Multigrid Analysis

Joachim Schöberl

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Abstract

The discretization of a partial differential equation leads to a large system of linear equations. Multigrid methods are iterative solvers for these equations. The idea is not to use only one finite element mesh, but a whole hierarchy of grids. The algorithm combines cheap iterative methods on each level. The result is an equation solver of optimal arithmetic complexity $O(N)$.

While the principle is very simple, a rigorous analysis is quite involved. It requires results from partial differential equations, finite element analysis, Hilbert space theory, as well as linear algebra. The topics of the lecture is to discuss the design and analysis of multigrid methods.

In the first part, we consider various techniques for a simple model problem. This chapter is split into no-regularity techniques and techniques based on shift theorems. The second part discusses extensions to important real-life problems including elasticity and Maxwell equations.

1 Overview of Finite Elements

Multigrid analysis is strongly connected to finite element analysis. Therefore, we start with a short overview of finite elements. We focus on results relevant to multigrid, for general fem theory please contact one of the available textbooks.

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ be an open, bounded, polyhedral domain. We consider the Poisson problem with homogenous Dirichlet boundary conditions:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Its weak form is to search u in a suitable Hilbert function space V such that

$$A(u, v) = f(v) \qquad \forall\, v \in V, \tag{1}$$

where the symmetric bilinear-form $A(.,.)$ and the linear form $f(.)$ are defined by

$$A(u, v) := (\nabla u, \nabla v) \qquad \text{and} \qquad f(v) := (f, v). \tag{2}$$

Here and in the following, $(.,.)$ and $\|.\|$ denote the $L_2(\Omega)$ inner product and $L_2(\Omega)$-norm, respectively. The bilinear-form $A(.,.)$ defines a second inner product. The corresponding norm

$$\|v\|_A = A(v,v)^{1/2}$$

is called energy norm.

1.1 Sobolev Spaces

We define a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, its absolut value $|\alpha| = \sum \alpha_i$, and the derivatives $\partial^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d}$.

For $k \in \mathbb{N}_0$, we define the Hilbert-space (semi)norms

$$|v|_k^2 := \sum_{|\alpha|=k} \|\partial^\alpha v\|^2$$

and norms

$$\|v\|_k^2 := \sum_{l=0}^{k} |v|_l^2.$$

The corresponding inner products are

$$(u,v)_k = \sum_{|\alpha| \leq k} (\partial^\alpha u, \partial^\alpha v).$$

Let $C^\infty(\overline{\Omega})$ be the function space of infinitely differentiable functions on $\overline{\Omega}$, and $C_0^\infty(\Omega)$ its subspace with compact support (=functions vanish in neighbourhood of $\partial\Omega$) in Ω.

Define the Sobolev spaces

$$H^k = \overline{C^\infty}^{\|\cdot\|_k} \quad \text{and} \quad H_0^k = \overline{C_0^\infty}^{\|\cdot\|_k}.$$

On the boundary, we define almost everywhere the normal derivative $\partial_n = \sum n_i \partial_{x_i}$. If the domain has Lipschitz continuous boundary, then

$$H_0^k = \{v \in H^k : v = \partial_n v = \ldots \partial_n^{k-1} v = 0 \text{ on } \partial\Omega\}$$

There holds Friedrichs inequality

$$\|v\| \leq c_F \, |v|_1 \qquad \forall \, v \in H_0^1,$$

where the constant c_F depends only on the domain Ω. The dual space of H_0^k is called H^{-k}. It consists of continuous linear functionals

$$\begin{array}{rcl} f(.) \; : \; H_0^k & \rightarrow & \mathbb{R} \\ v & \rightarrow & f(v) \end{array}$$

For example, a function $f \in L_2$ generates the linear functional $v \in H_0^1 \to (f, v)$. The dual norm is

$$\|f\|_{-k} = \sup_{v \in H_0^k} \frac{f(v)}{\|v\|_k}$$

The duality product

$$\begin{aligned} \langle ., . \rangle_{H^{-k} \times H_0^k} \; : \; H^{-k} \times H_0^k \;&\to\; \mathbb{R} \\ (f, v) \;&\to\; f(v) \end{aligned}$$

is a generalization of the L_2-inner product.

The Sobolev spaces form a nested sequence of spaces

$$\ldots H^2 \subset H^1 \subset H^0 = L_2 \subset H^{-1} \subset H^{-2} \ldots$$

Later, we will define also Sobolev spaces of fractional order.

Friedrichs' inequality proves norm equivalence $\|.\|_A \simeq \|.\|_1$. Thus,

$$V := (H_0^1, \|.\|_A, A(., .))$$

is a Hilbert space.

Unique and stable solvability of (1) follows directly from the Riesz' theorem:

Theorem 1 (Riesz' representation theorem). *For any continuous linear functional f on a Hilbert space V there exists an unique $u_f \in V$ such that*

$$(u_f, v)_V = f(v) \qquad \forall\, v \in V$$

and

$$\|u_f\|_V = \|f\|_{V^*}$$

On the other hand, any $u \in H_0^1$ is the solution of a weak problem with $f \in H^{-1}$: Take $f(.) := A(u, .)$.

Regularity, shift theorem: If the right hand side f belongs to a more regular function space than H^{-1}, the solution might be more regular than H^1, too. If Ω is convex, and $f \in L_2$, then the solution u belongs to H^2, and the shift theorem

$$\|u\|_2 \preceq \|f\|_0$$

is valid. Shift theorems are very specific for each problem class.

Notation: We write $a \preceq b$ if there exists an in principle computable constant c *of moderate value* ≈ 1 such that $a \leq cb$. In particular, the constant c does not depend on the number of elements in the discretization. We write $a \simeq b$ if $a \preceq b$ and $b \preceq a$.

If not stated otherwise, we will use the above definitions of V and A.

1.2 Finite Element Spaces

The finite element method (FEM) provides a computable approximation to the solution of (1).

A triangulation $\mathcal{T}$ is a set of (closed) simplicial (triangular, tetrahedral) elements T

$$\mathcal{T} = \{T\}.$$

Each element T is seen as a one to one, affine-linear mapping from the reference element T^R, i.e.,

$$T = F_T(T^R).$$

We define the local mesh size $h_T = \|F_T'\|$. This measure is equivalent to diam(T). The triangulation is called

- *regular* if two elements T and T' are either identic, or have a common face (only 3D), or a common edge, or a common vertex, or are distinct.
- *shape regular* if it is regular, and $\operatorname{cond}(F_T') = \|F_T'\| \cdot \|(F_T')^{-1}\| \preceq 1$.
- *quasi uniform* if it is shape regular, and $h_T \simeq h$, where h is a global mesh size parameter.

Next, we define the FE sub-space

$$V_h = \{v \in V : v|_T \circ F_T \in P^k(T^R)\},$$

where $P^k(T^R)$ is the set of polynomials up to total order $k \geq 1$ on the reference element. Functions in V_h are continuous.

On shape regular meshes there holds the following approximation estimate:

Lemma 2 (Approximation). *Let $k = 1$ or $k = 2$. For given $v \in H^k \cap V$, there exists a $v_h \in V_h$ such that*

$$\sum_{T \in \mathcal{T}} \left\{ h_T^{-2} \|v - v_h\|_{0,T}^2 + \|\nabla(v - v_h)\|_{0,T}^2 \right\} \preceq \sum_{T \in \mathcal{T}} h_T^{2(k-1)} |v|_{k,T}^2$$

Proof: For $k = 2$, define the nodal interpolation operator I_h and choose $v_h = I_h v$. For $k = 1$, replace I_h by the Clément quasi-interpolation operator (see literature or Section ... below). $\square$

Lemma 3 (Inverse inequality). *On shape regular triangulations there holds*

$$\|\nabla v_h\| \preceq \|h_T^{-1} v_h\| \qquad \forall\, v_h \in V_h.$$

Proof: By transformation to the reference element.

The FEM defines the approximation $u_h \in V_h$ as unique solution of

$$A(u_h, v_h) = f(v_h) \qquad \forall\, v_h \in V_h. \tag{3}$$

FEM theory, as well as multigrid analysis, is heavily based on orthogonality relations. Subtracting (3) from (1) leads to

$$A(u - u_h, v_h) = 0 \qquad \forall\, v_h \in V_h,$$

i.e., the error is orthogonal to V_h

$$u - u_h \perp_A V_h.$$

The FEM approximation u_h is the A-orthogonal projection of u onto V_h:

$$u_h = P_{V_h} u$$

with $P_{V_h} : V \to V_h$ defined by

$$A(Pw, v_h) = A(w, v_h) \qquad \forall\, w \in V \;\; \forall\, v_h \in V_h.$$

(Picture orthogonality)

An equivalent definition of the projection is

$$P_h u \in V_h \quad \text{such that:} \qquad \|u - P_h u\|_A \leq \min_{v_h \in V_h} \|u - v_h\|_A.$$

Proof: For any $v_h \in V_h$ there holds

$$\begin{aligned}
\|u - u_h\|_A^2 &\leq \|u - u_h\|_A^2 + \|u_h - v_h\|_A^2 \\
&= \|u - u_h\|_A^2 + 2\,(u - u_h, u_h - v_h)_A + \|u_h - v_h\|_A^2 \\
&= \|u - u_h + u_h - v_h\|_A^2 = \|u - v_h\|_A^2
\end{aligned}$$

This is Cea's Lemma for symmetric bilinear-forms.

Theorem 4. *On shape regular meshes there holds the a priori error estimate*

$$\|u - u_h\|_A^2 \preceq \sum_T h_T^2 |u|_2^2. \tag{4}$$

On quasi-uniform meshes and convex domains there holds

$$\|u - u_h\|_A \preceq h \|f\|_0 \tag{5}$$

Proof. Follows immediately from orthogonality, approximation, and shift theorem. □

The shift theorem provides a better rate of convergence in a weaker norm:

Theorem 5 (Aubin Nitsche). *Let $u_h = P_{V_h} u$. On quasi-uniform meshes and convex domains there holds*

$$\|u - u_h\|_0 \preceq h \, \|u - u_h\|_A$$

Proof. Pose the additional variational problem

$$A(w, v) = (u - u_h, v)_0 \qquad \forall \, v \in V$$

Due to regularity there holds $w \in H^2$ and $\|w\|_2 \preceq \|u - u_h\|_0$. Define w_h according to Lemma 2. The choice $v = u - u_h$ gives

$$\begin{aligned}
\|u - u_h\|_0^2 &= A(w, u - u_h) = A(w - w_h, u - u_h) \\
&\leq \|w - w_h\|_A \, \|u - u_h\|_A \\
&\preceq h \, |w|_2 \, \|u - u_h\|_A \\
&\preceq h \, \|u - u_h\|_0 \, \|u - u_h\|_A.
\end{aligned}$$

Dividing one factor proves the result. $\square$

This technique is essential for many types of multigrid proofs. The following theorem proves a multi-level decomposition using the Aubin-Nitsche trick. It relates the H^1 norm to scaled L_2-norms on different levels:

Theorem 6. *Let $L \in \mathbb{N}$, $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_L$ a family of hierarchically refined quasi-uniform triangulations on a convex domain Ω. The mesh-size of $\mathcal{T}_l$ is $h_l = 2^{-l}$. The generated fe-spaces $V_0 \subset V_1 \subset \ldots \subset V_L$ are nested. Let $P_l : V_L \to V_l$ be the A-orthogonal projections.*

Take $u_L \in V_L$ and its decomposition

$$u_L = w_0 + \sum_{l=1}^{L} w_l \qquad \textit{with} \qquad w_0 = P_0 u_L, \quad w_l = (P_l - P_{l-1}) u_L.$$

Then there holds

$$\|u_L\|_A^2 \simeq \|w_0\|_A^2 + \sum_{l=1}^{L} h_l^{-2} \|w_l\|_0^2.$$

Proof. From $(w_l, v_{l-1})_A = (P_l u_L, v_{l-1})_A - (P_{l-1} u_L, v_{l-1})_A = 0$ there follows $w_l \perp_A V_{l-1}$. The whole decomposition is A-orthogonal. Thus

$$\|u_L\|_A^2 = \| \sum_{l=0}^{L} w_l \|_A^2 = \sum_{l=0}^{L} \sum_{k=0}^{L} (w_l, w_k)_A = \sum_{l=0}^{L} \|w_l\|_A^2$$

The inverse estimate Lemma 3 applied to $w_l \in V_l$ claims $\|w_l\|_A \preceq h_l^{-1} \|w_l\|_0$, and the Aubin-Nitsche Lemma (applied to $w_l = (I - P_{l-1}) w_l$ proves the opposite estimate $\|w_l\|_0 \preceq h_l \|w_l\|_A$. $\square$

After choosing a basis $(\varphi_1, \ldots, \varphi_n)$ for V_h, $n = \dim V_h$, one ends up with the linear system

$$A\underline{u} = \underline{f}, \tag{6}$$

with $A_{ij} = A(\varphi_j, \varphi_i)$ and $\underline{f}_i = f(\varphi_i)$. The FEM approximation is $u_h = \sum_{i=1}^n \underline{u}_i \varphi_i$. We use underbars for vectors in $\mathbb{R}^n$, and sub-scripts h for fe functions.

