Triple Variational Principles for Operator Functions

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Operator functions

Let $T$ be an operator function in a Hilbert space $\mathcal{H}$ defined on $\Omega \subset \mathbb{C}$:

$$T(\lambda)$$ is a closed operator in $\mathcal{H}$ for every $\lambda \in \Omega$.

For example, $T(\lambda) = \lambda^2 A + \lambda B + C$ with bounded operators $A, B, C$, or $T(\lambda) = A - \lambda I$ with a closed operator $A$. 
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Eigenvalues.

$\lambda_0 \in \Omega$ is called eigenvalue of $T$ $\iff$ $\exists x_0 \in \text{dom}(T(\lambda_0)) \setminus \{0\}$:

$$T(\lambda_0)x_0 = 0$$

$\iff$ 0 is eigenvalue of $T(\lambda_0)$
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Spectrum.

$\lambda_0 \in \sigma(T)$ $\iff$ $0 \in \sigma(T(\lambda_0))$

$\lambda_0 \in \sigma_{\text{ess}}(T)$ $\iff$ $0 \in \sigma_{\text{ess}}(T(\lambda_0))$
Assumptions I

- $\Omega \subseteq \mathbb{C}$ domain; $\Delta \subseteq \Omega \cap \mathbb{R}$ interval
- $T(\lambda)$ is m-sectorial for each $\lambda \in \Omega$
- $T(\lambda)$ is self-adjoint for $\lambda \in \Delta$
- Let $t(\lambda)$ be the closed quadratic form: $t(\lambda)[x] = (T(\lambda)x, x)$; assume that $\mathcal{D} := \text{dom}(t(\lambda))$ is independent of $\lambda$
- for each $x \in \mathcal{D}$: $\lambda \mapsto t(\lambda)[x]$ is analytic on $\Omega$
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Eigenvalues and the function $t(\cdot)$

Let $\lambda_0$ be an eigenvalue of $T$ with eigenvector $x_0$. Then

$$t(\lambda_0)[x_0] = (T(\lambda_0)x_0, x_0) = 0,$$

i.e. the function $\lambda \mapsto t(\lambda)[x_0]$ has a zero at $\lambda_0$. 
‘Hyperbolic’ or ‘overdamped’ case

Let $\Delta = [\alpha, \beta)$ and assume that for each $x \in \mathcal{D} \setminus \{0\}$:

- $t(\alpha)[x] > 0$,
- $t(\cdot)[x]$ has exactly one zero in $(\alpha, \beta)$.

Denote this unique zero by $p(x)$.

The mapping $x \mapsto p(x)$ is called generalised Rayleigh functional.
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In the case when $T(\lambda) = A - \lambda I$ for a self-adjoint operator $A$ with quadratic form $a$ we have $t(\lambda)[x] = a[x] - \lambda \|x\|^2$ and hence

$$p(x) = \frac{a[x]}{\|x\|^2}.$$
Variational principle for eigenvalues

In situations as above (or more general situations) variational principles were proved by

Duffin 1955
Rogers 1964
Turner 1967
Hadeler 1968
Werner 1971
Barston 1974
Binding, Eschwé, H. Langer 2000
Eschwé, M.L. 2004
Voss 2015
Jacob, M.L., Trunk 2016

...
Theorem.
Consider the situation as above and assume that
\[ \sigma_{\text{ess}}(T) \cap [\alpha, \beta) = \emptyset, \quad \alpha \in \rho(T). \]
Then the spectrum of \( T \) in \( (\alpha, \beta) \) consists of eigenvalues:
\[ \lambda_1 \leq \lambda_2 \leq \cdots \]
and
\[ \lambda_n = \min_{L \subset D, \dim L = n} \max_{x \in L \setminus \{0\}} p(x) = \max_{L \subset \mathcal{H}, \dim L = n-1} \inf_{x \in D \setminus \{0\}, x \perp L} p(x). \]
Comparison of two operator functions

