High Order Nédélec Elements with local complete sequence properties

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***Abstract*:**
The goal of the presented work is the efficient computation of Maxwell boundary and eigenvalue problems using high order $H(\mathbf{curl})$ finite elements. We discuss a systematic strategy for the realization of arbitrary order hierarchic H(curl)-conforming finite elements for triangular and tetrahedral element geometries. The shape functions are classified as lowest-order Nédélec, higher-order edge-based, face-based (only in 3D) and element-based ones.

Our new shape functions provide not only the global complete sequence property, but also local complete sequence properties for each edge-, face-, element-block. This local property allows an arbitrary variable choice of the polynomial degree for each edge, face, and element. A second advantage of this construction is that simple block-diagonal preconditioning gets efficient. Our high order shape functions contain gradient shape functions explicitly. In the case of a magnetostatic boundary value problem, the gradient basis functions can be skipped, which reduces the problem size, and improves the condition number.

We successfully apply the new high order elements for a 3D magnetostatic boundary value problem, and a Maxwell eigenvalue problem showing severe edge and corner singularities.

***Keywords*: edge elements, high order finite elements, eigenvalue problems**

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I. Introduction

Electromagnetic problems are formulated in the function space

$$H(\mathrm{curl}) := \{u \in [L^2(\Omega)]^3 : \mathrm{curl}\, u \in [L^2]^3\}.$$

It naturally contains the continuity of the tangential components across sub-domain interfaces. This property has lead to the construction of finite elements with tangential continuity. The most prominent one is the edge element, which is the lowest order member of the first family of Nédélec elements [1]. A recent monograph is [2]. Up from the lowest order element all of them contain edge-based degrees of freedom. Higher order ones also contain unknowns in the element faces, and in the interior of the elements. To match the functions over the element-interfaces, the orientation of the edges and faces has to be taken into account. While on edges this is just changing the sign, the orientation of triangular faces is more involved. In [3], the problem was solved by defining rotational invariant sets of basis functions for the various orders. In [4] rotational symmetry was given up, which resulted in several reference elements.

Together with $H^1 := \{u \in L^2 : \nabla u \in [L^2]^3\}$ and $H(\mathrm{div}) := \{u \in [L^2]^3 : \mathrm{div}\; u \in L^2\}$, the function spaces form the sequence

$$H^1 \xrightarrow{\nabla} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2.$$

Moreover, this sequence is complete, i.e.

$$\begin{aligned} \mathrm{range}(\nabla) &= \ker(\mathrm{curl}), \\ \mathrm{range}(\mathrm{curl}) &= \ker(\mathrm{div}). \end{aligned}$$

This complete sequence property is inherited on the finite element level, if continuous elements $(W_{h,\,p+1})$, Nédélec elements $(V_{h,\,p})$, Raviart-Thomas elements $(Q_{h,\,p-1})$, and discontinuous elements $(S_{h,\,p-2})$ of proper polynomial order are chosen for the four spaces, respectively. This relation is visualized in the De Rham complex:

$$\begin{array}{ccccccc} H^1 & \xrightarrow{\nabla} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\ \bigcup & & \bigcup & & \bigcup & & \bigcup \\ W_{h,\,p+1} & \xrightarrow{\nabla} & V_{h,\,p} & \xrightarrow{\mathrm{curl}} & Q_{h,\,p-1} & \xrightarrow{\mathrm{div}} & S_{h,\,p-2}. \end{array}$$

The complete sequence property is essential for the convergence of the finite element approximation, in particular for eigenvalue problems [5], [6]. Also the kernel-preserving multigrid preconditioners of [7], [8] are based on the knowledge of the curl-free functions.

In this paper, we construct arbitrary order basis functions allowing individual polynomial orders for each edge, face, and element of the mesh. The resulting finite element spaces fulfill the complete sequence property. We show that for our new basis functions, block-Jacobi preconditioners are kernel-preserving and thus effective. Numerical experiments show the moderate growth of the condition number for increasing polynomial order. An extended version of the present paper is in preparation [11].

II. Hierarchical high-order shape functions

We construct hierarchical high order basis functions on triangular and tetrahedral elements. The extension to all other common element geometries is in [11]. We start with the construction for H^1 finite elements, and derive the corresponding $H(\mathrm{curl})$ elements.

