

# **Tangential Continuous Displacement and Normal-Normal Continuous Stress Mixed Finite Elements for Linear Elasticity**



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FWF Start Project Y-192  
"3D hp-Finite Elements: Fast Solvers and Adaptivity"  
RICAM, Linz

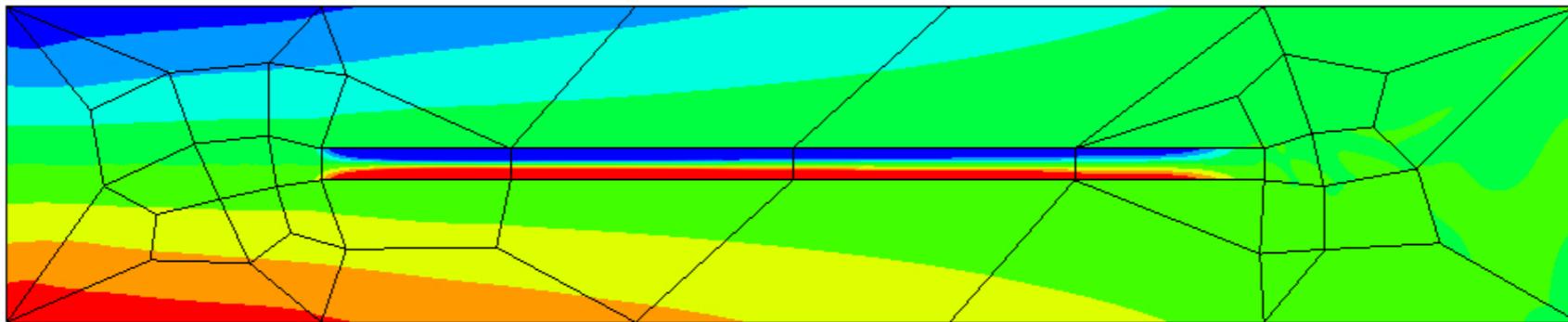


*HO-FEM Workshop, Ammersee, May 17-19, 2007*

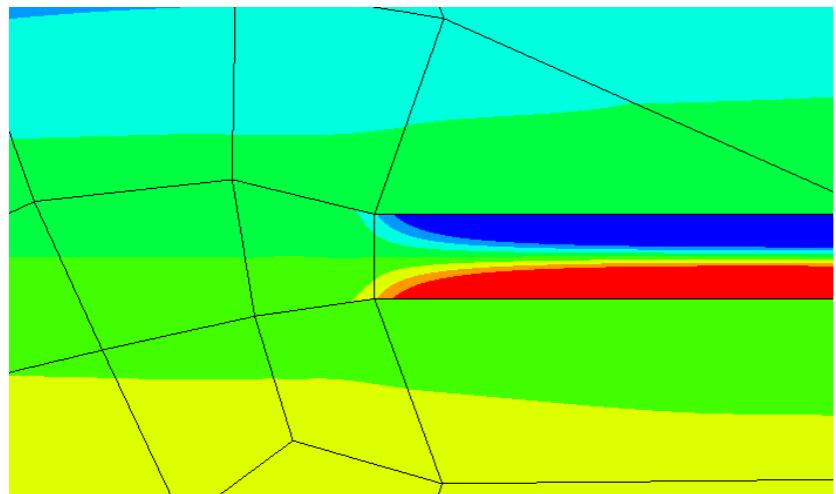
## Contents

- Motivation for another mixed formulation
- The Hellinger Reissner Principle
- Function Spaces and Finite Elements
- The 3-Step Exact Sequence Property
- Anisotropic Estimates for Thin Structures

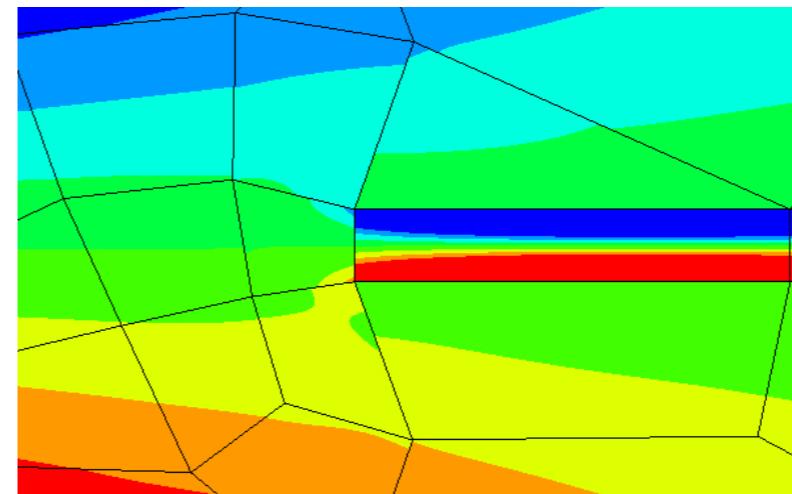
## A beam in a beam



Reenforcement with  $E = 50$  in medium with  $E = 1$ .



