# Multigrid Methods for Anisotropic Edge Refinement*


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**Abstract.** A finite element method with optimal convergence on non-smooth three dimensional domains requires anisotropic mesh refinement towards the edges. Multigrid methods for anisotropic tensor product meshes are available and are based either on line (or plane) smoothers or on semi-coarsening strategies. In this paper we suggest and analyze a new multigrid scheme combining semi-coarsening and line smoothers to obtain a solver of optimal algorithmic complexity for anisotropic meshes along edges.



**Key Words.** Multigrid, line Jacobi smoother, edge singularity, anisotropic mesh

**AMS(MOS) subject classification.** 65N55


## 1 Introduction

The finite element simulation of three dimensional problems described by partial differential equations is a challenging task. To keep the simulation time low at least two aspects have to be taken into account. First, the underlying triangulation has to be efficient for approximating the (unknown) solution, and, second,


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the chosen algorithm for solving the large scale system of equations should be of optimal algorithmic complexity.

For two dimensional elliptic problems optimal triangulations for low order finite elements can be achieved by isotropic mesh refinement based on *a posteriori* error indicators [Ver96]. The corresponding approach in three dimensions does in general not lead to optimal triangulations in the sense of an energy error of order $N^{-p/3}$, $p$ being the polynomial degree. Besides local refinement towards the corners optimal triangulations require *anisotropic* mesh refinement towards the edges of the geometry [Ape99, AN98, ANS00].

Multigrid methods (see [Hac85, Bra93] and many references therein) are algorithms of optimal (this means linear) complexity for the solution of the systems of linear equations obtained by the finite element method. Multigrid methods have been suggested and analyzed for anisotropic problems with tensor product structure. One approach is to take care of the strong connections by properly designed line or plane smoothers [Wit89, Hac89, Ste93, BZ00], another is to build up the hierarchy of triangulations by semi-coarsening [Zha95, GO95b, MXZ95].

Semi-coarsening and line/plane smoothing can be combined. In [BH99], for example, a certain class of singular perturbed problems is considered, and it is suggested to use semi-coarsening with respect to the “harmless” coordinate and line relaxation in the direction of the singular perturbation. In the case of edge singularities, the edge direction could be considered as the harmless direction but then we need a good plane smoother in the orthogonal direction. Since this strategy is not easy to implement for a hierarchical smoother, we propose to use a line smoother in edge direction and semi-coarsening in the orthogonal plane which turns out to be easy to implement and efficient in application. In this paper we prove robust V-cycle convergence rates of the suggested scheme. The framework is due to Braess and Hackbusch [BH83].

We note that this multigrid method is essentially a two-dimensional standard multigrid where the third dimension is treated only in the smoother. The two-dimensional method with mesh refinement towards singular corners is analyzed in [Yse86]. While in that paper regularity and interpolation results have been cited from [BKP79, Kon67] we cannot use results from literature immediately. The reason is that the two-dimensional plane with mesh refinement is only a trace of the three-dimensional domain where the problem is posed. In order to circumvent the loss of regularity due to trace theorems we introduce an intermediate semi-discrete space $\widetilde{V}$, see (15), and prove regularity of an auxiliary problem and interpolation results ourselves.

The rest of the paper is organized as follows. In Section 2 the investigated problem is formulated. Section 3 introduces the multigrid scheme. The multigrid analysis is performed in Section 4, two proofs are postponed to Sections 5 and 6. In Section 7 we give numerical results confirming our theory and show further applications of the developed multigrid scheme.

## 2 Problem Formulation and Discretization

Let $\Omega = G \times Z$ where $G \subset \mathbb{R}^2$ is a polygonal domain and $Z$ is a real interval. By the local nature of corner singularities (and then edge ones for $\Omega$), we may suppose that $G$ has possibly one corner with interior angle $\omega > \pi$ at the origin, the other interior angles being smaller than $\pi$. The corresponding edge of $\Omega$ is part of the $z$-axis and will be called the singular edge of $\Omega$. Spatial variables are written as $(x, z) = (x_1, x_2, z)$ with $x \in G$ and $z \in Z$. Accordingly, the gradient is split into partial derivatives as $\nabla = (\partial_x, \partial_z)$. Let $V := H_0^1(\Omega)$ be the usual Sobolev space. We consider the Poisson equation with Dirichlet boundary conditions whose variational form is: Find $u \in V$ such that

$$A(u, v) = f(v) \qquad \forall\, v \in V \tag{1}$$

with the symmetric, continuous and elliptic bilinear form $A(.,.)$, and the continuous linear form $f(.)$ on $V$, namely

$$A(u, v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \qquad \text{and} \qquad f(v) := \int_\Omega f v \,\mathrm{d}x.$$

The energy norm is defined as $\|u\|_A := A(u, u)^{1/2}$.

The domain $\Omega$ is covered by a tensor product triangulation $\mathcal{T} = \mathcal{T}_x \otimes \mathcal{T}_z$, where $\mathcal{T}_x$ and $\mathcal{T}_z$ are conforming triangulations of $G$ and $Z$, respectively [Cia78]. The two dimensional triangulation $\mathcal{T}_x$ is assumed to fulfill the bounded minimal angle condition. The triangulation $\mathcal{T}_z$ is arbitrary. We define the mesh size functions

$$h_{L,x} = h_{L,x}(x, z) \;=\; \operatorname{diam} T_x \qquad \text{for } x \in T_x \in \mathcal{T}_x,\ z \in Z, \tag{2}$$

$$h_{L,z} = h_{L,z}(x, z) \;=\; \operatorname{diam} T_z \qquad \text{for } x \in G,\ z \in T_z \in \mathcal{T}_z \tag{3}$$

for plane and edge directions. The positive integer $L$ denotes the final refinement level of the multigrid hierarchy defined below. We do not assume relations between $h_{L,x}$ and $h_{L,z}$ and thus anisotropic triangulations are included.

