Error Estimation for MSFEM for elliptic problems in 2D

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Summary. We follow multi-scale techniques for layered medium as described in [1]. The goal is to find error estimates for the homogenized solution using modifications of flux reconstruction techniques described in [2]. Since these estimators require integration over the layered domain, efficient integration techniques will be demonstrated.

1 Problem Setting

Consider a domain $\Omega \in \mathbb{R}^2$ which is composed of an outer domain Ω_0 and a layered inner domain Ω_m as demonstrated in Fig. 1. In our examples we take Ω_0 to be the surrounding air domain and Ω_m to consist of parallel rectangular iron layers of width d_1 separated by air gaps of width d_2. The unit cell width $d := d_1 + d_2$ is defined as the width of one iron layer including half of each surrounding gap. In the studied applications, usually $d_1 = 0.9d$.

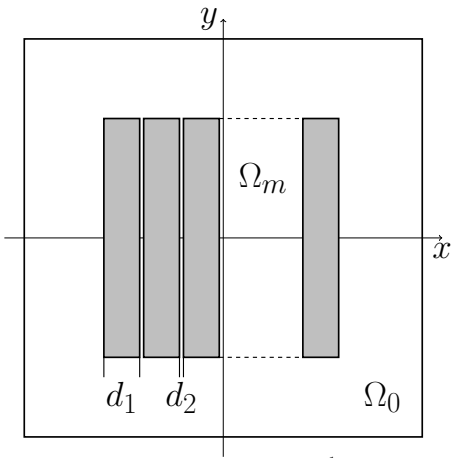

Fig. 1. The layered domain Ω

Two simplifications of Maxwell's equations are taken into consideration. Equation (1) shows the simplest setting where the real valued function u corresponds to the electric scalar potential and λ to the electric permittivity. In the numerical examples $\lambda = 1000$ in iron and $\lambda = 1$ in air. This setting will mainly be used to fix ideas and to develop the main tools which will later be modified to fit into the extended, complex valued setting (2).

$$\nabla \cdot (\lambda \nabla) u = f \tag{1}$$

$$\nabla \cdot (\rho \nabla u) + \mathrm{i}\omega\mu u = f \tag{2}$$

In the second equation ρ corresponds to the electric resistivity.

In the following we will outline the main ideas for the setting (1), since the details for (2) become rather technical.

1.1 Multi-Scale Ansatz

Studying the behaviour of the solution of (1), one observes that in Ω_m u can be split into a "mean function" and a periodic perturbation. These observations suggest the ansatz

$$u = u_0 + \phi u_1, \tag{3}$$

where $u_0 \in \mathrm{H}^1(\Omega)$, $u_1 \in \mathrm{H}^1(\Omega_m)$ and ϕ chosen as a linear zigzag function with a period of d. The main idea is to calculate u_0 and u_1 on a coarser mesh which does not incorporate each layer. Equation (1) becomes in weak form

$$\int_\Omega \bar{\lambda} \nabla u_0 \nabla v_0 + \overline{\lambda \phi_x} (u_{0,x} v_1 + v_{0,x} u_1) + \overline{\lambda \phi_x^2} u_1 v_1 + \overline{\lambda \phi^2} \nabla u_1 \nabla v_1 \, d\Omega = 0. \tag{4}$$

where an index x means the partial derivative with respect to the x coordinate and the bars indicate arithmetic means over one unit cell width.

2 Error Estimation

The base for a posteriori error estimation is the theorem of Prager and Synge [2].

Theorem 1. *Let u be the solution of* (1)*, $\sigma \in \mathrm{H}(\mathrm{div})$, $\sigma \cdot n = 0$ on Γ_N and $v \in \mathrm{H}^1$ with $v = 0$ on Γ_D, then*

$$\|\nabla u - \nabla v\|^2 + \|\nabla u - \sigma\|^2 = \|\nabla v - \sigma\|^2. \tag{5}$$

In application v is taken as the FEM-solution and σ as a cheaply calculated approximate flux, so the left hand sides becomes the energy error plus a small

positive perturbation while the right hand side can be computed directly.