New mixed FEM,  $p = 2$



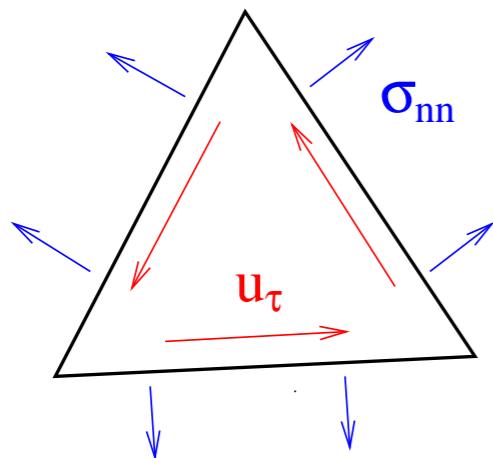
Primal FEM,  $p = 3$

## Degrees of freedom for TD-NNS elements

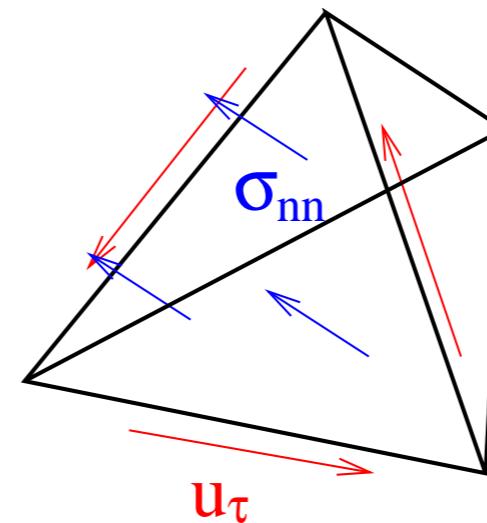
Mixed elements for approximating displacements and stresses.

- tangential components of displacement vector
- normal-normal component of stress tensor

Triangular Finite Element:

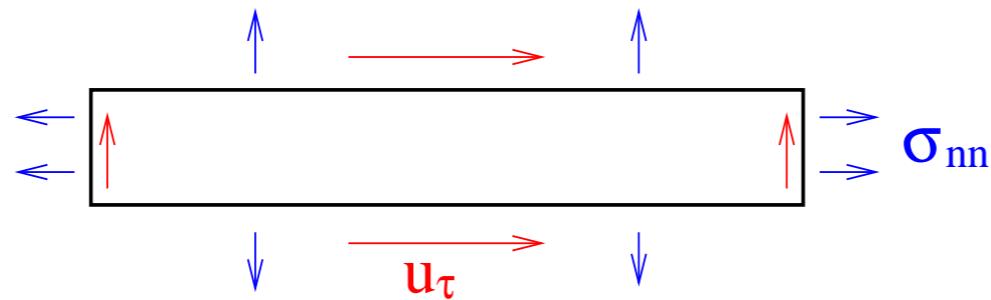


Tetrahedral Finite Element:



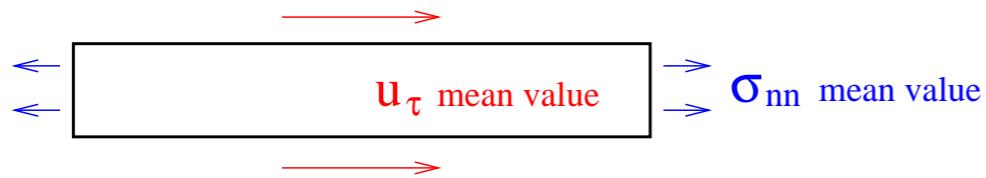
# The quadrilateral element

Dofs for general quadrilateral element:

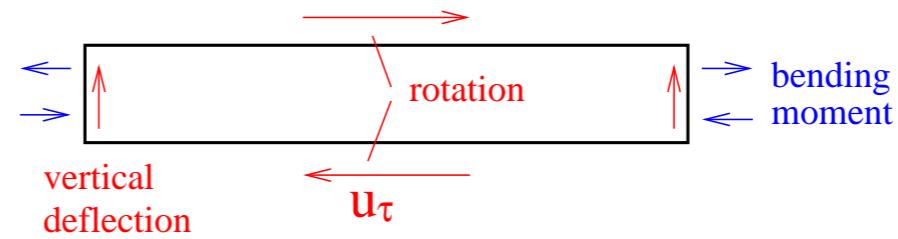


Thin beam dofs ( $\sigma_{nn} = 0$  on bottom and top):

Beam stretching components:



Beam bending components:



## Hellinger Reissner mixed methods for elasticity

Primal mixed method:

Find  $\sigma \in L_2^{sym}$  and  $u \in [H^1]^2$  such that

$$\begin{aligned} \int A\sigma : \tau - \int \tau : \varepsilon(u) &= 0 & \forall \tau \\ - \int \sigma : \varepsilon(v) &= - \int f \cdot v & \forall v \end{aligned}$$

Dual mixed method:

Find  $\sigma \in H(\text{div})^{sym}$  and  $u \in [L_2]^2$  such that

$$\begin{aligned} \int A\sigma : \tau + \int \text{div } \tau \cdot u &= 0 & \forall \tau \\ \int \text{div } \sigma \cdot v &= - \int f \cdot v & \forall v \end{aligned}$$

[Arnold+Falk+Winther]

## Reduced Symmetry mixed methods

Decompose

$$\varepsilon(u) = \nabla u + \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \operatorname{curl} u = \nabla u + \omega$$

Impose symmetry of the strain tensor by an additional Lagrange parameter:

