_H) = a(u_H, v_H) + \varepsilon^{-1}c(P_H Bu_H, P_H Bv_H)$$

$$V_{H,0} = \text{kern } P_H B$$

Fine grid bilinear form:

$$A_h^\varepsilon(u_h, v_h) = a(u_h, v_h) + \varepsilon^{-1}c(P_h Bu_h, P_h Bv_h)$$

$$V_{h,0} = \text{kern } P_h B$$

To be robust, the prolongation operator  $E_H : V_H \rightarrow V_h$  has to satisfy

$$E_H : V_{H,0} \rightarrow V_{h,0}.$$

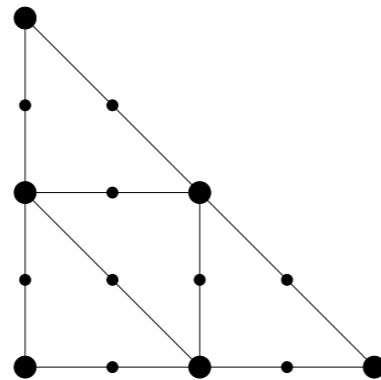
Consider  $u_H \in V_{H,0}$

$$\|E_H u_H\|_{A_h^\varepsilon}^2 = \|E_H u_H\|_a^2 + \frac{1}{\varepsilon} \|P_h B E_H u_H\|_c^2 \quad \text{and} \quad \|u_H\|_{A_H^\varepsilon}^2 = \|u_H\|_a^2$$

## Robust prolongation for nearly incompressible materials

$$u_H \in \text{kern}(B_H) \Leftrightarrow \int_{\partial T} n^T u_H \, ds = 0$$

$$E_H u_H \in \text{kern}(B_h) \Leftrightarrow \int_{\partial t_i} n^T (E_H u_H) \, ds = 0, \quad i = 1 \dots 4$$



$$T = \bigcup_{i=1}^4 t_i$$

1. Conforming (quadratic) prolongation at  $\partial T$
2. Adjust inner nodes by solving local Dirichlet problems

JS: proceedings to EMG96

## Multigrid Analysis for Augmented Stokes

*Approximation Property:*

$$\|u_h - E_H(A_H^\varepsilon)^{-1}E_H^T A_h^\varepsilon\|_{\tilde{0}} \preceq \|u_h\|_{A_h^\varepsilon}$$

with the norm

$$\|u_h\|_{\tilde{0}}^2 = h^{-2}\|u_h\|_0^2 + \frac{1}{\varepsilon}\|P_h \operatorname{div} u_h\|_0^2 + \|\frac{1}{\varepsilon}P_H \operatorname{div} u_h\|_0^2$$

*Smoothing Property:*

$$\|(I - D_h^{-1}A_h^\varepsilon)^m u_h\|_{A_h^\varepsilon} \preceq \frac{1}{\sqrt{m}} \|u_h\|_{[A_h^\varepsilon, D_h]_{1/2}}$$

The link:

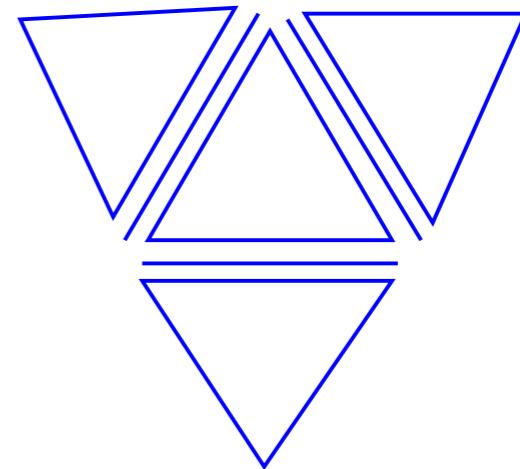
$$\|u_h\|_{[A_h^\varepsilon, D_h]_{1/2}} \preceq \|u_h\|_{\tilde{0}}$$

Stokes case: JS Numer. Math. 99, Reissner Mindlin: PhD-Thesis, 99

## Hybrid Discontinuous Galerkin (HDG) Methods

Hybrid versions of interior penalty methods / Nitsche methods [Cockburn+Gopalakrishnan+Lazarov]

- unknowns  $u$  in elements and unknowns  $\lambda = \text{trace } u$  on facets.
- more unknowns, but less matrix-entries
- fits into standard element-based assembling
- allows condensation of element unknowns