Efficient construction of such a σ is described in [2]. The idea is to use this technique to first reconstruct a "mean flux" depending only on u_0 and then adding further correctors of curl-type which incorporate the oscillations without changing the divergence. According to classical homogenization results, as found for example in [4], the natural mean flux has the form

$$\left(\overline{\lambda}^h \frac{\partial u_0}{\partial x}, \overline{\lambda} \frac{\partial u_0}{\partial y}\right)^{\mathrm{T}} \tag{6}$$

with the arithmetic mean $\overline{\lambda}$ and the harmonic mean $\overline{\lambda}^h$ of λ over one unit cell. Defining σ_0 as the reconstructed mean flux, we set $\sigma := \sigma_0 + \mathrm{curl}(\phi w)$ with an unknown function w so that the energy norm $\|\sigma\|^2_{\lambda^{-1}}$ becomes optimal. A good candidate for w can be found analytically without requiring further numerical calculations.

3 Highly Oscillatory Integrals

Calculating the estimation given by (5) requires integration over products of highly oscillating functions with functions defined on the course mesh. Such integrals have been extensively analyzed for example in [3]. However, the methods presented in the literature require the oscillator to be smooth. Since ϕ is only continuous, they cannot be applied directly.

3.1 Method

Let φ be a highly oscillating, not necessarily differentiable function and f be smooth. We propose the asymptotic expansion

$$\int_a^b \varphi(x) f(x)\, dx = \sum_{n=0}^{\infty} \int_a^b \overline{\varphi_n} f^{(n)}(x)\, dx. \tag{7}$$

The constants $\overline{\varphi_n}$ are calculated a priori such that (7) is exact for polynomials up to order n. Note that only the first integral in the expansion has to be calculated numerically. The others are given by evaluating $f^{(n-1)}$ at a and b.

This one dimensional setup can be applied to the two dimensional case via

$$\begin{aligned}\int_\Omega \varphi(x) f(x,y)\, d\Omega &= \int_a^b \varphi(x) \int_{c(x)}^{d(x)} f(x,y)\, dy\, dx \\ &=: \int_a^b \varphi(x) \tilde{f}(x)\, dx \end{aligned} \tag{8}$$

where $c(x)$ and $d(x)$ are parametrizations of the boundaries of the integration domain. Using additional information about the domain (i.e. that it is decomposed into triangles) it is possible to significantly reduce the number of evaluations of f.

While in the FEM setting $\tilde{f}$ is a polynomial and therefore (7) could be used to compute the exact integral, $\tilde{f}$ is of order $2n+1$ with n being the order of f in both coordinates, which would lead to an impractical number of evaluations of derivatives as we do not assume to have direct control over the shape functions. However, Fig. 2 shows that a few terms in the expansion (7) are enough to obtain satisfying results even for greater cell widths d, with each expansion term netting an additional order in d. Here $f = x^3 y$, $\varphi = \lambda(\phi^2 + \phi) + 1$ with ϕ and the integration domain Ω given in section 1.

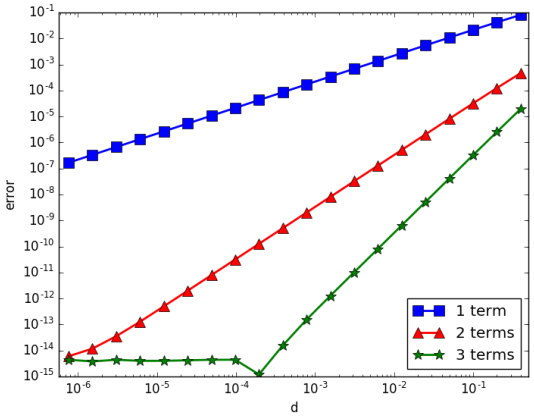

Fig. 2. Absolute error of $\int \varphi f$ for decreasing unit cell width

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