The isomorphism between $\mathbb{R}^n$ and V_h is denoted by

$$\begin{aligned} \Phi &: \mathbb{R}^n \to V_h \\ &: \underline{v} \to \sum_{i=1}^n \underline{v}_i \varphi_i \end{aligned}$$

Its dual $\Phi^* : V_h^* \to \mathbb{R}^n$ is defined by

$$(\Phi^* d_h)^T \underline{v} = \langle d_h, \Phi \underline{v} \rangle_{V_h^* \times V_h} \qquad \forall\, d_h \in V_h^* \ \ \forall\, \underline{v} \in \mathbb{R}^n$$

By choosing $\underline{v} = e_i$, the i^{th} unit vector, we observe

$$(\Phi^* d_h)_i = (\Phi^* d_h)^T e_i = \langle d_h, \Phi e_i \rangle_{V_h^* \times V_h} = \langle d_h, \varphi_i \rangle_{V_h^* \times V_h} = d_h(\varphi_i)$$

Instead of matrices in $\mathbb{R}^{n \times n}$, we will prefer to work with operators between V_h and its dual V_h^*. For this, define $A_h : V_h \to V_h^*$ implicitly by

$$\langle A_h u_h, v_h \rangle = A(u_h, v_h) \qquad \forall\, u_h, v_h \in V_h.$$

There holds

$$A = \Phi^* A_h \Phi,$$

since

$$e_j^T A e_i = A(\varphi_i, \varphi_j) = \langle A_h \varphi_i, \varphi_j \rangle = \langle A_h \Phi e_j, \Phi e_j \rangle = e_j^T \Phi^* A_h \Phi e_j$$

for $i, j = 1, \ldots, n$.

We are interested in efficient solution methods for the linear system (6).

2 Iterative methods

Let C be a regular matrix. The preconditioned Richardson iteration is

Choose u^1.
For $k = 1, 2, \dots$
$d^k = f - Au_k$
$w^k = C^{-1}d_k$
$u^{k+1} = u^k + \tau w^k$

The game is to define matrices C such that the iteration converges fast, and the application of C^{-1} is efficient.

The iteration can be written as

$$u^{k+1} = u^k + \tau C^{-1}(f - Au^k).$$

Define the error as $e^k = u^k - u$ and use $f = Au$ to obtain the error transition relation

$$\begin{aligned} e^{k+1} &= u^{k+1} - u = u^k - u + \tau C^{-1}A(u - u^k) \\ &= \underbrace{(I - \tau C^{-1}A)}_{M} e^k. \end{aligned}$$

The goal is to prove estimates for $\|M\|$ in a proper norm.

It is useful to choose symmetric and positive definite preconditioning matrices C. Then the iteration matrix $M = (I - \tau C^{-1}A)$ is self-adjoint w.r.t. the energy inner product $(u, v)_A = u^T Av$:

$$\begin{aligned} (Mu, v)_A &= ((I - \tau C^{-1}A)u)^T Av = u^T(A - \tau AC^{-1}A)v \\ &= u^T A(I - \tau C^{-1}A)v = (u, Mv)_A \end{aligned}$$

If a matrix is self-adjoint in some norm, its corresponding matrix norm is equal to the spectral radius (=the absolute value of its largest eigen-value).

The following two statements are equivalent:

- λ_i is an eigen-value of $Ax = \lambda Cx$
- $\mu_i := 1 - \tau\lambda_i$ is an eigen-value of $(I - \tau C^{-1}A)x = \mu x$

Thus

$$\|M\|_A = \sup_{\lambda \in \sigma(C^{-1}A)} |1 - \tau\lambda|$$

Let $\sigma(C^{-1}A) \subset [\lambda_1, \lambda_n]$ with $\lambda_1 > 0$. Then the optimal choice $\tau = \frac{2}{\lambda_1+\lambda_n}$ leads to $\|M\| \le 1 - \frac{2}{1+\lambda_n/\lambda_1}$.

Thus, the goal is to prove spectral estimates

$$\lambda_1 \|v\|_C^2 \le \|v\|_A^2 \le \lambda_n \|v\|_C^2.$$

In practice, one uses conjugate gradient iterations instead of the Richardson iteration. Also there, the spectral estimates are the basis for estimating the rate of convergence.

2.1 Representation in FE space

A simple preconditioner is the Jacobi preconditioner, i.e., choose

$$C = \operatorname{diag} A.$$

The goal is to rewrite the preconditioning matrix operation C^{-1} as operator in the finite element space, namely

$$C_h^{-1} : V_h^* \to V_h$$

The definition is

$$C_h^{-1} = \Phi C^{-1} \Phi^*$$

We start with $d_h \in V_h^*$ and compute $w_h = C_h^{-1} d_h$. Intermediate steps are $\underline{d} = \Phi^* d_h \in \mathbb{R}^n$, $\underline{w} = C^{-1}\underline{d} \in \mathbb{R}^n$ and $w_h = \Phi \underline{w}$. The matrix preconditioning operation is

$$\underline{w} = \underline{C}^{-1}\underline{d} = \sum_{i=1}^{n} e_i (e_i^T A e_i)^{-1} e_i^T \underline{d}.$$

Let $\underline{d}_i = e_i^T \underline{d}$ and $\underline{w}_i = (e_i^T A e_i)^{-1} \underline{d}_i$. This scalar equation can be written in variational form:

$$\underline{v}_i^T (e_i A e_i) \underline{w}_i = \underline{d}_i \underline{v}_i \qquad \forall\, v_i \in \mathbb{R},$$

Now, using the definition of the matrix and the vector $\underline{d}$, we have

$$\underline{v}_i A(\varphi_i, \varphi_i) \underline{w}_i = d(\varphi_i) \underline{v}_i \qquad \forall\, \underline{v}_i \in \mathbb{R}$$

This is a variational problem on $V_i := \operatorname{span}\{\varphi_i\}$: The finite element function $\underline{w}_i \varphi_i$ is the unique solution $w_i \in V_i$ of

$$A(v_i, w_i) = d(v_i) \qquad \forall\, v_i \in V_i.$$

Finally, we get

$$w_h = \Phi \underline{w} = \sum \Phi e_i \underline{w}_i = \sum \varphi_i \underline{w}_i = \sum w_i.$$

Combining the steps above, we have derived the preconditioning operator

$$C_h^{-1} : V_h^* \to V_h : d(.) \to w$$

$$w = \sum w_i \qquad \text{with} \qquad w_i \in V_i \;\text{ s.t. } A(w_i, v_i) = d(v_i) \quad \forall\, v_i \in V_i.$$

The error reduction operator translated to the finite element space, $M_h : V_h \to V_h$, is

$$\begin{aligned} M_h &= \Phi M \Phi^{-1} = \Phi(I - \tau C^{-1} A)\Phi^{-1} \\ &= \Phi(I - \tau(\Phi^{-1} C_h^{-1} [\Phi^*]^{-1} \Phi^* A_h \Phi))\Phi^{-1} \\ &= I - \tau C_h^{-1} A_h. \end{aligned}$$

Lemma 7. *Let $P_i : V_h \to V_i$ be the A-orthogonal projection. Then*

$$M_h = I - \tau \sum_{i=1}^{n} P_i$$

Proof. Set $u_h^2 = M_h u_h^1 = u_h^1 - \tau w_h$, where $w_h = C_h^{-1} A_h u_h^1$. By the results above,

$$w_h = \sum w_i, \qquad \text{with} \qquad A(w_i, v_i) = \langle A_h u^1, v_i \rangle \qquad \forall\, v_i \in V_i.$$

In other words, $w_i = P_i u^1$ □

Now, we started with a preconditioner in matrix-form, and translated the operation into fe notation. The analysis is performed in the fe notation. In the following, we will work in the fe notation. Only, when it comes to implementation, one has to think about the matrix-vector representation.

Some more examples:

- The Gauss-Seidel iteration is ($k \in \mathbb{N}, i \in \{1, \dots, n\}$):

  $$u^{k+i/n} = u^{k+(i-1)/n} - e_i A_{ii}^{-1} e_i^T (f - A u^{k+(i-1)/n})$$

  The iteration matrix of one full step $u^k \to u^{k+1}$ is

  $$M = M_n \dots M_2 M_1 \qquad \text{with} \qquad M_i = I - e_i A_{ii}^{-1} e_i^T A$$

  In fe form, one step is

  $$M_i = (I - P_i),$$

  the product is

  $$M_h = (I - P_n) \dots (I - P_2)(I - P_1)$$

  P_i is an A-orthogonal projection, so also $I - P_i$. The norm of an orthogonal projection is 1 (or, in the trivial case, it is 0). Thus, the Gauss-Seidel iteration without damping is non-expansive in A-norm. Later we will see whether it is convergent.

  In general, the multiplicative iteration is not A-self-adjoint, namely

  $$M_h^* = (I - P_1) \dots (I - P_n)$$

  Only if $P_1 = P_n$, $P_2 = P_{n-1}, \dots$, it is A-self-adjoint. Such an iteration is called symmetric.

- Block version: Let $i = 1, \dots, N$, and $E_i \in \mathbb{R}^{N \times m_i}$ be a full-rank matrix. Now, define the block-Jacobi preconditioner

$$C^{-1} := \sum_{i=1}^{N} E_i (E_i^T A E_i)^{-1} E_i^T$$

The usual case is $E = (e_{i_1}, e_{i_2}, \dots, e_{i_N})$. The embedding matrices $\mathbb{R}^{m_i}$ generate *small* spaces

$$V_i = \Phi E_i \mathbb{R}^{m_i}$$

The translation of the iteration matrix is the same as in the Jacobi case.

$$M_h = I - \tau \sum_{i=1}^{N} P_i.$$

Examples: Block relaxation of some nodes, local or global, anisotropic, high order blocks, systems of pdes, ...

The iteration does not depend on the specific choice of the basis, it depends on the sub-spaces V_i, only.

- Two-level preconditioner: Let V_H be a finite element sub-space of V_h on a coarser grid. Set $N_H = \dim V_H$. Basis functions are φ_i^H. The coarse grid basis functions can be expressed as linear combination of fine grid basis functions:

$$\varphi_j^H = \sum_{i=1}^{N} c_{ij} \varphi_i$$

Define transformation matrix

$$E_H = (c_{ij})_{\substack{i=1\dots N \\ j=1\dots N_H}}$$

Example: 1D hat functions.

The corresponding finite element function is

$$u_H = \sum_{j=1}^{N_H} \underline{u}_j^H \varphi_j^H = \sum_{j=1}^{N_H} \sum_{i=1}^{N} \underline{u}_j^H c_{ij} \varphi_i = \sum_{i=1}^{N} (E_H \underline{u}^H)_i \varphi_i = \Phi E_H \underline{u}^H.$$

The matrix E_H transforms the coarse-grid coefficient vector $\underline{u}_H$ to the fine grid coefficient vector representing the same finite element function. It is called prolongation.