Find  $\sigma \in [H(\operatorname{div})]^2$ ,  $u \in [L_2]^2$ , and  $\omega \in L_2^{skew}$  such that

$$\begin{aligned} \int A\sigma : \tau &+ \int u \operatorname{div} \tau + \int \tau : \omega &= 0 & \forall \tau \\ \int v \operatorname{div} \sigma & &= - \int fv & \forall v \\ \int \sigma : \gamma & &= 0 & \forall \gamma \end{aligned}$$

The solution satisfies  $u \in L_2$  and  $\omega = \operatorname{curl} u \in L_2$ , i.e.,

$$u \in H(\operatorname{curl})$$

Arnold+Brezzi, Stenberg,... 80s

## Choices of spaces

$\int \operatorname{div} \sigma \cdot u$  understood as

$$\langle \operatorname{div} \sigma, u \rangle_{H^{-1} \times H^1} = -(\varepsilon(u), \sigma)_{L_2} \quad (\operatorname{div} \sigma, u)_{L_2}$$

### Displacement

$$u \in [H^1]^2$$

continuous f.e.

$$u \in [L_2]^2$$

non-continuous f.e.

### Stress

$$\sigma \in L_2^{sym}$$

non-continuous f.e.

$$\sigma \in H(\operatorname{div})^{sym}$$

normal continuous ( $\sigma_n$ ) f.e.

The mixed system is well posed for all of these pairs.

## Choices of spaces

$\int \operatorname{div} \sigma \cdot u$  understood as

$$\langle \operatorname{div} \sigma, u \rangle_{H^{-1} \times H^1} = -(\varepsilon(u), \sigma)_{L_2} \quad \langle \operatorname{div} \sigma, u \rangle_{H(\operatorname{curl})^* \times H(\operatorname{curl})} \quad (\operatorname{div} \sigma, u)_{L_2}$$

### Displacement

$u \in [H^1]^2$   
continuous f.e.

$u \in H(\operatorname{curl})$   
tangential-continuous f.e.

$u \in [L_2]^2$   
non-continuous f.e.

### Stress

$\sigma \in L_2^{sym}$   
non-continuous f.e.

$\sigma \in L_2^{sym}, \operatorname{div} \operatorname{div} \sigma \in H^{-1}$   
normal-normal continuous ( $\sigma_{nn}$ ) f.e.

$\sigma \in H(\operatorname{div})^{sym}$   
normal continuous ( $\sigma_n$ ) f.e.

The mixed system is well posed for all of these pairs.

## The space $H(\text{div div})$

The dual space of  $H(\text{curl})$  is  $H^{-1}(\text{div})$ :

$$\begin{aligned}\|f\|_{H(\text{curl})^*} &= \sup_{v \in H(\text{curl})} \frac{\langle f, v \rangle}{\|v\|_{H(\text{curl})}} \simeq \sup_{\varphi \in H^1, z \in [H^1]^2} \frac{\langle f, \nabla \varphi + z \rangle}{\|\varphi\|_{H^1} + \|z\|_{H^1}} \simeq \|\text{div } f\|_{H^{-1}} + \|f\|_{H^{-1}} \\ &\simeq H^{-1}(\text{div})\end{aligned}$$

We search for  $\sigma \in L_2^{sym}$  and  $\text{div } \sigma \in H^{-1}(\text{div})$ . This is equivalent to

$$\sigma \in H(\text{div div}) := \{\sigma \in L_2^{sym} : \text{div div } \sigma \in H^{-1}\},$$

where

$$\text{div div} \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \text{div} \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \end{pmatrix} = \sum_{ij} \frac{\partial^2 \sigma_{ij}}{\partial x_i \partial x_j} \in H^{-1}$$

This implies continuity and LBB for  $\langle \text{div } \sigma, u \rangle$ .

## Continuity properties of the space $H(\operatorname{div} \operatorname{div})$

*Lemma:* Let  $\sigma$  be a piece-wise smooth tensor field on the mesh  $\mathcal{T} = \{T\}$  such that  $\sigma_{nt} \in H^{1/2}(\partial T)$ . Assume that  $\sigma_{nn} = n^T \sigma n$  is continuous across element interfaces. Then there holds  $\operatorname{div} \sigma \in H(\operatorname{curl})^*$ .

*Proof:* Let  $v$  be a smooth test function.

$$\begin{aligned}
 \langle \operatorname{div} \sigma, v \rangle &:= - \int \sigma : \nabla v = \sum_T \left\{ \int_T \operatorname{div} \sigma \cdot v - \int_{\partial T} \sigma_n \cdot v \right\} \\
 &= \sum_T \left\{ \int_T \operatorname{div} \sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \right\} + \sum_E \int_E \underbrace{[\sigma_{nn}]}_{=0} v_n \\
 &\leq \sum_T \| \operatorname{div} \sigma \|_{L_2(T)} \| v \|_{L_2(T)} + \| \sigma_{n\tau} \|_{H^{1/2}(\partial T)} \| v_\tau \|_{H^{-1/2}(\partial T)} \\
 &\preceq C(\sigma) \| v \|_{H(\operatorname{curl})}
 \end{aligned}$$

By density, the continuous functional can be extended to the whole  $H(\operatorname{curl})$ :

$$\langle \operatorname{div} \sigma, v \rangle = \sum_T \left\{ \int_T \operatorname{div} \sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \right\}$$