Let $T_1$ and $T_2$ be two operator functions as above and assume that $\mathcal{D}_1 \supseteq \mathcal{D}_2$ and

$$t_1(\lambda)[x] \leq t_2(\lambda)[x], \quad x \in \mathcal{D}_2, \; \lambda \in \Delta.$$
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Then

$$p_1(x) \leq p_2(x), \quad x \in \mathcal{D}_2,$$

and hence

$$\lambda_n^{(1)} = \min_{L \subseteq \mathcal{D}_1 \atop \dim L = n} \max_{x \in L \setminus \{0\}} p_1(x) \leq \lambda_n^{(2)}.$$
Dropping the assumption that $T$ is hyperbolic

We relax the conditions that $t(\alpha)[x] > 0$ and that $t(\cdot)[x]$ has exactly one zero.

Assume that for each $x \in \mathcal{D} \setminus \{0\}$ and $\lambda \in \Delta$:

$$t(\lambda)[x] = 0 \quad \Rightarrow \quad t'(\lambda)[x] < 0.$$
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We relax the conditions that $t(\alpha)[x] > 0$ and that $t(\cdot)[x]$ has exactly one zero.

Assume that for each $x \in \mathcal{D} \setminus \{0\}$ and $\lambda \in \Delta$:

$$ t(\lambda)[x] = 0 \implies t'(\lambda)[x] < 0. $$

A mapping $p : \mathcal{D} \setminus \{0\} \to \mathbb{R} \cup \{\pm\infty\}$ is called a generalised Rayleigh functional for $T$ if for all $x \in \mathcal{D} \setminus \{0\}$:

- $p(x) = \lambda_0$ if $t(\lambda_0)[x] = 0$,
- $p(x) < \alpha$ if $t(\lambda)[x] < 0$ on $\Delta$,
- $p(x) \geq \beta$ if $t(\lambda)[x] > 0$ on $\Delta$. 
**Triple variational principle**

A subspace $\mathcal{M} \subset \mathcal{D}$ is called $t(\lambda)$-non-negative if $t(\lambda)[x] \geq 0$ for all $x \in \mathcal{M}$; it is called maximal $t(\lambda)$-non-negative if it is maximal with this property. Denote by $\mathcal{M}_\alpha^+$ the set of all maximal $t(\alpha)$-non-negative subspaces of $\mathcal{D}$. 

-Theorem-

Assume that $\mathcal{M} \subset \mathcal{D}$, $t(\lambda)$, $T = (T_1, T_2)$ such that $\text{ess}(T) = \emptyset$ and $t(\lambda)$ are bounded.

Then the spectrum of $T$ in $(\lambda, \mu)$ consists of eigenvalues: 

$\lambda_1, \lambda_2, \ldots, \lambda_n$ and 

$\lambda = \sup_{\mathcal{M}^2} \text{dim} \mathcal{L}_{\mathcal{M}} = n$, 

$\lambda = \inf_{x \in \mathcal{M}^2} g(x)$.

(The inequality was proved in [Eschwe, H. Langer 2002] when $T$ are bounded.)
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**Theorem.** [M.L., Strauss 2016] Assume that

$$[\alpha, \beta) \cap \sigma_{\text{ess}}(T) = \emptyset, \quad \alpha \in \rho(T).$$

Then the spectrum of $T$ in $(\alpha, \beta)$ consists of eigenvalues: $\lambda_1 \leq \lambda_2 \leq \cdots$ and

$$\lambda_n = \sup_{\mathcal{M} \in \mathcal{M}_\alpha^+} \sup_{\text{dim } L = n-1} \inf_{x \in \mathcal{M}\setminus\{0\}} x \perp L \ p(x).$$

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**Theorem.** [M.L., Strauss 2016]

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$$\lambda_n = \sup_{\mathcal{M} \in \mathcal{M}_\alpha^+} \sup_{\dim L = n-1} \inf_{x \in \mathcal{M} \setminus \{0\}} \{p(x) : x \perp L\}.$$

(The inequality ‘$\geq$’ was proved in [Eschwé, H. Langer 2002] when $T(\lambda)$ are bounded.)
Virozub–Matsaev condition

We say that $T$ satisfies the condition (VM) if for every compact $I \subseteq \Delta$ there exist $\varepsilon, \delta > 0$:

$$|t(\lambda)[x]| \leq \varepsilon \quad \Rightarrow \quad t'(\lambda)[x] \leq -\delta$$

for every $\lambda \in I$ and every $x \in \mathcal{D}$ with $\|x\| = 1$. 
A perturbation of a linear function

Let $A$ be a self-adjoint operator that is bounded below with quadratic form $a$ and $\text{dom}(a) = \mathcal{D}$.