Find  $u \in P^k(T)$  and  $\lambda \in P^k(F)$  such that for all  $v \in P^k(T)$  and  $\mu \in P^k(F)$ :

$$\sum_T \left\{ \int_T \nabla u \cdot \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} (v - \mu) - \int_{\partial T} \frac{\partial v}{\partial n} (u - \lambda) + \frac{\alpha p^2}{h} \int_{\partial T} (u - \lambda)(v - \mu) \right\} = \int f v$$

consistency   symmetry   stability

## Convection - Diffusion Problems

$$\begin{aligned}-\varepsilon \Delta u + b \cdot \nabla u &= f && \text{in } \partial\Omega \\ u &= 0 && \text{on } \partial\Omega\end{aligned}$$

HDG Formulation:

$$A^d(u, \lambda; v, \mu) + A^c(u, \lambda; v, \mu) = \int f v$$

with diffusive term  $A^d(.,.)$  from above and upwind-discretization for convective term

$$A^c(u, \lambda; v, \mu) = \sum_T \left\{ - \int b u \cdot \nabla v + \int_{\partial T} b_n \{u/\lambda\} v \right\}$$

with upwind choice

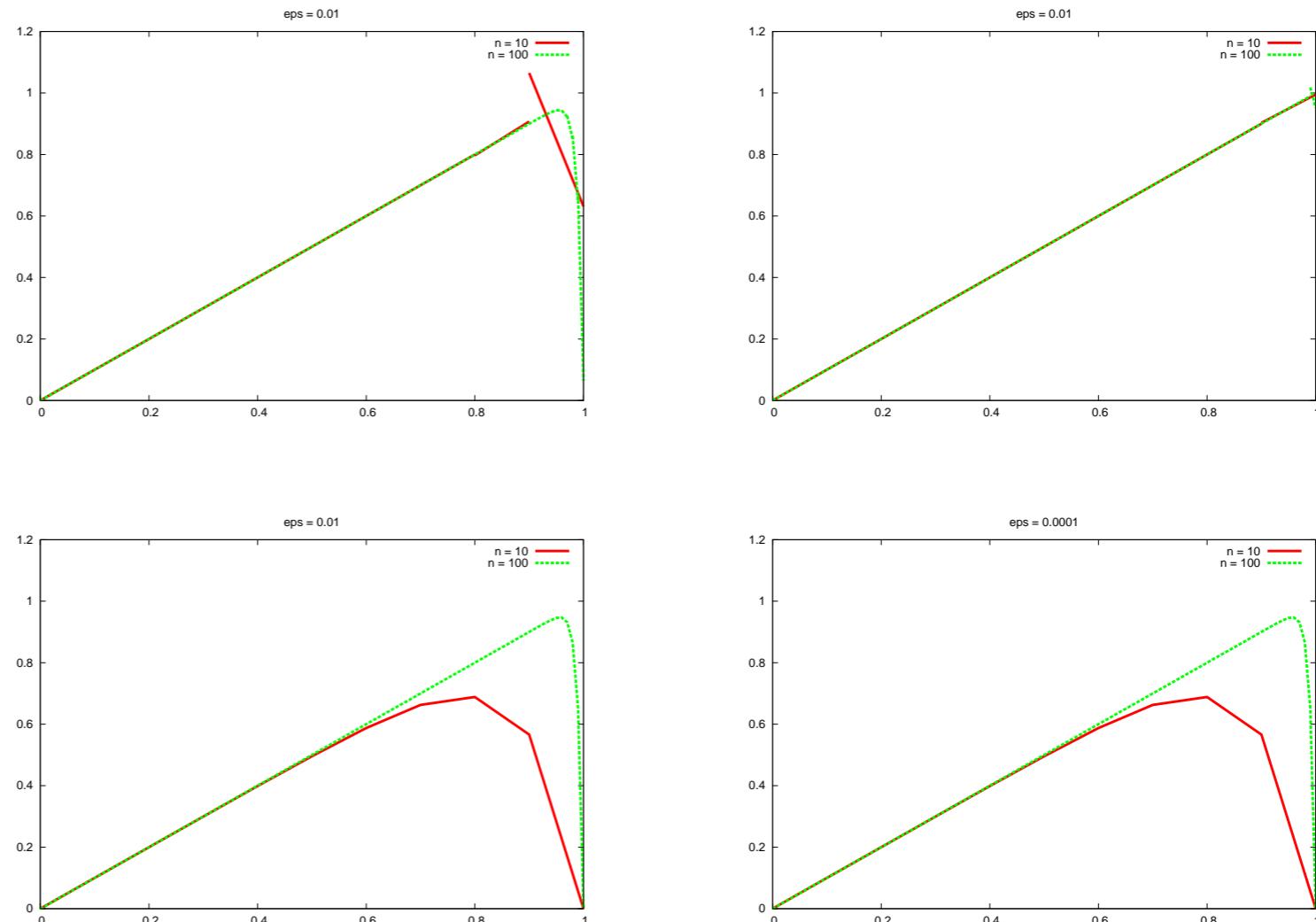
$$\{u/\lambda\} = \begin{cases} \lambda & \text{if } b_n < 0, \text{ i.e. inflow edge} \\ u & \text{if } b_n > 0, \text{ i.e. outflow edge} \end{cases}$$