## The TD-NNS-continuous mixed method

Assuming piece-wise smooth solutions, the elasticity problem is equivalent to the following mixed problem:  
 Find  $\sigma \in H(\text{div div})$  and  $u \in H(\text{curl})$  such that

$$\begin{aligned} \int A\sigma : \tau &+ \sum_T \left\{ \int_T \text{div } \tau \cdot u - \int_{\partial T} \tau_{n\tau} u_\tau \right\} = 0 & \forall \tau \\ \sum_T \left\{ \int_T \text{div } \sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \right\} &= - \int f \cdot v & \forall v \end{aligned}$$

*Proof:* The second line is equilibrium, plus tangential continuity of the normal stress vector:

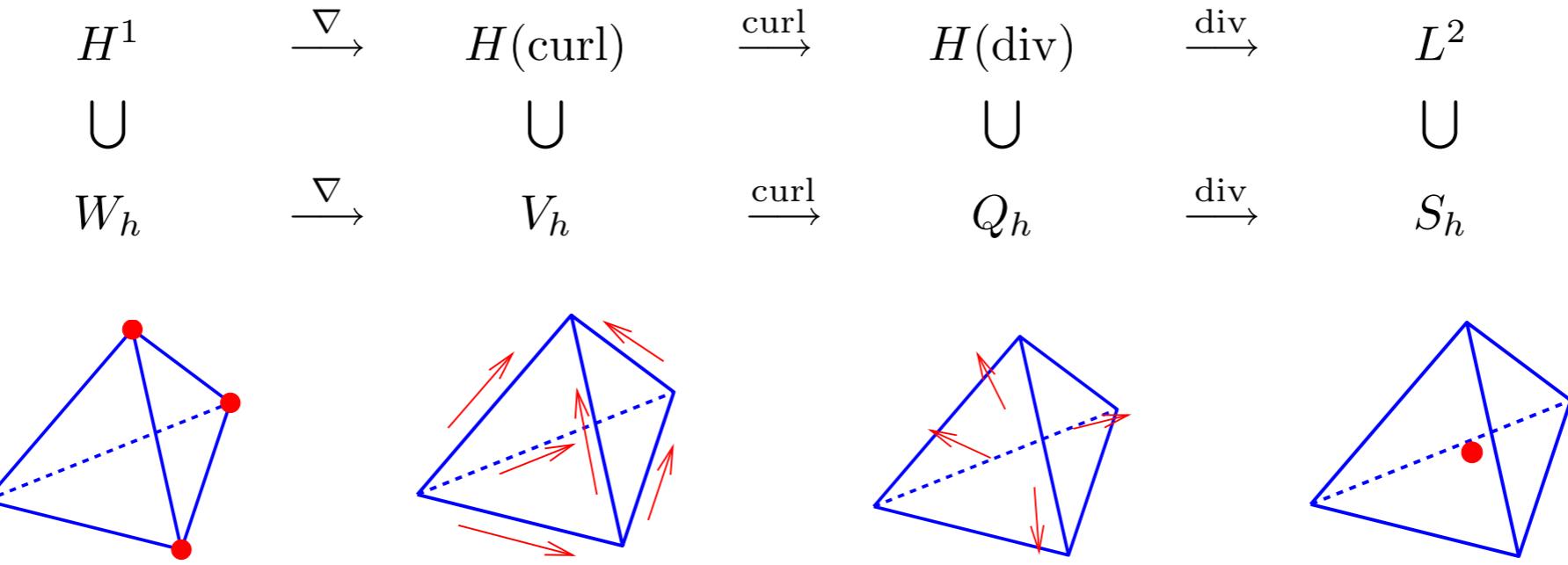
$$\sum_T \int_T (\text{div } \sigma + f)v + \sum_E \int_E [\sigma_{n\tau}]v_\tau = 0 \quad \forall v$$

Since the space requires continuity of  $\sigma_{nn}$ , the normal stress vector is continuous.  
 Element-wise integration by parts in the first line gives

$$\sum_T \int_T (A\sigma - \varepsilon(u)) : \tau + \sum_E \int_E \tau_{nn}[u_n] = 0 \quad \forall \tau$$

This is the constitutive relation, plus normal-continuity of the displacement. Tangential continuity of the displacement is implied by the space  $H(\text{curl})$ .

## The de Rham complex



satisfies the **complete sequence property**

$$\begin{aligned}
 \text{range}(\nabla) &= \ker(\text{curl}) \\
 \text{range}(\text{curl}) &= \ker(\text{div})
 \end{aligned}$$

on the continuous and the discrete level.

Important for stability, error estimates, preconditioning, ...

## Finite elements

$H^1$ -basis functions on the triangle

$$\text{Vertex basis functions} \quad \varphi^V = \lambda_V$$

$$\text{Edge basis functions} \quad \varphi_i^E$$

$$\text{Element basis functions} \quad \varphi_{ij}^T = u_i(x, y)v_j(y)$$

$H(\text{curl})$ -basis functions on the triangle

$$\text{Edge-element basis functions} \quad \mathcal{N}_0^{\alpha\beta} = \lambda_\alpha \nabla \lambda_\beta - \lambda_\beta \nabla \lambda_\alpha$$

$$\text{High-order edge basis functions} \quad \psi_i^E = \nabla \varphi_{i+1}^E$$

$$\text{Element basis functions} \quad \nabla(u_i v_j), u_i \nabla v_j - v_j \nabla u_i, \mathcal{N}_0^{\alpha\beta} v_j$$

Finite elements with local complete sequence property: [JS+S.Zaglmayr, 05]