We introduce the piecewise affine finite element spaces

$$\mathcal{M}_0^1(\mathcal{T}_x) = \{u \in C^0(G) : u|_{\partial G} = 0, u|_{T_x} \in \mathcal{P}^1\ \forall\, T_x \in \mathcal{T}_x\},$$
$$\mathcal{M}_0^1(\mathcal{T}_z) = \{u \in C^0(Z) : u|_{\partial Z} = 0, u|_{T_z} \in \mathcal{P}^1\ \forall\, T_z \in \mathcal{T}_z\}$$

with the nodal bases $\{\varphi_{L,x}^i\}_{i=1}^{N_{L,x}}$ and $\{\varphi_{L,z}^i\}_{i=1}^{N_{L,z}}$ and space dimensions $N_{L,x}$ and $N_{L,z}$. Then the tensor product bilinear finite element space is defined by

$$V_L := \mathcal{M}_0^1(\mathcal{T}_x) \otimes \mathcal{M}_0^1(\mathcal{T}_z) = \left\{ u = \sum_{i,j} u_{i,j} \varphi_{L,x}^i(x) \varphi_{L,z}^j(z) \right\}.$$

The finite element approximation $u_L \in V_L$ of the variational problem (1) is defined by Galerkin projection

$$A(u_L, v_L) = f(v_L) \qquad \forall\, v_L \in V_L. \tag{4}$$

Finally, we define the distance to the singular edge of $\Omega$ (the singular point of $G$, respectively) by

$$r = r(x,z) = r(x) = |x|. \tag{5}$$

For the following *a priori* estimate we refer to [Ape99, AN98]:

**Theorem 1 (A priori estimate).** *Let $(x_T, z_T)$ denote the center of the element $T \in \mathcal{T}$. Assume that the mesh sizes fulfill*

$$\begin{array}{rcll} h_{L,x}(x,z) & \simeq & h_L\, r(x_T)^\beta & \forall\,(x,z) \in T \in \mathcal{T}, \\ h_{L,z}(x,z) & \simeq & h_L & \forall\,(x,z) \in T \in \mathcal{T} \end{array} \tag{6}$$

*with the global (positive) mesh size parameter $h_L$. The grading parameter $\beta$ is fixed and is assumed to fulfill*

$$1 - \frac{\pi}{\omega} \cdot \frac{p}{2p-2} < \beta < 1. \tag{7}$$

*Then there holds the* a priori *error estimate*

$$\|u - u_h\|_1 \preceq h_L \|f\|_{0,p} \tag{8}$$

*for $p > 2$. The number of elements is of optimal order $h_L^{-3}$.*

The condition (7) shortens for $p = 2$ to the slightly weaker assumption

$$1 - \frac{\pi}{\omega} < \beta < 1, \tag{9}$$

but the estimate (8) has been proved in this case in [Ape99] for certain mixed boundary conditions only. For the Dirichlet problem just the result as stated in the theorem has been obtained yet. But we underline that our multigrid theory is also valid under the weaker assumption (9).

# 3 Multigrid Algorithm

The multigrid algorithm requires a sequence of triangulations $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_L$. We may and will assume that the triangulations and the generated finite element spaces are nested. The proposed refinement strategy is to perform first full refinement in $z$-direction, and then generate the hierarchy of meshes by refinement in the $x$-plane. Each triangulation in the hierarchy has the tensor product structure

$$\mathcal{T}_l = \mathcal{T}_{l,x} \otimes \mathcal{T}_z \qquad 1 \le l \le L.$$

This means that there is the full refinement in $z$-direction for all levels $l$, $1 \le l \le L$. We define the mesh size functions $h_{l,x}$ and $h_{l,z}$ in analogy to (2) and (3). We assume that the triangulations $\mathcal{T}_{l,x}$ fulfill the bounded minimal angle condition. Further, the grading of the meshes fulfills

$$h_{l,x} = h_{l,x}(x,z) \simeq h_l\, r(x_T)^\beta \qquad \forall\,(x,z) \in T \in \mathcal{T}_l \tag{10}$$

with the global mesh size parameter $h_l$ of level $l$, and $\beta$ from (9). The ratio of successive parameters $h_{l-1}/h_l$ is assumed to be bounded.

We mention two methods to generate the sequence of meshes fulfilling (10). The first one is to split each triangle of $\mathcal{T}_{l-1}$ into 4 triangles, and move only nodes along edges towards the singular corner. Another possibility, the so-called *dyadic partitioning*, is to use local mesh refinement of $\mathcal{T}_x$, where the elements with $h_{l,x} r(x_T)^{-\beta} > C_0 h_l$ (with a suitably defined constant $C_0$) are marked for refinement. Both methods have advantages. The first one enables a more efficient data structure, the second one is related to *a posteriori* mesh size control.