[H. Egger + JS, submitted]

## Results for 1D

$$-\varepsilon u'' + u' = 1, \quad u(0) = u(1) = 0$$

DG Discretization:  
left:  $\varepsilon = 1E - 2$   
right:  $\varepsilon = 1E - 4$



conforming elements with  
SUPG stabilization

## $H(\text{div})$ -conforming elements for Navier Stokes

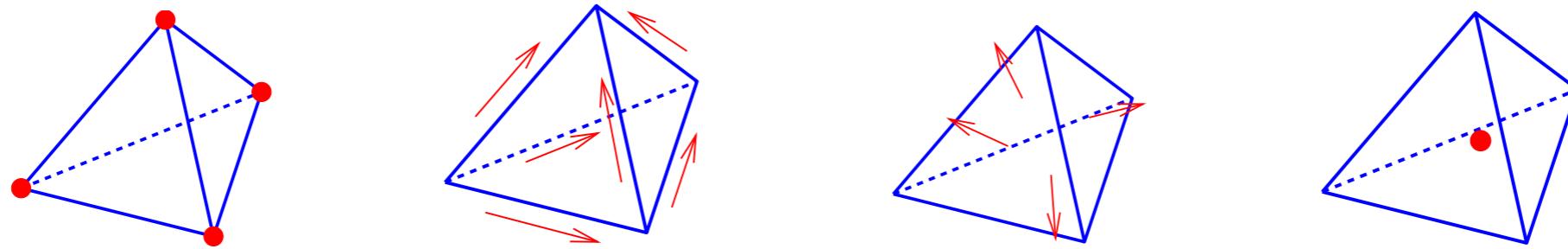
[Cockburn, Kanschat, Schötzau]:

$$\begin{aligned} \left( \frac{1}{\tau} M + A^\nu \right) \hat{u} + B^T \hat{p} &= f - \frac{1}{\tau} M u - A^c(u) \\ B \hat{u} &= 0 \end{aligned}$$

- $u_h \in V_h := BDM_k \subset H(\text{div})$ ,  $p \in Q_h := P_{k-1} \subset L_2$ .
- $u$  is exactly div-free ( $\Rightarrow$  stability of discrete time-dependent equation)
- viscosity term by hybrid DG (facet element with tangential component)
- convective term by upwinding
- stability for kinetic energy ( $\frac{d}{dt} \|u\|_0^2 \preceq \frac{1}{\nu} \|f\|_{L_2}^2$ )
- allows kernel-preserving smoothing and grid-transfer

## The de Rham Complex

$$\begin{array}{ccccccc}
 H^1 & \xrightarrow{\nabla} & H(\text{curl}) & \xrightarrow{\text{curl}} & H(\text{div}) & \xrightarrow{\text{div}} & L^2 \\
 \cup & & \cup & & \cup & & \cup \\
 W_h & \xrightarrow{\nabla} & V_h & \xrightarrow{\text{curl}} & Q_h & \xrightarrow{\text{div}} & S_h
 \end{array}$$



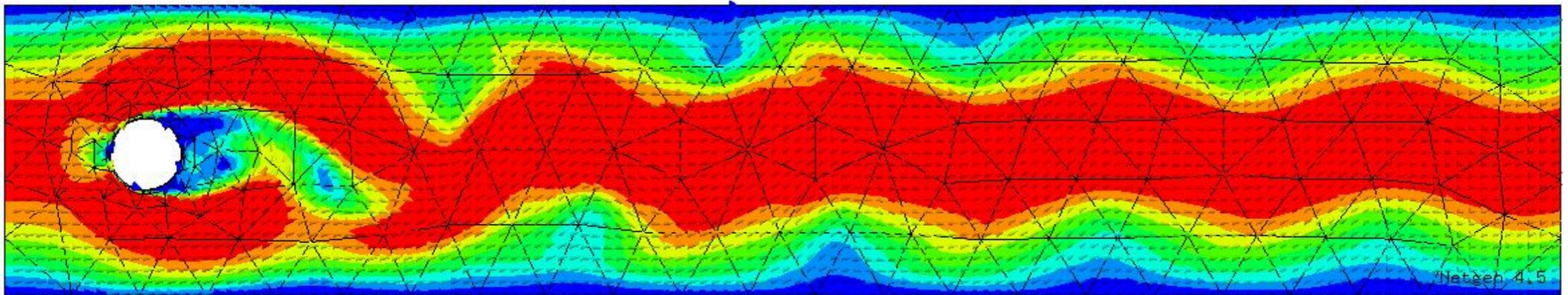
Used for constructing high order finite elements [JS+S. Zaglmayr, '05, Thesis Zaglmayr '06]

