Magnetic Force Formulae for Magnets at Small Distances

Nikola Popović, Dirk Praetorius, and Anja Schlömerkemper

1 Boston University, Department of Mathematics and Statistics and Center for BioDynamics, 111 Cumington Street, Boston, MA 02215, U.S.A.
2 Vienna University of Technology, Institute for Analysis and Scientific Computing, Wiedner Hauptstraße 8-10, A-1040 Vienna, Austria
3 Max Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, D-04103 Leipzig, Germany

In [7], the magnetic force on subregions of rigid magnetized bodies was studied as a discrete-to-continuum limit. The derived force formula includes a new term, which depends on the underlying crystalline lattice structure \( \mathcal{L} \). It originates from contributions of magnetic dipole-dipole interactions of dipole moments close to the boundary of the considered subregion.

Further studies of this new term have led to the question of how the magnetic force between two idealized magnets, which are a distance \( \varepsilon > 0 \) apart, depends on \( \varepsilon \) as \( \varepsilon \to 0 \). In this article, analytical aspects of this question are discussed, cf. [5], where also numerical experiments are performed.

1 Introduction

Ferromagnetic shape memory alloys have some potential as new micro-devices, cf. e.g. [1]. In order to construct such devices, a better fundamental understanding of the dynamics of moving interfaces is of interest, cf. e.g. [3] for the study of a micro-scale cantilever. In this context, the question arises which mathematical formula describes the force between two parts of a magneto-elastic material best. There is a long list of related literature on this, cf. the references in [7].

To get started, we assume the magnetized material to be rigid. Several formulae are known for the magnetic force that is exerted by one subbody on another one [6, 7]. While one can bring the formulae in a form such that the volume force densities are the same, the surface force densities are different. Therefore there is some need for experiments that clarify which formula is the most appropriate. Unfortunately, it is not possible or at least not obvious how to measure magnetic forces in the interior of a magnetic body. To circumvent this, we suggest to consider the force between two magnetic bodies – instead of looking at two subregions of one magnetic body. In particular, we discuss the force between two polygonal bodies \( A \) and \( B \) whose boundaries have at least a set of positive surface measure in common. Even in this case one finds several different force formulae.

The experimental idea which drove the analytical and numerical studies in [5] is the following: Take two cuboidal permanent magnets \( A, B \) that are a distance \( \varepsilon > 0 \) apart and measure the force \( \mathbf{F}^{(\text{sep}, \varepsilon)} \) as \( \varepsilon \) gets smaller. Then compare the experimental results as \( \varepsilon \to 0 \) with the different mathematical formulae. To prepare and motivate such experiments, we derived the corresponding magnetic force formulae under these geometrical assumptions and performed numerical experiments.

In this article we discuss some analytical results from [5], to which we refer for details. For brevity we focus on polygonal three-dimensional domains \( A \) and \( B \) with finitely many edges such that \( A \cap B = \emptyset \) and the surface measure of \( \partial A \cap \partial B \) is positive. Let \( \mathbf{m}_A : A \to \mathbb{R}^3 \) denote the magnetization corresponding to \( A \) which is a given Lipschitz-continuous vector field and trivially extended to the entire space \( \mathbb{R}^3 \), i.e., \( \mathbf{m}_A = \mathbf{m}_A \chi_A \). By \( \mathbf{H}_A \) we denote the magnetic field which is generated by the magnetization in \( A \) and which is obtained from the magnetostatic Maxwell equations \( \text{curl} \mathbf{H}_A = 0 \) and \( \text{div} (\mathbf{H}_A + \mathbf{m}_A) = 0 \) in some suitable physical units. We adopt the same notation for \( B \) and denote with \( \mathbf{H}_{A \cup B} \) the magnetic field generated by \( \mathbf{m} := \mathbf{m}_A + \mathbf{m}_B \).

2 Formula derived from a classical formula for magnets not being in contact

Following the above mentioned experimental idea, we firstly move the body \( B \) a distance \( \varepsilon \) apart; for definiteness we move \( B \) along the \((1, 0, 0)\)-axis and set \( B_\varepsilon = \{ x + \varepsilon (1, 0, 0) \mid x \in B \} \). Then we apply the classical, well-accepted magnetic force formula for separated bodies (cf. e.g. [4])

\[
\mathbf{F}^{(\text{sep}, \varepsilon)} = \int_A (\mathbf{m}_A \cdot \nabla) \mathbf{H}_{B_\varepsilon} \, dV.
\]

Several numerical experiments are performed in [5]. There, we also prove that the limit \( \mathbf{F}^{(\text{sep}, 0)} \) for \( \varepsilon \to 0 \) exists and all forces \( \mathbf{F}^{(\text{sep}, \varepsilon)} \) for \( \varepsilon \geq 0 \) can be computed by closed formulae.