The idea of two-level preconditioning is to add a correction step on the coarse grid:

$$\begin{aligned}
&d^k = f - Au_k \\
&w_H^k = E_H (E_H^T A E_H)^{-1} E_H^T d_k \\
&u^{k+1} = u^k + w_H^k + \dots
\end{aligned}$$

The matrix $E_H^T A E_H$ is the fe - matrix on the coarse grid space w.r.t the basis φ_i^H.
The additive 2-Level iteration is defined as

$$M_h = I - \tau(P_H + \sum_{i=1}^{N} P_i)$$

The multiplicative one with additive smoother is

$$M_h = (I - P_H)(I - \tau \sum_{i=1}^{N} P_i),$$

and the multiplicative one with multiplicative smoother is

$$M_h = (I - P_H)(I - P_1) \dots (I - P_N)$$

- Multi-level preconditioner: Let $V_0 \subset V_1 \subset \dots \subset V_L$ a nested sequence of fe spaces. Let $V_l = \operatorname{span}\{\varphi_{l,i}, i = 1, \dots n_l\}$. Then the additive multi-level iteration is

$$M_h = I - \tau \sum_{l=0}^{L} \sum_{i=1}^{n_l} P_{l,i}$$

The multiplicative counterpart is the conventional **multigrid** iteration.

3 Additive Schwarz theory

Let $(V, A(.,.))$ be a Hilbert space. Let $\{(V_i, C_i(.,.))\}$ be a countable set of Hilbert spaces. Denote embedding operators $E_i : V_i \to V$. Then, the

inexact additive Schwarz preconditioner $C^{-1} : V^* \to V, d(.) \to w$ is defined by

$$w = \sum_i E_i w_i \qquad \text{with} \qquad C_i(w_i, v_i) = d(E_i v_i) \;\; \forall\, v_i \in V_i.$$

The following theorem analyzes whether C^{-1} is indeed the inverse of an operator $C : V \to V^*$.

Theorem 8 (Additive Schwarz Lemma). *Define the splitting norm*

$$|||u|||^2 := \inf_{\substack{u=\sum_i E_i v_i \\ v_i \in V_i}} \sum_i \|v_i\|_{C_i}^2.$$

Assume that $|||.|||$ is an equivalent norm to $\|.\|_A$. Then C is an isomorphism between V and V^*, and

$$\|u\|_C = |||u|||. \tag{7}$$

Proof. The right hand side of (7) is a constrained minimization problem on $X := V_1 \times V_2 \ldots$ with constraint $\sum E_i v_i = u$. We will formulate it as Kuhn-Tucker system (a saddle point problem). First, rewrite the constrained minimization problem as unconstrained one using the characteristic function of the feasible set:

$$|||u|||^2 = \inf_{\substack{v_i \in V_i \\ \sum E_i v_i = u}} \sum \|v_i\|_{C_i}^2 = \inf_{v_i \in V_i} \sup_{\mu \in V^*} \underbrace{\sum_i \|v_i\|_{C_i}^2 + 2\left\langle \mu, u - \sum_i E_i v_i \right\rangle}_{:=L(v,\lambda)}$$

We search for the saddle-point (u, λ) of the strictly-convex/concave Lagrange functional $L(v, \lambda)$. The condition $\partial_{V_i} L(v, \mu) = 0$ is

$$C_i(u_i, v_i) + \left\langle \lambda, E_i v_i \right\rangle = 0 \qquad \forall\, v_i \in V_i,\; i = 1, 2, \ldots \tag{8}$$

the partial derivative w.r.t. $\mu \in V^*$ is the constraint

$$\left\langle \mu, \sum_i E_i u_i \right\rangle = \langle \mu, u \rangle \qquad \forall\, \mu \in V^*. \tag{9}$$

Existence and uniqueness of a solution follows from saddle-point theory. The essential LBB condition follows from the assumption $V = \sum E_i V_i$ is stable.

Equations (8) and equation (9) state

$$u = C^{-1} \lambda$$

Thus, u is in the domain of C. Testing (8) with $v_i = u_i$ gives

$$|||u|||^2 = \sum \|u_i\|_{C_i}^2 = \sum \langle \lambda, E_i u_i \rangle = \langle \lambda, u \rangle = \langle Cu, u \rangle$$

□

The ASM Lemma reduces the analysis of the condition number $\kappa(C^{-1}A) = \lambda_n/\lambda_1$ to the norm estimates

$$\lambda_1 |||u|||^2 \leq \|u\|_A^2 \leq \lambda_n |||u|||^2.$$

Usually, the left inequality requires some work. The technique is to construct an explicit decomposition $u = \sum E_i u_i$. Often, the right estimate is simply the Lemma below:

Lemma 9. *Define the interaction matrix $G = (g_{ij})$ by*

$$g_{ij} = \sup_{u_i \in V_i, v_j \in V_j} \frac{A(E_i u_i, E_j v_j)}{\|u_i\|_{C_i} \|u_j\|_{C_j}}.$$

Let $\rho(G)$ denote the spectral radius of G. Then

$$\|u\|_A^2 \leq \rho(G)\, |||u|||^2.$$

Proof. Let $u = \sum E_i v_i$ (with $v_i \in V_i$) be an arbitrary decomposition. Then

$$\|u\|_A^2 = \|\sum_i E_i v_i\|_A^2 = \sum_{i,j} A(E_i v_i, E_j v_j) \leq \sum_{i,j} g_{ij} \|v_i\|_{C_i} \|v_j\|_{C_j}.$$

From $c^T G c \leq \rho(G)\|c\|^2$ applied to $c_i = \|v_i\|_{C_i}$ there follows

$$\|u\|_A^2 \leq \rho(G) \sum_i \|v_i\|_{C_i}^2.$$

Since the decomposition was arbitrary, the estimate is true for the infimum as well. □

3.1 Overlapping domain decomposition

In this section we apply the abstract ASM theory to domain decomposition methods.

Decompose the domain Ω into overlapping sub-domains Ω_i of (local) diameter H_i. The overlap is of order H_i. Only a finite number of domains overlap.

This allows to define a partition of unity $\{\psi_i\}, \psi_i \in C^\infty(\Omega)$ with the following properties:

$$\sum \psi_i = 1,$$

$$\|\psi_i\|_{L_\infty} \preceq 1 \qquad \text{and} \qquad \|\nabla \psi_i\|_{L_\infty} \preceq H_i^{-1}$$

The functions ψ_i live inside Ω_i

$$\operatorname{supp} \psi_i \subset \Omega_i.$$

For technical reasons we will need that ψ_i are strictly inside:

$$\operatorname{dist}(\operatorname{supp}(\psi_i), \partial\Omega_i \setminus \partial\Omega) \succeq H_i$$

Now, let V_h be a finite element space on a shape-regular triangulation. The local mesh size fulfills $h \leq H$. Define sub-spaces

$$V_i = \{v_i \in V_h : v_i = 0 \ \text{ in } \Omega \setminus \Omega_i\}$$

We assume that everything was chosen s.t. $V = \sum V_i$. The operator $E_i : V_i \to V_h$ is trivial embedding, and the local forms are the same as the global:

$$C_i(u_i, v_i) = A(Eu_i, Ev_i) = A(u_i, v_i)$$

Some remarks:

- The implementation of the additive Schwarz preconditioner requires the solution of local Dirichlet problems in V_i.
- A special case with $h \simeq H$ is the Jacobi preconditioner (or a block-Jacobi preconditioner with small blocks).

We prove the splitting estimate required by the ASM theory:

Lemma 10. *There holds*

$$|||u_h||| \preceq \min\{H_i\}^{-1} \|u_h\|_A \qquad \forall\, u_h \in V_h. \tag{10}$$

Proof. Let $I_h : H^1 \to V_h$ be a Clément-type operator with the following properties:

$$I_h v_h = v_h \qquad (\textit{projection})$$

and

$$\|\nabla I_h v_h\|_0 \preceq \|\nabla v_h\|_0 \qquad (A - \textit{continuity}).$$

Then, for given $u_h \in V_h$ we chose the decomposition

$$u_i = I_h(\psi_i u).$$

By linearity, $\sum_i u_i = \sum_i I_h(\psi_i u_h) = I_h((\sum_i \psi_i) u_h) = I_h u_h = u_h$. The assumption $\text{supp}\{\psi_i\}$ strictly inside Ω_i ensures that $u_i \in V_i$.

Thus, (u_i) is a feasible candidate for the minimization problem.

We start to estimate

$$|||u|||^2 \leq \sum \|u_i\|_A^2 = \sum \|\nabla I_h(\psi_i u_h)\|_0^2 \preceq \sum \|\nabla(\psi_i u_h)\|_0^2$$

The involved functions are smooth enough to apply the product rule (together with $(a+b)^2 \leq 2(a^2+b^2)$):

$$|||u|||^2 \preceq \sum \{\|(\nabla \psi_i) u_h\|_0^2 + \|\psi_i (\nabla u_h)\|_0^2\}$$

Next, using L_∞ estimates and local support of ψ_i:

$$|||u|||^2 \preceq \sum_i \{\|H_i^{-1} u_h\|_{\Omega_i}^2 + \|\nabla u_h\|_{0,\Omega_i}^2\}$$

Since a finite number of domains are overlapping, parts of the norms are duplicated a finite number of times:

$$|||u|||^2 \preceq \|H_i^{-1} u_h\|_{0,\Omega}^2 + \|\nabla u_h\|_{0,\Omega}^2\}$$

Finally, Friedrichs inequality gives the result

$$|||u|||^2 \preceq \min\{H_i\}^{-2} \|u_h\|_{0,\Omega}^2 \preceq \min\{H_i\}^{-2} \|\nabla u_h\|_0^2.$$

□

The other estimate, $\|u\|_A \preceq |||u|||$ follows from Theorem 9. Since only a finite number of domains overlap, each row of G has the same finite number of non-zero entries. The spectral radius is bounded by the number of overlapping sub-domains.

Remark:

- For $H \simeq h$, there follows from the proof of Lemma 10 the equivalence

$$|||u_h||| \simeq \|h^{-1} u_h\|_{L_2} \tag{11}$$

3.2 Overlapping Domain Decomposition with Coarse Grid System

Now, we improve the overlapping domain decomposition algorithm by adding a global coarse grid space. This will give optimal condition number estimates.

Let $V_H \subset V$. Let $E_H : V_H \to V_h$ be an embedding operator (usually called prolongation). In the case $V_H \subset V_h$ we choose $E_h = id$. We assume that the prolongation operator has the following properties:

$$\|\nabla E_H u_H\|_0 \preceq \|\nabla u_H\|_0 \qquad (A - continuity)$$

$$\|H_i^{-1}(u_H - E_H u_H)\|_0 \preceq \|\nabla u_H\|_0 \qquad (approximation)$$

The coarse-grid form C_H is defined by $C_H(.,.) = A_H(.,.) = A(.,.)$. (An alternative possibility would be $C_H(.,.) = A(E_H.,E_H.)$).

Now, let the DD preconditioner with coarse grid system be defined as ASM method with respect to the set of triplets

$$\{(V_H, E_H, A_H(.,.),\ \cup_i(V_i, id, A(.,.))\}.$$

Lemma 11. *The DD preconditioner with CG fulfills the stable splitting estimate*

$$|||u_h||| \preceq \|u_h\|_A \qquad \forall\, u_h \in V_H$$

Proof. Let additionally $I_H : V \to V_H$ be a Clément-type interpolation operator into the coarse grid space fulfilling $|.|_1$-continuity and L_2 approximation

$$\|H_i^{-1}(u - I_H u)\|_{L_2} + \|\nabla I_H u\|_{L_2} \preceq \|\nabla u\|_{L_2} \qquad \forall\, u \in V.$$

From the proof of Lemma 10 we know that

$$\inf_{v_h = \sum_i v_i} \sum \|v_i\|_A^2 \preceq \|H_i^{-1} v_h\|_0^2 + \|\nabla v_h\|_0^2 \qquad \forall\, v_h \in V_h. \tag{12}$$

We choose the 2-level decomposition

$$u_h = E_H u_H + u_f$$

with

$$u_H = I_H u_h \qquad \text{and} \qquad u_f = u_h - E_H I_H u_H$$

(with index f as *f*ine).