Hierarchical finite elements are usually based on orthogonal polynomials. Let $(\ell)_{i=0,\dots p}$ denote the Legendre-polynomials up to order p. This is a basis for $P^p([-1,1])$. Furthermore, define the integrated Legendre-polynomials $L_{i=2,\dots,p}$ via

$$L_i(x) := \int_{-1}^{x} \ell_{l-1}(s)\, ds.$$

They form a basis for $P_0^p([-1,1]) := \{\psi \in P^p([-1,1]) : \psi(-1) = \psi(1) = 0\}$. The edges of the element are E_m, where $m = 1,2,3$ for triangles, and $m = 1,\dots,6$

Hierarchical triangular H^1-element of order p using Scaled Legendre Polynomials

Vertex-based functions

$\phi_i^V = \lambda_i \qquad$ for $i = 1, 2, 3$

Edge-based functions

for edge $E_m = \{e_1, e_2\}$, $m = 1, 2, 3$

for $0 \leq i \leq p-2$

$$\phi_i^{E_m} = L_i^{\mathcal{S}}(\lambda_{e_1} - \lambda_{e_2}, \lambda_{e_1} + \lambda_{e_2})$$

Interior functions

for $i, j \geq 0$, $i + j \leq p-3$

$$\phi_{ij}^I = L_i^{\mathcal{S}}(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2)\, \lambda_3\, \ell_j(\lambda_3 - \lambda_1 - \lambda_2)$$

for tetrahedra. An edge E^m has two vertices denoted by (e_1, e_2). We assume that all edges are oriented such that $e_1 < e_2$. A tetrahedron has four faces denoted by $F_1, \ldots F_4$. A face is represented by three vertex indices (f_1, f_2, f_3). We assume that they are ordered such that $f_1 < f_2 < f_3$.

The basis functions are expressed in terms of the barycentric coordinates λ_1, λ_2, and λ_3, and λ_4 for the 3D element. Following [9], we exploit a tensor product structure on the simplicial elements.

We introduce the scaled Legendre polynomials

$$\ell_n^{\mathcal{S}}(s,t) := t^n \ell_n(\frac{s}{t})$$

and scaled integrated Legendre polynomials

$$L_n^{\mathcal{S}}(s,t) := t^n L_n(\frac{s}{t})$$

on the triangular domain with vertices in $(-1, 1)$, $(1, 1)$, $(0, 0)$

$$t \in (0, 1],\ s \in [-t, t].$$

Both can be evaluated by division-free 3 term recurrences. We observe that L_n^S vanishes on the two edges $s = -t$ and $s = t$, and corresponds to $L_n(s)$ on the third edge $t = 1$. By means of two barycentric coordinates $\lambda_i, \lambda_j \geq 0$, $\lambda_i + \lambda_j \leq 1$, this domain can be parameterized as

$$(s,t) = (\lambda_i - \lambda_j, \lambda_i + \lambda_j).$$

A. High-order scalar shape functions

We start with presenting hierarchical high order finite elements for approximating scalar H^1 functions. These functions must be continuous over element boundaries. This means that the point values in element vertices, and the polynomials along edges (and faces in 3D) must be the same. This is obtained by defining basis functions associated with vertices, edges, (faces), and the interior of elements. The vertex basis functions are the standard linear nodal functions being one in one vertex, and vanishing in the other ones. The edge based functions have to span P_0^p on the edge, and must vanish at the other two edges. The scaled integrated Legendre polynomials fulfill this property. In 2D, the interior basis functions must vanish on the boundary of the triangle. They are defined as tensor product of 1D basis functions. One factor has to vanish on

Hierarchical tetrahedral H^1-element of order p using Scaled Legendre Polynomials

Vertex-based functions

$\phi_i^V = \lambda_i \qquad$ for $i = 1, 2, 3, 4$

Edge-based functions

for edge $E_m = \{e_1, e_2\}$, $m = 1, ...6$

for $2 \leq i \leq p-2$

$$\phi_i^{E_m} = L_i^{\mathcal{S}}(\lambda_{e_1} - \lambda_{e_2}, \lambda_{e_1} + \lambda_{e_2})$$