Let $T$ satisfy Assumptions I and (VM) and assume that

$$t(\lambda)[x] = a[x] - \lambda\|x\|^2 + t_1(\lambda)[x], \quad x \in \mathcal{D}, \ \lambda \in \Omega,$$

where $t_1(\lambda)$ is a quadratic form that satisfies

$$0 \leq t_1(\lambda)[x] \leq a(\lambda)\|x\|^2 + b(\lambda)a[x], \quad x \in \mathcal{D}, \ \lambda \in \Delta = (\alpha, \beta).$$

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\]
with $a(\lambda) \in \mathbb{R}$, $b(\lambda) \geq 0$.

Assume that $(\alpha, \beta) \cap \sigma_{\text{ess}}(A) = \emptyset$ and that there exists $\hat{\alpha} \in (\alpha, \beta)$ so that $\forall \varepsilon > 0 \ \exists \gamma \in (\hat{\alpha}, \hat{\alpha} + \varepsilon)$:
\[
a(\gamma) + \alpha(1 + b(\gamma)) < \gamma.
\]

Then $(\hat{\alpha}, \beta) \cap \sigma_{\text{ess}}(T) = \emptyset$. 

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Assume that $(\alpha, \beta) \cap \sigma_{\text{ess}}(A) = \emptyset$ and that there exists $\hat{\alpha} \in (\alpha, \beta)$ so that $\forall \varepsilon > 0 \; \exists \gamma \in (\hat{\alpha}, \hat{\alpha} + \varepsilon)$:

$$a(\gamma) + \alpha(1 + b(\gamma)) < \gamma.$$

Then $(\hat{\alpha}, \beta) \cap \sigma_{\text{ess}}(T) = \emptyset$.

If $\mu_1 \leq \mu_2 \leq \ldots$ are the eigenvalues of $A$ in $(\alpha, \beta)$ and $\lambda_1 \leq \lambda_2 \ldots$ the eigenvalue of $T$ in $(\hat{\alpha}, \beta)$, then $\mu_n \leq \lambda_n$. 
An operator matrix

Let

$$A = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

where $A$ and $D$ are self-adjoint; $A$ bounded below; $D$ bounded;

$$\|B^* x\|^2 \leq a_0 \|x\|^2 + b_0 a[x], \quad x \in \text{dom}(a),$$

with $a_0 \in \mathbb{R}$, $b_0 \geq 0$. 
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with \( a_0 \in \mathbb{R}, \ b_0 \geq 0. \)

**Corollary.** Let \( \max \sigma(D) = d_+ < \alpha < \beta \) such that \( (\alpha, \beta) \cap \sigma_{\text{ess}}(A) = \emptyset. \) Set

\[ \hat{\alpha} = \frac{\alpha + d_+}{2} + \sqrt{\left(\frac{\alpha - d_+}{2}\right)^2 + a_0 + b_0\alpha}. \]

Then \( (\hat{\alpha}, \beta) \cap \sigma_{\text{ess}}(A) = \emptyset \) provided that \( \hat{\alpha} < \beta. \)

If \( \mu_1 \leq \mu_2 \leq \ldots \) are the eigenvalues of \( A \) in \( (\alpha, \beta) \)
and \( \lambda_1 \leq \lambda_2 \ldots \) the eigenvalue of \( A \) in \( (\hat{\alpha}, \beta) \), then \( \mu_n \leq \lambda_n. \)

Proof uses the Schur complement \( T(\lambda) = A - \lambda - B(D - \lambda)^{-1}B^*. \)