We define the sequence of nested finite element spaces

$$V_l := \mathcal{M}_0^1(\mathcal{T}_{l,x}) \otimes \mathcal{M}_0^1(\mathcal{T}_z).$$

and the linear operator $A_l : V_l \to V_l$ by

$$(A_l u_l, v_l)_0 = A(u_l, v_l) \qquad \forall\, u_l, v_l \in V_l,$$

$l = 1, 2, \ldots, L$. Additionally, we define for $u_L \in V_L$ the $L_2$ and energy projections $Q_l : V_L \to V_l$ and $P_l : V_L \to V_l$ by

$$\begin{aligned} (Q_l u_L, v_l)_0 &= (u_L, v_l)_0 \quad \forall v_l \in V_l, \\ A(P_l u_L, v_l) &= A(u_L, v_l) \quad \forall v_l \in V_l, \end{aligned}$$

$l = 1, \ldots, L$. For $1 \le l \le k \le L$ there holds the equation

$$P_l = A_l^{-1} Q_l A_k \qquad \text{on } V_k.$$

The smoother of the considered multigrid scheme is a line Jacobi or symmetric line Gauss-Seidel iteration along mesh lines in $z$-direction. Let $\{\varphi_{l,i}^x\}_{i=1}^{N_{l,x}}$ be the nodal basis of $S_0^1(\mathcal{T}_x)$ and $N_{l,x} = \dim\, S_0^1(\mathcal{T}_x)$. We define on each level $l$ the subspaces

$$V_{l,i} := \operatorname{span}\{\varphi_{l,i}\} \otimes \mathcal{M}_0^1(\mathcal{T}_z), \quad i = 1, \ldots, N_{l,x},$$

and the corresponding energy projections $P_{l,i} : V_l \to V_{l,i}$, determined for $u_l \in V_l$ by

$$A(P_{l,i} u_l, v_{l,i}) = A(u_l, v_{l,i}) \qquad \forall v_{l,i} \in V_{l,i}.$$

Then the (damped) line Jacobi smoother

$$S_l := I - \tau \sum_{i=1}^{N_{l,x}} P_{l,i}$$

with a suitable damping parameter $\tau \simeq 1$ can be written as

$$S_l = I - \tau D_l^{-1} A_l.$$

The operator $D_l^{-1}$ is indeed the inverse of a selfadjoint and positive definite operator $D_l : V_l \to V_l$. It leads to the inner product

$$D_l(u_l, v_l) := (D_l u_l, v_l)_0 \qquad \forall\, u_l, v_l \in V_l$$

and the associated norm $\|u_l\|_{D_l} := D_l(u_l, u_l)^{1/2}$. By the technique of [BP92] the analysis of the present paper hands over to the symmetric multiplicative counterpart. The multiplicative version does not need damping at all.

Since the spaces are nested, the grid transfer operators are canonically defined by embedding. As usual, we define the V-cycle multigrid preconditioning operators $C_l^{-1} : V_l \to V_l$ by induction beginning with $C_1 = A_1$. For $l > 1$ and $f \in V_l$ we define $C_l^{-1} f = x_{2m+1}$ where $x_0 = 0$,

$$\left.\begin{array}{rcl} x_i &=& x_{i-1} + \tau D_l^{-1}(f - A_l x_{i-1}), \quad i = 1, 2, \ldots, m, \\ x_{m+1} &=& x_m + C_{l-1}^{-1} Q_{l-1}(f - A_l x_m) \\ x_i &=& x_{i-1} + \tau D_l^{-1}(f - A_l x_{i-1}), \quad i = m+2, m+3, \ldots, 2m+1. \end{array}\right\} \qquad (11)$$

First $m$ steps of pre-smoothing are performed, then the coarse grid correction takes place, finally $m$ steps of post-smoothing are applied. The selfadjoint operator $C_L^{-1}$ can be used in the multigrid iteration with iteration matrix $I - C_L^{-1} A_L$ or as preconditioner in the conjugate gradient iteration.

# 4 Multigrid Analysis

In this section we analyze the convergence of the multigrid scheme formulated above. In order to apply the multigrid framework of Braess and Hackbusch [BH83], see also Theorem 3.6 in [Bra93], we need first to verify the approximation property

$$\|u_l - P_{l-1} u_l\|_{D_l}^2 \le C\, \|u_l\|_A^2 \qquad \forall\, u_l \in V_l,\ l = 2, 3, \ldots L, \qquad (12)$$

see Theorem 5. The V-cycle convergence rate estimate is then a corollary. We start with three lemmata.

**Lemma 2 (Representation of $D_l$-norm).** *For the norm induced by the line Jacobi preconditioner $D_l$ there holds the following equivalence:*

$$\|u_l\|_{D_l}^2 \simeq \|h_{l,x}^{-1} u_l\|_0^2 + \|\partial_z u_l\|_0^2 \qquad \forall\, u_l \in V_l. \qquad (13)$$

*Proof.* Let $u_l \in V_l$. The decomposition $u_l = \sum_{i=1}^{N_{l,x}} u_{l,i}$ with $u_{l,i} \in V_{l,i}$ is unique. By the additive Schwarz method ([GO95a] use the most similar notation) we obtain

$$\|u_l\|_{D_l}^2 = \sum_{i=1}^{N_{l,x}} \|u_{l,i}\|_A^2.$$

Inverse inequalities applied to the basis functions $\varphi_{l,x}^i$ give

$$\begin{aligned} \|u_l\|_{D_l}^2 &= \sum_{i=1}^{N_{l,x}} \left( \|\partial_x u_{l,i}\|_0^2 + \|\partial_z u_{l,i}\|_0^2 \right) \\ &\simeq \sum_{i=1}^{N_{l,x}} \left( \|h_{l,x}^{-1} u_{l,i}\|_0^2 + \|\partial_z u_{l,i}\|_0^2 \right). \end{aligned}$$