## The 3-step 'exact sequence'

$$H^1 \cap H^2(\mathcal{T}) \xrightarrow{\nabla} H(\text{curl}) \cap [H^1(\mathcal{T})]^2 \xrightarrow{\sigma_{\mathcal{T}}(\cdot)} H(\text{div div}) \xrightarrow{\text{div}} H^{-1}(\text{div}) \xrightarrow{\text{div}} H^{-1}$$

with the stress operator

$$\sigma(v) = \begin{pmatrix} \frac{\partial v_y}{\partial y} & -\frac{1}{2} \left\{ \frac{\partial v_y}{\partial x} + \frac{\partial v_y}{\partial x} \right\} \\ \text{sym} & \frac{\partial v_x}{\partial x} \end{pmatrix}.$$

The composite operators are

$$\begin{aligned} \text{airy}(w) = \sigma(\nabla w) &= \begin{pmatrix} \frac{\partial^2 w}{\partial y^2} & -\frac{\partial w}{\partial x \partial y} \\ \text{sym} & \frac{\partial w}{\partial x^2} \end{pmatrix} \\ \text{div } \sigma(v) &= \frac{1}{2} \text{Curl curl } v \end{aligned}$$

There holds

$$\begin{aligned} \text{range}(\sigma(\nabla \cdot)) &= \ker(\text{div}) \\ \text{range}(\text{div } \sigma(\cdot)) &= \ker(\text{div}) \end{aligned}$$

## Finite elements for $H(\text{div div})$

Start with  $C^0$ -continuous finite elements for  $H^1 \cap H^2(\mathcal{T})$

Finite elements for  $H(\text{div div})$  can be built with

edge basis functions:  $\sigma(\nabla\varphi^E)$

ad hoc internal basis functions:  $\text{Sym}[\nabla\lambda_\alpha^\perp \otimes \nabla\lambda_\beta^\perp] \lambda_\gamma P^{k-1}$

Alternative: Take airy functions of internal  $C^0$ -continuous f.e., plus some more.

Potential to save dofs for subdomains with  $\text{div } \sigma = 0$ .

## Finite Element Analysis

Analysis in discrete norms:

$$\begin{aligned}\|v\|_{V_h}^2 &= \sum_T \|\varepsilon(v)\|_T^2 + \sum_E h^{-1} \| [v_n] \|_{L_2(E)}^2 \\ \|\tau\|_{\Sigma_h}^2 &= \|\tau\|_{L_2}^2 + \sum_E h \|\tau_{nn}\|_{L_2(E)}^2.\end{aligned}$$

Continuous and inf-sup stable. By saddle-point theory:

$$\|u - u_h\|_{V_h} + \|\sigma - \sigma_h\|_{\Sigma_h} \leq ch^m \|\varepsilon(u)\|_{H^m}$$

for  $m \leq k$ .

By adding the stabilization term

$$\sum_T h_T^2 (\operatorname{div} \sigma, \operatorname{div} \tau)_T,$$

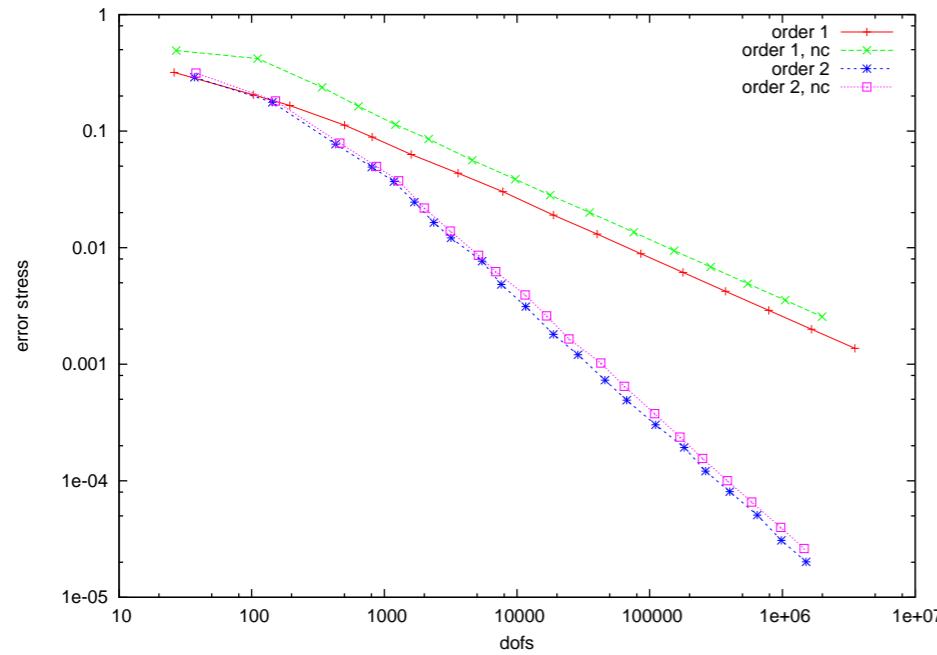
to the bilinear-form (and consistent rhs) the method is robust as  $\nu \rightarrow 1/2$ .