Face-based functions

for face $F_m = \{f_1, f_2, f_3\}$, m=1,...,4

for $i, j \geq 0$, $i + j \leq p-3$

$$\begin{aligned}\phi_{ij}^{F_m} &= L_i^{\mathcal{S}}(\lambda_{f_1} - \lambda_{f_2}, \lambda_{f_1} + \lambda_{f_2}) \\ &\quad \times \lambda_{f_3}\, \ell_j^{\mathcal{S}}(\lambda_{f_3} - \lambda_{f_1} - \lambda_{f_2}, \lambda_1 + \lambda_2 + \lambda_3)\end{aligned}$$

Interior-based function

for $0 \leq i + j + k \leq p-4$

$$\begin{aligned}\phi_{ijk}^I &= L_i^{\mathcal{S}}(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2) \\ &\quad \times \lambda_3\, \ell_j^{\mathcal{S}}(\lambda_3 - \lambda_1 - \lambda_2, \lambda_1 + \lambda_2 + \lambda_3) \\ &\quad \times \lambda_4\, \ell_k(\lambda_4 - \lambda_1 - \lambda_2 - \lambda_3)\end{aligned}$$

two edges, the second one has to eliminate the third edge. In 3D, the face shape functions are defined by means of scaled Legendre polynomials, which ensures that the argument takes values on the whole interval $(-1, 1)$, and thus improves the conditioning.

B. High-order vector shape functions

Next, we explain our new vector-valued basis functions. The basic idea is to include the gradients of the scalar basis functions into the set of basis functions. If the scalar function is continuous, then the gradient has continuous tangential components. First, we take the lowest order Nédélec elements, which have one basis function for each edge. The additionally needed edge-based basis functions for spanning P^p on the edge can be taken exactly as the gradients of the scalar edge-based functions which span P_0^{p+1}. Since the edge-based functions are gradients, the curl of them vanishes. The interior basis functions for the triangle need vanishing tangential components. This is obtained by taking the gradients of the H^1-interior functions. But now, the gradient fields are not enough to span all vector valued functions. Recall that the scalar basis is built as tensor product $u_i v_j$. The evaluation of the gradient gives $\nabla(u_i v_j) = (\nabla u_i) v_j + u_i \nabla v_j$. Note that not only the sum, but both individual terms have vanishing tangential boundary values. Thus, also different linear combinations like $(\nabla u_i) v_j - u_i \nabla v_j$ can be taken as basis functions. By counting the dimension one finds that still $p-1$ functions are missing. These can be chosen linearly independent as the product of the edge-element function, and a polynomial v_j. By means of the scaled Legendre polynomials, the 2D construction is easily extended to 3D elements.

Hierarchical triangular H^{curl}-element of order p using Scaled Legendre Polynomials

Edge-based shape functions

for edge $E^m = \{e_1, e_2\}$, $m = 1, 2, 3$

Edge-element shape function

$$\phi_1^{E_m} = \nabla\lambda_{e_1}\,\lambda_{e_2} - \lambda_{e_1}\nabla\lambda_{e_2}$$

High order edge-based functions

for $2 \le i \le p+1$

$$\phi_{i-1}^{E_m} = \nabla L_i^{\mathcal{S}}(\lambda_{e_1} - \lambda_{e_2}, \lambda_{e_2} + \lambda_{e_1})$$

Element-based functions

We set
$$\begin{aligned} u_i &:= L_{i+2}^{\mathcal{S}}(\lambda_2 - \lambda_1, \lambda_1 + \lambda_2) \\ v_j &:= \lambda_3 \ell_j(\lambda_3 - \lambda_1 - \lambda_2) \end{aligned}$$

Type 1: gradient fields

for $i, j \ge 0$, $i + j \le p - 2$

$$\phi_{ij}^{I_1} = \nabla(u_i\, v_j)$$

Type 2:

for $i, j \ge 0$; $i + j \le p - 2$

$$\phi_{ij}^{I_2} = (\nabla u_i)\, v_j - u_i\, \nabla(v_j)$$

Type 3:

for $0 \le j \le p - 2$

$$\phi_j^{I_3} = (\nabla\lambda_1\,\lambda_2 - \lambda_1\,\nabla\lambda_2)\, v_j$$