By mapping techniques one verifies the $L_2$ stability of the splitting (see [Bra93], Chapter 5)

$$\sum_{i=1}^{N_{l,x}} \|c_i \varphi_{l,x}^i\|_{0,T_x}^2 \simeq \left\| \sum_{i=1}^{N_{l,x}} c_i \varphi_{l,x}^i \right\|_{0,T_x}^2 \qquad \forall\, c \in \mathbb{R}^{N_{l,x}},\ \forall\, T_x \in \mathcal{T}_{l,x}. \tag{14}$$

Since the equivalence is local, we may insert the element-wise constant weight $h_{l,x}$. Summing over the elements $T_x \in \mathcal{T}_{l,x}$ gives

$$\sum_{i=1}^{N_{l,x}} \|h_{l,x}^{-1} c_i \varphi_{l,x}^i\|_{0,G}^2 \simeq \left\| \sum_{i=1}^{N_{l,x}} h_{l,x}^{-1} c_i \varphi_{l,x}^i \right\|_{0,G}^2.$$

Set now $u_{l,i} =: c_i(z)\varphi_{l,x}^i(x)$. Integration over $z \in Z$ leads to

$$\sum_{i=1}^{N_{l,x}} \|h_{l,x}^{-1} u_{l,i}\|_0^2 \simeq \|h_{l,x}^{-1} u_l\|_0^2.$$

By inserting $\partial_z u_{l,i} =: c_i(z)\varphi_{l,x}^i$ into (14), summing over the elements $T_x \in \mathcal{T}_{l,x}$, and integrating over $z \in Z$ we obtain

$$\sum_{i=1}^{N_{l,x}} \|\partial_z u_{l,i}\|_0^2 \simeq \|\partial_z u_l\|_0^2,$$

and the proof is complete. □

The sequence of nested spaces $V_l$ is contained in the *semi-discrete* space

$$\widetilde{V} := H_0^1(G) \otimes \mathcal{M}_0^1(\mathcal{T}_z). \tag{15}$$

For our analysis we consider a subspace of $\widetilde{V}$,

$$\begin{aligned} V^+ &:= \{u \in \widetilde{V} : \|u\|_{V^+} < \infty\} \\ \|u\|_{V^+}^2 &:= \|r^\beta \partial_x \nabla u\|_0^2 + \|r^{\beta-1}\partial_x u\|_0^2 \end{aligned}$$

with $\beta \in \mathbb{R}$ and $r$ defined in (5).

We remark that $\partial_{zz}u$ does not appear in $\| \, . \, \|_{V^+}$ since this derivative is not contained in $L^2(\Omega)$. Moreover, the first order term is stronger than $\|\partial_x \nabla u\|_0$ since we are interested in the case $\beta < 1$. The following two lemmata could also be proved without this term. The regularity result was then slightly shorter to prove but the prize consisted in more effort for proving the interpolation result. We decided to use the norm as defined above because it is the simpler of the two versions.

**Lemma 3 (Regularity).** *Let $u \in \widetilde{V}$ be the solution of the variational problem*

$$A(u,v) = (f,v)_0 \qquad \forall\, v \in \widetilde{V} \tag{16}$$

*with $f$ such that $r^\beta f \in L_2$, $1 - \pi/\omega < \beta < 1$. Then there holds the regularity estimate*

$$\|u\|_{V^+} \preceq \|r^\beta f\|_0. \tag{17}$$

Note that the restriction $\beta < 1$ ensures that $f \in H^{-1}(\Omega)$ and therefore the right hand side of (16) makes sense.

**Lemma 4 (Interpolation error estimate).** *There exists an interpolation operator $I_l : V^+ \to V_l$ such that the interpolation error satisfies*

$$\|u - I_l u\|_A \preceq h_l \, \|u\|_{V^+} \qquad \forall\, u \in V^+.$$

The proofs of Lemmata 3 and 4 are postponed to Sections 5 and 6.

**Theorem 5 (Approximation Property).** *The approximation property (12) is fulfilled for the considered multigrid method (11).*

*Proof.* We use the equivalence (13) and obtain

$$\|u_l - P_{l-1}u_l\|_{D_l}^2 \simeq \|h_{l,x}^{-1}(u_l - P_{l-1}u_l)\|_0^2 + \|\partial_z(u_l - P_{l-1}u_l)\|_0^2.$$

The second term of the right hand side is simply estimated by

$$\|\partial_z(u_l - P_{l-1}u_l)\|_0 \le \|u_l - P_{l-1}u_l\|_A \le \|u_l\|_A.$$

It remains to show

$$\|h_{l,x}^{-1}(u_l - P_{l-1}u_l)\|_0 \preceq \|u_l\|_A. \tag{18}$$

As usual we formulate a dual problem. Since $V_l \subset \widetilde{V}$ for all $l$, we can define $w \in \widetilde{V}$ by

$$A(w, v) = (h_{l,x}^{-2}(u_l - P_{l-1}u_l), v)_0 \qquad \forall\, v \in \widetilde{V} \hookrightarrow V.$$

Since by definition (10) of $h_{l,x}$

$$\|r^\beta h_{l,x}^{-2}(u_l - P_{l-1}u_l)\|_0 \preceq h_l^{-1}\|h_{l,x}^{-1}(u_l - P_{l-1}u_l)\|_0 < \infty \tag{19}$$

we can apply Lemma 3 and conclude, by using again (19),

$$\|w\|_{V^+} \preceq h_l^{-1}\|h_{l,x}^{-1}(u_l - P_{l-1}u_l)\|_0.$$

Here, no special consideration of the origin is necessary. We continue with Galerkin orthogonality, approximation, and regularity:

$$\begin{aligned}
\|h_{l,x}^{-1}(u_l - P_{l-1}u_l)\|_0^2 &= A(w, u_l - P_{l-1}u_l) \\
&= A(w - I_{l-1}w, u_l - P_{l-1}u_l) \\
&\le \|w - I_{l-1}w\|_A \,\|u_l - P_{l-1}u_l\|_A \\
&\preceq h_l\|w\|_{V^+}\,\|u_l\|_A \\
&\preceq \|h_{l,x}^{-1}(u_l - P_{l-1}u_l)\|_0\,\|u_l\|_A
\end{aligned}$$