We bound the minimal decomposition by this candidate:

$$\begin{aligned} |||u_h|||^2 &= \inf_{u_h = E_H v_H + \sum v_i} \{\|v_H\|_A^2 + \sum \|v_i\|_A^2\} \\ &= \inf_{v_H \in V_H} \{\|v_H\|_A^2 + \inf_{\substack{v_i \in V_i \\ u_h - E_H v_H = \sum v_i}} \sum \|v_i\|_A^2\} \\ &\leq \|u_H\|_A^2 + \inf_{u_f = \sum v_i} \sum \|v_i\|_A^2 \end{aligned}$$

We apply (12) with $v_h = u_h - E_H u_H = u_f$:

$$|||u_h|||^2 \preceq \|u_H\|_A^2 + \|H_i^{-1} u_f\|_0^2 + \|u_f\|_A^2.$$

From $|.|_1$-continuity of I_H and E_H we get

$$\|u_H\|_A + \|u_f\|_A \preceq \|u_H\|_A$$

L_2-approximation of E_H and I_H proofs

$$\|H_i^{-1} u_f\|_0 = \|H_i^{-1}(u_h - E_H I_H u_h)\|_0 \leq \|H_i^{-1}(u_h - I_H u_h)\|_0 + \|H_i^{-1}(id - E_H) I_H u_h)\|_0 \preceq \|u_h\|_A.$$

□

We denote the ASM preconditioner associated with the fine-grid spaces V_i by $D_h^{-1} : V_h^* \to V_h$. By the ASM Lemma, the energy norm $\langle D_h u_h, v_h \rangle_{V_h^* \times V_h}$ generated by D_h is exactly the splitting norm:

$$\|v_h\|_{D_h}^2 := \inf_{v_H = \sum v_i} \sum \|v_i\|_A^2.$$

It is a Hilbert-space norm, the inner product is denoted by $D_h(.,.)$.

The 2-level method can be seen as a ASM with two local spaces (namely V_H and V_h), and inexact bilinear-forms A_H and D_h:

$$|||u_h|||_{2-level}^2 = \inf_{u_h = E_H u_H + u_f} \|u_H\|_{A_H}^2 + \|u_f\|_{D_h}^2.$$

Lemma 12. *The estimate*

$$\|u_h\|_A \preceq |||u_h|||_{2-level} \qquad \forall\, u_h \in V_h$$

is valid.

Proof. We apply Lemma 9 for 2 sub-spaces and bound all entries of G:

$$g_{HH} = \sup_{u_H, V_H \in V_H} \frac{A(E_H u_H, E_H v_H)}{\|u_H\|_A \|v_H\|_A} \preceq c,$$

which is due to continuity od E.

$$g_{ff} = \sup_{u_f, v_f \in V_h} \frac{A(u_f, v_f)}{\|u_f\|_D \|v_f\|_D} \preceq c,$$

which follos from finite overlap of local spaces implying $\|v_h\|_A \preceq \|v_h\|_D$. The off-diagonal value g_{Hf} is bounded by an additional Cauchy-Schwarz inequality. □

3.3 Clément-type quasi-interpolation operators

We used several times local quasi-interpolation operators fulfilling various continuity and approximation estimates. Now, we are going to construct and analyse such operators.

Let V_h be a finite element sub-space (or order p_h) of $H^1_{0,D}(\Omega)$ on a shape-regular triangulation $\{\mathcal{T}\}$. Choose the nodal basis $\{\varphi_i\}$ for the set of nodes $\mathcal{N} = \{N_i\}$. The nodes are assigned to vertices, edges, faces and elements.

To each node N_i define a set ω_i such that $dist(N_i, \omega_i) \preceq h_i$, and function $f_i \in L_\infty(\omega_i)$ such that the following is true:

$$\|f_i\|_{L_\infty} \preceq h_i^{-d} \qquad \|f_i\|_{L_1} \preceq 1 \tag{13}$$

Assume that $f_i(.)$ coincides with point evaluation in the node when applied to polynomials up to order p:

$$(f_i, v)_{L_2(\omega_i)} = v(N_i) \qquad \forall\, v \in \Pi^p$$

Then, the Clément-type operator is defined as

$$\begin{aligned} I_h \ &: \ L_2 \to V_h \\ & \quad I_h v = \sum_{N_i \in \mathcal{N}} (f_i, v)_{L_2(\omega_i)} \varphi_i \end{aligned}$$

Lemma 13. *On finite element spaces, the L_2 norm and the H^1 semi-norm are equivalent to the discrete norms, respectively:*

$$\|v_h\|_{L_2}^2 \simeq \sum_{N_i} h_i^d \, |v_h(N_i)|^2$$

$$\|\nabla v_h\|_{L_2}^2 \simeq \sum_{\substack{N_i, N_j \in \mathcal{N} \\ \exists T: N_i \in T, N_j \in T}} h_i^{d-2} \, |v_h(N_i) - v_h(N_j)|^2$$

Proof. Both estimates are proven by transformation techniques. □

Theorem 14 (Continuity). *For $p \geq -1$, the operator I_h is continuous in L_2 norm. For $p \geq 0$, the operator I_h is continuous in the H^1 semi-norm.*

Proof. By means of Lemma 13 it is enough to establish

$$(I_h v)(N_i) = (f_i, v)_{L_2(\omega_i)} \preceq h_i^{-d/2} \|v\|_{L_2(\omega_i)}$$

to prove the L_2 estimate. This follows immediately from (13) by $\|f_i\|_{L_2(\omega_i)}^2 \leq \|f_i\|_{L_1(\omega_i)} \|f_i\|_{L_\infty(\omega_i)} \preceq h_i^{-d}$.

To establish he H^1 estimate, we start with

$$(I_h v)(N_i) - (I_h v)(N_j) = \int_{\omega_i} f_i(x) v(x) \, dx - \int_{\omega_j} f_j(y) v(y) \, dy.$$

The assumptino $p \geq 0$ onto f_i ensures that $\int f_i \, dx = 1$. Thus,

$$(I_h v)(N_i) - (I_h v)(N_j) = \int_{\omega_i} \int_{\omega_j} f_i(x) f_j(y) [v(x) - v(y)] \, dy \, dx.$$

The difference $v(x) - v(y)$ is expressed as integral

$$v(x) - v(y) = \int_y^x \partial_\tau v(\xi) \, d\xi = \int_0^1 (x-y)^T (\nabla v)(y + s(x-y)) \, ds.$$

Changing order of integration (i.e. $\int_0^1 \int_{\omega_i} \int_{\omega_j}$), and a couple of C.-S. estimates, proves that

$$|(I_h v)(N_i) - (I_h v)(N_j)| \preceq h_i^{(-d+2)/2} \|\nabla v\|_{L_2([\omega_i, \omega_j])},$$

with the convex hull $[\omega_i, \omega_j]$ of $\omega_i \cup \omega_j$. □

Theorem 15 (Approximation). *For $1 \le q \le \min\{p, p_h\}$, there holds the approximation estimate*

$$\|h_i^q(v - I_h v)\|_0 + \|h_i^{q-1}\nabla(v - I_h v)\|_0 \preceq \|\nabla^q v\|_0$$

Proof. Since I_h preservers locally polynomials up to order $\min\{p, p_h\}$ (in the sense of $(I_h w)|_T = w|_T$ if $w|_{\omega_T}$ is a polynomial).

We split the global norm into element terms, and insert an arbitrary polynomial w of order q:

$$h_i^q\|v - I_h v\|_{L_2(T)} = h_i^q\|(id - I_h)(v - w)\|_{L_2(T)} \preceq h_i^q\|v - w\|_{L_2(\omega_T)}.$$

The rest is the approximation $\inf_{w\in\Pi^q}\|v - w\|_{L_2(\omega_T)} \preceq h_i^q\|\nabla^q\|_{L_2(\omega_T)}$. The H^1 estimate is the same argument. □

There are many possibilities to choose the local domains ω_i and weighting functions f_i. Thus various properties can be achieved:

- The operators I_h can be constructed as projections onto V_h. For this, choose ω_i such that $N_i \in \overline{\omega_i}$. Then, take the restiction of V_h onto ω_i, i.e. $V_i = \{v_h|_{\omega_i} : v_h \in V_h\}$. The linear functional

$$\begin{aligned} f_i(v) \quad &: \quad V_i \to \mathbb{R} \\ & \quad f_i(v) = (P_{L_2}^{V_i} v)(N_i) \end{aligned}$$

  (with $P_{L_2}^{V_i}$ the L_2 projection of $L_2(\omega_i)$ onto V_i) is continous on L_2. Thus, it can be represented as L_2 function f_i. In general, the norm of f_i depends on the choice of ω_i. The original construction by Clément used

$$\omega_i = \{T \in \mathcal{T} : N_i \in \overline{T}\},$$

  an alternative version (by Scott and Zhang) uses

$$\omega_i = T_{N_i} \qquad T_{N_i} \text{ some element s.t. } N_i \in T_{N_i}$$

- The choice

$$f_i = \left(\int \varphi_i \, dx\right)^{-1}\varphi_i \tag{14}$$

  is consistent of order 0. The corresponding quasi-interpolation operator is L_2 self-adjoint:

$$\begin{aligned} (I_h u, v)_{L_2} &= \Big(\sum_i \Big(\int \varphi_i\Big)^{-1}(\varphi_i, u)_{L_2}\varphi_i, v\Big)_{L_2} \\ &= \sum_i \Big(\int \varphi_i\Big)^{-1}(\varphi_i, u)_{L_2}(\varphi_i, v)_{L_2} \\ &= (u, I_h v)_{L_2} \end{aligned}$$

- The case of jumping coefficients accross sub-domains requires special care. The influence domain ω_i must be chosen as sub-set of elements with large coefficient. Then, under the so called quasi-monotonicity assumption, the quasi-interpolation operator is continuous in energy norm.
- It is possible to choose lower dimensional manifolds ω_i. Then, of coarse, I_h is not defined on L_2 anymore, but the H^1 estimates may stay valid. Scott and Zhang used boundary faces for ω_i to preserver polynomial boundary conditions.

4 Multi-level and multigrid methods

Multi-level and multigrid methods can be seen as extension of 2-level methods. Instead of one fine and one coarse grid, one works with a hierarchy of many grids. On each grid (except maybe the coarsest), one applies a cheap (local) preconditioner.

Let $L \in \mathbb{N}$ denote the number of levels,

$$\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_L,$$

be a family of nested triangulations, and

$$V_0 \subset V_1 \subset \ldots \subset V_L$$

the generated family of nested finite element spaces.

On each level $l, 0 \leq l \leq L$, we need a (cheap) preconditioner D_l, i.e. an operation

$$D_l^{-1} : V_l^* \rightarrow V_l.$$

It shall be defined by means of the symmetric bilinear form $D_l(.,.) : V_l \times V_l \rightarrow \mathbb{R}$ via

$$D_l(D_l^{-1} g_l, v_l) = g_l(v_l) \qquad \forall\, g_l \in V_l^* \;\forall\, v_l \in V_l.$$

The simplest (and typical) choice is a Jacobi preconditioner. For computations, a (symmetric) Gauss Seidel preconditioner is favourable. In terms of the last section we will call D an additive (or multiplicative) Schwarz preconditioner.

Since $V_l \subset V_L$, every functional in V_L^* has a canonical restriction onto V_l^*, and we can apply D_l^{-1} on the whole V_L^* without special notation.

One possibility to combine the preconditioners is to add them all up (ASM), i.e. define the preconditioning operation $C^{-1} : V_L^* \rightarrow V_L$ as

$$C^{-1} = \sum_{l=0}^{L} D_l^{-1}. \tag{15}$$

This method is called multi-level preconditioner. The case of Jacobi preconditioners D_l is called BPX preconditioner (after Bramble,Pasciak and Xu), or MDL (multilevel diagonal scaling).

An other possibility is to run the individual preconditioners sequentially (MSM):

$$u^{k+(l+1)/(L+1)} = u^{k+l/(L+1)} + D_l^{-1}(f - A_L u^{k+l/(L+1)}). \tag{16}$$

The corresponding iteration operator M is

$$(I - D_L^{-1}A_L)(I - D_{L-1}^{-1}A_L)\dots(I - D_0^{-1}A_L).$$

To obtain an A-symmetric iteration, one should run the symmetric version. This iteration is the classical multigrid V-1-1 - cycle.

4.1 Implementation

The implementation of the additive and the multiplicative preconditioners use the hierarchical structure. Let $N_l = dim\{V_l\}$.

Define, for $0 \leq l < k \leq L$, the embedding matrices $E_l^k : \mathbb{R}^{N_l} \to \mathbb{R}^{N_k}$. For $l < m < k$, there holds $E_l^k = E_m^k E_l^m$.

The finite element matrix $A_l \in \mathbb{R}^{N_l \times N_l}$ on level l fulfills the Galerkin relation

$$A_l = (E_l^L)^T A_L E_l^L.$$

On each level, there is defined the preconditioning matrix $D_l^{-1} \in \mathbb{R}^{N_l \times N_l}$ (e.g., $D_l =$ diag A_l).