Hierarchical tetrahedral H^{curl}-element of order p using Scaled Legendre Polynomials

Edge-based functions

for edge $E^m = \{e_1, e_2\}$, $m = 1, \ldots, 6$

Edge-element shape function

$$\phi_1^{E_m} = \nabla\lambda_{e_1}\,\lambda_{e_2} - \lambda_{e_1}\nabla\lambda_{e_2}$$

High order edge-based functions

for $2 \le i \le p+1$

$$\phi_{i-1}^{E_m} = \nabla L_i^{\mathcal{S}}(\lambda_{e_1} - \lambda_{e_2}, \lambda_{e_2} + \lambda_{e_1})$$

Face-based functions

for face $F_m = \{f_1, f_2, f_3\}$, m=1,...,4

$$\begin{aligned} u_i &:= L_{i+2}^{\mathcal{S}}(\lambda_{f_1} - \lambda_{f_2}, \lambda_{f_1} + \lambda_{f_2}) \\ v_j &:= \lambda_3 \ell_j^{\mathcal{S}}(\lambda_{f_3} - \lambda_{f_1} - \lambda_{f_2}, \lambda_{f_3} + \lambda_{f_1} + \lambda_{f_2}) \end{aligned}$$

Type 1: gradient fields

for $i, j \ge 0$, $i + j \le p - 2$

$$\phi_{i,j}^{F_{m,1}} = \nabla(u_i\, v_j)$$

Type 2:

for $i, j \ge 0$, $i + j \le p - 2$

$$\phi_{ij}^{F_{m,2}} = (\nabla u_i)\, v_j - u_i\, \nabla(v_j)$$

Type 3:

for $0 \le j \le p - 2$

$$\phi_j^{F_{m,3}} = (\nabla\lambda_{f_1}\,\lambda_{f_2} - \lambda_{f_1}\,\nabla\lambda_{f_2})\, v_j$$

Element-based functions

$$\begin{aligned} u_i &:= L_{i+2}^{\mathcal{S}}(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2) \\ v_j &:= \lambda_3 \ell_j^{\mathcal{S}}(\lambda_3 - \lambda_1 - \lambda_2, \lambda_1 + \lambda_2 + \lambda_3) \\ w_k &:= \lambda_4 \ell_k(\lambda_4 - \lambda_1 - \lambda_2 - \lambda_3) \end{aligned}$$

Type 1: gradient fields

for $i, j, k \ge 0$, $i + j + k \le p - 3$

$$\phi_{ijk}^{I_1} = \nabla(u_i\, v_j\, w_k)$$

Type 2:

for $i, j, k \ge 0$, $i + j + k \le p - 3$

$$\begin{aligned} \phi_{ijk}^{I_{2a}} &= \nabla u_i\, v_j\, w_k - u_i\, \nabla v_i\, w_k + u_i\, v_j\, \nabla w_k \\ \phi_{ijk}^{I_{2b}} &= \nabla u_i\, v_j\, w_k - u_i\, \nabla v_i\, w_k - u_i\, v_j\, \nabla w_k \end{aligned}$$

Type 3:

for $j, k \ge 0$, $j + k \le p - 3$

$$\phi_{jk}^{I_3} = (\nabla\lambda_1\,\lambda_2 - \lambda_1\,\nabla\lambda_2)\, v_j\, w_k$$

C. *Local complete sequence properties*

The global scalar finite element space can be decomposed into vertex, edge, (face,) and element based spaces:

$$W_{p+1} = W_V + \sum_{E_i} W_{E_i} + \sum_{F_i} W_{F_i} + \sum_{T_i} W_{T_i} \quad \subset H^1$$

Similarly, the $H(\mathrm{curl})$ finite element space is split into the edge-element basis, the high-order edge, (face,) and element blocks:

$$V_p = V_{\mathcal{N}_0} + \sum_{E_i} V_{E_i} + \sum_{F_i} V_{F_i} + \sum_{T_i} V_{T_i} \quad \subset H(\mathrm{curl}).$$