Dividing by one factor gives (18) and thus the desired approximation property. □

**Theorem 6 (Convergence rate estimate).** *For the V-cycle multigrid algorithm (11) with $m$ pre- and $m$ post-smoothing steps there holds the convergence rate estimate*

$$\|I - C_L^{-1}A_L\|_A \le \frac{C}{C + 2m}. \tag{20}$$

*Proof.* The result follows from the general multigrid theory of Braess and Hackbusch [BH83], see also Theorem 3.6 in [Bra93], by using the approximation property (12) which is proved in Theorem 5. □

# 5 Regularity

*Proof of Lemma 3.* The lemma will be proved in three steps: partial Fourier decomposition, regularity of the Fourier coefficients, and Fourier composition. Let $\{e_i\}_{i=1}^{N_z}$ be the Fourier basis in $\mathcal{M}_0^1(\mathcal{T}_z)$, that means, $e_i = e_i(z)$ are the eigensolutions of

$$(e_i', v')_{0,Z} = \lambda_i^2(e_i, v)_{0,Z} \qquad \forall\, v \in \mathcal{M}_0^1(\mathcal{T}_z)$$

with $(e_i, e_j)_{0,Z} = \delta_{ij}$ and $(e_i', e_j')_{0,Z} = \lambda_i^2\delta_{ij}$. Inserting $u = \sum_{i=1}^{N_z} u_i(x)e_i(z)$ into (16) yields that $u_i(x)$ is solution of

$$(\partial_x u_i, \partial_x v)_{0,G} + \lambda_i^2(u_i, v)_{0,G} = (f_i, v)_{0,G} \qquad \forall\, v \in H_0^1(G),$$

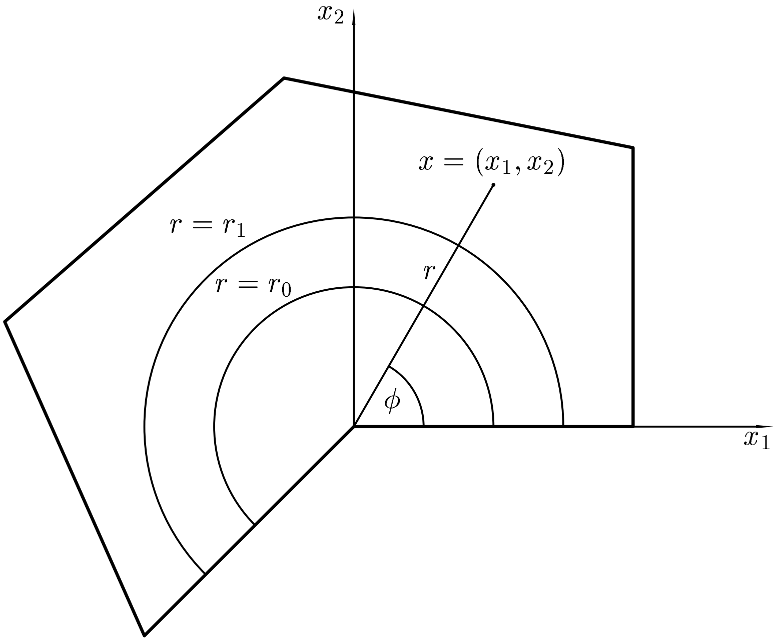


Figure 1: Illustration of the notation.

with $f_i = (f, e_i)_{0,Z}$ and $r^\beta f_i \in L_2(G)$.

Introduce now a cut-off function $\xi \in C^\infty(R_+)$, $\xi(r) \in [0,1]$, $\xi(r) = 1$ for $r \leq r_0$, $\xi(r) = 0$ for $r \geq r_1 > r_0$, see Figure 1 for an illustration. Then we consider the function $\tilde{u}_i = \xi(r) u_i$ which satisfies

$$(\partial_x \tilde{u}_i, \partial_x v)_{0,G} + \lambda_i^2 (\tilde{u}_i, v)_{0,G} = (\tilde{f}_i, v)_{0,G} \qquad \forall\, v \in H_0^1(G),$$

with $\tilde{f}_i = \xi(r) f_i - 2\partial_x u_i \partial_x \xi(r) - u_i \partial_x^2 \xi(r)$ and $r^\beta \tilde{f}_i \in L_2(G)$. Observe that $\tilde{u}_i = \tilde{f}_i = 0$ for $r \geq r_1$. Thus we can extend $\tilde{u}_i$ and $\tilde{f}_i$ by 0 and consider instead of $G$ the infinite cone $K := \{(r\cos\phi, r\sin\phi) \in \mathbb{R} : 0 < r < \infty, 0 < \phi < \omega\}$. After the change of variables $\eta = \lambda_i x$ we obtain that $U_i(\eta) = \tilde{u}_i(x)$ is solution of the following problem:

$$(\partial_\eta U_i, \partial_\eta V)_{0,K} + (U_i, V)_{0,K} = (\lambda_i^{-2} F_i, V)_{0,K} \qquad \forall\, V \in H_0^1(K).$$