The additive Schwarz preconditioner in matrix notation is

$$C^{-1} = \sum_{l=0}^{L} E_l^L D_l^{-1} (E_l^L)^T.$$

We define the intermediate preconditioners

$$C_l^{-1} = \sum_{k=0}^{l} E_k^l D_k^{-1} (E_k^l)^T$$

Clearly, there is $C^{-1} = C_L$.

Theorem 16. *Starting with $C_0^{-1} = D_0^{-1}$, the preconditioners can be computed recursively:*

$$C_l^{-1} = D_l^{-1} + E_{l-1}^l C_{l-1}^{-1} (E_{l-1}^l)^T. \tag{17}$$

Proof. Per induction. Assume the relation is true for $l - 1$. Then

$$\begin{aligned}
& D_l^{-1} + E_{l-1}^l C_{l-1}^{-1} E_{l-1}^T = \\
&= D_l^{-1} + \sum_{k=0}^{l-1} E_{l-1}^l E_k^{l-1} D_k^{-1} (E_k^{l-1})^T (E_{l-1}^l)^T \\
&= D_l^{-1} + \sum_{k=0}^{l-1} E_k^l D_k^{-1} (E_k^l)^T \\
&= C_l^{-1}
\end{aligned}$$

□

The computational complexity $CPU(C_l^{-1})$ can be estimated from (17). The operations D_l^{-1}, E_{l-1}^l and $(E_{l-1}^l)^T$ are all of linear complexity $O(N_l)$. Thus

$$CPU(C_l^{-1}) = O(N_l) + CPU(C_{l-1}^{-1}).$$

If the number of unknowns grows geometrically (i.e. $N_l = O(\beta^l)$ with $\beta > 1$), one obtains *optimal !* complexity

$$CPU(C_l^{-1}) = O(N_l).$$

For the multiplicative version we define C_l^{-1} per recursion as follows: $C_0^{-1} = D_0^{-1}$, and $C_l^{-1} : d_l \to w_l$ is defined by the algorithm:

$$\left.\begin{array}{rcl} w_l^0 &=& E_{l-1}^l C_{l-1}^{-1} (E_{l-1}^l)^T d_l \\ w_l &=& w_l^0 + D_l^{-1}(d_l - A_l w_l^0) \end{array}\right\} \tag{18}$$

Theorem 17. *The iteration defined in (16) can be written as*

$$M = I - C_L^{-1} A_L,$$

where C_L is defined by (18).

Proof. One step of the iteration (16) is $m_l = I - E_l^L D_l^{-1} (E_l^L)^T A_L$, and, by recursion, we define

$$\begin{array}{rcl} M_0 &=& m_0 \\ M_l &=& m_l M_{l-1} \end{array}$$

The multiplicative iteration defined in (16) is $M = M_L$. The operation C_l^{-1} defined in (18) is

$$\begin{array}{rcl} C_l^{-1} &=& E_{l-1}^l C_{l-1}^{-1} (E_{l-1}^l)^T + D_l^{-1}(I - A_l E_{l-1}^l C_{l-1}^{-1} (E_{l-1}^l)^T) \\ &=& (I - D_l^{-1} A_l) E_{l-1}^l C_{l-1}^{-1} (E_{l-1}^l)^T + D_l^{-1} \end{array}$$

Now, we proof by induction

$$M_l = I - E_l^L C_l^{-1} (E_l^L)^T A_L.$$

Assume, the relation is true for $l-1$. Then

$$\begin{array}{rcl} I - E_l^L C_l^{-1} (E_l^L)^T A &=& I - E_l^L \left[(I - D_l^{-1} A_l) E_{l-1}^l C_{l-1}^{-1} (E_{l-1}^l)^T + D_l^{-1} \right] (E_l^L)^T A_L \\ &=& I - E_l^L D_l^{-1} (E_l^L)^T A_L - E_{l-1}^L C_{l-1}^{-1} (E_{l-1}^L)^T A_L \\ && \quad + E_l^L D_l^{-1} (E_l^L)^T A_L E_{l-1}^L C_{l-1}^{-1} (E_{l-1}^L)^T A_L \\ &=& (I - E_l^L D_l^{-1} (E_l^L)^T A_L)(I - E_{l-1}^L C_{l-1}^{-1} (E_{l-1}^L)^T A_L) \\ &=& m_l M_{l-1} \\ &=& M_L \end{array}$$

$\square$

Other versions of multigrid cycles can be computed similarly to the above V-cycle with post-smoothing. A symmetric V-cycle with pre-smoothing and post-smoothing is defined as:

$$\begin{aligned} w_l^0 &= D_l^{-1} d_l \\ w_l^1 &= w_l^0 + E_{l-1}^l C_{l-1}^{-1} (E_{l-1}^T)(d_l - A_l w_l^0) \\ w_l &= w_l^1 + D_l^{-1}(d_l - A_l w_l^1) \end{aligned}$$

One can run several steps of the smoothing iterations.

4.2 Analysis of the additive multi-level method

The additive multi-level method is an ASM method with the set of triples

$$\{(V_l, id, D_l(.,.))\}$$

Thus, the norm generated by the preconditioner is exactly the splitting norm

$$\|u\|_C^2 = |||u|||^2 = \inf_{\substack{u=\sum v_l \\ v_l \in V_l}} \sum_{l=0}^{L} \|w_l\|_{D_l}^2$$

What is the D_l-norm ? For the bilinear-form $A(u,v) = (\nabla u, \nabla v)$, and D_l is a Jacobi preconditioner, it is the corresponding splitting norm

$$\|u_l\|_{D_l}^2 = \inf_{\substack{u=\sum v_i \\ v_i \in \text{span}\{\varphi_i\}}} \sum \|v_i\|_A^2 \simeq \inf_{\substack{u=\sum v_i \\ v_i \in \text{span}\{\varphi_i\}}} \sum h_{l,i}^{-2} \|v_i\|_{L_2}^2.$$

One verifies that this local norm is equivalent to the global L_2-norm, i.e.

$$\|u_l\|_{D_l}^2 \simeq \|h_l^{-1} u_l\|_{L_2}^2. \tag{19}$$

Lemma 18. *For the additive multi-level preconditioner C with Jacobi smoothers there holds the following norm equivalence:*

$$\|u\|_C^2 \simeq \inf_{\substack{u=\sum v_l \\ v_l \in V_l}} \sum_{l=0}^{L} \|h_l^{-1} v_l\|_{L_2}^2. \tag{20}$$

Next, we will investigate the bounds of the norm estimates $\lambda_1 \|u\|_A^2 \leq \|u\|_C^2 \leq \lambda_2 \|u\|_C^2$. Especially, we are interested in the (in)dependency of the number of levels L.

Lemma 19. *Assume that Ω is convex, and the triangulation is quasi-uniform. Then there holds*

$$\|u\|_C^2 \preceq \|u\|_A^2$$

The proof is given in Theorem 6. It used the A-orthogonal decomposition $v_l := (P_l^A - P_{l-1}^A)u$.

Lemma 20. *Assume that the family of triangulations is shape-regular. Further, assume $h_l \simeq h_{l-1}$. Then*

$$\|u\|_C^2 \preceq L\,\|u\|_A^2$$

Proof. We choose $w_0 = I_0 u$, and $w_l = (I_l - I_{l-1})u$ for $1 \leq l \leq L$, where I_l is a Clément type quasi-interpolation operator. Then

$$\|h_l^{-1} w_l\|_0 \preceq \|h_l^{-1}(u - I_l u)\|_0 + \|h_l^{-1}(u - I_{l-1}u)\|_0 \preceq \|u\|_A^2.$$

If we consider h_0 to be a constant, then $\|h_0^{-1} w_0\|_0 \preceq \|u\|_A$. □

Remark: If the coarse grid $\mathcal{T}_0$ is already fine, i.e., it is not appropriate to consider h_0 to be a constant, one should use $D_0(.,.) = A(.,.)$. Then, h_0 does not enter the generic *constant* c.

Lemma 21. *There holds*

$$\|u\|_A^2 \preceq L\|u\|_C^2$$

Proof. Follows from Lemma 9. Since $\|u_l\|_A \preceq \|u_l\|_{D_l}$ implies $g_{ij} \preceq 1$, and $G \in \mathbb{R}^{L\times L}$, the spectral radius $\rho(G)$ is bounded by cL. □

The above, quite simple, norm estimates depend on the number of levels L. This might be acceptable for the analysis of preconditioners, since, in practice, the number of levels is not too large (maybe, 5 to 10). But, it is not optimal. An improved analysis can remove the factors L in both estimates. This allows to push the number of levels to infinity, and prove theorems about H^1.

Theorem 22. *Let $\{\mathcal{T}_l\}$ be a familiy of quasi-uniform triangulations of mesh-sizes $h_l \simeq 2^{-l}$. Let V_l be the piece-wise linear finite element space. Then there holds*

$$\|u\|_A^2 \preceq \|u\|_C^2 \qquad \forall\, v \in V_L. \tag{21}$$

Proof. We will establish the sharper estimate for the coefficients of the interaction matrix G:

$$g_{ij} = \sup_{\substack{u\in V_i\\ v\in V_j}} \frac{A(u,v)}{\|u\|_{D_i}\|v\|_{D_j}} \preceq \gamma^{|i-j|} \tag{22}$$

for some $\gamma \in (0,1)$. Then, the row sum (and thus, the spectral radius), is bounded independently of L:

$$\sum_j g_{ij} \leq \sum_{j\in Z} \gamma^{|i-j|} \leq 2\frac{1}{1-\gamma}.$$

Assume that $i < j$, and choose $u_i \in V_i$ and $v_j \in V_j$. Define the union of edges at the coarser grid

$$\mathcal{E}_i = \cup \partial T : T \in \mathcal{T}_i.$$

First, we verify

$$A(u_i, w) = 0 \quad \forall\, w \in V \text{ s.t. } w = 0 \,\text{on}\, \mathcal{E}_i$$

by integration by parts: $(\nabla u_i, \nabla w) = \sum_T (-\Delta u, w)_{L_2(T)} + (\partial_n u, w)_{L_2(\partial T)} = 0$.

Next, define the fine-grid finite element function $\tilde{v}_j \in V_j$ as

$$\tilde{v}_j(x) = \begin{cases} v_j(x) & x \text{ a vertex in } \mathcal{E}_i \\ 0 & x \text{ a vertex not in } \mathcal{E}_i. \end{cases}$$

Since $v_j - \tilde{v}_j = 0$ on $\mathcal{E}_i$, there holds

$$A(u_i, v_j) = A(u_i, \tilde{v}_j).$$

Define the strip

$$S_{ij} = \cup T_j : T_j \in \mathcal{T}_j \text{ and } T_j \cap \mathcal{E}_i \neq \emptyset.$$

There holds

$$|S_{ij} \cap T_i| \leq c 2^{j-i} |T_i| \qquad \forall\, T_i \in \mathcal{T}_i.$$

Using the above observations, and the fact that $\nabla u_i = \textit{const}$ on each T_i, we estimate

$$\begin{aligned}
A(u_i, v_j) &= A(u_i, \tilde{v}_j) = \sum_{T \in \mathcal{T}_i} (\nabla u_i, \nabla \tilde{v}_j)_{L_2(S_{ij} \cap T)} \\
&\leq \{\sum_T \|\nabla u_i\|^2_{L_2(S_{ij} \cap T)}\}^{1/2} \{\sum_T \|\nabla \tilde{v}_j\|^2_{L_2(S_{ij} \cap T)}\}^{1/2} \\
&\preceq 2^{(i-j)/2} \{\sum_T \|\nabla u_i\|^2_{L_2(T)}\}^{1/2} \{\sum_T \|\nabla \tilde{v}_j\|^2_{L_2(S_{ij} \cap T)}\}^{1/2} \\
&\preceq \gamma^{|i-j|} \|\nabla u_i\|_{L_2} \|\nabla \tilde{v}_j\|_{L_2} \\
&\preceq \gamma^{|i-j|} h_i^{-1} \|u_i\|_{L_2} h_j^{-1} \|\tilde{v}_j\|_{L_2} \\
&\preceq \gamma^{|i-j|} h_i^{-1} \|u_i\|_{L_2} h_j^{-1} \|v_j\|_{L_2} \\
&\simeq \gamma^{|i-j|} \|u_i\|_{D_i} \|v_j\|_{D_j}
\end{aligned}$$

□

For the reverse estimate, $\|u\|_C \preceq \|u\|_A$, we will improve the estimates onto the decomposition $\sum h_l^{-2} \|(I_l - I_{l-1})u\|^2_{L_2}$. Some of the terms will depend more on the smooth parts of u, while other terms will depend more on the high frequency part. The idea is similar to Fourier decomposition of u.