## Unit square, left side fixed, vertical load, adaptive refinement

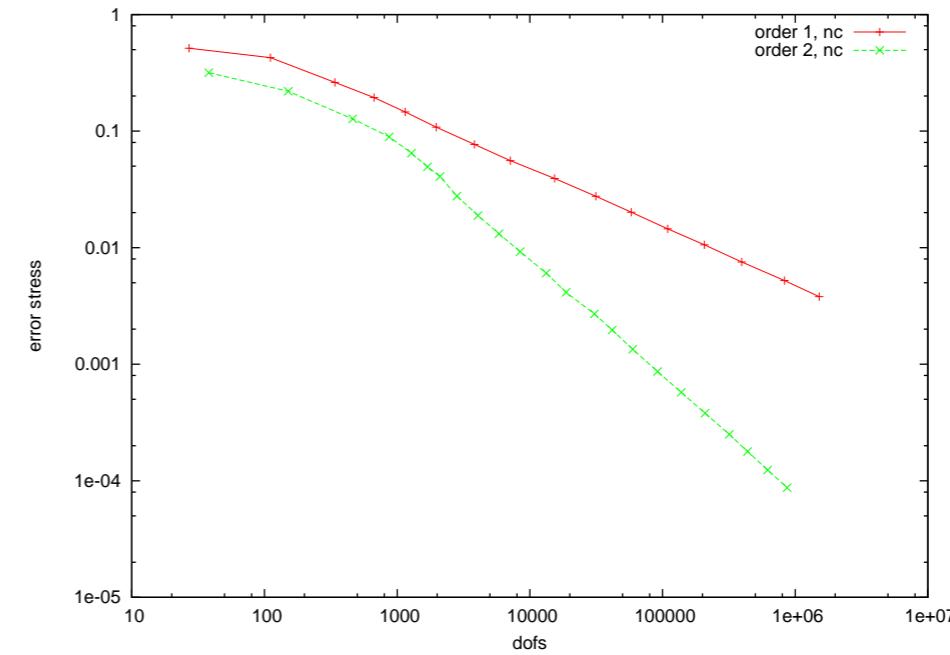
$\sigma \in P^1$   
 2 dof  $\sigma_{nn}$  per edge

	conforming	non-conforming
$u \in P^1$	2 dof $u_\tau$	1 dof $u_\tau$
$u \in P^2$	3 dof $u_\tau$	2 dof $u_\tau$

$\nu = 0.3:$

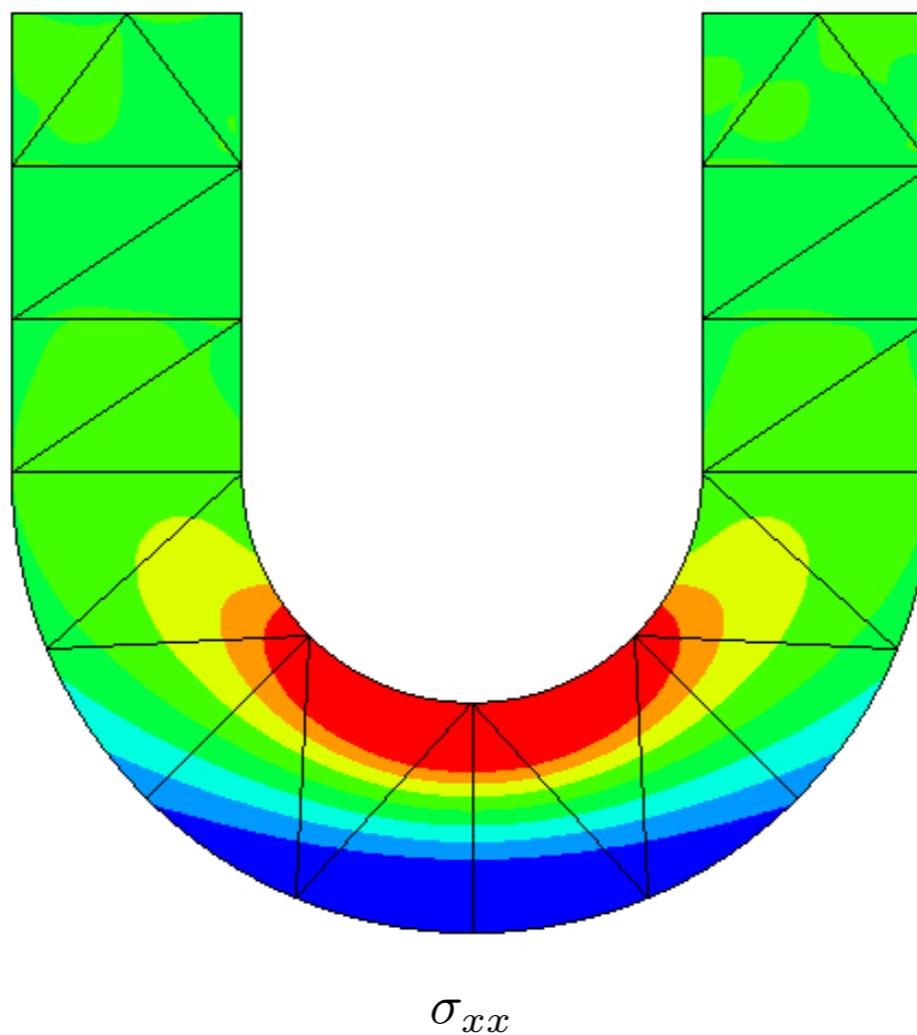


$\nu = 0.4999:$



## Curved elements

fixed left top, pull right top  
Elements of order 5



## Reissner Mindlin Plates

Energy functional for vertical displacement  $w$  and rotations  $\beta$ :

$$\|\varepsilon(\beta)\|_{A^{-1}}^2 + t^{-2}\|\nabla w - \beta\|^2$$

MITC elements with Nédélec reduction operator:

$$\|\varepsilon(\beta)\|_{A^{-1}}^2 + t^{-2}\|\nabla w - R_h\beta\|^2$$

Mixed method with  $\sigma = A^{-1}\varepsilon(\beta) \in H(\text{div div})$ ,  $\beta \in H(\text{curl})$ , and  $w \in H^1$ :