We can now use the regularity result in Proposition 1.1 of [NP94, page 385],

$$\|U_i\|_{E^2_\beta(K)} \preceq \|\lambda_i^{-2} F_i\|_{E^0_\beta(K)} \qquad \text{if } |\beta - 1| < \frac{\pi}{\omega},$$

where the space $E^\ell_\beta(K)$ is the completion of $C_0^\infty(\overline{K} \setminus 0)$ with respect to the norm

$$\|U\|^2_{E^\ell_\beta(K)} := \sum_{|\alpha| \leq \ell} \int_K \rho^{2\beta}(1 + \rho^{|\alpha| - \ell})^2 |D^\alpha U|^2 \,\mathrm{d}\eta$$

[NP94, page 300], $\rho := |\eta|$. By transforming the norms back one obtains

$$\begin{aligned}
\|\xi(r)\, u\|_{V^+}^2 &= \sum_{i=1}^{N_z} \int_G \left(r^{2\beta}|\partial_x^2 \tilde{u}_i|^2 + (r^{2\beta}\lambda_i^2 + r^{2\beta-2})|\partial_x \tilde{u}_i|^2\right)\, \mathrm{d}x \\
&= \lambda_i^{-2\beta+2} \sum_{i=1}^{N_z} \int_K \rho^{2\beta} \left(|\partial_\eta^2 U_i|^2 + (1+\rho^{-2})|\partial_\eta U_i|^2\right)\, \mathrm{d}\eta \\
&\preceq \lambda_i^{-2\beta+2} \sum_{i=1}^{N_z} \int_K \rho^{2\beta} |\lambda_i^{-2} F_i|^2\, \mathrm{d}\eta \\
&= \sum_{i=1}^{N_z} \int_G r^{2\beta} |\tilde{f}_i|^2\, \mathrm{d}x \\
&\preceq \|\xi(r)\, r^\beta f\|_0^2.
\end{aligned}$$

By using that $(1-\xi(r))u$ vanishes near the corner, and hence is regular, the desired result is proved. □

# 6 Interpolation

*Proof of Lemma 4.* Let $\mathrm{Z}_h : H^1(G) \to \mathcal{M}_0^1(\mathcal{T}_{l,x})$ be the Scott-Zhang interpolation operator [SZ90]. For an arbitrary triangle $T_x \in \mathcal{T}_{l,x}$ and for $m = 0, 1$, $\ell = 1, 2$, $p \in [1, \infty]$ the error estimate

$$|u - \mathrm{Z}_h u|_{m,T_x} \preceq |T_x|^{1/2-1/p} h_{l,x}^{\ell-m} |u|_{\ell,p,\tilde{T}_x} \tag{21}$$

is satisfied [SZ90], with $\tilde{T}_x$ being the union of $T_x$ and the triangles adjacent to $T_x$.

Denote by $\{\varphi_i\}_{i=1}^{N_z}$ the nodal basis in $\mathcal{M}_0^1(\mathcal{T}_z)$ and split $u$ with respect to this basis,

$$u = \sum_{i=1}^{N_z} u_i(x)\varphi_i(z).$$

Note that the $u_i$ are here different from them in Section 5. Then we define the interpolation operator $\mathrm{I}_l : \widetilde{V} \to V_l$ by

$$\mathrm{I}_l u = \sum_{i=1}^{N_z} (\mathrm{Z}_h u_i)(x)\varphi_i(z).$$

For an arbitrary element $T = T_x \times (z_j, z_{j+1})$, $T_x \in \mathcal{T}_{l,x}$, $(z_j, z_{j+1}) \in \mathcal{T}_z$, introduce $\tilde{T} := \tilde{T}_x \times (z_j, z_{j+1})$. Divide now the set $\mathcal{T}_l$ into two subsets, $\mathcal{T}_l = \mathcal{T}_{l,R} \cup \mathcal{T}_{l,S}$,

$$\begin{aligned}
\mathcal{T}_{l,R} &:= \{T \in \mathcal{T}_l : \inf_{(x,z)\in\tilde{T}} |x| > 0\}, \\
\mathcal{T}_{l,S} &:= \{T \in \mathcal{T}_l : \inf_{(x,z)\in\tilde{T}} |x| = 0\},
\end{aligned}$$

namely elements away from the edge and elements close to the edge.

For elements $T \in \mathcal{T}_{l,R}$ we obtain from (21) the estimates

$$\begin{aligned}
\|\partial_x(u - \mathrm{I}_l u)\|_{0,T} &\simeq \sum_{i=j}^{j+1} h_z^{1/2} \|\partial_x(u_i - \mathrm{Z}_h u_i)\|_{0,T_x} \\
&\preceq \sum_{i=j}^{j+1} h_z^{1/2} h_{l,x} \|\partial_x^2 u_i\|_{0,\tilde{T}_x} \\
&\simeq h_{l,x} \|\partial_x^2 u\|_{0,\tilde{T}}, \\
\|\partial_z(u - \mathrm{I}_l u)\|_{0,T} &= h_z^{-1/2} \|(u_{j+1} - u_j) - \mathrm{Z}_h(u_{j+1} - u_j)\|_{0,T_x} \\
&\preceq h_z^{-1/2} h_{l,x} \|\partial_x(u_{j+1} - u_j)\|_{0,\tilde{T}_x} \\
&= h_{l,x} \|\partial_x \partial_z u\|_{0,\tilde{T}}.
\end{aligned}$$

That means, by using (10) and $r(x_T) \simeq r(x)$ for $x \in \tilde{T}$,

$$\sum_{T \in \mathcal{T}_{l,R}} \|u - \mathrm{I}_l u\|_A^2 \preceq \sum_{T \mathcal{T}_{l,R}} h_{l,x}^2 \|\nabla \partial_x u\|_{0,\tilde{T}}^2 \preceq h_l^2 \|u\|_{\tilde{V}}^2, \tag{22}$$

where we have also used that only a finite number of $\tilde{T}$ overlap in any point.