We define the so-called K-functionals, $K_\Omega : \mathbb{R}^+ \times H^1(\Omega) \to \mathbb{R}$ as

$$K_\Omega(t, u) = \inf_{v \in H^2(\Omega)} \{\|u - v\|^2_{L_2(\Omega)} + t^2 \|v\|^2_{H^2(\Omega)}\}^{1/2}$$

For rough functions $u \in L_2$, there is the trivial bound $K(t,u) \le \|u\|_{L_2}$, and for smooth functions, there tholds $K(t,u) \le t\|u\|_{H^2}$. The asymptotic decay as $t \to 0$ is a measure of smoothness.

Let $I_l : L_2 \to V_l$ be quasi-interpolation operators preserving locally linear polynomials. Since for arbitrary $v \in H^2$ there holds

$$\begin{aligned} \|(I_l - I_{l-1})u\|_{L_2}^2 &\preceq \|(I_l - I_{l-1})(u-v)\|_{L_2}^2 + \|(I_l - I_{l-1})v\|_{L_2}^2 \\ &\preceq \|u-v\|_{L_2}^2 + h_l^4 \|v\|_{H^2}^2, \end{aligned}$$

if follows that

$$\|(I_l - I_{l-1})u\|_{L_2}^2 \preceq K(h_l^2, u)^2,$$

and, after summation,

$$\|u\|_C^2 \preceq \sum h_l^{-2} \|(I_l - I_{l-1})u\|_{L_2}^2 \preceq \sum h_l^{-2} K(h_l^2, u)^2.$$

Lemma 23. *Let Ω be a Lipschitz domain. For $\gamma \in \mathbb{R}^+ \setminus \{1\}$ there holds*

$$\sum_{l \in \mathbb{Z}} \gamma^{-l} K(\gamma^l, u)^2 \preceq \|\nabla u\|_{L_2}^2. \tag{23}$$

Proof. First, we verify estimate (23) for the domain $\Omega = \mathbb{R}^d$ by Fourier analysis. Let

$$\hat{u}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}^d} e^{i\xi \cdot x} u(x)\, dx.$$

Then $\|u\|_{L_2} = \|\hat{u}\|_{L_2}$, $\|\nabla u\|_{L_2} = \||\xi|\hat{u}\|_{L_2}$, etc. The K-functional is

$$K(t,u) = \inf_{v \in H^2} \{\|\hat{u} - \hat{v}\|_{L_2}^2 + t^2 \||\xi|^2 \hat{v}\|_{L_2}^2\}^{1/2}.$$

The global optimization splits into the one dimensional, quadratic minimization problems $|\hat{u}(\xi) - \hat{v}(\xi)|^2 + t^2|\xi|^4|\hat{v}|^2$, which solution is taken at

$$\hat{v}(\xi) = \frac{1}{1 + t^2|\xi|^4} \hat{u}(\xi),$$

and takes the value

$$\frac{t^2|\xi|^4}{1 + t^2|\xi|^4} |\hat{u}(\xi)|^2.$$

Integrating ξ over $\mathbb{R}^d$, one obtains

$$K(t,u)^2 = \int_{\mathbb{R}^d} \frac{t^2|\xi|^4}{1 + t^2|\xi|^4} \, |\hat{u}(\xi)|^2 \, d\xi.$$

The quantity of interest is

$$\begin{aligned}\sum_{l\in\mathbb{Z}} \gamma^{-l} K(\gamma^l,u)^2 &= \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}^d} \frac{\gamma^l|\xi|^4}{1+\gamma^{2l}|\xi|^4}\,|\hat{u}(\xi)|^2\,d\xi \\ &\le \sup_{\xi\in\mathbb{R}^d}\left\{\sum_{l\in\mathbb{Z}} \frac{\gamma^l|\xi|^2}{1+\gamma^{2l}|\xi|^4}\right\}\int |\xi|^2|\hat{u}(\xi)|^2\,d\xi.\end{aligned}$$

The second factor is exactly $\|\nabla u\|^2$, the first factor is bounded by a constant. To prove this, let $l_0\in\mathbb{R}$ such that $\gamma^{-l_0}=|\xi|^2$. Assume $|\gamma|>1$. Then, the first factor is

$$\sum_{l\in\mathbb{Z}} \frac{\gamma^{l-l_0}}{1+\gamma^{2(l-l_0)}} = \sum_{l\in\mathbb{Z}} \frac{1}{\gamma^{l_0-l}+\gamma^{l-l_0}} \le \sum_{l>l_0} \frac{1}{\gamma^{l-l_0}} + \sum_{l\le l_0} \frac{1}{\gamma^{l_0-l}} \le \frac{2\gamma}{\gamma-1}$$

Thus, we have proved

$$\sum_{l\in\mathbb{Z}} \gamma^{-l} K_{\mathbb{R}^d}(\gamma^l,u)^2 \preceq \|\nabla u\|^2_{L_2(\mathbb{R}^d)} \tag{24}$$

We are left to prove estimate (23) on the Lipschitz domain Ω. Let $E: H^1(\Omega)\to H^1(\mathbb{R}^d)$ be a continous extension operator (which is available for Lipschitz domains). From

$$\begin{aligned}K_\Omega(t,u) &= \inf_{v\in H^2(\Omega)} \{\|u-v\|^2_{L_2(\Omega)} + t^2\|v\|^2_{H^2(\Omega)}\}^{1/2} \\ &\le \inf_{v\in H^2(\mathbb{R}^d)} \{\|Eu-v\|^2_{L_2(\mathbb{R}^d)} + t^2\|v\|^2_{H^2(\mathbb{R}^d)}\}^{1/2} \\ &= K_{\mathbb{R}^d}(t,Eu),\end{aligned}$$

together with (24), there follows estimate (23). □

Theorem 24. *On Lipschitz domains Ω, spaces $V=H^1(\Omega)$, and quasi-uniform triangulations $\{\mathcal{T}_l\}$ of mesh-sizes $h_l=2^{-l}$, there holds the norm estimate*

$$\|u\|_C \preceq \|u\|_A \qquad \forall\, u\in V_L.$$

Proof. Follos immediately form the collected results above:

$$\|u\|_C^2 \preceq \sum_{l=0}^{L} h_l^{-2}\|(I_l-I_{l-1})u\|^2_{L_2} \preceq \sum_{l=0}^{L} h_l^{-2}K(h_l^2,u)^2 \preceq \|\nabla u\|^2_{L_2(\Omega)}.$$

□

4.3 Analysis of the multigrid V-cycle

In this section, we analyse the multiplicative version of the multi-level iteration. This is the popular V-cycle multigrid iteration.

Theorem 25. *Assume that there exists s.p.d. bilinear forms $\widetilde{D}_l(.,.) : V_l \times V_l \to \mathbb{R}$ sucht that*

- *The smoothers are properly scaled:*
$$\|v_i\|_A \leq \|v_i\|_{D_i} \qquad \forall\, v_i \in V_i \tag{25}$$
- *The smoothers are bounded by the forms $\widetilde{D}_l(.,.)$:*
$$\|v_i\|_{D_i} \preceq \|v_i\|_{\widetilde{D}_i} \qquad \forall\, v_i \in V_i \tag{26}$$
- *There exists $\gamma \in (0,1)$ such that*
$$A(u_i, v_j) \preceq \gamma^{|i-j|} \|u_i\|_A \, \|v_j\|_{\widetilde{D}_j} \qquad \forall\, u_i \in V_i \,\forall\, v_j \in V_j \; 0 \leq i \leq j \leq L \tag{27}$$
- *The lower bound of the ASM preconditioner with $\widetilde{D}_l(.,.)$ is uniform in L:*
$$\|u_L\|_A^2 \preceq \sum_{i=0}^{L} A(\widetilde{D}_i^{-1} A u_L, u_L) \qquad \forall\, u_L \in V_L \tag{28}$$

Then the convergence rate of the multigrid V-cycle is independent of the number of levels, i.e.
$$\|(I - D_L^{-1}A) \ldots (I - D_1^{-1}A)(I - D_0^{-1}A)\|_A \leq C, \tag{29}$$
with $C \in (0,1)$ independent of L.

Some comments:

- The idea of introducing $\widetilde{D}_l$ is to compare the smoother D_l with a simple smoother $\widetilde{D}_l$.
- Assumption (27) is proven as in the proof of Theorem 22. One inverse inequality is skipped.
- Condition (28) follows form $C_{ASM} \preceq A$, which implies $A(C_{ASM}^{-1} A u, u) \geq A(u,u)$.

Proof. We define per induction

$$\begin{aligned} M_{-1} &= I \\ M_i &= (I - D_i^{-1} A) M_{i-1} \quad 0 \leq i \leq L. \end{aligned}$$

The goal is to estimate

$$A(M_L u, M_L u) \leq C A(u,u),$$

with $C \in (0,1)$. This is equivalent to

$$A(u,u) \preceq A(u,u) - A(M_L u, M_L u). \tag{30}$$

The inductive definition of M_l immediately gives (for $0 \leq l \leq L$)

$$M_{l-1} - M_l = D_l^{-1} A M_{l-1},$$

and

$$\begin{aligned} & A(M_{l-1}u, M_{l-1}u) - A(M_l u, M_l u) \\ = \;& A((2D_l^{-1}A - D_l^{-1}AD_l^{-1}A)M_{l-1}u, M_{l-1}u) \\ \geq \;& A(D_l^{-1}AM_{l-1}u, M_{l-1}u). \end{aligned} \tag{31}$$

The last inequality follows from $A \leq D_l$.

Using the ASM estimate

$$A(u,u) \preceq \sum_{l=0}^{L} A(\widetilde{D}_l^{-1}Au, u),$$

rewriting (30) as telescopic sum and substituting (31),

$$A(u,u) - A(M_L u, M_L u) = \sum_{l=0}^{L} A(M_{l-1}u, M_{l-1}u) - A(M_l u, M_l u) \geq \sum_{l=0}^{L} A(D_l^{-1}AM_{l-1}u, M_{l-1}u),$$

reduces the proof to verify

$$\sum_{l=0}^{L} A(\widetilde{D}_l^{-1}Au, u) \preceq \sum_{l=0}^{L} A(D_l^{-1}AM_{l-1}u, M_{l-1}u). \tag{32}$$

The left hand side is bounded by ($(a+b)^2 \leq 2(a^2+b^2)$):

$$\sum_{l=0}^{L} A(\widetilde{D}_l^{-1}Au, u) \leq 2\sum_{l=0}^{L} A(\widetilde{D}_l^{-1}AM_{l-1}u, M_{l-1}u) + 2\sum_{l=0}^{L} A(\widetilde{D}_l^{-1}A(I - M_{l-1})u, (I - M_{l-1})u).$$

The first term is simply bounded by $D_l^{-1} \preceq D_l$. The key point to handle the second term is to bound the interaction of the correction on level l, with smoothing on coarser levels. We telescope $I - M_{l-1}$, namely

$$I - M_{l-1} = M_{-1} - M_{l-1} = \sum_{j=0}^{l-1} M_{j-1} - M_j = \sum_{j=0}^{l-1} D_j^{-1}AM_{j-1}.$$

Inserting this expansion into the second term above, we obtain

$$\sum_{l=0}^{L} A(\widetilde{D}_l^{-1}A(I-M_{l-1})u,(I-M_{l-1})u) = \sum_{l=0}^{L}\sum_{j=0}^{l-1}\sum_{k=0}^{l-1} A(\widetilde{D}_l^{-1}AD_j^{-1}AM_{j-1}u, D_k^{-1}AM_{k-1}u).$$

To simplify the notation, we introduce $w_j := D_j^{-1}AM_{j-1}u \in V_j$.