Similarly, one defines high order basis functions for $H(\mathrm{div})$ as lowest order Raviart-Thomas $Q^{\mathcal{RT}}$, high order face, and high order element-based functions, and the L_2 spaces as constants, and high order element functions. We constructed the basis functions such that each one of the blocks satisfies a **local complete sequence property**:

$$\begin{array}{lllllll}
W^V_{h,\,p+1=1} & \xrightarrow{\nabla} & V_h^{\mathcal{N}_0} & \xrightarrow{\mathrm{curl}} & Q_h^{\mathcal{RT}_0} & \xrightarrow{\mathrm{div}} & S_{h,\,0} \\
W^E_{p_E+1} & \xrightarrow{\nabla} & V^E_{p_E} & & & & \\
W^F_{p_F+1} & \xrightarrow{\nabla} & V^F_{p_F} & \xrightarrow{\mathrm{curl}} & Q^F_{p_F-1} & & \\
W^I_{p_I+1} & \xrightarrow{\nabla} & V^I_{p_I} & \xrightarrow{\mathrm{curl}} & Q^I_{p_I-1} & \xrightarrow{\mathrm{div}} & S^I_{p_I-2}
\end{array}$$

Some of the advantages of this local property are

- An arbitrary individual polynomial order can be chosen on each edge, face and element. The global complete sequence property is satisfied automatically.
- Simple edge-face-element block-diagonal preconditioning becomes efficient.
- The basis functions contain high order gradient fields explicitly. A simple way of gauging (as needed, e.g., for the magnetostatic boundary value problem) can be performed by skipping the high order gradient fields. Note that the lowest order gradient fields are still in the system.
- The implementation of the gradient-operator from H^1 to $H(\mathrm{curl})$ finite element spaces becomes trivial.

This kind of basis-functions has been implemented in our software package NgSolve for common 2D and 3D element geometries (triangles, quadrilaterals, tetrahedra, prisms and hexahedra).

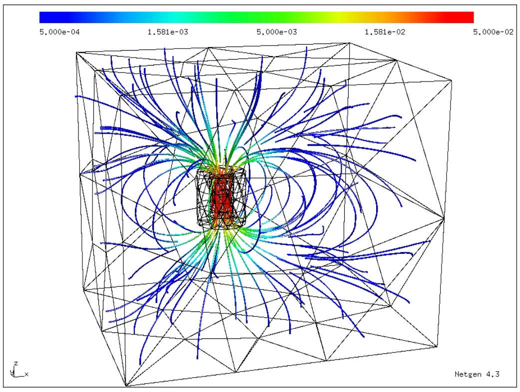

Figure 1: Magnetic field induced by the coil, order p=6.

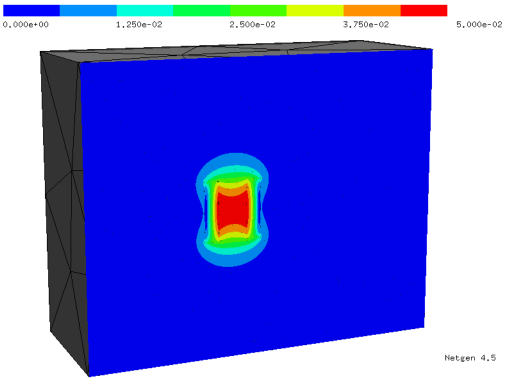

Figure 2: $|\operatorname{curl} A| = |B|$, order p=6.

III. MAGNETOSTATIC BOUNDARY VALUE PROBLEM

We consider the magnetostatic boundary value problem

$$\operatorname{curl} \mu^{-1} \operatorname{curl} A = j,$$

where j is the given current density, and A is the vector potential for the magnetic flux $B = \operatorname{curl} A$. For gauging, we add a small (6 orders of magnitude smaller than μ^{-1}) zero-order term to the operator. In order to show the performance of the constructed shape functions we compute the magnetic field induced by a cylindrical coil.

The mesh generated by Netgen [12] contains 2035 curved tetrahedral elements. The two pictures below show the magnetic fieldlines and the absolute value of the magnetic flux simulated with $H(\operatorname{curl})$-elements of uniform order 6.