For elements $T \in \mathcal{T}_{l,S}$ we derive estimates in a weighted space, namely

$$\begin{aligned}
\|\partial_x(u - \mathrm{I}_l u)\|_{0,T} &\simeq \sum_{i=j}^{j+1} h_z^{1/2} \|\partial_x(u_i - \mathrm{Z}_h u_i)\|_{0,T_x} \\
&\preceq \sum_{i=j}^{j+1} h_z^{1/2} \|\partial_x u_i\|_{0,\tilde{T}_x} \\
&\preceq \|\partial_x u\|_{0,\tilde{T}} \\
&\preceq h_{l,x}^{1-\beta} \|r^{\beta-1} \partial_x u\|_{0,\tilde{T}_x} \\
&\simeq h_l \|r^{\beta-1} \partial_x u\|_{0,\tilde{T}_x},
\end{aligned}$$

which is valid due to $r \preceq h_{l,x}$, $r(x_T) \simeq h_{l,x}$ (thus $h_{l,x}^{1-\beta} \simeq h_l$), and $\beta \leq 1$. Moreover we get

$$\begin{aligned}
\|\partial_z(u - \mathrm{I}_l u)\|_{0,T} &\simeq h_z^{-1/2} \|(u_{j+1} - u_j) - \mathrm{Z}_h(u_{j+1} - u_j)\|_{0,T_x} \\
&\preceq h_z^{-1/2} |T_x|^{-1/2} h_{l,x} \|\partial_x(u_{j+1} - u_j)\|_{0,1,\tilde{T}_x} \\
&\preceq h_z^{-1/2} |T_x|^{-1/2} h_{l,x} \|r^{-\beta}\|_{0,\tilde{T}_x} \|r^{\beta} \partial_x(u_{j+1} - u_j)\|_{0,\tilde{T}_x} \\
&\preceq h_{l,x}^{1-\beta} \|r^{\beta} \partial_x \partial_z u\|_{0,\tilde{T}} \\
&\simeq h_l \|r^{\beta} \partial_x \partial_z u\|_{0,\tilde{T}}.
\end{aligned}$$

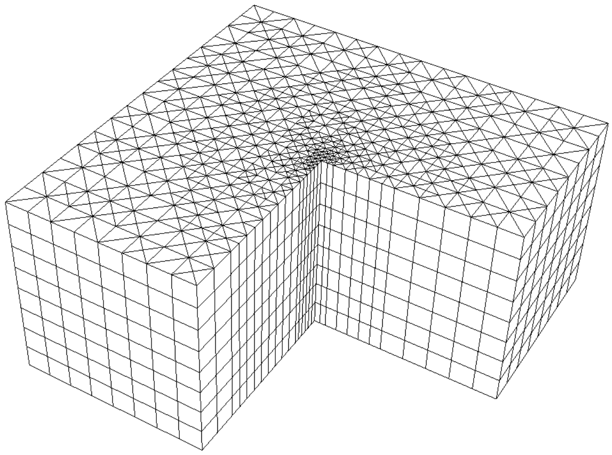

Figure 2: Mesh with 8 uniform layers and 5013 nodes

Consequently, we get

$$\sum_{T\in\mathcal{T}_{l,S}} \|u - \mathrm{I}_l u\|_A^2 \preceq h_{l,x}^{1-\beta}\|u\|_{\tilde{V}}^2. \tag{23}$$

With (22) and (23) the lemma is proved. □

# 7 Numerical Results

For verification of the analysis and to demonstrate the performance of the method, we present the following numerical results. We consider the three dimensional L-shaped domain

$$\Omega = G \times (0,1), \qquad \text{with} \qquad G = (-1,1)^2 \setminus [0,1]^2.$$

An initial triangulation was generated with 16 nodes and 6 prismatic elements. For the first tests (Tables 1 and 2), the elements were successively bisected in vertical direction until the triangulation $\mathcal{T}_1$ was obtained. For further tests (Table 3), we first split the prisms at $z = 0.05$, and proceeded with bisecting as before. The hierarchy of triangulations was obtained by bisecting the whole stack of elements based on a priori element markers. All those elements $T \in \mathcal{T}_l$ were refined for which

$$h_{T,x} r_T^{-\beta} \geq 0.3 \max_{T'\in\mathcal{T}_l}\{h_{T',x} r_{T'}^{-\beta}\}$$

holds. We chose the refinement factor $\beta = 1/2$, which fulfills the condition $\beta > 1/3$ to ensure an asymptotically optimal discretization error. Pictures of the meshes are shown in Figure 2 and Figure 3.