The next, intermediate step is to bound

$$A(\widetilde{D}_l^{-1}Aw_j, w_k) \preceq \gamma^{l-j}\|w_j\|_A\gamma^{l-k}\|w_k\|_A.$$

This follows by Cauchy-Schwarz w.r.t. the spd form $A(\widetilde{D}_l^{-1}A.,.)$, and assumption (25)

$$A(\underbrace{\widetilde{D}_l^{-1}Aw_j}_{\in V_l}, \underbrace{w_j}_{\in V_j}) \preceq \gamma^{|l-j|}\|\widetilde{D}_l^{-1}Aw_j\|_{\widetilde{D}_l}\,\|w_j\|_A = \gamma^{|l-j|}A(\widetilde{D}_l^{-1}Aw_j,w_j)^{1/2}\,\|w_j\|_A,$$

and dividing one factor completes the step.

We continue

$$\begin{aligned}
\sum_{l=0}^{L} A(\widetilde{D}_l^{-1}A(I-M_{l-1})u,(I-M_{l-1})u) &= \sum_{l=0}^{L}\sum_{j=0}^{l-1}\sum_{k=0}^{l-1} A(\widetilde{D}_l^{-1}Aw_j, w_k)\\
&\preceq \sum_{l=0}^{L}\sum_{j=0}^{l-1}\sum_{k=0}^{l-1}\gamma^{l-j}\gamma^{l-k}\|w_j\|_A\|w_k\|_A\\
&\leq 1/2\sum_{l=0}^{L}\sum_{j=0}^{l-1}\sum_{k=0}^{l-1}\gamma^{l-j}\gamma^{l-k}\{\|w_j\|_A^2+\|w_k\|_A^2\}\\
&= \sum_{l=0}^{L}\sum_{j=0}^{l-1}\sum_{k=0}^{l-1}\gamma^{l-j}\gamma^{l-k}\|w_j\|_A^2\\
&\leq \sum_{l=0}^{L}\sum_{j=0}^{l-1}\sum_{k=0}^{l-1}\gamma^{l-j}\gamma^{l-k}\|w_j\|_{D_j}^2\\
&\preceq \sum_{j=0}^{L-1}\sum_{l=j+1}^{L}\gamma^{l-j}\|w_j\|_{D_j}^2\\
&\preceq \sum_{j=0}^{L-1}\|w_j\|_{D_j}^2 = \sum_{j=0}^{L-1}\|D_j^{-1}AM_{j-1}u\|_{D_j}^2\\
&= \sum_{j=0}^{L-1}A(D_j^{-1}AM_{j-1}u, M_{j-1}u)
\end{aligned}$$

This is the right hand side of (32), and thus, the proof is complete. □

5 Multigrid analysis based on Smoothing and Approximation

A different approach to multigrid analysis is the Hackbusch theory based on the smoothing property and the approximation property. Here, just the interaction of two levels is explored.

Consider a multiplicative 2-level method. One step is the coarse grid correction

$$(I - P_{l-1}^A),$$

the other one consists of m smoothing steps

$$(I - D_l^{-1} A)^m.$$

If D_l is properly scaled, then both steps are non-expansive. We will show that under regularity assumptions the product is a contraction. We consider second order problems. There holds $\|.\|_A \simeq \|.\|_{H^1}$, and $\|.\|_D \simeq h^{-1}\|.\|_{L_2}$.

Lemma 26 (Approximation property). *If the underlying pde provides the full regularity shift theorem ($\|u\|_{H^2} \preceq \|f\|_{L_2}$), then the approximation property is fulfilled:*

$$\|u_l - P_{l-1}^A u_l\|_{D_l} \preceq \|u_l\|_A.$$

Proof: Follows from Aubin Nitsche lemma, and the scaling of the smoother $\|u_l\|_{D_l} \leq h_l^{-1}\|u_l\|$. □

The coarse grid correction step is measured in $\|.\|_{A\to D}$. Accordingly, the smoothing steps are measured in $\|.\|_{D\to A}$:

Lemma 27 (Smoothing property). *Assume that $\sigma(D_l^{-1}A) \subset [0,1]$. Then*

$$\|(I - D_l^{-1}A)^m u_l\|_A^2 \leq \frac{1}{2m}\|u_l\|_{D_l}^2 \tag{33}$$

Proof. The estimate is rewritten as

$$(D^{-1}A(I - D_l^{-1}A)^{2m} u_l, u_l)_D \leq \frac{1}{2m}\|u_l\|_D^2.$$

Since $D_l^{-1}A$ is self-adjoint in $(.,.)_D$, so is also $D^{-1}A(I - D_l^{-1}A)^{2m}$, and one can apply spectral theory:

$$\|D^{-1}A(I - D_l^{-1}A)^{2m}\|_D \leq \sup_{a\in\sigma(D_l^{-1}A)} a(1-a)^{2m}.$$

The maximum of $a(1-a)^{2m}$ on $[0,1]$ is attained at $\overline{a} = 1/(1+2m)$, and is less than $1/(2m)$. □

5.1 V-cycle analysis with full regularity

The V-cylce analysis needs the stronger full-regularity assumption than the multi-level type analysis, but, the result here is also stronger: More smoothing steps improve the rate of convergence. The following theorem is due to Braess-Hackbusch:

Theorem 28. *Assume that*

$$\|u_l - P_{l-1}^A u_l\|_{D_l}^2 \leq C\|u_l\|_A^2 \qquad \forall\, u_l \in V_l \;\; 1 \leq l \leq L. \tag{34}$$

Then

$$\|(I - D_0^{-1}A)^m \dots (I - D_L^{-1}A)^m\|_A^2 \leq \frac{C}{C + 2m}$$

Proof. Define $S_l = I - D_l^{-1}A$ and

$$M_0 = I \qquad \text{and} \qquad M_l = M_{l-1}S_l^m$$

Observe that

$$D_l^{-1}Av = 0 \qquad \forall\, v \in V_L : v \perp_A V_l,$$

and thus

$$M_l(I - P_l^A) = (I - P_l^A)$$

Now, we prove by induction

$$A(M_l u_l, M_l u_l) \leq \delta A(u_l, u_l) \qquad \forall\, u_l \in V_l,$$

where $\delta = C/(C + 2m)$. The hypothesis is true for $l = 0$ since $D_0(.,.) = A(.,.)$. Now, assume that the hypothesis is true for $l - 1$. Then

$$\begin{aligned}
\|M_l u\|_A^2 &= \|M_{l-1}S_l^m u\|_A^2 = \|M_{l-1}(I - P_{l-1}^A + P_{l-1}^A)S_l^m u\|_A^2 \\
&= \|M_{l-1}(I - P_{l-1}^A)S_l^m u\|_A^2 + \|M_{l-1}P_{l-1}S_l^m u\|_A^2 \\
&\quad +2(M_{l-1}(I - P_{l-1}^A)S_l^m u, M_{l-1}P_{l-1}S_l^m u)_A \\
&= \|(I - P_{l-1}^A)S_l^m u\|_A^2 + \|M_{l-1}P_{l-1}S_l^m u\|_A^2.
\end{aligned}$$

The last step is due to $M_{l-1} : V_{l-1} \to V_{l-1}$, and $M_{l-1}(I - P_{l-1}^A) = (I - P_{l-1}^A)$. We continue by using the induction hypothesis

$$\begin{aligned}
\|M_l u\|_A^2 &\leq \|(I - P_{l-1}^A)S_l^m u\|_A^2 + \delta\|P_{l-1}S_l^m u\|_A^2 \\
&= (1 - \delta)\|(I - P_{l-1})S_l^m u\|_A^2 + \delta\|S_l^m u\|_A^2
\end{aligned}$$

Next, we establish the smoothing+approximation result

$$\|(I - P_{l-1})S_l^m u\|_A^2 \leq \frac{C}{2m}(\|u\|_A^2 - \|S_l^m u\|_A^2) :$$

$$
\begin{aligned}
\|(I-P_{l-1})S_l^m u\|_A^2 &= D_l((I-P_{l-1})S_l^m u, D_l^{-1}AS_l^m u) \\
&\le \|(I-P_{l-1})S_l^m u\|_{D_l}\|D_l^{-1}AS_l^m u\|_{D_l} \\
&\le \sqrt{C}\|(I-P_{l-1})S_l^m u\|_A\|D_l^{-1}AS_l^m u\|_{D_l}.
\end{aligned}
$$

We used that $(I-P_{l-1})$ is a projector, and assumption (34). Dividing one factor gives

$$
\|(I-P_{l-1})S_l^m u\|_A^2 \le C\|D_l^{-1}AS_l^m u\|_{D_l}^2.
$$

Again, using spectral theory $(a^2(1-a)^{2m} \le (2m)^{-1}(a-a(1-a)^{2m})$ we obtain

$$
\|D_l^{-1}AS_l^m u\|_{D_l} \le \frac{1}{2m}(\|u\|_A^2 - \|S_l^m u\|_A^2),
$$

and thus the statement. We conclude the proof by

$$
\begin{aligned}
\|M_l u\|_A^2 &\le (1-\delta)\frac{C}{2m}(\|u\|_A^2 - \|S^l u\|_A^2) + \delta\|S_l u\|_A^2 \\
&= \frac{C(1-\delta)}{2m}\|u\|_A^2 + (\delta - \frac{C(1-\delta)}{2m})\|S_l^u\|_A^2 \\
&= \delta\|u\|_A^2.
\end{aligned}
$$

□

6 Multigrid analysis with partial regularity

The goal of this chapter is multigrid analysis under partial regularity. We will consider the variable V-cycle, and the classical W-cycle. The results are weaker in comparison to Section 4.3 since the spaces and/or forms are not necessarily nested.

6.1 Interpolation spaces

Let V_0 and V_1 be two Hilbert spaces with compact embedding $V_1 \subset V_0$. Then the eigen value problem

$$
(x_i, v)_{V_1} = \lambda_i^2 (x_i, v)_{V_0} \qquad \forall\, v \in V_1 \tag{35}
$$

leads to a sequence of eigen values $\lambda_i \to \infty$. The eigen vectors are normalized in $\|\cdot\|_{V_0}$. They form an orthonormal basis in V_0, and an orthogonal basis in V_1. There holds for $u \in V_0$

$$
u = \sum_{i=0}^{\infty} (u, x_i)_{V_0} x_i,
$$

and

$$
\|u\|_{V_0}^2 = \sum_{i=0}^{\infty} (u, x_i)_{V_0}^2
$$

If $u \in V_1$, then

$$\|u\|_{V_1}^2 = \sum_{i=0}^{\infty} \lambda_i^2 (u, x_i)_{V_0}^2.$$

For $s \in (0,1)$, we define the interpolation norm (Hilbert space interpolation)

$$\|u\|_s^2 := \|u\|_{[V_0,V_1]_s}^2 := \sum_{i=1}^{\infty} \lambda_i^{2s} (u, x_i)_{V_0}^2, \tag{36}$$

and the (Hilbert) space

$$V_s := [V_0, V_1]_s := \{v \in V_0 : \|v\|_s < \infty\}.$$

It is interesting to consider operators between spaces and interpolation spaces:

Theorem 29. *Let $V_1 \subset V_0$ (compact) and $W_1 \subset W_0$ (compact). Let $T : V_0 \to W_0$, linear, with norm $\|T\|_0 = \|T\|_{V_0 \to W_0}$, as well as $T : V_1 \to W_1$ with norm $\|T\|_1 = \|T\|_{V_1 \to W_1}$. Then*

$$T : V_s \to W_s,$$

and

$$\|T\|_{V_s \to W_s} \leq \|T\|_0^{1-s} \|T\|_1^s.$$

The proof will use the real method of interpolation and is given below.