We have chosen different polynomial orders p, and compared the number of unknowns (dofs), the condition numbers of the preconditioned system, the required iteration numbers of the PCG for an error reduction by 10^{-10}, and the required computation time on an Acer notebook with Pentium Centrino 1600 MHz CPU. We run the experiments once with keeping the gradient shape functions, and a second time with skipping them. The computed B-field is the same for both versions. One can observe a considerable improvement of the required solver time in the Table I.

p	dofs	grads	$\kappa(C^{-1}A)$	iter	solvertime
2	19719	yes	7.9	20	1.9 s
2	10686	no	7.9	21	0.7 s
3	50884	yes	24.2	32	9.8 s
3	29130	no	18.2	31	2.9 s
4	104520	yes	71.4	48	40.5 s
4	61862	no	32.3	40	10.7 s
5	186731	yes	179.9	69	137.9 s
5	112952	no	55.5	49	31.9 s
6	303625	yes	421.0	97	427.8 s
6	186470	no	84.0	59	87.4 s
7	286486	no	120.0	68	209.6 s

TABLE I: PERFORMANCE OF THE SOLVER

A. *Maxwell Eigenvalue Problem*

We consider the Maxwell eigenvalue problem: find $\omega \neq 0$ such that

$$\begin{aligned} \operatorname{curl} E &= i\omega\mu H, \\ \operatorname{curl} H &= -i\omega\varepsilon E. \end{aligned}$$

The corresponding weak form is to find eigenvalues $\omega > 0$ and eigenvectors $E \in H(\operatorname{curl}), E \neq 0$ such that there holds

$$\int_\Omega \mu^{-1} \operatorname{curl} E \cdot \operatorname{curl} v \, dx = \omega^2 \int_\Omega \varepsilon E \cdot v \, dx$$

for all $v \in H(\operatorname{curl})$. Finite element discretization leads to the generalized matrix eigenvalue problem

$$Au = \omega^2 Mu.$$

It is well known that this system contains many zero-eigenvalues which correspond to the gradient fields. A standard eigenvalue solver such as the inverse power iteration would suffer from first computing all the unwanted zero-eigenvalues. Following [10], we perform an *inexact inverse iteration with inexact projection*: Given u_n, we compute

$$\begin{aligned} \lambda_n &= \frac{(Au_n, u_n)}{(Mu_n, u_n)} \\ \widetilde{u}_{n+1} &= u_n - C_{A+\sigma M}^{-1}(Au_n - \lambda_n Mu_n) \\ u_{n+1} &= (I - \widetilde{P})\widetilde{u}_n \end{aligned}$$

Here, $C_{A+\sigma M}^{-1}$ is a preconditioner for the shifted $H(\operatorname{curl})$ problem, and $I - \widetilde{P}$ is an inexact projection into the complement of gradient fields. It is realized by performing k steps of the inexact projection

$$u_{n+1} = (I - B_\nabla C_\Delta^{-1} B_\nabla^T M_V)^k \, \widetilde{u}_{n+1}$$

Here, B_∇ is the matrix representing the gradient, M_V is the mass matrix for the vector-elements, and C_Δ is a Poisson-preconditioner. As mentioned above, the gradient operator is very simple for the presented basis functions.

We have chosen to compute the Neumann-eigenvalues on the Fichera domain $[-1, 1]^3 \setminus [0, 1]^3$. It shows severe singularities along the non-convex edges and at the vertex in the origin. We have chosen a priori a mesh refinement with 3 levels of anisotropic hp-refinement to resolve the

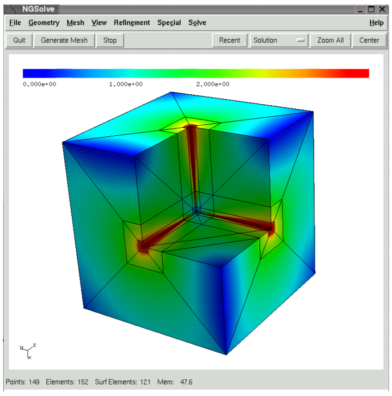

Figure 3: First Maxwell eigenvector

singularities. Fig. 3 shows the first non-trivial eigenvector approximated by elements of order $p = 4$.

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