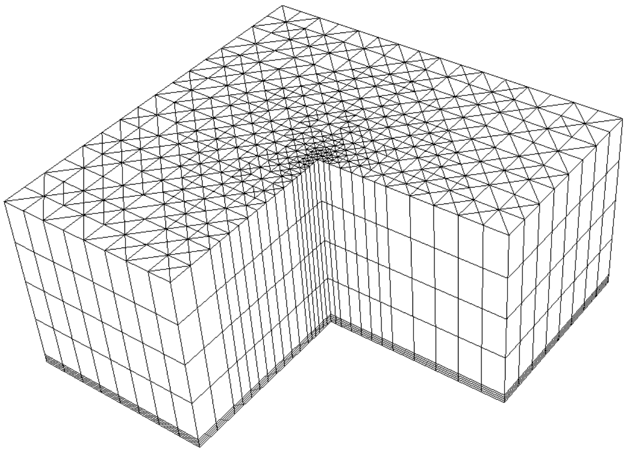

Figure 3: Mesh with boundary layer at $z = 0.05$ and 5013 nodes

For preconditioning the resulting finite element system, the multigrid scheme (11) was applied with one multiplicative pre-smoothing and one reverse-order multiplicative post-smoothing step. For comparison, we did all computations also with a multigrid method with the standard point smoother on the same hierarchy of meshes.

First we computed the condition numbers $\kappa\{C_L^{-1}A_L\}$ of the preconditioned matrix. In addition, we solved the Poisson problem

$$-\Delta u = 1 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

and used the multigrid preconditioner in the conjugate gradient method to reduce the residual error (measured by $\sqrt{r^T C_L^{-1} r}$) by a factor of $10^{-8}$. Tables 1, 2 and 3 show the results for various numbers of layers in vertical direction and numbers of nodes. Processor time refers to an SGI Octane R 10000, 250 MHz.

The tests show the excellent performance of our multigrid method. The iteration numbers are independent of the refinement depth, and also independent of the mesh in edge direction. In comparison, the point smoother has problems with strongly anisotropic meshes, expressed through a large condition number and a large number of CG iterations.

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| | Point Smoother | | | Line Smoother | | |
|---|---|---|---|---|---|---|
| Nodes | $\kappa\{C_L^{-1}A_L\}$ | CG its. | Time[sec] | $\kappa\{C_L^{-1}A_L\}$ | CG its. | Time [sec] |
| 136 | 1 | 2 | | 1 | 2 | |
| 289 | 7.0 | 18 | | 1.1 | 5 | |
| 816 | 4.8 | 17 | 0.1 | 1.3 | 7 | 0.1 |
| 1717 | 2.7 | 13 | 0.2 | 1.5 | 8 | 0.2 |
| 3536 | 2.3 | 12 | 0.4 | 1.6 | 9 | 0.6 |
| 7480 | 2.1 | 11 | 1.0 | 1.7 | 10 | 1.5 |
| 20060 | 2.1 | 12 | 3.1 | 1.9 | 11 | 4.3 |
| 40766 | 2.0 | 12 | 7.2 | 1.9 | 11 | 9.9 |
| 111027 | 2.5 | 13 | 20.2 | 1.9 | 11 | 26.1 |
| 241536 | 3.2 | 14 | 50.3 | 2.2 | 11 | 61.9 |
| 320093 | 2.8 | 13 | 78.6 | 2.0 | 11 | 104.6 |

Table 1: Results for uniform refinement in $z$-direction, 16 layers of elements.

| | Point Smoother | | | Line Smoother | | |
|---|---|---|---|---|---|---|
| Nodes | $\kappa\{C_L^{-1}A_L\}$ | CG its. | Time[sec] | $\kappa\{C_L^{-1}A_L\}$ | CG its. | Time [sec] |
| 520 | 1 | 2 | | 1 | 2 | |
| 1105 | 100.8 | 63 | 0.4 | 1.1 | 5 | 0.2 |
| 3120 | 65.6 | 63 | 1.4 | 1.3 | 8 | 0.6 |
| 6565 | 29.9 | 47 | 3.2 | 1.5 | 8 | 1.4 |
| 13520 | 19.9 | 40 | 6.7 | 1.6 | 9 | 3.4 |
| 28600 | 14.6 | 32 | 12.4 | 1.7 | 10 | 8.3 |
| 76700 | 12.2 | 29 | 29.9 | 1.9 | 11 | 22.8 |
| 155870 | 10.9 | 27 | 62.4 | 1.9 | 11 | 50.6 |
| 424515 | 10.3 | 26 | 155.7 | 1.9 | 11 | 129.1 |
| 923520 | 9.9 | 25 | 346.6 | 2.2 | 11 | 298.3 |

Table 2: Results for uniform refinement in $z$-direction, 64 layers of elements.

|  | Point Smoother |  |  | Line Smoother |  |  |
|---|---|---|---|---|---|---|
| Nodes | $\kappa\{C_L^{-1}A_L\}$ | CG its. | Time[sec] | $\kappa\{C_L^{-1}A_L\}$ | CG its. | Time [sec] |
| 520 | 1 | 2 |  | 1 | 2 |  |
| 1105 | 261.2 | 84 | 0.6 | 1.1 | 5 | 0.2 |
| 3120 | 237.7 | 139 | 3.0 | 1.3 | 8 | 0.6 |
| 6565 | 169.5 | 122 | 8.1 | 1.5 | 8 | 1.4 |
| 13520 | 149.1 | 110 | 18.3 | 1.6 | 9 | 3.4 |
| 28600 | 114.8 | 96 | 36.6 | 1.7 | 10 | 8.3 |
| 76700 | 87.9 | 81 | 82.1 | 1.9 | 11 | 22.8 |
| 155870 | 62.6 | 64 | 145.2 | 1.9 | 11 | 50.7 |
| 424515 | 36.1 | 50 | 293.4 | 1.9 | 11 | 128.6 |
| 923520 | 30.6 | 45 | 610.7 | 2.2 | 11 | 297.7 |

Table 3: Results for mesh with non-uniform refinement in $z$-direction (boundary layer at $z = 0$), 64 layers of elements.


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