A different approach to interpolation spaces is the so called *real method of interpolation* based on K-functionals. This is defined for Banach spaces. Let $B_1 \subset B_0$ be Banach spaces. The K-functional $K : \mathbb{R}^+ \times B_0 \to \mathbb{R}$ is defined as

$$K(t,u) := \inf_{\substack{u=u_0+u_1 \\ u_i \in V_i}} \{\|u_0\|_{B_0}^2 + t^2 \|u_1\|_{B_1}^2\}^{1/2}.$$

Note that $K(t,u) \leq \|u\|_{B_0}$, and $K(t,u) \leq t\|u\|_{B_1}$ for $u \in B_1$. The decay of $K(t,u)$ for $t \to 0$ can be used to measure the smoothness of u. One possibility is to define the interpolation norm is

$$\|u\|_{B_{s,\infty}} := \sup_{t>0} t^{-s} K(t,u).$$

Other weightings are possible ($1 \leq p < \infty$):

$$\|u\|_{B_{s,p}} := \Big(\int_0^\infty t^{-sp} K^p(t,u) \, dt/t \Big)^{1/p}$$

Note that $K(t,u) \simeq K(ct,u)$, there holds for fixed $\gamma \in \mathbb{R}^+ \setminus \{1\}$

$$\|u\|_{B_{s,p}} = \Big(\sum_{k \in \mathbb{Z}} \int_{\gamma^k}^{\gamma^{k+1}} t^{-sp} K^p(t,u) \, dt/t \Big)^{1/p} \simeq \sum_{k \in \mathbb{Z}} (\gamma^{-sk} K(\gamma^k, u))^p \Big)^{1/p}$$

This interpolation norm has been used already in Section 4.2 with $s = 1/2$ and $p = 2$.

If B_i are Hilbert spaces, and $p = 2$, then both methods of interpolation coincide:

Lemma 30. *Let $V_1 \subset V_0$ (compact) be Hilbert spaces. Define $\|u\|_s$ as Hilbert space interpolation norm (36). Then*

$$\|u\|_s = C_s \|u\|_{B_{s,2}}$$

with $C_s = (\int_0^\infty (t^{1-2s})/(t^2+1)\,dt)^{-1/2} = \sqrt{2/\pi \sin(\pi s)}$.

Proof. There is

$$K(t,u)^2 = \inf_{u_1 \in V_1} \{\|u-u_1\|_{V_0}^2 + t^2 \|u_1\|_{V_1}^2\}.$$

With $u = \sum a_i x_i$ and $u_1 = \sum b_i x_i$ there holds

$$\|u-u_1\|_{V_0}^2 + \|u_1\|_{V_1}^2 = \sum_i [(a_i - b_i)^2 + t^2 \lambda_i^2 b_i^2].$$

We choose b_i to minimize $(a_i - b_i)^2 + t^2\lambda_i^2 b_i^2$, which is $b_i = a_i(t^2\lambda_i^2+1)^{-1}$. Hence

$$K^2(t,u) = \sum_{i=1}^\infty t^2 \lambda_i (t^2\lambda_i^2 + 1)^{-1} a_i^2.$$

Now,

$$\begin{aligned}
\int_0^\infty t^{-2s} K^2(t,u) dt/t &= \sum_i \int_0^\infty t^{1-2s} \lambda_i^2 (t^2\lambda_i^2+1)^{-1} dt a_i^2 \\
&= \sum_i \Big(\int_0^\infty t^{1-2s} \lambda_i^{1-2s} (t^2\lambda_i^2+1)^{-1} dt \lambda_i \Big) \lambda_i^{2s} a_i^2 \\
&= \sum_i \Big(\int_0^\infty \tau^{1-2s} (\tau^2+1)^{-1} d\tau \Big) \lambda_i^{2s} a_i^2 \\
&= C_s^{-2} \|u\|_s^2
\end{aligned}$$

□

Lemma 31. *Let $B_1 \subset B_0$ and $\widetilde{B}_1 \subset \widetilde{B}_0$ be Banach spaces. Let $T : B_i \to \widetilde{B}_i$, linear, with norm $C_i := \|T\|_i$. Then $T : [B_0, B_1]_{s,p} \to [\widetilde{B}_0, \widetilde{B}_1]_{s,p}$ with norm*

$$\|T\|_{s,p} \leq C_0^{1-s} C_1^s.$$

Proof. For $u = u_0 + u_1$, also $Tu_0 + Tu_1$ is a proper decomposition of Tu. Thus

$$\begin{aligned}
\|Tu\|_{B_{s,p}} &= \Big(\int_0^\infty t^{-sp} \widetilde{K}(t, Tu)^p dt/t \Big)^{1/p} \\
&\leq \Big(\int_0^\infty t^{-sp} \inf_{u=u_0+u_1} \{\|Tu_0\|_{\widetilde{B}_0}^2 + t^2 \|Tu_1\|_{\widetilde{B}_1}^2\}^{p/2} dt/t \Big)^{1/p} \\
&\leq \Big(\int_0^\infty t^{-sp} \inf_{u=u_0+u_1} \{C_0^2\|u_0\|_{B_0}^2 + t^2 C_1^2 \|u_1\|_{B_1}^2\}^{p/2} dt/t \Big)^{1/p} \\
&= C_0^{1-s} C_1^s \Big(\int_0^\infty (C_1 t/C_0)^{-sp} \inf_{u=u_0+u_1} \{\|u_0\|_{B_0}^2 + t^2 C_1^2/C_0^2 \|u_1\|_{B_1}^2\}^{p/2} dt/t \Big)^{1/p} \\
&= C_0^{1-s} C_1^s \Big(\int_0^\infty \tau^{-sp} \inf_{u=u_0+u_1} \{\|u_0\|_{B_0}^2 + \tau \|u_1\|_{B_1}^2\}^{p/2} d\tau/\tau \Big)^{1/p} \\
&= C_0^{1-s} C_1^s \|u\|_{B_{s,p}}.
\end{aligned}$$

□

The excursion to the real method of interpolation proofs Theorem 29:

Proof. of Theorem 29: Let $u \in V_s$. Then

$$\|Tu\|_s = C_s \|Tu\|_{B_{s,2}} \le C_s C_0^{1-s} C_1^s \|u\|_{B_{s,2}} = C_0^{1-s} C_1^s \|u\|_s$$

□

6.2 Finite element analysis in interpolation spaces

The fractional order Sobolev spaces $H^s = H^{k+\alpha}$ are interpolation spaces

$$H^{k+\alpha} = [H^k, H^{k+1}]_\alpha, \qquad \alpha \in (0,1).$$

A symmetric, second order elliptic problem $Lu = f$ on non-convex domains fulfulls (typically) a partial regularity shift theorem

$$\|u\|_{1+\alpha} \le \|f\|_{-1+\alpha}. \tag{37}$$

The rate of convergence in fractional order Sobolev spaces is obtained immediately by interpolation.

Lemma 32. *There holds the a priori estimate*

$$\|u - u_h\|_1 \preceq h^\alpha \|u\|_{1+\alpha}$$

Proof. Let $P_h : H^1 \to V_h$ be the energy projector into the finite element space. The coprojection $I - P_h$ is a linear operator from H^1 into H^1, and also from H^2 into H^1. The norms are

$$\|(I - P_h)\|_{H^1 \to H^1} \preceq 1,$$

and

$$\|(I - P_h)\|_{H^2 \to H^1} \preceq h.$$

Thus, there follows by interpolation

$$\|(I - P_h)\|_{H^{1+\alpha} \to H^1} \preceq h^\alpha.$$

□

Lemma 33 (Aubin-Nitsche technique in fractional Sobolev spaces). *Assume that the shift theorem (37) is available. Then there holds*

$$\|u - P_h u\|_{H^{1-\alpha}} \preceq h^\alpha \|u\|_{H^1}. \tag{38}$$

Proof. Pose the dual problem

$$a(\varphi, v) = g(v) \qquad \text{with} \qquad g(v) := (u - u_h, v)_{1-\alpha}.$$

There holds

$$\|g\|_{H^{-1+\alpha}} = \sup_{v \in H^{1-\alpha}} \frac{g(v)}{\|v\|_{1-\alpha}} = \sup_{v \in H^{1-\alpha}} \frac{(u - u_h, v)_{1-\alpha}}{\|v\|_{1-\alpha}} = \|u - u_h\|_{H^{1-\alpha}}.$$

Choose $v = u - u_h$ to obtain

$$\begin{aligned}
\|u - u_h\|_{1-\alpha}^2 &= a(\varphi - P_h\varphi, u - u_h) \leq \|(I - P_h)\varphi\|_1 \|u - u_h\|_1 \\
&\leq h^\alpha \|\varphi\|_{1+\alpha} \|u - u_h\|_1 \leq h^\alpha \|g\|_{-1+\alpha} \|u - u_h\|_1 \\
&= h^\alpha \|u - u_h\|_{1-\alpha} \|u - u_h\|_1.
\end{aligned}$$

□

6.3 Multigrid analysis with partial regularity assumption

The smoothing iteration $S_l = I - D_l^{-1} A : V_l \to V_l$ fulfills the norm estimates

$$\|S_l^m u\|_A \leq \|u\|_A$$

and

$$\|S_l^m u\|_A \leq \frac{1}{2m} \|u\|_{D_l}.$$

By interpolation, there follows (with $\alpha \in (0, 1)$):

$$\|S_l^m u\|_A \leq \frac{1}{(2m)^\alpha} \|u\|_{[A, D_l]_\alpha}.$$

Lemma 34. *There holds*

$$\|S_l^m u_l\|_A \preceq \frac{1}{(2m)^\alpha} h^{-\alpha} \|u_l\|_{H^{1-\alpha}}. \tag{39}$$

Proof. Note that, $(H^s, \|\cdot\|_{H^s})$ is defined as interpolation space between L_2 and H^1. The norm is different to the interpolation norm between $(V_l, \|\cdot\|_{L_2})$ and $(V_l, \|\cdot\|_{H^1})$. Let I_l be a Clément-type quasi-interpolation operator being a projection on V_l. Then

$$\|S_l^m I_l u\|_A \leq \|I_l u\|_A \preceq \|u\|_{H^1}$$

and

$$\|S_l^m I_l u\|_A \leq \frac{1}{2m} \|I_l u\|_D \preceq \frac{1}{2m} h_l^{-1} \|I_l u\|_{L_2} \preceq \frac{1}{2m} h_l^{-1} \|u\|_{L_2}$$

Now, using interpolation, there follows $\|S_l^m I_l u\|_A \leq \frac{1}{(2m)^\alpha} h^{-\alpha} \|u\|_{H^{1-\alpha}}$. In particular, the estimate is true for $u_l \in V_l$, where the interpolation operator I_l vanishes. □

Theorem 35 (Two-Grid Convergence). *Assume that the shift theorem (37) is valid. Then the norm of the two grid iteration*

$$M_{l,2g} = S_l^m(I - P_{l-1}^A)$$

is bounded by

$$\|M_{l,2g}\|_A \leq cm^{-\alpha}.$$

Proof. Combine smoothing property (39) and approximation property (38):

$$\|S_l^m(I - P_{l-1}^A)\|_{H^1 \to H^1} \leq \|S_l^m\|_{H^{1-\alpha} \to H^1} \|(I - P_{l-1}^A)\|_{H^1 \to H^{1-\alpha}} \leq cm^{-\alpha}$$

□

6.4 W-cycle analysis

Define the W-cycle multigrid iteration $\hat{u}_l = Mg_l(u_l, f_l)$ by $\hat{u}_0 = A_0^{-1} f_0$, and, for $l = 1, \dots L$,

$$
\begin{aligned}
u_l^0 &= u_l \\
u_l^k &= u_l^{k-1} + D_l^{-1}(f_l - A_l u_l^{k-1} \qquad k = 1, \dots m \qquad (presmoothing) \\
w_{l-1}^0 &= 0 \\
d_{l-1} &= E_l^T (f_l - A_l u_l^k) \\
w_{l-1}^k &= w_{l-1}^{k-1} + Mg_{l-1}(w_{l-1}, d_{l-1}), \dots k = 1, 2 \qquad (2 coarse grid correction steps) \\
u_l^{m+1} &= u_l^m + E_l w_{l-1}^2 \\
u_l^k &= u_l^{k-1} + D_l^{-1}(f_l - A_l u_l^{k-1} \qquad k = m+2, \dots k = 2m+1 \qquad (postsmoothing)
\end{aligned}
$$

Then, the iteration operator M fulfills the recursive definition

$$M_l = S_l^m (I - E_l (I - M_{l-1}^2) A_{l-1}^{-1} E_l^T A_l) S_l^m.$$

Theorem 36 (W-cycle analysis). *Assume that the two-grid iteration matrix is bounded in some norm $\|.\|_{V_l}$,*

$$\|M_{l,2g}\|_{V_l} \leq C \leq ?$$

and assume that the smoother is non-expansive in the same norm, i.e.,

$$\|S_l\|_{V_l} \leq 1.$$

Then, the norm of the W - cylce multigrid iteration is bounded by

$$\|M_l\| \leq ?$$