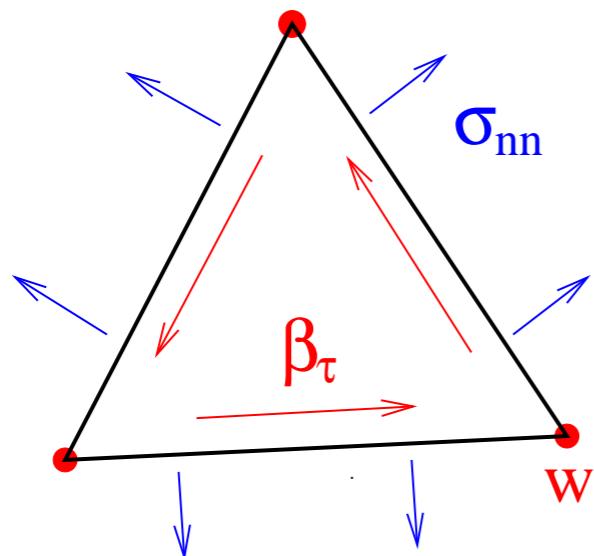
$$L(\sigma; \beta, w) = \|\sigma\|_A^2 + \langle \text{div } \sigma, \beta \rangle - t^{-2}\|\nabla w - \beta\|^2$$

## Reissner Mindlin Plates and Thin 3D Elements

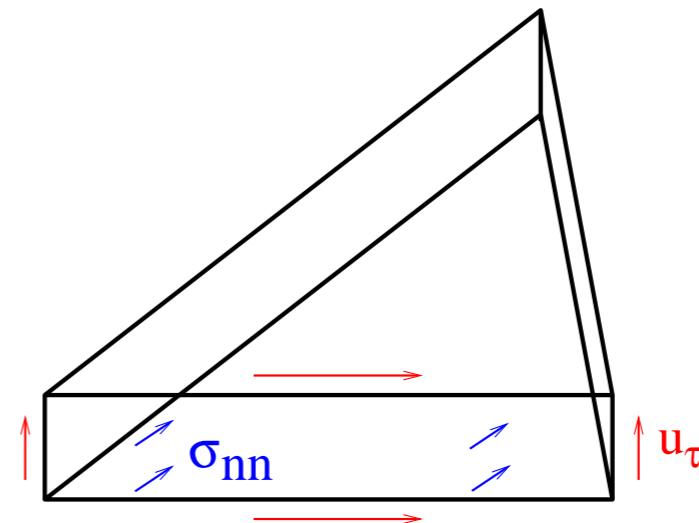
Mixed method with  $\sigma = A^{-1}\varepsilon(\beta) \in H(\text{div div})$ ,  $\beta \in H(\text{curl})$ , and  $w \in H^1$ :

$$L(\sigma; \beta, w) = \|\sigma\|_A^2 + \langle \text{div } \sigma, \beta \rangle - t^{-2} \|\nabla w - \beta\|^2$$

Reissner Mindlin element:



3D prism element:



## Tensor-product Finite Elements

Thin domain:  $\omega \subset \mathbb{R}^2$ ,  $I = (-t/2, t/2)$ ,  $\Omega = \omega \times I$ . FE-space for displacement:

$$\begin{aligned}\mathcal{L}_{k+1}^{xy} &= \{v \in H^1(\omega) : v|_T \in P^{k+1}\} & \mathcal{L}_{k+1}^z &= \{v \in H^1(I) : v|_T \in P^{k+1}\} \\ \mathcal{N}_k^{xy} &= \{v \in H(\text{curl}, \omega) : v|_T \in P^k\} & \mathcal{N}_k^z &= \{v \in L_2(I) : v|_T \in P^k\}\end{aligned}$$

Tensor-product Nédélec space:

$$V_k = \underbrace{\mathcal{N}_k^{xy} \otimes \mathcal{L}_{k+1}^z}_{u_{xy}} \times \underbrace{\mathcal{L}_{k+1}^{xy} \otimes \mathcal{N}_k^z}_{u_z}$$

Regularity-free quasi-interpolation operators (Clement) which commute (JS 2001):

$$I_{k+1}^{xy} : L_2(\omega) \rightarrow \mathcal{L}_{k+1}^{xy}, \quad Q_k^{xy} : L_2(\omega) \rightarrow \mathcal{N}_k^{xy} : \quad \nabla I_{k+1}^{xy} = Q_k^{xy} \nabla$$

Tensor product interpolation operator:

$$Q_k = \underbrace{Q_k^{xy} \otimes I_{k+1}^z}_{u_{xy}} \times \underbrace{I_{k+1}^{xy} \otimes Q_k^z}_{u_z}$$