$$\begin{aligned}
 W_{hp} &= W_{\mathcal{L}_1} + \text{span}\{\varphi_{h.o.}^W\} \\
 V_{hp} &= V_{\mathcal{N}_0} + \text{span}\{\nabla \varphi_{h.o.}^W\} + \text{span}\{\varphi_{h.o.}^V\} \\
 Q_{hp} &= Q_{\mathcal{R}T_0} + \text{span}\{\text{curl } \varphi_{h.o.}^V\} + \text{span}\{\varphi_{h.o.}^Q\} \\
 S_{hp} &= S_{\mathcal{P}_0} + \text{span}\{\text{div } \varphi_{h.o.}^S\}
 \end{aligned}$$

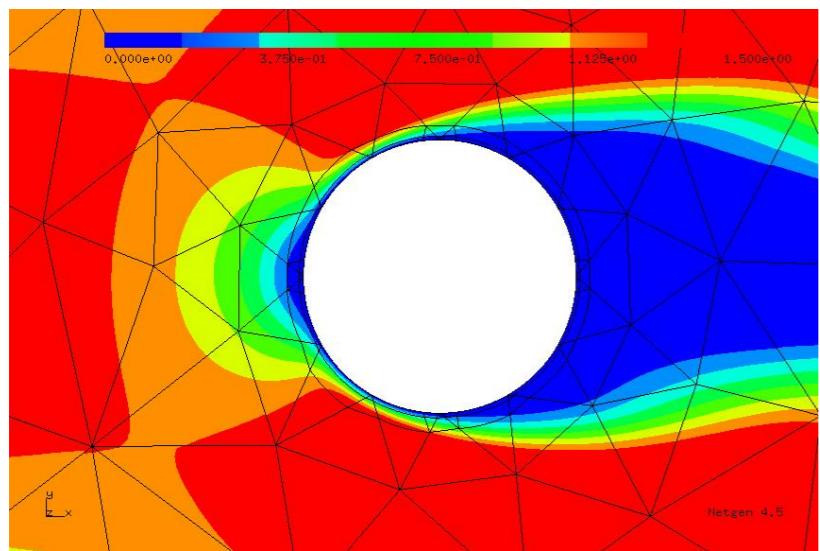
Allows to construct high-order-divergence free elements  $\{v \in BDM_k : \text{div } v \in P_0\}$

## Flow around a disk, 2D

$Re = 100$ , 5<sup>th</sup>-order elements

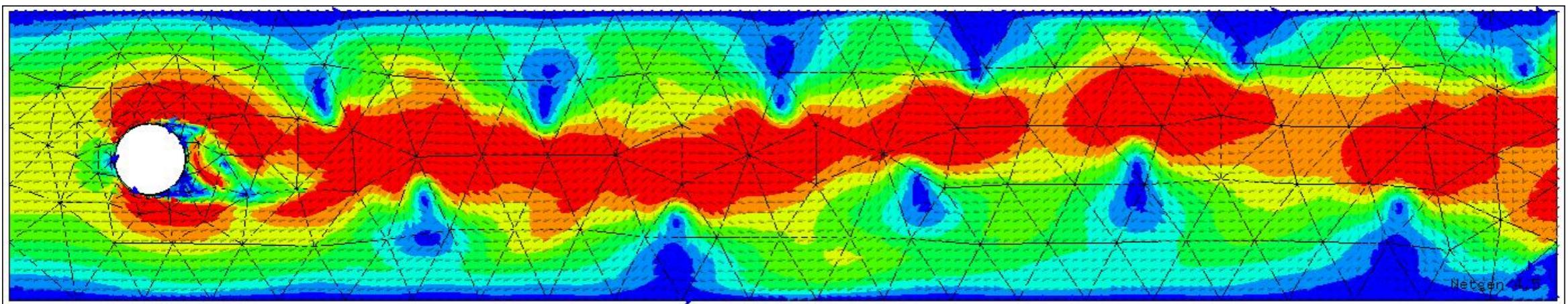


Boundary layer mesh around cylinder:

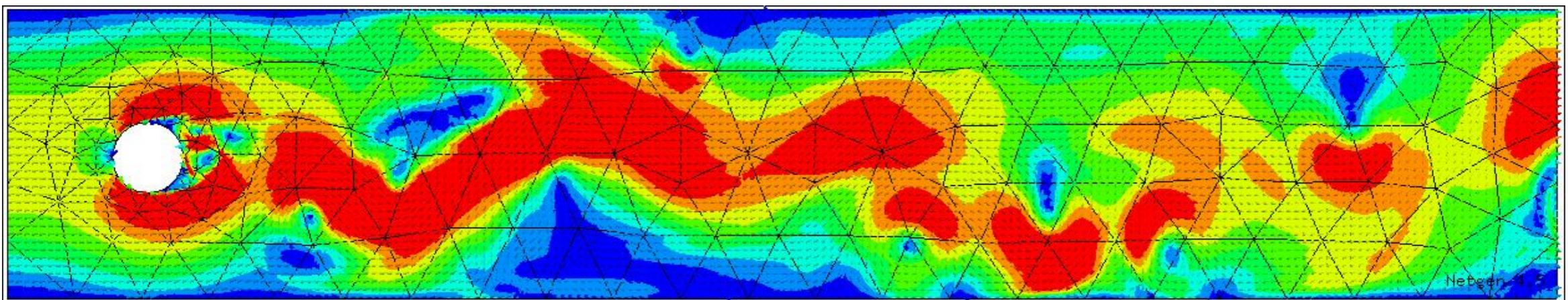


## Flow around a disk, 2D

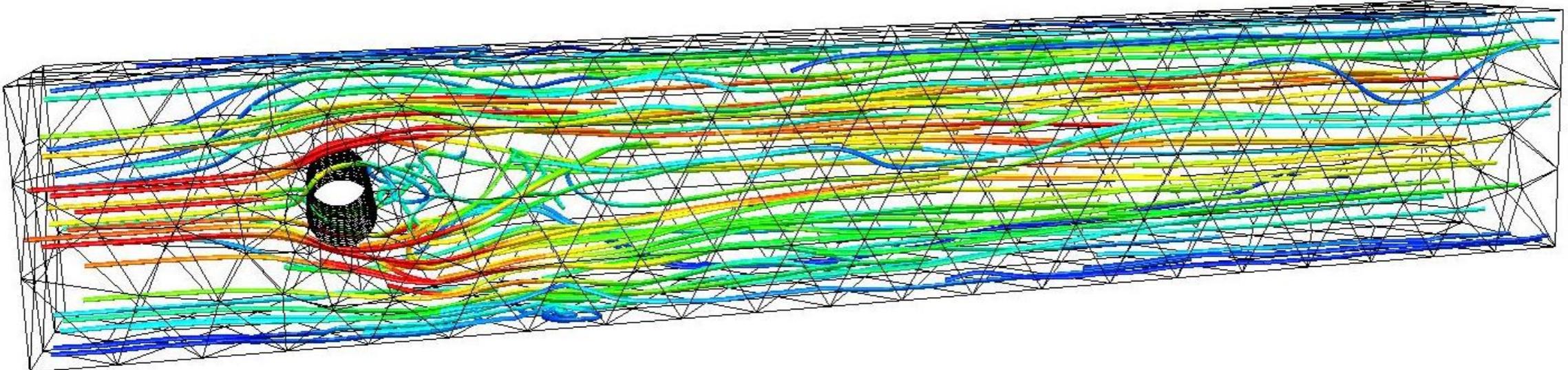
$Re = 1000$ :



$Re = 5000$ :



## Flow around a cylinder, Re = 100



Low-order / high-order two-level preconditioning:

Order	$N$	$\kappa(C^{-1}A)$	its (1E-8)	$N$	$\kappa(C^{-1}A)$	its (1E-8)
1	3046	6.2	11	28978	20.6	17
2	9369	21.1	19	92781	45.9	25
3	20052	31.8	22	202080	60.3	29
4	35965	33.9	22	-	-	-

Flow through a channel without cylinder

## Conclusion and Ongoing Work

Conclusions:

- Preconditioner for ill-conditioned  $A_{1,1}$ -block from augmented Lagrangian
- Hybrid-DG div-free Navier Stokes elements
- Robust discretization for large Reynolds numbers

Ongoing work:

- Turbulence models, e.g.. variational multiscale methods (vms)
- (Weakly) compressible flows: numerical fluxes, limiters
- Instationary 3D flow simulation needs parallel codes

Open source software Netgen/NGSolve freely available from [www.hpfem.jku.at/netgen](http://www.hpfem.jku.at/netgen)