* Corresponding author: e-mail: schloem@mis.mpg.de, Phone: +49 341 9959 547, Fax: +49 341 9959 633

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3 Formula derived from a discrete setting of magnetic dipoles

In this section we consider two magnets $A$ and $B$ being in contact, i.e., the surface measure of $\partial A \cap \partial B$ is positive. We focus on a force formula which is obtained from a discrete setting of magnetic dipole moments, cf. [7]. Here we present a version of the theorem generalized to polygonal domains which are in contact but not necessarily nested. In [5] we consider a further estimate of the force terms is based on quite some idealizations, it seems to be worth to perform similar real-life experiments.

Let $L$ be a Bravais lattice describing the crystalline structure, e.g. $L = \mathbb{Z}^3$. For each $x \in \frac{1}{2}L$, $\ell \in \mathbb{N}$ we introduce a magnetic dipole moment $m^{(\ell)}(x) = \frac{1}{\ell^3} m(x)$ and denote the $i$-th component by $m_i^{(\ell)}(x)$. The magnetic force between all dipole moments in $A \cap \frac{1}{2}L$ and those in $B \cap \frac{1}{2}L$ is given by superposition of all dipole-dipole interactions [7, 5]

$$F^{(\ell)} = \frac{1}{4\pi} \sum_{i,j=1}^{3} \sum_{x \in A \cap \frac{1}{2}L} m_i^{(\ell)}(x) \sum_{y \in B \cap \frac{1}{2}L} \nabla \partial_i \partial_j (x - y)^{-1} m_j^{(\ell)}(y).$$

**Theorem 3.1 ([5])** Under the above assumptions on $A$, $B$, $m_A$ and $m_B$ the limit $\lim_{\ell \to \infty} F^{(\ell)} =: F^{(\text{lim})}$ exists and

$$F_{k}^{(\text{lim})} = \int_A (m_A \cdot \nabla)(H_{AUB})_k dV + \frac{1}{2} \int_{\partial A} ((m_A - m_B) \cdot n_A)(m_A \cdot n_A)(n_A)_k d\Gamma$$

$$+ \frac{1}{2} \sum_{i,j,p = 1}^3 S_{ijkp} \int_{\partial A \cap \partial B} (m_A)_i (m_B)_j (n_A)_p d\Gamma,$$

where $n_A$ denotes the outer normal to $\partial A$ and

$$S_{ijkp} := -\frac{1}{4\pi} \lim_{\delta \to 0} \lim_{\ell \to \infty} \int_{z \in B_{\ell} \cap \frac{1}{2}L \setminus \{0\}} (\partial_k \partial_i (\varphi^{(\delta)}(z)|z|^{-1})) (z) \frac{1}{z \delta^3},$$

for an arbitrary smooth function $\varphi^{(\delta)} : \mathbb{R}^3 \to [0, 1]$ with $\varphi(z) = 1$ if $|z| < \frac{\delta}{2}$ and $\varphi(z) = 0$ if $|z| > \delta$.

Theorem 3.1 states that the limit force $F^{(\text{lim})}$ is a good formula to describe the magnetic force between two magnets being in contact.

**Discussion and further development**

Our analytical and numerical studies [5] are driven by two questions: (i) Which is the best force formula of those, which are derived within the continuum theory? That is, we compare $F^{(\text{Brown})}$ with $F^{(\text{sep.0})} =$ $F^{(\text{long})}$. (ii) Is the additional surface force contribution $F^{(\text{short})}$ of physical relevance? That is, we ask whether the discrete-to-continuum limit $F^{(\text{lim})}$ is a good formula to describe the magnetic force between two magnets being in contact.

In the numerical experiments in [5] we consider two idealized magnets of cuboidal shape with constant magnetization at distance $\varepsilon > 0$ and $\varepsilon = 0$, respectively. We give here a first basic observation: To be specific, let $A$ and $B$ be two unit cubes which touch at one face with normal $(1, 0, 0)$. Moreover, let $m_A = (1, 0, 0)$ and $m_B = (1, 0, 0)$. Then $F_{1}^{(\text{long})} \approx 5.5$. According to (2), we need to fix some underlying lattice structure to calculate the short range force term. We choose the cubic lattice $L = \mathbb{Z}^3$ and obtain $F_{1}^{(\text{short})} = \frac{1}{2} S_{1111} |\partial A \cap \partial B| \approx 0.67$. Hence $(\lim_{\ell \to \infty} F_{1}^{(\text{lim})}) - F_{1}^{(\text{long})} / F_{1}^{(\text{long})} = F_{1}^{(\text{short})} / F_{1}^{(\text{long})} \approx 12.2\%$. This indicates that $F^{(\text{short})}$ is of some relevance for the total force that one magnet exerts on the other. Though this estimate of the force terms is based on quite some idealizations, it seems to be worth to perform similar real-life experiments.

In Theorem 3.1 the domains $A$ and $B$ are assumed to be in contact. The above observation together with the numerical experiments for $\varepsilon > 0$ in [5] raise the following question: How does the short range contribution $F^{(\text{short})}$ changes if the bodies $A$ and $B$ are not in contact but a small distance $\varepsilon > 0$ apart? However, this requires new analytical techniques for the passage from the discrete setting to the continuum.

**References**