## Anisotropic Estimates

**Thm:** There holds

$$\sum_T \|\varepsilon(u - u_h)\|_T^2 + \sum_F h_{op}^{-1} \| [u_n] \|_F^2 + \|\sigma - \sigma_h\|^2 \leq c \{ h_{xy}^m \|\nabla_{xy}^m \varepsilon(u)\| + h_z^m \|\nabla_z^m \varepsilon(u)\| \}^2$$

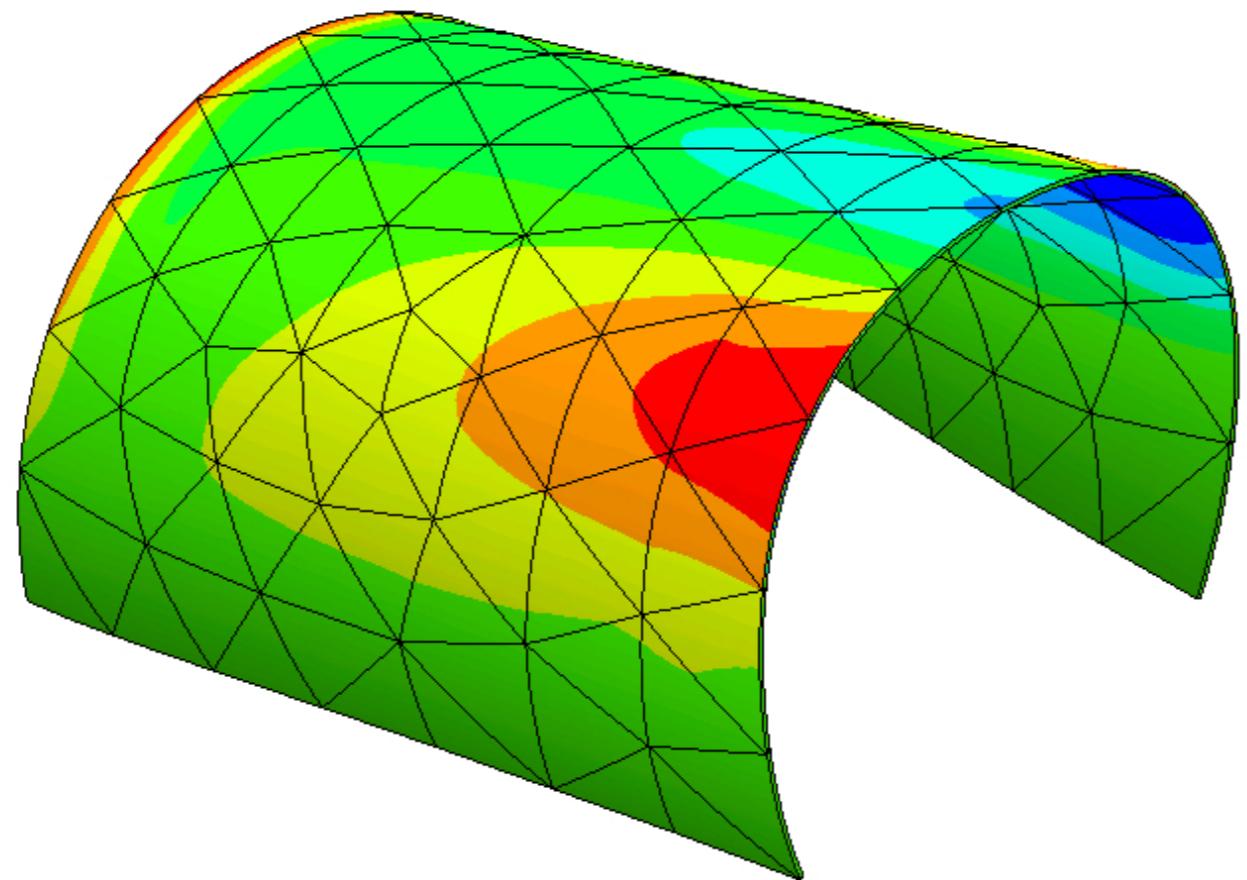
Proof: Stability constants are robust in aspect ratio (for tensor product elements)

Anisotropic interpolation estimates ( $H^1$ : Apel). E.g., the shear strain components

$$\begin{aligned} 2\|\varepsilon_{xy,z}(u - Q_k u)\|_{L_2} &= \|\nabla_z(u_{xy} - I^z \otimes Q^{xy} u_{xy}) + \nabla_{xy}(u_z - I^{xy} \otimes Q^z u_z)\|_{L_2} \\ &= \|(I - Q^{xy} \otimes Q^z)(\nabla_z u_{xy} + \nabla_{xy} u_z)\|_{L_2} \\ &\preceq h_{xy}^m \|\nabla_x^m \varepsilon_{xy,z}(u)\|_0 + h_z^m \|\nabla_z^m \varepsilon_{xy,z}(u)\|_{L_2} \end{aligned}$$

## Shell structure

$R = 0.5, t = 0.005$   
 $\sigma \in P^2, u \in P^3$



Netgen 4.5

stress component  $\sigma_{yy}$

## Concluding Remarks

- Mixed elements with mixed continuity properties of arbitrary order
- Robust estimates for anisotropic structures/elements
- Locking free w.r.t. volume locking
- Implementation as displacement formulation by hybridization
- Wire-basket preconditioner for high order elements
- Alternative approach: Hybrid - Discontinuous Galerkin

Open-source Netgen/NgSolve software from [www.hpfem.jku.at/netgen](http://www.hpfem.jku.at/netgen)